Student part of talk in Columbia Number theory

1.) A comm. ring, I ~~ideal(A)~~. Form

$$\hat{A} = \varprojlim A/I^n \qquad \text{complete + separ. for } I\cdot\hat{A}\text{-adic topology.}$$

often easier structure than A itself.

Example: ~~A~~ A local ring, I = maximal ideal. If A regular + contains a field k ~~projecting isom. to A/I~~, then

$$\hat{A} \simeq k[[X_1, \dots, X_n]] \quad \text{power series}, \; n = \dim A$$
k = residue field

2.) Globalisation: pass from Spec A to arbitrary, not nec. affine, scheme X ($X \supset Y = V(I)$). Let (A, I) as before, let $X / \mathrm{Spec}\, A$ scheme $\longmapsto \hat{X} = (X_n)_n$, $X_n = X \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A/I^n$.

If $X = \mathrm{Spec}\, A$, get back previous, $\mathrm{Spf}\, \hat{A}$.

leads to concept of formal scheme $(X_n)_n$. Hence any Spec A-scheme X defines a formal scheme $\hat{X}$.

Theorem: The ~~functor~~ assoc. $X \longmapsto \hat{X}$ defines a fully faithful functor

$$\left(\text{proper } \operatorname{Spec} A\text{-schemes}\right) \longrightarrow \left(\text{formal schemes} / \operatorname{Spf} A\right)$$

In general, image is not known. But proper smooth

$$\begin{pmatrix}\text{proper curves of} \\ \text{genus } g \geq 2 \ /A\end{pmatrix} \simeq \begin{pmatrix}\text{proper curves } X_n / \\ \text{of genus } g \geq 2 \text{ over } \operatorname{Spf} A\end{pmatrix}$$

used by Grothendieck in def. $\hat{\pi}_1^p(X)$

3.) <u>Big disadvantage at first sight of formal schemes</u>: Assume

that $A = DVR$, $I = m$ max. ideal. Would like to

define $\quad \mathfrak{X} \times_{\operatorname{Spec} \hat{A}} \operatorname{Spec} \hat{K}$, $\quad K = \operatorname{Frac}(\hat{A})$.

Let $\mathfrak{X} = \hat{X}$, where $X = \operatorname{Spec} B$ with B A-alg. of f.t.

Then we would take $\mathfrak{X} \otimes_A K = \operatorname{Spm}(\hat{B} \otimes_{\hat{A}} \hat{K})$

This is difficult to globalize (naive non-arch. open discs are closed).

<u>Key</u>: $\operatorname{Spm}(\hat{B} \otimes_{\hat{A}} \hat{K}) \leftrightarrow \operatorname{Spf} V$ closed formal subscheme of $\operatorname{Spf} \hat{B}$, finite flat integral.

RHS makes sense for all formal schemes ($\mathfrak{X}_m$ over $\operatorname{Spf} \hat{A}$.

Can put (Grothendieck) topol. + structure sheaf on this set ~>

$\mathfrak{X}^{rig}$ rigid-analytic variety

Close analogy with complex-analytic spaces (/ $\mathbb{C}$).

Recall Serre's GAGA-functor: X = alg. var. / $\mathbb{C}$

$\mapsto X^{cx}$ = complex space

<u>Theorem</u>: The assoc. $X \mapsto X^{cx}$ induces a fully faithful

$$\left(\text{proper alg. var.} / \mathbb{C}\right) \longrightarrow \left(\text{compact complex spaces}\right).$$

In general, image not known.

<u>Analogously</u>: X = alg. var. / $\hat{K}$ $\mapsto X^{rig}$ = rigid space over $\hat{K}$

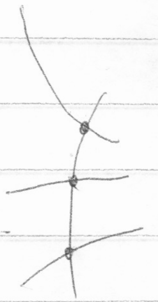
$$\left(\text{proper alg. var} / \hat{K}\right) \longrightarrow \left(\text{proper rigid spaces} / \hat{K}\right).$$

$\angle$: If $X = \operatorname{Spec} B$. Can do 2 things:

$$\left.\begin{array}{l} \operatorname{Spec}(B) \longmapsto \operatorname{Spf} \hat{B} \longmapsto (\operatorname{Spf}(\hat{B}))^{rig} \\ \operatorname{Spec} B \longmapsto (\operatorname{Spec} B)^{rig} \end{array}\right\} \text{ \underline{not} the same thing!}$$

4.) <u>An example of non-algebraic rigid space</u>

Start with $\overset{V=\mathbb{Z}_p}{V \in \hat{\mathcal{U}}}$ and $X^0 = \mathbb{P}^1_{\mathbb{Z}_p}$. ~~Assume~~ The $X^0(\mathbb{F}_p)$ has finitely many pts. Let $X^1 = bl_{X^0(\mathbb{F}_p)}(X^0)$.



Repeat in special points, continue to get $X^0, X^1, X^2, \dots$

$\hat{X}^0, \hat{X}^1$ stabilizes, get $\hat{\Omega}^2_{\mathbb{Q}_p} = \varinjlim \hat{\mathcal{U}}^i$.

$\hat{\mathcal{U}}^0 \subseteq \hat{\mathcal{U}}^1$

Drinfeld's formal scheme.

<u>Observation 1</u>: $(\hat{\Omega}^2_{\mathbb{Q}_p})^{rig} = (\mathbb{P}^1_{\mathbb{Q}_p})^{rig} \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

Indeed, a point of $\mathbb{P}^1(\mathbb{Q}_p)$ can be extended to $\mathbb{P}^1(\mathbb{Z}_p)$, but specializ. is never left in peace!

<u>Observation 2</u>: $(\hat{\Omega}^2_{\mathbb{Q}_p})_{red}$ = union of $\mathbb{P}^1_k$'s. Each ~~line~~ $\mathbb{P}^1$ meets $p+1$ other $\mathbb{P}^1$'s, namely $\mathbb{P}^1(\mathbb{F}_p)$. Two $\mathbb{P}^1$'s intersect $\Rightarrow$ in 1 point, each ~~into~~ point of $\mathbb{P}^1(\mathbb{F}_p)$ lies on 2 lines.

In fact, dual graph of $(\hat{\Omega}^2_{\mathbb{Q}_p})_{red}$ = BT-tree $(PGL_2(\mathbb{Q}_p))$

$\Gamma \subset PGL_2(\mathbb{Q}_p)$ discrete cocompact $\Rightarrow$ acts prop. disc. for Z-top. on $\hat{\Omega}_{\mathbb{Q}_p}$, quotient is p-adic completion of proj. curve, cf. supra