Paris, June 14

Talk: Around the AFL of Wei Zhang, 2

> AFL: calculate both sides explicitly, and compare.

Today: F/F_0 ramified (but still $p \neq 2$), $\pi^2 = \varpi$.

<u>Start with geometric side:</u> Let $n \geq 1$. Consider $(X, \iota, \lambda)/S$,

where $S \in \mathrm{Nilp}_{O_{\breve F}}$. Here

X = p-divis. formal O_{F_0}-module of ht $2n$, <u>as before</u>

$\iota : O_F \to \mathrm{End}(X)$ <u>as before</u> with char $= (T-a)(T-\bar a)^{n-1}$

$\lambda : X \to X^\vee$ polariz. <u>as before</u> with $\mathrm{Rosati}_\lambda : a \mapsto \bar a$.

But this time impose

$$\mathrm{Ker}\,\lambda = \begin{cases} X[\pi] & n \text{ even} \\ \subset X[\pi], \ |\mathrm{Ker}\,\lambda| = q^{n-1} & n \text{ odd.} \end{cases}$$

<u>Furthermore impose:</u>

* $\bigwedge^2 (\iota(\pi) + \pi \mid \mathrm{Lie}\, X) = 0$, $\bigwedge^n (\iota(\pi) - \pi \mid \mathrm{Lie}\, X) = 0$

* $\iota(\pi) \mid \mathrm{Lie}\, X$ nowhere zero endo.

Again, if $S = \mathrm{Spec}\, \bar k$, (X, ι, λ) unique up to isogeny respecting ι and λ. Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.

As before define functor $\mathcal{N}_n = \{(X, \iota, \lambda, \varrho)\}$.

<u>Theorem:</u> Let $F_0 = \mathbb{Q}_p$. $\mathcal{N}_n$ is formally smooth of relative dim $n-1$ over $\mathrm{Spf}\, O_{\breve F}$. If n even, then $\mathcal{N}_n$ is essentially proper.

<u>Remark:</u> If $n \geq 3$ is odd, then $\mathcal{N}_n$ not essent. proper. But $\exists$ open embedding $\mathcal{N}_n \subset \overline{\mathcal{N}}_n$ s.t. $\overline{\mathcal{N}}_n$ is formally smooth + ess. proper (Arzdorf, Richarz, Smithling) (<u>exotic smoothness</u>)

<u>Examples:</u> • $\mathcal{N}_1 = \mathrm{Spf}\, O_{\breve{F}}$: again $\mathbb{E}$, with $\iota : O_F \hookrightarrow \mathrm{End}(\mathbb{E})$, $\mathcal{E}$

• $\mathcal{N}_2 \simeq \mathrm{Spf}\, O_{\breve{F}}[[t]]$

• $(\mathcal{N}_3)_{red}$ 1-dimensional – structure?

In general, $(\overline{\mathcal{N}}_n)_{red}$ is on list of Görtz-He.

From now on let n odd, ≥ 3. Then get (sic!)

$$\delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n : (Y, \varrho) \longmapsto (Y \times \overline{\mathcal{E}}, \varrho_Y \times c(\varrho_Y)^{-1} \cdot \varrho_0)$$

and hence also

$$\Delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n .$$

Let $g \in \mathrm{Aut}(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) \rightsquigarrow \Delta_g = (1 \times g)^*(\Delta)$ and

$$\mathrm{Int}_g = \langle \Delta, \Delta_g \rangle$$

(This makes again sense since $\mathcal{N}_{n-1} \times \mathcal{N}_n$ regular.)

<u>Now pass to analytic side.</u> Use extension

$$\tilde{\eta} : F^\times \longrightarrow \mathbb{C}^\times \quad \text{of } \eta$$

to define <u>transfer factor</u>

$$\omega(\gamma) = \tilde{\eta}(\det(v, \gamma v, \dots, \gamma^{n-1} v))$$

Hence get matching

$$f' \in C_c^\infty(S) \longleftrightarrow (f_0, f_1) \in C_c^\infty(U_0) \times C_c^\infty(U_1).$$

Fix π-modular lattice $\Lambda_0 \subset W(J_0) \rightsquigarrow$

$$\tilde{\Lambda}_0 = \Lambda_0 \oplus O u \quad , \quad \tilde{K}_0 = \mathrm{Stab}_{U_0}(\tilde{\Lambda}_0).$$

(In matching, normalize Haar measures by $\mathrm{vol}(S(O)) = 1$, $\mathrm{vol}(\tilde{K}_0) = 1$, and arbitrary on U_1).

