Paderborn, Oct. 13

Talk: On formal moduli spaces of p-divisible groups

Notation: $\underbrace{D / F}_{d^2} \underbrace{/ \mathbb{Q}_p}_{m}$

$$r : \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p) \longrightarrow \mathbb{Z}_{\geq 0}, \quad \varphi \longmapsto r_\varphi .$$

Reflexfield: $\mathrm{Gal}(\overline{\mathbb{Q}}_p / E) = \{ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mid r_{\sigma\varphi} = r_\varphi, \forall \varphi \}$

$\breve{E}$, with residue field $\bar{k}$.

Fix $n \geq 1$, moduli pb. comes in two variants

EL-variant: For $S \in (\mathrm{Nilp}/O_{\breve{E}})$, consider pairs (X, ι), where

- X = p-divis. gp / S of height $n \cdot m \cdot d^2$ and dimension $d \cdot \sum_\varphi r_\varphi$

  (assume $d \cdot \sum_\varphi r_\varphi \leq n \cdot m \cdot d^2$)

- $\iota : O_D \longrightarrow \mathrm{End}(X)$, s.t.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = \prod_\varphi \varphi(\mathrm{chard}(a))^{r_\varphi}, \quad \forall a \in O_D$$

Fix such a pair $(\mathbb{X}, \iota_{\mathbb{X}})$ over $\mathrm{Spec}\, \bar{k}$ (framing object).

$$\mathcal{N}(S) = \{ \text{iso-classes of } (X, \iota, \rho) \mid \rho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \bar{S} \ \ O_D\text{-quis of height } 0 \}$$

Thm: $\mathcal{N}$ is represented by formal scheme, loc. formally f.t. / $\mathrm{Spf}\, O_{\breve{E}}$.

(Sometimes add further conditions on (X, ι)).

PEL-variant: Fix involution $*$ on D, let $F_0 = F^{\langle * \rangle}$.
$p \neq 2$

Assume $d(r_\varphi + r_{\varphi^*}) = n d^2$, $\forall \varphi$.

For $S \in (\mathrm{Nilp}/O_{\breve{E}})$, consider (X, ι, λ) where $\lambda : X \longrightarrow X^\vee$

polariz. s.t.

- $\mathrm{Rosati}_\lambda \mid O_D = *$
- $\mathrm{Ker}\,\lambda \subseteq X[\pi_D]$ of fixed order.

Fix framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathrm{Spec}\, \bar{k}$.

$$\mathcal{N}(S) = \{ \text{iso. cl. of } (X, \iota, \lambda) \mid \varrho^*(\lambda_{\mathbb{X}}) \sim \lambda \mid X \times_S \bar{S} \}.$$

Identification problem: What does $\mathcal{N}$ "look like"?

3 possible interpretations:

1.) What is $\mathcal{N}^{rig}$ = rigid space / $\breve{E}$

2.) What is $\mathcal{N}_{red}$ = alg. var. / $\bar{k}$

3.) What is $\mathcal{N}$?

Ad 1) When is $\mathcal{N}^{\mathrm{rig}} \neq \emptyset$?

<u>Example</u>: PEL-type, $D = F \neq F_0 = \mathbb{Q}_p$ ramified, $m = n = 2$,

$r_{\varphi_0} = r_{\varphi_1} = 1$, $|\mathrm{Ker}\,\lambda| = p^2$

Let $\mathcal{E}/\bar{k}$. Then $\mathrm{End}(\mathcal{E}) = O_B$, fix $O_F \hookrightarrow O_B$.

Let $\mathbb{X} = \mathcal{E} \times \mathcal{E}$, $\iota_{\mathbb{X}}(a) = \mathrm{diag}(a, \bar{a})$,

$\lambda_{\mathbb{X}} = (\lambda_{\mathcal{E}} \times \lambda_{\mathcal{E}}) \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \iota_{\mathcal{E}\times\mathcal{E}}(\pi_F \cdot \zeta) \cdot \mathrm{diag}(u_0, u_1)$,

where $\zeta \in O_B^\times$ s.t. $\zeta \cdot a \zeta^{-1} = a^*$, $\forall a \in O_F$

$u_0, u_1 \in \mathbb{Z}_p^\times$ s.t. $-u_0 u_1 \notin \mathrm{Nm}\, F^\times$.

