Maryland, Feb 16

p-adic uniformization of Shimura curves

<u>Complex uniformization</u>: Let X connected smooth projective alg. curve $/\mathbb{C}$ of genus ≥ 2.

<u>Poincaré</u>: ex. discrete co-compact subgp $\Gamma \subset PGL_2(\mathbb{R})$ s.t.

$$X^{an} \simeq \Omega_{\mathbb{R}} / \Gamma. \quad = X_\Gamma$$

Here

$$\Omega_{\mathbb{R}} = \mathfrak{h}^+ \subset \mathfrak{h}^- = (\mathbb{P}^1_{/\mathbb{R}})^{an} \setminus \mathbb{P}^1(\mathbb{R})$$

<u>Complicated history</u> ($\rightarrow$ Gray): Klein, Koebe, Hilbert

<u>Poincaré</u>: Systematic way of constructing such Γ (he used SO_3 – but comes to the same). Let

F = totally real field

w = fixed arithm. place

B = quaternion algebra $/F$ with $B \otimes_F F_{w'} = \begin{cases} M_2(\mathbb{R}) & w' = w \\ \mathbb{H} & w' \neq w. \end{cases}$

Get action of $B^\times$ on $\Omega_{\mathbb{R}}$ via

$$B^\times \longrightarrow (B \otimes_F F_w)^\times = GL_2(\mathbb{R}) \longrightarrow PGL_2(\mathbb{R})$$

Let $O_B \subset B$ order, then take $\Gamma =$ image of $O_B^\times$, discrete subgp.

If B quaternion division algebra, then Γ co-compact.

(automatic if $[F:\mathbb{Q}] > 1$). <u>Shimura curve</u>

<u>p-adic uniformization</u>: (Cherednik, 1976):

Let B/F as above. ~~Let~~ Fix p and

v = p-adic place of F such that $B \otimes_F F_v$ division alg.

Assume $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$ <u>unique maximal order</u>.

Then

$$\left(X_\Gamma^{alg} \otimes_{\mathbb{C}} \cancel{\mathbb{Q}_p} \overline{F}_v\right)^{an} \simeq \left(\Omega_{F_v} \otimes_{F_v} \overline{F}_v\right)/\overline{\Gamma} \,,$$

where

$$\overline{\Gamma} \subset PGL_2(F_v) \quad \text{discrete co-compact}$$

Here Ω_{F_v} p-adic analogue of $\Omega_{\mathbb{R}}$,

$$\Omega_{F_v} = \left(\mathbb{P}^1_{/F_v}\right)^{an} \setminus \mathbb{P}^1(F_v). \qquad \underline{\text{Drinfeld half plane}}.$$

~~<u>Interpretation</u>: LHS is algebraization of complex analytic var. (unique!)~~

after Drinfeld.

~~RHS rigid-analytic var (—)~~

Applications

Very mysterious, even today – Cherednik gave heuristic argument.
Varshavsky, RZ, Boutot-Zink,

<u>Drinfeld (1976)</u>: gave explanation + proof of Ch's theorem for $F = \mathbb{Q}$. Uses <u>adelic version</u>, i.e. finite union of X_Γ's:

~~Then~~ Let

$$G = B^\times \text{ as algebr. gp} / \mathbb{Q}.$$

Let

C open compact subgp of $G(\mathbb{A}_f)$.

$$X_C = G(\mathbb{Q}) \backslash \left[\Omega_{\mathbb{R}} \times G(\mathbb{A}_f)/C \right].$$

<u>Advantage of adelic point of view:</u>

- X_C represents a moduli pb.
- X_C "curve" defined over $\mathbb{Q}$.

<u>Moduli pb.</u>: X_C parametrizes triples (A, ι, α), where

A = abelian variety of dim 2

$\iota : O_B^\circ \longrightarrow \mathrm{End}(A)$ Here O_B° = max. order, fixed.

α level structure, i.e. $T(A) \simeq O_B \otimes \hat{\mathbb{Z}}$ mod C.

Now assume $B \otimes \mathbb{Q}_p$ div. algebra, $C = C^p \cdot C_p$ where C_p = unique max cpct. $G(\mathbb{Q}_p)$.

Then $\alpha \Leftrightarrow \alpha^p$. Hence get moduli scheme $\mathcal{M}_C$ over

$\mathrm{Spec}\, \mathbb{Z}_{(p)}$. To force flatness of $\mathcal{M}_C$ impose

(*) $\quad \mathrm{char}(\iota(x) \mid \mathrm{Lie}\, A) = \mathrm{chard}(x), \qquad \forall\, x \in O_B^\circ$.

<u>Theorem (Drinfeld)</u>: $\hat{\mathcal{M}}_C \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \simeq \underline{G}(\mathbb{Q}) \left[\hat{\Omega}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \times G(\mathbb{A}_f^p)/C^p \right]$

Here $G_- = B_-^\times$, where B_- is $\mathrm{roch}(B)$.

casting

Furthermore, $\hat{\Omega}_{\mathbb{Q}_p}$ <u>integral model of</u> $\Omega_{\mathbb{Q}_p}$ (Deligne, Drinfeld, Mumford) successive blow-up of $\mathbb{P}^1_{\mathbb{Z}_p}$.

local statement.