Again we consider $f' \in C_c^\infty(S)$ with transfer $(\mathbb{1}_{\tilde{K}_0}, 0)$.

<u>But this time no natural candidate.</u>

<u>Conjecture (WZ)</u>: (i) $\exists$ such f' s.t. $\forall \gamma \in S_{rs}$ with match $\delta \in U_{1,rs}$, have

$$\omega(\gamma) \cdot \partial \mathrm{Orb}_\gamma(f') = \mathrm{Int}_g \cdot \log q \qquad g \simeq \delta$$

(ii) $\forall$ such f' $\exists$ $f'_{corr} \in C_c^\infty(S)$ s.t. $\forall$ $\gamma \leftrightarrow \delta$ have

$$\omega(\gamma) \partial \mathrm{Orb}_\gamma(f') = \mathrm{Int}_g \cdot \log q + \omega(\gamma) \mathrm{Orb}_\gamma(f'_{corr}) \qquad g \simeq \delta.$$

<u>Remark</u>: Implicitly, we are asserting that RHS $< \infty$.

<u>Theorem désiré (RSZ)</u>: Let $F_0 = \mathbb{Q}_p$. Then conjecture holds for $n = 3$. More precisely, in this case ~~for~~ the intersection of Δ ad Δ_g is local artinian, in Int_g no higher Tor-terms and statements (i) and (ii) hold.

What about n even?

In this case, want to consider a different moduli functor $\widetilde{\mathcal{N}}_n$:

(X, ι, λ) where

$$\operatorname{Ker}\lambda \subset X[\pi], \qquad |\operatorname{Ker}\lambda| = q^{n-2}$$

+ other conditions s.t. $\widetilde{\mathcal{N}}_n \,/\, \operatorname{Spf} O_{\breve{F}}$ flat.

Conjecture: $\widetilde{\mathcal{N}}_n$ has semi-stable reduction, in fact smoothly equivalent to curve w. ordinary double point.

If this were true, the product

$$\mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \widetilde{\mathcal{N}}_n$$

would be regular, and we could pose same conjecture.

(But first non-trivial case is $n = 4$!).

In fact, statement (i) follows from (ii) by the following easy lemma

<u>Lemma</u>: $\forall f \in C_c^\infty(S)\ \exists f'' \in C_c^\infty(S)$ s.t.

$$\partial \mathrm{Orb}_\gamma(f) = \mathrm{Orb}_\gamma(f'') \quad , \quad \forall \gamma \in S_{rs}.$$

Furthermore, any such f'' has transfer $(0,0)$. //

<u>The proof uses a reduction to Lie algebra</u> (like W.Z.'s lecture):

For this note the following moduli description of Δ_g:

$\Delta_g = \{ (Y, \varrho_Y), (X, \varrho_X) \in \mathcal{N}_{n-1} \times \mathcal{N}_n \mid$ the quasi-homo's

$$\mathbb{X}_{n-1} \hookrightarrow \mathbb{X}_n \xrightarrow{g} \mathbb{X}_n \quad \text{extend to } Y \to X$$

$$\mathbb{X}_n \xrightarrow{g} \mathbb{X}_n \twoheadrightarrow \mathbb{X}_{n-1} \quad \text{——} \quad X \to Y$$

$$\overline{\mathbb{E}} \hookrightarrow \mathbb{X}_n \xrightarrow{g} \mathbb{X}_n \quad \text{——} \quad \overline{\mathcal{E}} \to X$$

$$\mathbb{X}_n \xrightarrow{g} \mathbb{X}_n \to \overline{\mathbb{E}} \quad \text{——} \quad X \to \overline{\mathcal{E}} \quad \}.$$

This definition makes sense for any endo of $\mathbb{X}_n$, in particular for $x \in \mathfrak{u}_1 = \mathrm{Lie}\, U_1$

<u>Lie algebra version</u>: Consider $\underline{s}$ and $\mathfrak{u}_0, \mathfrak{u}_1$. Have again