<u>Then</u>: $\mathcal{N} = \{pt\}$, $\mathcal{N}^{\mathrm{rig}} = \emptyset$.

see paper w. Viehmann for precise conjectural condition of $\mathcal{N}^{\mathrm{rig}} \neq \emptyset$.

Ad 2) Let $J = \mathrm{End}^\circ(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})^\times$: LAG $/\mathbb{Q}_p$.

<u>Question</u>: When $\exists$ stratification of $\mathcal{N}_{\mathrm{red}}$ s.t.

- combinatorics of strata $\overset{\text{"descr. in terms of"}}{\longleftrightarrow}$ $\mathcal{B}(J, \mathbb{Q}_p)$
- each stratum is isom. to DL-variety for some LAG $G'/\mathbb{F}_q$.
  (varies w. stratum)

Always take $\mathbb{X}$ <u>basic</u> ("=" <u>supersingular</u>).

label	case	names
$GL^*_{d-1,1}$ (EL)	$D = D_{\frac{1}{d}}$, $F = \mathbb{Q}_p$, $r_{\varphi_0} = 1$, $n = 1$	Drinfeld
$GL_{2,2}$	$D = F = \mathbb{Q}_p$, $r_{\varphi_0} = 2$, $n = 4$	Görtz / He
$GU_{d-1,1}$ (PEL)	$D = F \neq F_0 = \mathbb{Q}_p$, $r_{\varphi_0} = 1$, $r_{\varphi_1} = d-1$ λ principal $F/\mathbb{Q}_p$ un. $F/\mathbb{Q}_p$ ram.	Vollaard / Wedhorn Rapo / Terstiege / Wilson
$GU_{2,2}$	$D = F \neq F_0 = \mathbb{Q}_p$, $r_{\varphi_0} = r_{\varphi_1} = 2$ λ principal $F/\mathbb{Q}_p$ un.	Howard / Pappas
GSp_4	$D = F = F_0 = \mathbb{Q}_p$, $r_{\varphi_0} = 2$ λ principal	Katsura / Oort, Kaiser, Kudla / Rapo

In context of ADLV: essent. complete list (Görtz / He)

Transpose this list! Makes sense since Wansu Kim. Haifeng Wu, semi-stable case. AFL

Non-examples: GSp_{2n}, $n \geq 3$, $GU(p,q)$,

Ad 3): Fix $\varphi_0 : F \longrightarrow E$ (assume exists).

Definition: Let $(X, \iota_F)/S$. Sc Formal O_F-module $\overset{=}{Df}$

X = formal p-divis. gp s.t.

$$\iota_F(a) \mid \mathrm{Lie}\, X = \varphi_0(a) . \mathrm{id}_{\mathrm{Lie}\, X}, \quad \forall a \in O_F.$$

Hence $r_\varphi = 0$, $\forall \varphi \neq \varphi_0$, and $E = F$.

Remark: Converse holds if $F/\mathbb{Q}_p$ unramified.

Now let $D = D_{\frac{1}{d}}/F$. Consider EL-type $\mathcal{N} = \mathcal{N}_{Dr, F}$:

here $n = 1$ and

s.f. O_D-module

$$r_\varphi = \begin{cases} 1 & \varphi = \varphi_0 \\ 0 & \varphi \neq \varphi_0. \end{cases}$$

and $\iota \mid O_F$ defines structure of formal O_F-module.

Thm (Drinfeld): a) framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ "unique".

b) $\mathcal{N}$ isomorphic to $\hat{\Omega}^d_F \hat{\otimes}_{O_F, \varphi_0} O_{\breve{E}}$

Here $\hat{\Omega}^d_F$ = Deligne / Drinfeld rel. to F :

- π-adic formal scheme, w. semi-stable reduction.
- $(\hat{\Omega}^d_F)_{red}$ satisfies criterion in 2.)
- $(\Omega)^{rig} = \mathbb{P}^{d-1}_F \setminus \bigcup H$

<u>formal O_F-modules from abelian varieties:</u> rare

<u>Joint w. Zink</u>: extend Drinfeld's result to more realistic situations.

