Bielefeld, July 17

Talk: On the tame covering of the Drinfeld upper halfspace

§1 Statement of the main result

Two important RZ-space which we know explicitly:

1) LT-space

2) Drinfeld space.

__Let us define them__: Let $d \geq 2$, let $F/\mathbb{Q}_p$ finite extension, Define functors on $\mathrm{Nilp}_{O_{\breve{F}}}$:

LT: $S \mapsto \{$ iso-classes of $(X, \varrho) \mid X$ strict formal O_F-module of rel. height d, dim. 1; $\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\,\bar{k}} \bar{S}$ O_F-quasi-isog. of height 0 $\}$.

Drinfeld: $S \mapsto \{$ iso-cl. of $(X, \iota, \varrho) \mid X$ strict formal O_F-module of rel. height d^2, dim. d; $\iota : O_D \to \mathrm{End}(X)$ special action; $\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\,\bar{k}} \bar{S}$ O_D-quasi-isog. of height 0 $\}$.

Here $\mathrm{inv}(D) = 1/d$, __special__: let $F' \subset D$ unram. ext of degree d, cont. in $\breve{F}$. Get eigenspace decomp

$$\mathrm{Lie}\, X = \bigoplus_{i \in \mathbb{Z}/d} \mathrm{Lie}_i\, X$$

__Demand__: $\mathrm{rk}\, \mathrm{Lie}_i\, X = 1, \ \forall i$.

Known: $\mathcal{M}_{LT} \simeq \mathrm{Spf}\, O_{\breve{F}} [\![t_1, \dots, t_{d-1}]\!]$ → regular

$\mathcal{M}_{Dr} \simeq \hat{\Omega}^d_F \hat{\otimes}_{O_F} O_{\breve{F}}$.

System of coverings: Let $X / \mathcal{M}$ universal object, with generic fiber $X_\eta / \mathcal{M}_\eta = \mathcal{M}^{rig}$. For $K \subset G(F)$ get covering via

$$\alpha : X_\eta [\pi^n] \simeq V_{\mathbb{Z}} / \pi^n V_{\mathbb{Z}} \quad \text{mod } K. \quad , n >> 0.$$

Here

$$G = \begin{cases} GL_d \\ \underline{D}^\times \end{cases}$$

$$V_{\mathbb{Z}} \subset V = \begin{cases} O_F^d \subset F^d \\ O_D \subset D. \end{cases}$$

Get finite étale covering $\mathcal{M}_{K,\eta} \longrightarrow \mathcal{M}_\eta$. Via normalization, get finite flat covering $\mathcal{M}_K \longrightarrow \mathcal{M}$.

Problems: a) Structure of $\mathcal{M}_K$

b) Find moduli problem solved by $\mathcal{M}_K$.

Answer in LT case (Drinfeld):

Theorem: Let $K = K_n$. Then

a) $\mathcal{M}_{LT, K_n}$ is $\mathrm{Spf}\, A_{K_n}$, where A_{K_n} regular complete local ring

b) $\mathcal{M}_{LT, K_n}$ represents moduli pb. over $\mathcal{M}_{LT}$:

$$S \longmapsto \{ \alpha : O_F^d / \pi^n O_F^d \longrightarrow X[\pi^n] \text{ Drinfeld level structure} \}.$$

Last definition is quite subtle: $\mathbb{X}[\pi^n]$ has only zero section. Uses in an essential way that $\dim X = 1$.

<u>Variant:</u> Similar statements hold if K_n replaced by Iwahori in K_n, resp. pro-unip. radical, resp. mirahoric variants (Z: references ?).

Corresponding facts unknown in Drinfeld case — a) is more complicated (as we will see).

<u>Theorem (Vanhaecke):</u> Let $K = K_1 = \{ g \in \mathcal{O}_D^\times \mid g \equiv 1 \bmod \Pi \}$ Einseinheiten.

