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Talk: On variants of the LT and the Drinfeld moduli pbs

Mitte

Notation: $F/\mathbb{Q}_p$ of degree d. $\bar{k}$ = alg closure of residue field.

Def: Let $S \in (Sch/O_F)$. A formal O_F-module over S = p-divis. formal gp X/S + $\iota: O_F \to End(X)$ s.t. induced O_F-action on $Lie\, X$ = via structure morphism $O_F \to O_S$

Comes up e.g. in proof of existence thm of LCF of Lubin-Tate.

Fix $n \geq 2$.

Moduli pb. of LT-type, resp. D-type

Links

LT: Consider (X, ι) formal O_F-module of height nd, dim 1.

Easy: Over $\bar{k}$ unique upto O_F-isogeny of height 0 $\leadsto (\mathbb{X}, \iota_{\mathbb{X}})$.

Let $O_{\breve{F}} \to \bar{k}$, and $Nilp_{O_{\breve{F}}}$. Functor

$$\mathcal{N}^{LT}: S \mapsto \{(X, \iota_X, \varrho) \mid \varrho: X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{Spec\, \bar{k}} \bar{S} \ O_F\text{-quis ht } 0\}$$

Thm (LT): $\mathcal{N}^{LT}$ is represented by a formal scheme isom. to

$$Spf\, O_{\breve{F}}[[t_1, \ldots, t_{n-1}]].$$

rechts

D: Fix central div-algebra D over F, with inv$(D)=\frac{1}{n}$.

Consider (X, ι_D), where X formal O_F-module of ht $n^2 d$, dim n, with $\iota_D : O_D \to \mathrm{End}(X)$ prolongs O_F-action and s.t.

$$\mathrm{char}_{O}(T; \iota(a) \mid \mathrm{Lie}\, X) = \mathrm{chard}(a)(T), \quad \forall a \in O_D.$$

Easy: Over $\bar{k}$ unique upto O_D-isogeny of ht 0: $(\mathbb{X}, \iota_D)$.

Functor on $\mathrm{Nilp}_{O_{\breve F}}$:

$$\mathcal{N}^D : S \mapsto \{ (X, \iota_D, \varrho) \mid \varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar k} \bar{S} \ O_D\text{-quasi ht }0 \}$$

Thm (D): $\mathcal{N}^D$ is represented by $\hat{\Omega}^n_F \hat{\otimes}_{O_F} O_{\breve F}$.

Mitte

Note: formal O_F-modules rare in nature. E.g., let $A/\bar{k}$ be abel var. with action by O_F, F tot-real s.t. p inert in F.

Kudla!

→ If (A, ι) lifts to char 0 $\Rightarrow$ Lie A is free $O_F \otimes k$-module.

Aim: Enlarge cat. of formal O_F-modules s.t. LT/D-thms still valid.

Joint w. Zink.

fixed

Let $\Phi = \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$. Fix $\varphi_0 \in \Phi$.

Fix $r : \Phi \longrightarrow \mathbb{Z}$ s.t.

$$r_\varphi = \begin{cases} 1 & \varphi = \varphi_0 \\ n \text{ or } 0 & \varphi \neq \varphi_0. \end{cases}$$

$\rightsquigarrow$ reflex field $E = E_r \subset \overline{\mathbb{Q}}_p$ with

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p / E) = \{ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\sigma\varphi} = r_\varphi, \ \forall \varphi \in \Phi \}$$

Let $S \in (Sch/O_E)$. Consider

LT: $(X, \iota)/S$ again ht $X = nd$, but now

$$\mathrm{char}(T, \iota(a) \mid \mathrm{Lie}\, X)(T) = \prod_{\varphi \in \Phi} (T - \varphi(a))^{r_\varphi}, \quad a \in O_F.$$

D: $(X, \iota_D)/S$

$$\mathrm{char}(T, \iota(a) \mid \mathrm{Lie}\, X)(T) = \prod_{\varphi \in \Phi} \mathrm{chard}(a)(T)^{r_\varphi}, \quad a \in O_D.$$

Get analogues $\mathcal{N}_r^{LT}$, resp. $\mathcal{N}_r^{D}$ ($/\bar{k}$-uniqueness by Kottwitz)

Note: $F \xhookrightarrow{\varphi_0} E$, $\breve{F} \hookrightarrow \breve{E}$

$$F \otimes_{F^t} W \to E \otimes_{F^t} W \to E \otimes_{E^t} W.$$

Thm 1: Let $F/\mathbb{Q}_p$ unramified. Then

(i) $\mathcal{N}_r^{LT} \simeq \mathcal{N}^{LT} \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{E}}$, $\mathcal{N}_r^{D} \simeq \mathcal{N}^{D} \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{E}}$

(ii) $\mathcal{N}_{r_0}^{LT} = \mathcal{N}^{LT}$ and $\mathcal{N}_{r_0}^{D} = \mathcal{N}^{D}$ (equality!).

Remarks: (i) The construction of the iso is very indirect, uses theory of relative displays (Ahsendorf, Zink)

(ii) If $F/\mathbb{Q}_p$ ramified, then $\mathcal{N}_r^{LT}$ and $\mathcal{N}_r^{D}$ not flat.

But generic fibers are OK:

Thm 2 (Scholze): $\left(\mathcal{N}_r^{D}\right)^{rig} \simeq \left(\mathcal{N}^{D} \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{E}}\right)^{rig}$.