<u>Thm (R/Zink)</u>: Consider EL-case, where $D = D_{\frac{1}{d}}$, $n = 1$, and

$$r_\varphi = \begin{cases} 1 & \varphi \neq \varphi_0 \\ 0 \text{ or } d & \varphi \neq \varphi_0 \end{cases}.$$

Assume $F/\mathbb{Q}_p$ unramified. Then

a) $(\mathbb{X}, \iota_{\mathbb{X}})$ unique

b) $\mathcal{N} = \mathcal{N}_{D, F}$, hence $\mathcal{N} \simeq \hat{\Omega}^d_F \hat{\otimes}_{O_F} O_{\breve{E}}$.

Uses displays (<u>relative version</u>): the process that assoc. to (X, ι) a s.f. O_D-module is <u>totally mysterious</u>! Uses Ahsendorf / Zink.

When $F/\mathbb{Q}_p$ ramified, $\mathcal{N}$ is not good.

Consider $\overbrace{F / F^u}^{e} \overbrace{/ \mathbb{Q}_p}^{f}$. Let $\pi \in O_F$, with Eisenstein

$$Q(T) \in O_{F^u}[T] \quad \text{of degree } e.$$

Let σ^i $(i \in \mathbb{Z}/f)$ powers of Frobenius on F^u. Set

$$A_i = \{ \varphi \mid \varphi_{|F^u} = \sigma^{-i}, \ r_\varphi = d \}$$

$$B_i = \{ \text{———————} , \ r_\varphi = 0 \}.$$

Get decompositions in $O_E[T]$:

$$\varphi_0(Q(T)) = Q_0(T) \cdot Q_{A_0}(T) \cdot Q_{B_0}(T)$$

$$\varphi_0 \circ \sigma^{-i}(Q(T)) = Q_{A_i}(T) \cdot Q_{B_i}(T), \qquad i \neq 0.$$

Here $Q_0(T) = T - \varphi_0(\pi)$, $\quad Q_{A_i}(T) = \prod_{\varphi \in A_i} (T - \varphi(\pi))$, ...

<u>Eisenstein conditions:</u> Via $O_{F^u} \otimes_{\mathbb{Z}_p} O_F \simeq \bigoplus_{i \in \mathbb{Z}/f} O_F$,

get

$$\mathrm{Lie}\, X = \bigoplus_{i \in \mathbb{Z}/f} \mathrm{Lie}_i X, \quad O_D\text{-acts on } \mathrm{Lie}_i X.$$

<u>Impose:</u>

- $\bigwedge^{d+1} Q_{A_0}(\iota(\pi) \mid \mathrm{Lie}_0 X) = 0.$
- $Q_0(\iota(\pi) \mid \mathrm{Lie}_0 X) \cdot Q_{A_0}(\iota(\pi) \mid \mathrm{Lie}_0 X) = 0$
- $Q_{A_i}(\iota(\pi) \mid \mathrm{Lie}_i X) = 0, \qquad \forall i \neq 0.$

Denote corresp. formal scheme by $\mathcal{N}^*$.

<u>Remark</u>: If $F/\mathbb{Q}_p$ unramified, then (Eis) automatic, i.e. $\mathcal{N} = \mathcal{N}^*$.

<u>Thm (R/Zink)</u>: a) $(\mathbb{X}, \iota_{\mathbb{X}})$ unique.

b) $\mathcal{N}^*$ is flat over $\mathrm{Spf}\, O_{\breve{E}}$, and π-adic, and

$$\mathcal{N}^* \otimes_{O_{\breve{E}}} k \simeq \mathcal{N}_{Dr, F} \otimes_{O_{\breve{E}}} k.$$

c) $$(\mathcal{N}^*)^{rig} = \Omega_F^d \otimes_F \breve{E}.$$

<u>Corollary</u>: $\mathcal{N}^*$ indep't of π.

<u>Conjecture</u>: $\mathcal{N}^* \simeq \mathcal{N}_{Dr, F}$.

In the proof, we use the following nice lemma.

<u>Lemma</u>: Let N loc. free R-module of rank n. Let $f: N \to N$ s.t. $\mathrm{Ker} f$ loc. direct summ. of rank r. Let $M \subset N$ loc. direct summand of rank m, which is f-stable. Assume

$$\Lambda^{s+1}(f_{|M}) = 0, \quad s = m - r \ (\geq 0).$$

Then $\mathrm{Ker} f \subset M$.