- notion of rs (both via gp action and via linear algebra)
- transfer factor (ess. same factor)
- transfer of functions $\quad \phi' \in C_c^\infty(s) \longleftrightarrow (\phi_0, \phi_1) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$

Again consider ϕ' with transfer $(\mathbb{1}_{\tilde{\mathfrak{a}}_0}, 0)$ (no natural choice)

<u>Conjecture (WZ)</u>: (i) $\exists$ such ϕ' s.t. $\forall y \in \mathfrak{z}_{rs} \Leftrightarrow x \in \mathfrak{a}_{1,rs}$

$$\omega(y)\, \partial \mathrm{Orb}_y(\phi') = \mathrm{Int}_x \cdot \log q.$$

(ii) $\forall$ such ϕ', $\exists\, \phi'_{corr}$ s.t. ______

______ $+\, \omega(y)\, \mathrm{Orb}_y(\phi'_{corr})$

<u>Now let $n = 3$.</u> Then $\mathcal{N}_2(\bar{k}) = \{pt\}$. Hence via δ

get natural point of $\mathcal{N}_3(\bar{k})$ and hence

$$\tilde{K}_1 \subset U_1(F_0), \quad \text{max. open compact subgp.}$$

A) <u>Calculation of Int in both versions</u>: 1° Reduction to Lie algebra

<u>And therefore</u>: Let $g \in U_1$, resp. $x \in \mathfrak{a}_1$. If

$$\mathrm{Int}_g \neq 0 \rightarrow g \in \tilde{K}_1 \quad (\text{same for } x \in \tilde{\mathfrak{a}}_1).$$

For $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$ define Cayley transform

$$c_\xi : \mathfrak{u}_1 \longrightarrow U_1, \quad x \mapsto \xi(1+x)(1-x).$$

<u>Lemma</u>: (i) $c_\xi \mid \tilde{\mathfrak{a}}_1$ is well-defined, ~~with image~~

$$c_\xi : \tilde{\mathfrak{a}}_1 \longrightarrow \tilde{K}_1$$

and images for varying ξ cover all of $\tilde{K}_1$.

(ii) $\Delta_x = \Delta_{c_\xi(x)}$, $\forall \xi$ and hence

$$\mathrm{Int}_x = \mathrm{Int}_{c_\xi(x)}, \quad \forall x \in \tilde{\mathfrak{a}}_1, \ \forall \xi$$

Hence calculation of Int reduced from group case to Lie algebra case

~~Step 11~~: 2°/ Calculation of Int_x, for $x \in \tilde{\mathcal{E}}_1$.

Based again on connection to Kudla world. Let $\mathcal{M}$ = formal moduli of formal O_{F_0}-modules of ht 2 and dim 1
space ~~$\mathcal{M}_2$ of KR~~ ~~Lehsa~~, hence

$$\mathcal{M} \simeq \mathrm{Spf}\, O_{\breve{F}_0}[[t]] \quad ; \text{ universal object } Y$$

Proposition: Have isom. given by Serre construction,

$$\varphi : \mathcal{M} \otimes_{O_{\breve{F}_0}} O_{\breve{F}} \xrightarrow{\sim} \mathcal{N}_2 \ , \qquad \mathcal{E} \mapsto (O_F \otimes_{O_{F_0}} \mathcal{E},\ \iota \otimes id,\ O_F \otimes \lambda_0)$$

(Note that $\mathrm{Ker}(O_F \otimes \lambda_0) = X[\pi]$, because different $= \pi \cdot O_F$).

On $\mathcal{N}_2$ have analogues of KR-divisors: Fix $i : O_F \hookrightarrow \mathrm{End}(\mathbb{E})$

For $c : \mathbb{X}_2 \longrightarrow \mathbb{E}$ O_F-linear $\mapsto$ $Z(c)$ = locus inside $\mathcal{N}_2$

where c lifts to $X \longrightarrow \mathcal{E}$. ($O_F$-linear)

Proposition: Let $c^\flat \in \mathrm{End}(\mathbb{E})$ correspond to c under adjunction,

$$\mathrm{Hom}(\mathbb{E}, \mathbb{E}) \simeq \mathrm{Hom}_{O_F}(O_F \otimes_{O_{F_0}} \mathbb{E}, \mathbb{E}) = \mathrm{Hom}_{O_F}(\mathbb{X}_2, \mathbb{E}).$$

