Talk: ~~On~~ Around Wei Zhang's AFL-conjecture, ~~and its variants~~, I.

Some general notation: $p \neq 2$

F_0 finite extension of $\mathbb{Q}_p$, O_{F_0}, ϖ.

F/F_0 quadratic extension, k'/k, $a \mapsto \bar{a}$.

$\eta = \eta_{F/F_0}$ avoid E because needed later.

Fix $n \geq 1$. Let

$$u = (0, \dots, 0, 1) \in F_0^n \rightsquigarrow GL_{n-1} \hookrightarrow GL_n \,/F_0.$$

Let

$$S = S_n = \{ s \in \mathrm{Res}_{F/F_0}(GL_{n/F}) \mid s \cdot \bar{s} = 1 \}.$$

~~Have action of $GL_{n-1}(F_0)$~~,

Recall from WZ's lectures: Let J_0, resp. $J_1 \in \mathrm{Herm}_{n-1}(F/F_0)$

s.t. J_0 split and J_1 non-split. ~~Let U_0, resp. U_1 corresp. unitary groups~~ Get hermitian structure on F^n, via

$$\tilde{J}_0 \overset{\text{orthog.}}{=} J_0 \overset{\perp}{\oplus} 1, \quad \text{resp. } \tilde{J}_1 = J_1 \overset{\perp}{\oplus} 1,$$

i.e. add orthog. vector u with $(u,u) = 1$.

Let U_0, resp. U_1 corresp. unitary groups of $\tilde{J}_0$, resp. $\tilde{J}_1$ ~~size n~~.

(notation) $U_{0,1,rs}$

Then have bijection between orbit spaces

$$[U_0(F_0)_{rs}] \amalg [U_1(F_0)_{rs}] \overset{\text{matching}}{=} [S(F_0)_{rs}]$$

Here on the left: orbits under $U(J_0)(F_0)$, resp. $U(J_1)(F_0)$, by conj.

On the right: —— $GL_{n-1}(F_0)$, by conj.

Example: Let $n = 2$.

- $S(F_0)_{rs} = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(F_0) \;\middle|\; b \cdot c \neq 0 \right\}$.

$$= \left\{ \gamma = \begin{pmatrix} a & b \\ \frac{1-N(a)}{\bar{b}} & -\bar{a} \cdot \frac{b}{\bar{b}} \end{pmatrix} \;\middle|\; b \neq 0,\ 1-N(a) \neq 0 \right\}.$$

- Let $J_0 \hat{=} \varepsilon_0 = 1$, $J_1 \hat{=} \varepsilon_1 \in F_0^\times \setminus \mathrm{Nm}\, F^\times$. Then

γ matches ~~δ~~ $g \in U_{\varepsilon_0}(F)$ resp. $g \in U_{\varepsilon_1}(F)$ dep. on whether $1-N(a) \in \mathrm{Nm}$ or $\notin \mathrm{Nm}$

$$g = \begin{pmatrix} a & x \\ \frac{\bar{x} \cdot \cancel{b}}{\varepsilon_i \bar{b}} & -\bar{a} \frac{b}{\bar{b}} \end{pmatrix} \in U_{\varepsilon_i}(F_0)$$

where ~~if~~ $N(x) = \varepsilon_i \cdot (1 - N(a))$,

Note that here ~~U_0~~ $U(J_0)(F_0) = U(J_1)(F_0) = F^{\times,1}$.

Recall also from WZ's lectures:

For $f' \in C_c^\infty(S) \mapsto$

$$\mathrm{Orb}_\gamma(f'), \quad \gamma \in S(F_0)_{rs}.$$

Here Haar measure normalized by $\mathrm{vol}(GL_{n-1}(\mathcal{O}_{F_0})) = 1$.

1 → bw. Similarly for $f_i \in C_c^\infty(U_i) \mapsto$

$$\mathrm{Orb}_g(f_i), \quad g \in U_i(F_0)_{rs}.$$

Here Haar measure normalized how? For $i = 0$, by selfdual $\mathrm{vol}(U(J_0)(\mathcal{O}_{F_0})) = 1$. For $i=1$ doesn't matter.