Sketch of proof: Via crystalline period maps, both sides are open adic subsets of $\mathbb{P}^{n-1}_{\breve{E}}$. Hence suffices to see that C-valued points are identical. Now use description of p-div. grps $/\mathcal{O}_C$ by Scholze-Weinstein, via modific. of trivial vb on FF-curve. For r add trivial modific.

$$\begin{cases} \mathcal{E} \twoheadrightarrow \mathcal{E} \otimes \kappa(\varphi) & r_\varphi = n \\ \mathcal{E} \twoheadrightarrow 0 & r_\varphi = 0. \end{cases}$$

For integral theory, need to define closed formal subschemes $(\mathcal{N}_r^{LT})' \subset \mathcal{N}_r^{LT}$, resp. $(\mathcal{N}_r^{D})' \subset \mathcal{N}_r^{D}$:

Further notation: Let F^t = max. unr. sub., let

$$\Psi = \mathrm{Hom}_{\mathbb{Q}_p}(F^t, \overline{\mathbb{Q}}_p) \ni \psi_0$$

Pb: rep'n of $\pi \in F$ not semi-simple if $F/\mathbb{Q}_p$ ramified.

Let $\pi \in F$ uniformizer, with min. polyn. $Q(T) \in O_{F^t}[T]$.

Fix $\psi \in \Psi \mapsto \psi(Q(T)) = \prod_{\substack{\varphi \in \Phi \\ \varphi|F^t = \psi}} (T - \varphi(\pi))$

fact. in $O_E[T]$

For $\psi \neq \psi_0$ $\quad \psi(Q(T)) = Q_{A_\psi}(T) \cdot Q_{B_\psi}(T)$, where

$$A_\psi = \{ \varphi \in \Phi \mid \varphi|_{F^t} = \psi,\ r_\varphi = n \}$$
$$B_\psi = \{ \qquad\qquad\qquad = 0 \}$$

For ψ_0 $\quad \psi_0(Q(T)) = Q_0(T) \cdot Q_{A_{\psi_0}}(T) \cdot Q_{B_{\psi_0}}(T)$,

with $Q_0(T) = T - \varphi_0(\pi)$.

For $S \in (Sch/O_E)$ have $\quad O_{F^t} \otimes_{\mathbb{Z}_p} O_S = \bigoplus_{\psi \in \Psi} O_S$.

Eisenstein conditions on $\mathrm{Lie}\, X = \bigoplus \mathrm{Lie}_\psi X$ are

For $\psi \neq \psi_0$: $Q_{A_\psi}(\iota(\pi) \mid \mathrm{Lie}_\psi X) = 0$

For ψ_0 : $\begin{cases} Q_0 \cdot A_{\psi_0}(\iota(\pi) \mid \mathrm{Lie}_{\psi_0} X) = 0 \quad \text{and} \\ \text{for LT : } \operatorname{rk} Q_{A_{\psi_0}}(\iota(\pi) \mid \mathrm{Lie}_{\psi_0} X) \leq 1 \\ \text{for D} \qquad\qquad\qquad\qquad \leq n. \end{cases}$

<u>Thm 3</u>: a) $(\mathcal{N}_r^{LT})' \simeq \mathcal{N}^{LT} \hat{\otimes}_{O_{\breve{F}}} O_{\breve{E}}$.

b) $(\mathcal{N}_r^D)'$ is a flat normal p-adic formal scheme over $O_{\breve{E}}$, with special fiber isom. to $\mathcal{N}^D \otimes \bar{k}$. Furthermore, the $'$-subscheme of $\mathcal{N}_r^D$ indep't of π, and is equal to $\mathcal{N}_r^D$ if $F/\mathbb{Q}_p$ unramified.

<u>Remarks</u>: Uses display theory + theory of local models.

<u>Conjecture</u>: $(\mathcal{N}_r^D)' \simeq \mathcal{N}^D \hat{\otimes}_{O_{\breve{F}}} O_{\breve{E}}$.

Need to understand better relation between SW-theory + display theory.

"Proof" for $(\mathcal{N}_r^D)'$ over $\bar{k}$; via Dieudonné theory (linear algebra):

Assume $F/\mathbb{Q}_p$ tot. ramified. $\mathcal{D}$-module (M, F, V) with

$$pM \subset \underbrace{VM \subset M}_{an^2+n}$$

$\Leftarrow$ det-condition.

$Q_A(T) \equiv T^a \bmod p$. + Eisenstein

$$\Pi^{a+1} M \subset VM \subset \Pi^a M$$

Now form relative Dieudonné module (M', F', V') with

$$M' = M = W_{O_F}(\bar{k})\text{-module of rk } n^2$$

$$V' = \Pi^{-a} \cdot V, \quad F' = \Pi \cdot (V')^{-1}.$$

Then M' corresp. s.f. O_D-module.

<u>Lemma</u>: R commut. ring. Let W = loc. free R-module of rk n.

Let

$$f : W \longrightarrow W$$

s.t. Coker f loc. free of rank r. Let $V \subset W$ direct summand of rank m. Assume that $s \underset{\text{Def}}{=} m - r \geq 0$ and that

$$\Lambda^{s+1}(f|V) = 0$$

Then $\operatorname{Ker} f \subset V$.