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# Talk: Accessible + weakly accessible period domains

Motivation: I asked Scholze: consider LSV and corresp. crystalline period maps

$$\left(\mathbb{M}(G,b,\{\mu\})_K\right)_K \xrightarrow{\phi_K} \check{\mathcal{F}}(G,\{\mu\}).$$

When is it surjective? (image described by Faltings/Hartl/Scholze/Weinstein)

Scholze: only in LT-case (surjectivity due to Gross/Hopkins).

Scholze sketched proof in e-mail; I tried to work out details — but many things used in this proof of Scholze are not written down properly.

Different motivation: In a construction of Scholze, he used that Gross-Hopkins period map

$$\left(\mathbb{M}(G,b,\{\mu\})_K\right) \xrightarrow{\phi_K} \mathbb{P}^{n-1}$$

is surjective. I asked: is this the only time this can happen?

Scholze's answer: yes — I tried to understand his answer, (do not claim to understand all of it!)

## §1 Filtered isocrystals

$F/\mathbb{Q}_p$. Fix $\bar{F}$, ~~$\bar{F}/\mathbb{Q}_p$~~ get $(\breve{F}, \sigma)$ (e.g. $F = \mathbb{Q}_p$) and $\mathbb{C}_p = \hat{\bar{F}}$

<u>Isocrystal</u> (relative to $F$) $\underset{\mathrm{Df}}{=}$ $\breve{F}$-vs. $\breve{V}$ + $\sigma$-lin. auto $\varphi : \breve{V} \to \breve{V}$

<u>Filtration of isocrystal</u> over $K/\breve{F}$ $\underset{\mathrm{Df}}{=}$ monotone function

$$\mathcal{F}^{\bullet} : \mathbb{Z} \to \{ \text{subvectorsp. of } \breve{V}_K \}$$

s.t. $\mathcal{F}^i = (0)$ for $i \gg 0$ and $\mathcal{F}^i = \breve{V}_K$ for $i \ll 0$

<u>Newton, resp. Hodge vectors:</u> $n = \dim \breve{V}$

- $(\breve{V}, \varphi) \longmapsto$ Newton slope vector $\nu \in (\mathbb{Q}^n)_{\geq}$,

satisfies integrality property : $\nu = (\nu_1^{(m_1)}, \ldots, \nu_r^{(m_r)})$

Then $m_i \cdot \nu_i \in \mathbb{Z}$.

- $(\breve{V}, \mathcal{F}^{\bullet}) \longmapsto$ Hodge slope vector $\mu \in (\mathbb{Z}^n)_{\geq}$ : $\mu = (\mu_1^{(m_1)}, \ldots, \mu_r^{(m_r)})$

then $gr_j(\mathcal{F}^{\bullet}) \neq 0$ iff $j = \mu_i$ some $i$

$\dim gr_{\mu_i}(\mathcal{F}^{\bullet}) = m_i$.

<u>Numerical invariants of</u> $(\breve{V}, \varphi, \mathcal{F}^{\bullet})$ : $rk = \dim \breve{V}$, $\deg = \sum(\mu) - \sum(\nu)$

$(\breve{V}, \varphi, \mathcal{F}^{\bullet})$ <u>semi-stable</u> $\underset{\mathrm{Df}}{=}$ $\forall\ (\breve{V}', \varphi) \subset (V, \varphi)$ $\mathrm{slope}(\breve{V}') \leq \mathrm{slope}(\breve{V})$

<u>weakly ad</u> $=$ ss + $\mathrm{slope}(\breve{V}) = 0$.

## § 2 Period domains

Fix reductive group $G/F$, $b \in G(\breve{F})$, $\{\mu\}$. (PD-triple)

$\rightsquigarrow$ $E = E(G,\{\mu\}) \subset \bar{F}$ + partial flag var. $\mathcal{F} = \mathcal{F}(G,\{\mu\}) / E$.

Set $\breve{\mathcal{F}} = \mathcal{F} \otimes_E E.\breve{F}$.

Definition: A point $x \in \breve{\mathcal{F}}(K)$ is called s-s if the isocrystal $(\breve{\mathfrak{g}}, \mathrm{Ad}\, b\, \sigma)$ with filtration $\mathcal{F}_x^{\bullet}$ on $\mathfrak{g}_K$ is semi-stable.

wa if ss + "total degree" $= 0$, i.e. $\breve{\mathcal{F}}^{wa} = \breve{\mathcal{F}}^{ss}$ or $\emptyset$.