Transfer factor ω is a smooth function $\omega : S(F)_{rs} \to \mathbb{C}^\times$

$$\mathrm{Orb}_\gamma(f') = \int f'(h\cdot\gamma)\overset{\eta(h)}{\vphantom{x}}\,dh$$

$$= \cancel{\mathrm{Orb}}^{\mathrm{Int}}(h \mapsto h\cdot\gamma)$$

(1) Also let

$$\mathrm{Orb}_\gamma(f', s) = \mathrm{Orb}_\gamma(f' \cdot |\det h|^s).$$

$$\partial\mathrm{Orb}_\gamma(f') = \frac{d}{ds}\mathrm{Orb}_\gamma(f', s)\Big|_{s=0}.$$

s.t. $\quad \omega(h \cdot s) = \eta(h) \cdot \omega(s)$. $\qquad$ how unique ?

Here $\eta(h) = \eta(\det h)$.
Canonical choice dep. on extension η' of η to $F^\times$: ~~write down~~.
<u>Transfer of functions:</u> $\quad f' \leftrightarrow (f_0, f_1) \overset{=}{\underset{\text{Df.}}{}}$

$$\omega(\gamma) \cdot \mathrm{Orb}_\gamma(f') = \begin{cases} \mathrm{Orb}_{\delta_0}(f_0) & \text{if } \gamma \leftrightarrow \delta_0 \in U_0(F_0)_{rs} \\ \mathrm{Orb}_{\delta_1}(f_1) & \text{——} \quad \delta_1 \in U_1(F_0)_{rs}. \end{cases}$$

<u>Today</u>: Let F/F_0 be unramified. In this case get transfer factor by

$$\omega(\gamma) = \cancel{(-1)^{\mathrm{val}(\det(\gamma^i v)_{i=0,\dots,n-1})}} = \eta'(\det(\;)) \text{ where } \in \{\pm 1\}.$$

η' = obvious ext. of η, with $\eta'|_{O_F^\times} \equiv 1$.

§2: <u>We also introduce RZ spaces</u>: ~~f~~ consider ~~p-div. grps~~ formal O_{F_0}-module X of ht $2n$. ~~dim n~~ over $S \in \mathrm{Nilp}_{O_{\breve F}}$, with action $\iota : O_F \to \mathrm{End}(X)$, and polarization $\lambda : X \to X^\vee$. Assume:

- $\mathrm{Rosati}_\lambda \mid O_F \;:\; a \mapsto \bar a$
- λ principal polariz.
- $\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T-a) \cdot (T-\bar a)^{n-1} \in O_S[T]$, $\quad \forall a \in O_F$.

If $S = \mathrm{Spec}\, \bar{\mathbb{F}}_p$, triple (X, ι, λ) unique up to isogeny (respecting ι and λ). Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. Can ~~define formal scheme~~ take $\mathbb{E}$ = formal O_{F_0} ... ~~ht 2~~ $\hat{F} \subset \mathrm{End}(\mathbb{E})$,

$$\mathbb{X} = \mathbb{E} \times \bar{\mathbb{E}}^{n-1}, \quad \cancel{\iota_{\mathbb{X}} = (\iota_{\mathbb{E}}, \iota_{\bar{\mathbb{E}}})}, \quad \lambda_{\mathbb{X}} = (\lambda_0, \lambda_0^{n-1}).$$

Examples: i) $n=1$. Then $\mathcal{N}_1 = \mathrm{Spf}\, O_{\breve F}$. (Drinfeld) : $\mathbb{E}$ LT

ii) $n=2$. Then $\mathcal{N}_2 \simeq \mathrm{Spf}\, O_{\breve F}[[T]]$ (Lubin-Tate)

iii) $n=3$. Then $(\mathcal{N}_3)_{red}$ beautiful structure (Vollaard).

Now define $\mathcal{N}_n(S) = \{(X, \iota, \lambda; \rho) \mid \rho : X \times_S \bar S \longrightarrow \mathbb{X} \times_{\mathrm{Spec}(\bar k)} \bar S$ quasi-isog. of ht 0, resp. ι, λ up to $c \in O_{F_0}^\times$, $c = c(\rho)$ $\}$.

Thm: Formally smooth rel. $\dim = n-1$, loc. proper.

Now define

$$\delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_n$$
$$(X, \rho) \longmapsto (X \times \bar{\mathbb{E}},\ \rho \times \rho_0^{c^{-1}})$$

where $c = c(\rho)$.

