Chicago, March 15

<u>Talk</u>: On the AT conjecture for formal moduli spaces

<u>Motivation</u>: Understand better formal moduli spaces (RZ-spaces), in particular cycles on them.

- Kudla program: divisors
- Zhang program: half-dimension.

I. <u>Set-up</u>: F/F_0 quadr., $\eta = \eta_{F/F_0}$, q $(p \neq 2)$.

$n \geq 2$, $GL_{n-1} \hookrightarrow GL_n$, via $A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$.

$$S = \{ \gamma \in \mathrm{Res}_{F/F_0}(GL_n) \mid \gamma \cdot \bar{\gamma} = 1 \} \circlearrowleft GL_{n-1}$$

Let $J_0^\flat$, $J_1^\flat \in \mathrm{Herm}_{n-1}(F/F_0)$ s.t. J_0, resp. J_1 : add 1.

$$U_i = U(J_i) \circlearrowleft U(J_i^\flat).$$

<u>Matching bijection</u>

$$[U_0(F_0)_{rs}] \amalg [U_1(F_0)_{rs}] \simeq [S(F_0)_{rs}].$$

<u>Dual to matching</u>: For $f' \in C_c^\infty(S)$ and $\gamma \in S_{rs}$, and $s \in \mathbb{C}$,

$$\mathrm{Orb}(\gamma, f', s) = \int_{GL_{n-1}(F_0)} f'(h^{-1}\gamma h) |\det h|^s \cdot \eta(\det h)\, dh.$$

$\sim$ vol $GL_{n-1}(O_{F_0}) = 1$

$$\mathrm{Orb}(\gamma, f') = \mathrm{Orb}(\gamma, f', 0)$$

$$\partial \mathrm{Orb}(\gamma, f') = \frac{d}{ds} \mathrm{Orb}(\gamma, f', s)\Big|_{s=0.}$$

Similarly, for $f_i \in C_c^\infty(U_i)$

$$\mathrm{Orb}(g, f_i) = \int_{U_i^\flat} f_i(h^{-1} g h)\, dh.$$

<u>Definition:</u> f' transfers to (f_0, f_1) iff $\forall \gamma \in S_{rs}$

$$\omega(\gamma)\, \mathrm{Orb}(\gamma, f') = \begin{cases} \mathrm{Orb}(g_0, f_0) & \text{if } \gamma \Leftrightarrow g_0 \\ \mathrm{Orb}(g_1, f_1) & \text{if } \gamma \Leftrightarrow g_1 \end{cases}$$

Here $\omega(\gamma)$ = <u>transfer factor</u>, makes LHS constant on orbits.

FL-<u>Conjecture</u>: Let F/F_0 unramified. Take ω = "natural" transfer factor, normalize Haar measure on $U_0^\flat$ by $\mathrm{vol}(K_0^\flat) = 1$. Then

selfdual lattice: <u>special</u>

$$\mathbb{1}_{S(O)} \longleftrightarrow (\mathbb{1}_{K_0}, 0).$$

True for $p \gg 0$ (Yun, Gordon).

II. <u>AFL-conjecture</u>: $\omega(\gamma) \cdot \partial \mathrm{Orb}(\gamma, \mathbb{1}_{S(O)})$ when $\gamma \Leftrightarrow g_1 \in U_{1,rs}.$

$$\| $$

$$- \langle \Delta, \Delta_{g_1} \rangle \cdot \log q = -\mathrm{Int}(g_1) \cdot \log q.$$

<u>Explanation of RHS</u>: Let $\breve{F} = \widehat{F}^{un}$. For $S \in Sch/Spf\, O_{\breve{F}}$, consider (X, ι, λ)

- X formal O_{F_0}-module of rel ht $2n/S$
- $\iota : O_F \to End(X)$
- λ polariz.

s.t.

- $Rosati_\lambda : a \mapsto \bar{a}$
- Kottwitz cond. : $char(\iota(a) \mid Lie\, X) = (T-a)\cdot(T-\bar{a})^{n-1}$, $a \in O_F$
- $Ker\, \lambda = (0)$.

If $S = Spec\, \bar{k}_F$, then (X, ι, λ) unique up to isog., if supersingular.

Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.

$\mathcal{N}_n : S \mapsto \{(X, \iota, \lambda, \varrho)\}/\sim$ where $\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{Spec} \bar{S}$

O_F-quasi-iso of height 0 s.t. $\varrho^*(\lambda_{\mathbb{X}}) = \lambda|_{\bar{S}}$.

<u>Thm</u> (RZ): At least if $F_0 = \mathbb{Q}_p$, $\mathcal{N}_n$ repres. by formal scheme over $Spf\, O_{\breve{F}}$, formally smooth of rel. dim. $n-1$, and essent. proper.

~~Defin~~ <u>Examples</u>: If $n = 1$, then $\mathcal{N}_1 = Spf\, O_{\breve{F}}$ with $\mathcal{E}/\mathcal{N}_1$ universal object (Lubin, Tate, Drinfeld)

$\mathcal{N}_2 \simeq \mathrm{Spf}\, O_{\breve{F}}[[t]]$, $(\mathcal{N}_3)_{red}$ has dimension one.

Get $\delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n$, $Y \mapsto Y \times \bar{\mathcal{E}}$.