Fact: $\mathcal{F}(G, b, \{\mu\})^{ss} \subset \breve{\mathcal{F}}(G,\{\mu\})$ is admiss. open rigid/adic subset.

Examples: LT-case: $G = GL_n$, $\{\mu\} = (1, 0^{(n-1)})$, resp. $(1^{(n-1)}, 0)$ and $b$

E = F

then $\mathcal{F}^{ss} = \breve{\mathcal{F}} = \mathbb{P}^{n-1}_{\breve{F}}$.

Drinfeld case: $G = D^{\times}_{\pm 1/n}$, $\{\mu\}$ as before, $\mathcal{F}^{ss} = \Omega^n \subset \mathbb{P}^{n-1}_{\breve{F}}$.

All this depends only on $b$ up to $\sigma$-conjugacy.

## §3 Weakly ~~admissible~~ accessible period domains

<u>Def.</u>: A PD-triple $(G, b, \{\mu\})$ is <u>weakly ~~adm~~ accessible</u> $\overset{=}{\text{Df}}$

$$\mathcal{F}(G, b, \{\mu\})^{wa} = \breve{\mathcal{F}}(G, \{\mu\}).$$

<u>Theorem</u>: $(G, b, \{\mu\})$ weakly accessible $\Leftrightarrow$ $[b] \in A(G, \{\mu\})$ and

$J_b$ inner form of $G$, anisotropic modulo center.

<u>Explanation:</u> Kottwitz theory of $B(G) = G(\breve{F})/$ mod $\sigma$-conjugacy.

<u>2 invariants</u>:

<u>Newton map</u>: $\nu : B(G) \longrightarrow \mathfrak{a}^+_{\mathbb{Q}}$ generalizes $\nu$ above

<u>Kottwitz map</u>: $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ (not visible for $GL_n$)

<u>Acceptable set</u> $A(G, \{\mu\}) = \{[b] \in B(G) \mid \nu_{[b]} \overset{\text{dominance order}}{\leq} \bar{\mu}\}$ finite set

<u>Neutral acceptable set</u> $B(G, \{\mu\}) = \{[b] \in A(G, \{\mu\}) \mid \kappa([b]) = \mu^\natural\}$.

$J_b$ $\sigma$-<u>centralizer of $b$</u>.

Further, $\kappa$ has natural section ( image $(\kappa) = B(G)_{basic}$ ) and

$b$ <u>basic</u> $\Longleftrightarrow$ $J_b$ inner form of $G$.

<u>Note</u>: $\mathcal{F}(G, b, \{\mu\})^{wa} \neq \emptyset \iff [b] \in A(G, \{\mu\})$. (Fontaine inequ.)

## §4 $B(G)$ after Fargues + Scholze

The FF-curve $X_F$ is a 1-dimensional, noeth., regular scheme over Spec $F$,

$$X_F = \mathrm{Proj}\left( \bigoplus_{d=0}^{\infty} (B^+_{cris})^{\phi = \pi^d} \right)$$

It comes with a point $\infty \in X_F(\mathbb{C}_p)$, corresp. to $B^+_{cris} \to \mathbb{C}_p$.

Theorem (Fargues): Let $G/F$. Have a functor of groupoid categories

$$G(\breve{F}) \longrightarrow (G\text{-bundles on } X_F)$$

$\left( \text{for } G = GL_n,\ b \mapsto \text{graded module } \bigoplus (B^+_{cris} \otimes \breve{F}^n)^{\phi \otimes b\sigma = \pi^d} \right).$

Induces

a) bijection $B(G) \to \{\text{iso-cl. of } G\text{-bundles on } X_F\}$

b) equiv. of categories $G(\breve{F})_{basic} \longrightarrow (\text{semi-stable } G\text{-bdles})$.

Here semi-stable = Mumford-ss (deg + rank of vb on $X_F$ are defined, extend to $G$-bdls via Ad [Atiyah-Bott]).

Remark: The interpret. of Fargues/Scholze is dual to Groth./Kottwitz, e.g. in families locus where basic is $\begin{cases} \text{open in F-S} \\ \text{closed in R-R.} \end{cases}$

# § 5 The admissible set

Fix $(G, b, \{\mu\})$.