~~For $g \in U$~~ Note: $\mathrm{Aut}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) = U_1(F_0)$.

Also

$$\Delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_n$$

"middle dimension"

$$\Delta_g = (\mathrm{id} \times g)^*(\Delta)$$

Lemma (WZ): Let $g \in U_1(F_0)_{rs}$, $F_0 = \mathbb{Q}_p$. Then $\Delta \cap g^*(\Delta_g)$ $(\mathrm{id}\times g)^*$ is ~~proper and~~ a scheme (i.e. ~~defining~~ ideal of def. is nilpotent) which is proper (and has support in special fiber) non-degenerate.

We may thus define

$$\mathrm{Int}(g) = \chi(O_\Delta \otimes^{\mathbb{L}} O_{g^*(\Delta)}) \log q \quad \text{integer.}$$

$g^*(\Delta) = \Delta_g$

~~Conject~~

<u>Remark</u>: The scheme $\Delta \cap g^{*}(\cancel{\Delta})$ Δ_g is the following locus inside $\mathcal{N}_{n-1} \times \mathcal{N}_n$: locus of $(X, \varrho_Y, X, \varrho_X)$ where

$$X \simeq Y \times \bar{E}$$

and where $\mathrm{proj}_j \circ g \circ \mathrm{inj}_i$ lifts to endomorphism, e.g.

$$X_{n-1} \hookrightarrow X_{n-1} \times \bar{E} = X_n \xrightarrow{g} X_n \longrightarrow X_{n-1}$$

lifts to $Y \longrightarrow Y$.

Note that this makes sense $\forall g \in \mathrm{End}(X_n)$.

<u>Conjecture</u> ~~AFL~~

a) [JR] $f' = \mathbb{1}_{S(O)}$ transfers to (f_0, f_1) where $f_1 = 0$ and

$f_0 = \mathbb{1}_{U_0(O)}$, explain i.e. in part. $\mathrm{Orb}_\gamma(f') = 0$, falls $g \in U_{1,rs}$

b) [WZ] If $\gamma \overset{S_{rs}}{\longleftrightarrow} g \in U_{1,rs}$, then (in add. to $\mathrm{Orb}_\gamma(f') = 0$),

AFL ~~$\omega(\gamma)$~~ ?

$$\mathrm{Int}_g(l_g) = -\,\omega(\gamma)\, \partial \mathrm{Orb}_\gamma(\underbrace{\mathbb{1}_{S(O)}}_{f'}).$$

<u>Theorem 1</u> (WZ) Let $n = 2$ or 3. Then the intersection of Δ and $g^*(\Delta)$ is ~~non-degenerate~~ proper, i.e. $\Delta \cap g^*\Delta$ is artinian. There are no higher Tor-terms in X, and AFL holds.

Relation to Kudla program (cf. Kudla's lectures):

Recall the added vector u with $(u,u)=1$. If $g \in U_1(F_0)_{rs}$,

$$u, gu, \dots, g^{n-1}u \quad \text{basis of } F^n.$$

Now $u \in \mathbb{V} = \mathrm{Hom}(\bar{\mathbb{E}}, \mathbb{X}_n) = V(J_1 \otimes 1) \mapsto$

$$Z(u), Z(gu), \dots, Z(g^{n-1}u) \quad \text{special (KR-) divisors.}$$

explain?

And

$$\Delta \cap g^*(\Delta) \subset Z(u) \cap Z(gu) \cap \dots \cap Z(g^{n-1}u).$$

In general a strict inclusion, but gives upper bound.

Let

$$L = L_g = \mathrm{span}_{O_F} \langle u, gu, \dots, g^{n-1}u \rangle \quad \text{lattice in } F^n.$$

Let $L \subset L^\vee \mapsto \mathrm{inv}(g) = \text{elem. divisors of } (L^\vee, L)$

$$\in (\mathbb{Z}^n)_{\geq 0}$$

$\Rightarrow \mathrm{inv}(g) = (\mu_1, \mu_2, \dots, \mu_{n-1}, 0)$.

Thickness Conjecture (RTZ): $\varpi^{\mu_1} \cdot \mathcal{O}_{\cap Z} = (0)$.

