Pasadena, March 14

Towards a theory of local Shimura varieties

Recall: a complex torus = connected compact complex Lie gp.

$\Rightarrow$ commutative, because adjoint rep'n

$$T \longrightarrow GL(\mathcal{O}_{T,0}/\mathfrak{m}^n).$$

Consider $\mathfrak{t} = \mathrm{Lie}\, T$ and exponential map. Then

$$\mathfrak{t} \xrightarrow{\exp} T$$

is surjective, and is universal covering. Let $\Lambda = \mathrm{Ker}(\exp)$.

This induces equiv. of categories

$$\left\{ \text{complex tori} \right\} \longleftrightarrow \left\{ \begin{array}{c} (\mathfrak{t}, \Lambda), \text{ where} \\ \mathfrak{t} = \text{f.d. } \mathbb{C}\text{-vs}, \ \Lambda \text{ lattice} \end{array} \right\}.$$

Reformulation in terms of Hodge structures.

Recall: (i) A $\mathbb{Z}$-H.S. of weight -1 $\underset{\text{df}}{=}$ f.-free $\mathbb{Z}$-module Λ plus $\mathbb{C}$-subspace $V \subset \Lambda \otimes \mathbb{C}$, s.t. $V \oplus \overline{V} \simeq \Lambda \otimes \mathbb{C}$.

(ii) A polariz. of a $\mathbb{Z}$-HS (V, Λ) of wt -1 $\underset{\text{df}}{=}$

$$\psi : \Lambda \otimes \Lambda \longrightarrow 2\pi i\, \mathbb{Z}$$

(principal $\underset{\text{df}}{=}$)

s.t. $\psi(x, Cy)$ is symm. + pos.-def. on $\Lambda \otimes \mathbb{R}$.

Here C depends on V, equals i on V and $-i$ on $\overline{V}$.

Hence also

$$\{\text{complex tori}\} \longleftrightarrow \{(\Lambda, V) \text{ H.S. of wt } -1\}$$

$$V = \ker(\Lambda \otimes \mathbb{C} \to t).$$

Recall: complex abelian var $\underset{\text{def}}{=}$ complex torus that is a project. var.

Thm (Riemann): Equivalence of categ.

$$\{\text{complex abelian var.}\} \longleftrightarrow \{\text{polarizable } \mathbb{Z}\text{-H.S. of wt } -1\}$$

Fix (Λ, ψ). Then the set of all $V \subset \Lambda \otimes \mathbb{C}$ s.t. (Λ, V, ψ) polarized HS of wt -1 is a complex mf. + action of $Sp(\Lambda, \mathbb{R})$, in fact can be identified with $(g = \mathrm{rk}\,\Lambda)$

$$\mathfrak{h}_g = \{ Z \in M_g(\mathbb{C}) \mid \mathrm{Im}\, Z \gg 0 \}$$

with $\gamma \in Sp(2g, \mathbb{R})$ action by $Z \longmapsto (aZ+b)\cdot(cZ+d)^{-1}$.

Hence $\mathfrak{H}_g$ parametrizes set ~~of~~ of abelian varieties w. given polarization.

Want to algebraicize this: ~~...~~ ~~...~~ Fix g and let $n \geq 3$.

Consider foll. set-valued functor $\mathcal{M}$ on $(Sch/\mathbb{Q})$:

$\mathcal{M}(S) = \{$ iso-classes of (A, λ, α) | A abelian scheme / S

of dim. g, λ = principal polariz. on A,

$$\alpha : A[n] \simeq (\mathbb{Z}/n)^{2g}_S$$

sympl. similitude $\}$.

Theorem (Mumford): $\mathcal{M}$ is representable by a quasi-proj. scheme / $\mathbb{Q}$, with

$$\mathcal{M}(\mathbb{C}) = G(\mathbb{Q}) \backslash \left[X_\infty \times G(\mathbb{A}_f) / K(n) \right].$$

Here X_∞ = union of upper + lower Siegel space.

$G = GSp(\Lambda, \psi)$, $G(\mathbb{R})$ acts on X_∞,

$K(n) = \{ g \in GSp(\Lambda \otimes \hat{\mathbb{Z}}) \mid g \equiv 1 \bmod n \}$.

Leads to concept of Shimura varieties (Shimura, Deligne, Langlands)

Recall: A $\mathbb{Z}$-H.S. of wt -1 $\hat{=}$ an alg. homo

$$h : \underline{\mathbb{C}}^{\times} \longrightarrow GL(\Lambda \otimes \mathbb{R})$$

s.t. composition $\mathbb{R}^{\times} \xrightarrow{w} \mathbb{C}^{\times} \longrightarrow GL(\Lambda \otimes \mathbb{R})$

equals $t \longmapsto t^{-1} \, id_{\Lambda \otimes \mathbb{R}}$.