Let ${}^cF$ = image of F in $\mathrm{End}(\mathbb{E})$ conjugate under $c^\flat$

The closed formal subscheme $Z(c)$ is a relative divisor on $\mathcal{N}_2$

prove this!

and have equality of divisors on $\mathcal{M}$,

$$\varphi^{-1}(Z(c)) = \sum_{0 \le r \le v(c)} \mathcal{W}_{{}^cF, r}$$

where $c^\flat$ lifts to ~~locus in $\mathcal{M}$~~

locus in $\mathcal{M}$ where $c^\flat$ lifts to $Y \to \mathcal{E}$.

Here $v(c) = \mathrm{val}((c,c))$. ARGOS have quasi-canonical divisors for ${}^cF \hookrightarrow \mathrm{End}^\circ(\mathbb{E})$ and varying level r.

Now one interprets the entries of $x \in \mathfrak{a}_1$ as endomorphisms of $\mathbb{E}$ (via adjunction). Hence

$$\varphi^{-1}(\Delta \cap \Delta_x) = \text{locus in } \mathcal{M}, \text{ where certain endo's of } \mathbb{E} \text{ extend to } \begin{cases} \text{hom of } ~~\mathbb{X}~~ \text{ to } \mathcal{E} \\ \text{resp. endo's of } ~~\mathbb{X}~~ \end{cases}$$

$$= ~~\text{locus in } \varphi^{-1}(Z(c)) \text{ where endo's of } \mathbb{E} \text{ extend to endo's of } X~~$$

$$= \bigcup \text{ locus in } W_{F,r} \text{ where cert. endo's of } \mathbb{E} \text{ extend to endo's of } X_r$$

$$= \text{known by Keating.}$$

(1) Recall that

$$W_{F,r} = \operatorname{Spf} \mathcal{O}_{\breve{F}_r}, \text{ with pullback } X_r$$

of universal object such that $\operatorname{End}_{W_{F,r}}(X_r) = \mathcal{O}_{F_0} + \varpi^r \cdot \mathcal{O}_F$.

(Gross).

B) Calculation of $\omega(\gamma)\,\partial \mathrm{Orb}_\gamma(\phi')$

Cannot be calculated explicitly because ϕ' not known, only known through its transfer $(\mathbb{1}_{U_0(O)}, 0)$. But only need ~~k~~ this up to $\omega(\gamma)\cdot \mathrm{Orb}_\gamma(f_{corr})$!

1°/ Reduction to Lie algebra

Uses Cayley transform, just as Int-calculation – but more subtle.

2°/ Lie algebra version:

Uses proposition from WZ's talk: Consider

$$\pi : \underline{s} \longrightarrow Q = s / GL_{n-1} \simeq \mathbb{A}^{\bullet} \text{ affine space.}$$

Proposition: (WZ) Let φ be a C^∞ fct on Q_{rs} which vanishes outside a compact set of Q. Equivalent:

(i) $\exists\ \phi'' \in C_c^\infty(s)$ s.t.

$$\varphi(\pi(x)) = \omega(x)\,\mathrm{Orb}_x(\phi''), \quad \forall x \in s_{rs}$$

(ii) $\forall\ x_0 \in Q(F_0)$, $\exists$ open nbhd U of x_0 and $\phi'' \in C_c^\infty(\underline{s})$ s.t.

$$\varphi(\pi(x)) = \omega(x)\,\mathrm{Orb}_x(\phi''), \quad \forall x \in s_{rs} \cap \pi^{-1}(U)$$

Hence problem is localized on the base. Locally around x_0 can write (roughly)

$$\omega(\gamma)\,\partial \mathrm{Orb}_\gamma(\phi') = \varphi(\pi(\gamma)) + \sum_r \Gamma_r(\pi(\gamma))' \cdot \mathrm{Orb}_r(\phi')$$

where φ = local orbit. integral function

Γ_n' = derivative of some germ fcts (indep't of ϕ')

$\mathrm{Orb}_n(\phi')$ = nilpotent orbital integral of ϕ'

(known through transfer of ϕ').

<u>Now show:</u> 2^n summand − Int = local integral fct.

In dimension one lower, the comparison can be done directly on the group → see thesis of Mihatsch.