(corresp. cover $\mathcal{M}_{Dr,1}$ is maximal tame cover of $\mathcal{M}_{Dr}$ in generic fiber).

a) $\mathcal{M}_{Dr,1}$ is normal and CM, but not regular. The morphism

$$\mathcal{M}_{Dr,1} \otimes \bar{k} \longrightarrow \mathcal{M}_{Dr} \otimes \bar{k}$$

is a universal homeo.

b) $\mathcal{M}_{Dr,1}$ represents moduli pb. over $\mathcal{M}_{Dr}$:

$$S \longmapsto \{ x \in X[\Pi] \quad \text{a generator} \}$$

§2 <u>Generators and non-nul sections of Raynaud group schemes</u>

<u>Definition:</u> Let $S/W(\bar{k})$. Let $\mathbb{F}$ be a finite field with p^r elements. An $\mathbb{F}$-<u>Raynaud group scheme</u> over S = a finite flat $\mathbb{F}$-group scheme of f. p. over S such that in eigenspace decomp. of augmentation ideal

$$\mathcal{J} = \bigoplus_{\chi \in (\mathbb{F}^\times)^\wedge} \mathcal{J}_\chi$$

have $\operatorname{rk} \mathcal{J}_\chi = 1, \forall \chi$.

<u>Remark</u>: If $\mathbb{F}=\mathbb{F}_p$, ~~the condition is automatic and~~ only condition is $|G|=p$ (Oort–Tate)

$\mathbb{F}$-Raynaud's form a stack over $(\mathrm{Sch}/W(\bar{k}))$ that one can describe explicitly. Let $w \in p \cdot W(\bar{k})^\times$ be <u>Raynaud's constant</u>.

- $$\mathcal{M}_{\mathbb{F}\text{-Ray}} = \left[\operatorname{Spec} \frac{W(\bar{k})[X_1,\dots,X_r,Y_1,\dots,Y_r]}{\langle X_i Y_i - w\rangle_{i=1,\dots,r}} \Big/ \mathbb{G}_m^r \right]$$

where $\mathbb{G}_m^r(B)$ acts on $\operatorname{Spec} W[\underline{X},\underline{Y}](B)$, via

$$\underline{\lambda}\cdot(\underline{X},\underline{Y}) = \left(\lambda_i^p X_i \lambda_{i+1}^{-1},\ \lambda_i^{-p} Y_i \lambda_{i+1}\right)_{i\in\mathbb{Z}/r}.$$

- $$\mathcal{G}/\mathcal{M}_{\mathbb{F}\text{-Ray}} = \left[\operatorname{Spec} \frac{\mathcal{O}_{\mathcal{M}_{\mathbb{F}\text{-Ray}}}[Z_1,\dots,Z_r]}{\langle Z_i^p - X_i Z_{i+1}\rangle_{i\in\mathbb{Z}/r}} \Big/ \mathbb{G}_m^r \right]$$

$$\underline{\lambda}\cdot\underline{Z} = (\lambda_i Z_i)_i$$

Let $\mathcal{G}^\times \subset \mathcal{G}$ be closed subscheme defined by ideal

$$J_{\mathcal{G}} = \langle (Z_1\cdots Z_r)^{p-1} - X_1\cdots X_r\rangle.$$

<u>Definition</u>: Let G be $\mathbb{F}$-Raynaud over S, corresp. to

$$\varphi_G : S \longrightarrow \mathcal{M}_{\mathbb{F}\text{-Ray}},$$

so that $G = \varphi_G^*(\mathcal{G})$. Let $G^\times = \varphi_G^*(\mathcal{G}^\times)$. A section $x \in G(S)$ is called a <u>generator</u> if $x \in G^\times(S)$.

Proposition (Pappas): Let G be $\mathbb{F}$-Raynaud over S and let $x \in G(S)$.

Equivalent:

(i) x is a generator

(ii) x is non-nul

(iii) $\{\lambda x \mid \lambda \in \mathbb{F}\}$ is a full set of sections of G.