Construction (Scholze): To any $C/\breve{E}_p$ and $x \in \check{\mathcal{F}}(G,\{\mu\})(C) \rightsquigarrow G$-bdle $\mathcal{E}_{b,x}$ on $X_F \otimes_F C$ : modification of $\mathcal{E}_b$ at $\infty_x$ along $x$.

Example: $G = GL_n$, $\mathcal{E}_b$ vb. of rank $n$. Let $\{\mu\} = (1^{(r)}, 0^{(n-r)})$ minuscule. Then

$$\mathcal{E}_b \overset{r}{\subset} \mathcal{E}_{b,x} \overset{n-r}{\subset} \mathcal{E}_b \otimes_{\mathcal{O}} \mathcal{O}(\infty_x).$$

The general definition uses $B^+_{dR}$ - Grassm. of Scholze.

Definition: A point $x \in \mathcal{F}(C)$ is admissible (wrt. $b$) $\underset{\mathrm{df}}{\iff}$ $\mathcal{E}_{b,x}$ is semi-stable.

Fact: The admissible points form open adic subset $\mathcal{F}(G,b,\{\mu\})^a$ of $\mathcal{F}(G,b,\{\mu\})^{wa}$, induces bijection on classical points.

Proposition: Bijection of last § induces a bijection

$$\{\text{iso-cl. } G\text{-bdls of the form } \mathcal{E}_{1,x} \mid x \in \mathcal{F}(C)\} \simeq B(G,\{\mu^{-1}\}).$$

Proof: a) Let $[b] \in$ image. Then $\kappa([b]) = (\mu^{-1})^{\natural}$ (direct calculation).

b) Let $b \in G(\breve{F})$. Then $\mathcal{E}_b \simeq \mathcal{E}_{1,x}$ some $x \iff \exists\, x^* \in \mathcal{F}(G,\{\mu^{-1}\})$ s.t. $\mathcal{E}_{b,x^*}$ semi-stable $\iff \mathcal{F}(G,b,\{\mu^{-1}\})^a \neq \emptyset \iff \mathcal{F}(G,b,\{\mu^{-1}\})^{wa} \neq \emptyset \iff [b] \in A(G,\{\mu^{-1}\})$

see § 3 //

<u>Corollary</u>: Let $b \in G(\breve{F})_{basic}$. Then bijection

$$\{\text{iso.-cl. of } G\text{-bdls of form } \mathcal{E}_{b,x} \mid x \in \mathcal{F}(G,\{\mu\})\} \simeq B(J_b, \{\mu^{-1} \cdot \nu_b\}).$$

<u>Proof</u>: use translation by $b$. //

# §7 Accessible period domains.

Def.: $(G, b, \{\mu\})$ accessible $\overset{=}{\text{df}}$ $\mathcal{F}(G, b, \{\mu\})^a = \check{\mathcal{F}}(G, \{\mu\})$.

Theorem (Scholze): accessible $\Leftrightarrow$ $b$ basic and $(J_b, \{\mu^{-1}\})$ is uniform, i.e.

$B(J_b, \{\mu^{-1}\})$ has a single element.

Proof: First ass. follows from (ii) in §3, because access. $\Rightarrow$ weakly access.

Second assertion: accessible $\Leftrightarrow$ $\forall x \in \mathcal{F}(G, \{\mu\})$, $\mathcal{E}_{b,x}$ is semi-stable

$\Leftrightarrow$ $B(J_b, \{\mu^{-1}\nu_b\})$ singleton $\Leftrightarrow$ $B(J_b, \{\mu^{-1}\})$ singleton. //

Uniform classified by Kottwitz. In particular,

Corollary: Assume $G$ abs. simple + adjoint, $\{\mu\}$ non-trivial and $[b] \in B(G, \{\mu^{-1}\})$.

Then $(G, b, \{\mu\})$ accessible iff. $G = PGL_n$, $\mu = (1, 0^{(n-1)})$ or $(1^{(n-1)}, 0)$,

and $b$ basic (unique).

Corollary: (uses Faltings/Hartl/Scholze/Weinstein) Assume that $(G, b, \{\mu\})$ comes from RZ-data. Then period

map is surjective iff LT case or its dual. /

Open question: When is $\mathcal{F}(G, b, \{\mu\})^a = \mathcal{F}(G, b, \{\mu\})^{wa}$ ?