<u>Principle of Drinfeld's proof</u>: Let F be p-adic. Drinfeld shows that $\hat{\Omega}_F \hat{\otimes}_{O_F} O_{\breve{F}}$ represents a moduli pb. of p-divisi. gps of (X, ι) : here

X = p-div. group of $ht(X) = 4 \cdot |F : \mathbb{Q}_p|$ + strict O_F-action

$\iota : O_B \longrightarrow End(X)$ s.t. $(*)$.

Proof extremely difficult! Works also if B replaced by div.-algebra D with $inv_F(D) = 1/n$.

Leads to theory of general formal moduli spaces of p-divis. gps (RZ-spaces): I tried to "understand" these for the last 25 years.

What if $|F : \mathbb{Q}| > 1$? I don't know, X_C represents no known moduli pb. $\leadsto$ Boutot/Zink.

<u>The last part of talk</u>: work of KRZ on a variant.

use exc. iso $SL_2 \simeq SU_2$.

<u>New notation</u>:

K/F purely imag. quadratic extension.

V = 2-dim K-vs + hermitian form. ~~Assume~~

$$r : \mathrm{Hom}(K, \bar{\mathbb{Q}}) \longrightarrow \{0, 1, 2\} \quad \text{with } r_\varphi + r_{\bar\varphi} = 2 \text{ and}$$

such that

$$r_\varphi = \begin{cases} 1 & \varphi | ~~w~~ \; w \\ 0 \text{ or } 2 & \varphi | w' \neq w. \end{cases}$$

$\rightsquigarrow$ $E = E(r)$ and $F \hookrightarrow E$.

Demand that $\mathrm{sgn}(V_\varphi) = (r_\varphi, 2 - r_\varphi)$.

Let

$$G = GU(V) \quad (\text{multiplier in } \mathbb{Q}^\times) : \text{LAG over } \mathbb{Q}.$$

For $C \subset G(\mathbb{A}_f) \rightsquigarrow$

$$X_C = G(\mathbb{Q}) \backslash [\mathcal{H}_{\mathbb{R}} \times G(\mathbb{A}_f)/C].$$

Here use exceptional iso $SU_2 \simeq SL_2$ of LAG / $\mathbb{R}$.

Then X_C represents always a moduli problem : parametrizes tuples $(A, \iota, \lambda, \alpha)$ where

A abelian variety of dimension $2 \cdot |F : \mathbb{Q}|$

$\iota : O_K \longrightarrow \mathrm{End}(A)$ s.t.

$$\mathrm{char}(\iota(x) \mid \mathrm{Lie}\, A) = \prod_{\varphi} (T - \varphi(x))^{r_\varphi} \cdot \cancel{(T - \bar\varphi_{w'}(x))^{2-r_w}}.$$

λ a certain type of polarization, almost principal.

α level-C-structure.

<u>Impose</u> : $\quad \mathrm{inv}_{v'}(A, \iota, \lambda) = \mathrm{inv}_{v'}(V), \quad \forall v' < \infty$

Here LHS ~~can~~ is via $H_1(A, \mathbb{Q})$ + Riemann form.

Then X_C defined over E.

Now want to force p-adic uniformization. Fix

v over p, non-split in K and s.t. $\mathrm{inv}_v(V) = -1$.

Also fix $\nu : E \hookrightarrow \overline{\mathbb{Q}}$ inducing v. On moduli data impose

- λ_v principal if K_v/F_v ramified
- λ_v almost principal — unramified

$\Big\}$ + define inv at all places

Also fix $C = C^p . C_p$, where C_p is <u>maximal at v</u>.

<u>Theorem A:</u> ~~(i)~~ If $F_v/\mathbb{Q}_p$ unramified, the corresp. integral model $\mathcal{M}_C$

~~over $\mathcal{O}_{E_{(\nu)}}$ is flat and normal~~ satisfies

$$(\ast) \qquad \mathcal{M}_C \otimes_{\mathcal{O}_{E_{(\nu)}}} \mathcal{O}_{\breve{E}_\nu} \simeq G(\mathbb{Q}) \backslash \left(\hat{\Omega}_{F_v} \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\breve{E}_\nu} \right) \times G(\mathbb{A}_f)/C$$

If $F_v/\mathbb{Q}_p$ ramified, then need to impose Eisenstein conditions

on action of $\mathcal{O}_K$ on $\mathrm{Lie}\, A$ in $\mathcal{M}_C$.

KRZ $\Big\{$ <u>Theorem B:</u> (i) $\mathcal{M}_C$ flat and normal

(ii) $(\ast)$ holds in special fiber

S (iii) $(\ast)$ holds in generic fiber

(iv) (*) holds as stated

(Scholze, uses integral p-adic Hodge theory – on the server today): this is because (ii) uses crystalline coho. of abelian varieties (iii) uses p-adic étale coho. ——, so for (iv) need a relation between the two (B/M/S).