Explanation: a) If G étale, all these notions are the obvious ones.

b) The notion (ii) is due to Kottwitz / Wake and applies to any finite flat gp scheme of f.p. over any scheme S. If $S = \operatorname{Spec} R$ and $G = \operatorname{Spec} B$, consider the ideal J_G of $G^\vee$-invariant distributions,

$$J_G = \{ f \in B \mid f' f = f'(e) f, \ \forall f' \in B \} = \operatorname{Ann}_B(\mathcal{J}).$$

This is loc. free $\mathcal{O}_S$-module of rank 1. A section $x \in G(S)$ is called non-nul if $x \in V(J_G)$.

c) The notion (iii) is due to Katz / Mazur and applies to any scheme Z finite, flat of f.p. over any scheme S. Let $x_1, \dots, x_n \in Z(S)$. Then this is a full set of sections if $\forall f \in \mathcal{O}_Z$,

$$\operatorname{char}_f(T) = \prod_{i=1}^{n} (T - f(x_i))$$

Here if $S = \operatorname{Spec} R$ and $Z = \operatorname{Spec} B$, get for $f \in B$ endo $f : B \to B$ of projective R-module.

§3 Application to the Drinfeld moduli scheme

Lemma: Let (X, ι) be s.f. O_D-module over $S \in \mathrm{Nilp}_{O_{\breve{F}}}$.

(i) $X[\Pi]$ is an $\mathbb{F}$-Raynaud over S. Here $\mathbb{F} = O_D / \Pi O_D \cong \mathbb{F}_{q^d}$.

(ii) $X[\Pi]$ is a purely infinitesimal group scheme

Now let

$$\mathcal{M}_{Dr,1} = \{ (X, \iota, \varrho, x) \}, \quad \text{i.e., cartesian diagram,}$$

$$\begin{tikzcd} \mathcal{M}_{Dr,1} \arrow[r] \arrow[d] & \mathcal{L}^{\times} \arrow[d] \\ \mathcal{M}_{Dr} \arrow[r] & \mathcal{M}_{\mathbb{F}\text{-Ray}} \end{tikzcd}$$

Then:

- $\mathcal{M}_{Dr,1}$ is CM + normal: use regularity of $\mathcal{M}_{Dr}$ + $\mathcal{L}^{\times}$ + Serre crit.
- $\mathcal{M}_{Dr,1} \otimes \bar{k} \longrightarrow \mathcal{M}_{Dr} \otimes \bar{k}$ universal homeo: use (ii) in previous lemma.
- $(\mathcal{M}_{Dr,1})_\eta = \mathcal{M}_{K_1,\eta}$: use Explanation, a).
- $\mathcal{M}_{Dr,1}$ not regular: use Abhyankar's lemma + purity : ?

Remark: Let $(X, \iota) / S \in \mathrm{Nilp}_{O_{\breve{F}}}$ be s.f. O_D-module. For $i \in \mathbb{Z}/d$, let

$$S_i(X) = \{ \text{locus where } \Pi : \mathrm{Lie}_i X \to \mathrm{Lie}_{i+1} X \text{ is zero} \}$$

i-critical locus, $S_i(X) \subset \bar{S}$.

Let $G / S / W(\bar{k})$ be $\mathbb{F}$-Raynaud. For $j \in \mathbb{Z}/r$, let

$$S_j(G) = \{ \text{locus where } \bar{c}_j : \mathcal{J}_{\chi_{j+1}} \longrightarrow \mathcal{J}_{\chi_j}^{\otimes p} \text{ is zero} \}$$

Here χ_j = j-th fundam. character, and $\bar{c}_j$ induced by p-th power of co-mult. map.

j-<u>critical locus</u>, $S_j(G) \subset S \otimes_{W(\bar{k})} \bar{k}$.

<u>Proposition</u> (?): Let $(X, \iota)/S$. Then for $j \in \mathbb{Z}/r$

$$S_j(X[\pi]) = \begin{cases} \emptyset & \text{if } j \notin d\mathbb{Z}/r\mathbb{Z} \\ S_i(X) & \text{if } j = id \end{cases}.$$

Note that $r = d \cdot f$.