Let $\Delta = \Gamma_\delta \subset \mathcal{N}_{n-1} \times_{O_{\breve{F}}} \mathcal{N}_n$ middle dim.

~~Set~~ ~~$K(\Delta)$~~

Now $\mathrm{Aut}(\mathbb{X}_n) = U_{1,1}(F) \ni g \mapsto \Delta_g = (1 \times g) \cdot \Delta$.

Set $\mathrm{Int}(g) := \langle \Delta, \Delta_g \rangle = \chi(O_\Delta \overset{\mathbb{L}}{\otimes} O_{\Delta_g})$. // end of AFL

$F_0 = \mathbb{Q}_p$: Thm: (i) (Zhang - Mihatsch) AFL holds for $n = 2, 3$

(ii) (R.-Terstiege-Zhang) AFL holds for any n, but restrictive hypoth. on g.

III. Now let F/F_0 be ramified (π = uniformizer). Then no natural transfer factor, no natural K_0, etc. But

ST-conjecture: Any $f' \in C_c^\infty(S)$ has (non-unique) transfer (f_0, f_1); any (f_0, f_1) is transfer of (non-unique) f' (here measure, ω, etc are fixed).

True by Zhang (Annals).

<u>Formal moduli spaces have to be modified</u>: now consider

(X, ι, λ) such that as before, but

- $\mathrm{Ker}\,\lambda \subset X[\iota(\pi)]$, $|\mathrm{Ker}\,\lambda| = q^{2\cdot[\frac{n}{2}]}$

For $n > 1$ • $\mathrm{rk}(\iota(\pi) \mid \mathrm{Lie}\, X_s) = 1$, $\forall s \in \bar{S}$.

<u>Thm</u> (Arzdorf, Pappas/R): $\mathcal{N}$ formally smooth of relative dim $n-1/\mathrm{Spf}\, O_{\breve{F}}$.

Essent. proper, if n even.

<u>AT-conjecture</u>: Let n be odd. Have fairly natural transfer factor ω.

Let $K_0 = \mathrm{stab}(\Lambda_0)$, where $\Lambda_0 = \Lambda_0^{\flat} \oplus O_F e$
$\Lambda_0^{\flat}$ π-modular.

<u>Conjecture</u>: a) $\exists f'$ ~~st~~ $\leftrightarrow (\mathbb{1}_{K_0}, 0)$ s.t.

$$2 \cdot \omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = - \mathrm{Int}(g) \cdot \log q, \quad \forall \gamma \leftrightarrow g \in U_{1,rs}$$

b) $\forall f'$ as above, $\exists f'_{\mathrm{corr}}$ s.t.

$$\mathrm{LHS} - \mathrm{RHS} = \omega(\gamma)\, \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}), \quad \forall \gamma \leftrightarrow g \in U_{1,rs}$$

<u>Thm</u> (R/Smithling/Zhang): AT-conjecture holds for $n = 3$.

<u>What about n even?</u> <u>Two things.</u>

- Can embed $\mathcal{N}_{n-1} \hookrightarrow \tilde{\mathcal{N}}_{n-1}$, still formally smooth but now also essent. proper. (Smithling, Richarz).
- Define $\tilde{\mathcal{N}}_n$ by now considering (X, ι, λ), where
  - $\operatorname{Ker} \lambda \subset X[\iota(\pi)]$, $|\operatorname{Ker} \lambda| = q^{n-2}$.
  - plus conditions s.t. flat / $\operatorname{Spf} O_{\breve{F}}$.

<u>Conj.:</u> $\tilde{\mathcal{N}}_n$ has semi-stable reduction.

If true, then can define $\delta : \tilde{\mathcal{N}}_{n-1} \hookrightarrow \tilde{\mathcal{N}}_n$ and $\operatorname{Int}(g)$ as before, and AT-conjecture should be true.

<u>About ~~proof~~:</u> b) $\Rightarrow$ a) easy.

b) • reduce to Lie algebra via Cayley map $c : \mathfrak{a}_1 \to U_1$

$$c(X) = \frac{1+X}{1-X}.$$ Then $\Delta \cap \Delta_X = \Delta \cap \Delta_{c(X)}$.

- Serre construction gives $\mathcal{N}_2 = \mathcal{M}_{\operatorname{Spf} O} \amalg \mathcal{M}_{\operatorname{Spf} O}$,

where $\mathcal{M}$ = universal def space of O_{F_0}-module $E \otimes \bar{k}$.
Use theory of quasi-canonical ~~liftings~~ divisors (KR, ARGOS).

- Next, consider both sides as function on the regular semi-simple subset of common categorical quotient

$$\mathfrak{s}_n / GL_{n-1} = \mathfrak{u}_1 / U_1' .$$

(prolong by zero to elements matching with $U_{0,rs}$). Then check:

$\forall\, x_0 \in \mathfrak{s}_n / GL_{n-1}$ $\exists$ nbhd V_{x_0} s.t. $LHS - RHS|_{V_{x_0,rs}}$ loc. const.

- Show that "any" fct. on $(\mathfrak{s}_n / GL_{n-1})_{rs}$ with previous property is of the form $\omega(\gamma) \cdot Orb(\gamma, f_{corr})$, for suitable f_{corr}.