Now start with pair $(G, \{h\})$, where

- G = reductive alg. gp $/\mathbb{Q}$.
- $\{h\}$ = $G(\mathbb{R})$-conj.-class of homo's $\underline{\mathbb{C}}^{\times} \longrightarrow G_{\mathbb{R}}$

Axioms of Shimura data:

(i) $h \circ w$ is central

(ii) $Ad \circ h$ induces $\mathbb{R}$-H S of wt -1 on $Lie\, G_{\mathbb{R}}$.

(iii) axiom which implies that

Axioms have 2 important consequences:

a) $\{h\}$ is set of $\mathbb{C}$-points of complex var. on

which $G(\mathbb{R})$ acts by holo. auto : $X_{\{h\}}$

b) The cocharacter $\mu_h : \underline{\mathbb{C}}^\times \xrightarrow{(id,1)} \underline{\mathbb{C}}^\times \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{h} G_{\mathbb{C}}$

is minuscule.

Starting from $(G, \{h\})$ form complex variety

$$M_K = G(\mathbb{Q}) \backslash \left[X_{\{h\}} \times G(\mathbb{A}_f)/K \right] = \coprod_g \Gamma_g \backslash X^{\circ}_{\{h\}}$$

For varying K, get projective system, on which $G(\mathbb{A}_f)$ acts.

Theorem (BB/B): $\exists !$ structure of qu-projective algebr. varieties on this system.

Let $E = E(G, \{h\})$ = field of def. of conj.-class $\{\mu_h\}$

$\subset \mathbb{C}$.

Then whole system can be defined in a natural way over E (canonical model, Borovoi, Deligne, Milne, ...).

In Siegel case: $E = \mathbb{Q}$.

Interest: Have action of $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on coho.

$$\lim_K H^i(M_K \otimes_E \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell).$$

ms
- Best way to construct Langlands correspondences
  (Harris/Taylor, Henniart, Scholze)
- Gives a class of algebr. varieties for which can determine the HW-zeta fct. in terms of automorphic L-functions
  (Kottwitz, Morel, Shin, ...).

Next aim: analogue of such a theory for $\mathbb{Q}_p$ instead of $\mathbb{Q}$.

(local Shimura varieties). p fixed

Analogue of complex tori = p-divis. gp.

Fix a p-div. group $\mathbb{X} / \overline{\mathbb{F}}_p$. Let

Nilp = sub-cat. of $(Sch/\breve{O})$ s.t. $p \cdot O_S$ loc. nilp. ideal.

Define functor $\mathcal{M} : \mathrm{Nilp} \longrightarrow (\mathrm{Sets})$ with

$$\mathcal{M}(S) = \{ (X, \varrho) \mid X/S \text{ p-div. gp.}, \ \varrho : X \times_S \overline{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \overline{\mathbb{F}}_p} \overline{S} \text{ quasi-isog.} \}.$$

Thm (R/Zink): $\mathcal{M}$ is representable by a formal scheme loc. form. of f.t. over $\breve{O}$.

Variants, e.g.: Start with $(\mathbb{X}, \lambda_{\mathbb{X}})$ princ. polarized p-div.

Then consider

$$\mathcal{M}^{\#}(S) = \{ (X, \lambda, \varrho) \mid (X, \lambda)/S \text{ p.-p. p-div. gp}, \ \varrho : X \times_S \overline{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \overline{\mathbb{F}}_p} \overline{S} \text{ quasi-isog., s.t. } \lambda \sim \varrho^*(\lambda_{\mathbb{X}}) \}$$

Formal scheme $\mathcal{M}/\breve{O} \mapsto \mathcal{M}^{rig}$ = rigid-analytic space over $\breve{\mathbb{Q}}_p$.

Have system of coverings: let $X^{rig}/\mathcal{M}^{rig}$ universal object.

- In first ex., for $K \subset GL_n(\mathbb{Q}_p)$, let

$\mathbb{M}_K/\mathcal{M}^{rig}$: $\alpha: V_p(X) \simeq \mathbb{Q}_p^n$ mod K.

- In second ex., for $K \subset GSp_{2n}(\mathbb{Q}_p)$, let

$\mathbb{M}_K/\mathcal{M}^{rig}$: $\alpha: V_p(X) \simeq \mathbb{Q}_p^{2n}$ mod K

symplectic simil.