Thm (RTZ): $F_0 = \mathbb{Q}_p$. Assume $\mu_1 = 1$. Then

(i) Thick-Conjecture holds, and $\cap Z$ is a scheme.

(ii) $\Delta \cap g^*\Delta$ is artinian, there are no higher Tor-terms and AFL holds, provided that $p \geq \frac{n}{2} + 1$.

(iii) $\exists$ simple criterion (in terms of $\bar{g}: L^\vee/L \circlearrowleft$) to decide when $\mathrm{Int}(g) \neq 0$.

<u>Lie algebra version.</u> For $x \in \mathfrak{u}_1$ rs Δ_x and

$$\mathrm{Int}_x(\mathfrak{u}_1) = \chi(\mathcal{O}_\Delta \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\Delta_x}).$$

If $x \in \mathfrak{u}_{1,rs}$ ($\to$ WZ's lecture), then $\mathrm{Int}(x)$ finite.

Have transfer factor $\omega : \mathfrak{s}_{rs} \longrightarrow \mathcal{O}^\times$: "same" formula

<u>Conjecture</u>: a) $\varphi' = \mathbb{1}_{\mathfrak{s}(\mathcal{O})}$ transfers to (φ_0, φ_1), where

$$\varphi_0 = \mathbb{1}_{\mathfrak{u}_0(\mathcal{O})}, \quad \varphi_1 = 0$$

b) If $y \in \mathfrak{s}_{rs}$ matches $x \in \mathfrak{u}_1(F_0)_{rs}$, then

$$\omega(y)\, \partial \mathrm{Orb}_y(\varphi') = - \mathrm{Int}_x(\mathfrak{u}_1).$$

This conjecture seems easier to prove.

<u>Lemma</u>: (i) Let $\xi \in F^1$. The Cayley transform defines isomorphism

$$\alpha \quad \mathfrak{s}_n \setminus D_1 \xrightarrow{\sim} S_n \setminus D_\xi$$

$$\det(\xi \cdot 1 - s) \neq 0$$

Sth. similar holds for $(\mathfrak{u}_i, U_i)$, $i = 1, 2$.

(ii) Let $\xi_1, \dots, \xi_{n+1} \in F^1$ distinct elements.

Lie algebra version: Let $x \in \mathfrak{u}_1 \mapsto \Delta_X$, and

$$\mathrm{Int}(X) = \chi(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_X}).$$

again finite if X in $(\mathfrak{u}_1)_{rs}$.

x rs ($\to$ W. Zhang's lecture).

Transfer factor:

Conjecture:

a) $\varphi' = \mathbb{1}_{\mathfrak{s}(\mathcal{O})}$ transfers to (φ_0, φ_1) where $\varphi_1 = 0$ and

$$\varphi_0 = \mathbb{1}_{\mathfrak{u}_0(\mathcal{O})}.$$

b) If $y \in \mathfrak{s}(F_0)_{rs}$ matches $x \in \mathfrak{u}_1(F_0)_{rs}$, then

$$\omega(y)\, \partial \mathrm{Orb}_y(\varphi') = -\mathrm{Int}_x.$$

Transfer factor: only depends on extension η' of η to $F^\times$.

Lemma (W. Z.): i) Let $\xi \in E^1$. The map

$$\alpha_\xi : M_{n+1}(F) \setminus \{\det(1-x) \neq 0\} \longrightarrow GL_{n+1}(F)$$
$$x \longmapsto -\xi \cdot (1+x) \cdot (1-x)^{-1}$$

induces an isomorphism, which is $H'(F_0)$-equiv.,

$$\mathfrak{s}_n(F_0) \setminus \{\det(1-x) \neq 0\} \longrightarrow S_n(F_0) \setminus \{\det(\xi 1 - x) \neq 0\}$$

[Something analogous holds for $(\mathfrak{u}_i(F_0)$, resp. $U_i(F_0))$.

The map is compat. with transfer factors defined by η'.

ii) $\exists\ \xi_1, \dots, \xi_n \in E^1$ s.t. $\Rightarrow$ Im α_{ξ_i} open cover of $S_n(F_0)$.

(iii) Let E/F unramified. ~~The ξ^i can be chosen~~ in $\mathcal{O}_F / E^1$,

in ~~which case get $i\infty$'s~~