Projective system of finite étale morphisms over $\mathcal{M}^{rig}$.

Would like to have a formal set-up for this, similar to Deligne's.

Start with triple $(G, \{\mu\}, [b])$, where

- G reductive alg. group $/\mathbb{Q}_p$.
- $\{\mu\}$ conj.-class of cocharacters $\mathbb{G}_{m\,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, which is minuscule

- $[b]$ a set of elements $b \in G(\breve{\mathbb{Q}}_p)$ up to σ-conj'ing.

Make assumption $[b] \in B(G, \{\mu\})$ (Mazur's inequality)

— For $b \in [b]$, have

$$J_b(\mathbb{Q}_p) = \{ g \in G(\breve{\mathbb{Q}}_p) \mid g^{-1} \cdot b \cdot \sigma(g) = b \}.$$

(this ingredient missing in classical situation).

~~Conjecture: $\exists$ tower of rigid~~

— $E = E(G, \{\mu\}) \subset \overline{\mathbb{Q}}_p$.

<u>Conjecture:</u> $\exists$ tower of rigid-analytic varieties,

$$\mathbb{M} = \mathbb{M}(G, \{\mu\}, b) = \mathbb{M}(G, \{\mu\}, b)_K \quad , K \subset G(\mathbb{Q}_p)$$

with transition maps all finite étale, all members smooth of dimension $\langle \{\mu\}, 2\rho \rangle$ and equipped with action of $J_b(\mathbb{Q}_p)$ in a compatible way.

Furthermore, should be equipped with Weil-descent datum from $\breve{\mathbb{Q}}_p$ to E.

The construction should have many good properties, e.g. any $(G_1, \{\mu_1\}, [b_1]) \to (G_2, \{\mu_2\}, [b_2])$ should induce $\mathbb{M}_1 \to \mathbb{M}_2$, etc. $\longrightarrow$ paper w. Viehmann. Period map.

But: Don't know how to characterize $\mathbb{M}$ uniquely.

However, if $\mathbb{M}(G, \{\mu\}, [b])$ is known, then all $\mathbb{M}'$ for $(G', \{\mu'\}, [b']) \subset (G, \{\mu\}, [b])$ should be unique with above additional properties.

Above 2 examples correspond to

- $G = GL_n$, $\{\mu\} = (1^{(d)}, 0^{(n-d)})$, $[b]$ arbitrary in $B(G, \{\mu\})$
- $G = GSp_{2n}$, $\{\mu\} = (1^{(n)}, 0^{(n)})$, ——————— //

Even for those 2 examples not all projected properties known to hold (e.g. only depends on $(G, \{\mu\}, [b])$?).

Interest: Have action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ on

$$\lim_K H^i_c(\mathbb{M}_K \otimes_{\mathbb{Q}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell).$$

Gives local Jacquet-Langlands-Weil correspondences

(Kottwitz, Mantovan, Shin, Fargues, Mieda, Scholze, ...).

Why there is hope now: Scholze's analogue of Riemann's thm.

Let C be non-arch., complete alg. closed field.

Equiv. of categories

$$\left\{ \begin{matrix} p\text{-divis. grps} \\ \mathcal{G}/\mathcal{O}_C \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} (\Lambda, W) \mid & \Lambda \text{ finite free } \mathbb{Z}_p\text{-module} \\ W \subset \Lambda \otimes_{\mathbb{Z}_p} C & C\text{-subspace} \\ & W = \mathrm{Lie}\, \mathcal{G}^*(1) \end{matrix} \right\}$$

Using this, Scholze has perhaps given a linear alg. description of tower in 1^{st} example for $(GL_n, \{\mu_d\}, [b])$, and probably all subobjects (local Sh.-varieties of Hodge type) and charact. them. Perhaps even of abelian type.

Here are some examples of local Sh.-varieties which are not RZ-spaces

- Product of RZ-spaces are LSV, but not RZS.
- $G = T$ a torus, $\{\mu\} \in X_*(T)$ arbitrary, $[b]$ unique. Then

$$\mathbb{M}_K = \mathcal{T}(\mathbb{Q}_p)/K \qquad , \text{ with } Gal(\overline{\mathbb{Q}}_p/E)\text{-action}$$
à la Deligne.

- Let $(G, \{\mu\}, [b])$ LSV-data. Then for $c : \mathbb{G}_m \to Z(G)$

$(G, \{\mu \cdot c\}, [b \cdot c(p)])$ is again.

But not RZ-type in general. (e.g. level structure

$$T_p(X)(d) \simeq \mathbb{Z}_p^{2n} \quad \text{mod } K.$$