Zürich, 14.12.12

Talk: On the Drinfeld halfspace.

Old subject, but new light in joint work w. Kudla.

F = p-adic field , $(O_F, \pi, k = \mathbb{F}_q)$.

(1) Drinfeld halfplane assoc. to F

$$\Omega_F = \Omega_F^2 = \mathbb{P}_F^1 \setminus \mathbb{P}^1(F)$$

rigid-analytic variety / F : $\Omega_F = \varinjlim U(\varepsilon)$, $U(\varepsilon) = \mathbb{P}_F^1 \setminus \bigcup_{x \in \mathbb{P}^1(F)} B_x(\varepsilon)$.

admissible open covering.

Formal model $\hat{\Omega}_F$ of Ω_F: formal scheme / O_F, defined by Mumford, Deligne, Drinfeld.

Obtained by gluing open formal subschemes $\hat{\Omega}_\Delta$, with Δ running through simplices of B-T-tree of $PGL_2(F)$, with

$$\hat{\Omega}_\Delta \cap \hat{\Omega}_{\Delta'} = \begin{cases} \hat{\Omega}_{\Delta \cap \Delta'} & \text{if } \Delta \cap \Delta' \text{ simplex} \\ \emptyset & \text{if not.} \end{cases}$$

Furthermore, action of $PGL_2(F)$ on $\hat{\Omega}$ s.t.

$$g\hat{\Omega}_\Delta = \hat{\Omega}_{g\Delta}$$

Finally, $\hat{\Omega} \otimes_{O_F} k = \hat{\Omega}_{red}$ = union of copies of $\mathbb{P}^1$'s, s.t. dual graph $(\hat{\Omega}_{red})$ = $BT(PGL_2(F))$.

Typical open subsets $\hat{\Omega}_\Delta$:

$$\Delta = [\Lambda_0] : \quad \hat{\Omega}_\Delta = \left(\mathbb{P}(\Lambda_0) \setminus \mathbb{P}(\Lambda_0)(k)\right)^\wedge$$

$$\simeq \operatorname{Spf} \mathcal{O}_{\breve{F}}\left[T, (T^q - T)^{-1}\right]^\wedge \quad \pi\text{-adic.}$$

$\Delta = ([\Lambda_0], [\Lambda_1])$:

$$\hat{\Omega}_\Delta = \operatorname{Spf} \mathcal{O}_{\breve{F}}\left[T_0, T_1, (T_0^{q-1} - 1)^{-1}, (T_1^{q-1} - 1)^{-1}\right]^\wedge / (T_0 T_1 - \pi).$$

② Moduli problems attached to $\hat{\Omega}_F$: Let $\breve{F}$ = completion of max. unramif.

0.) Let Nilp = $\mathcal{O}_{\breve{F}}$-schemes S s.t. $\pi \cdot \mathcal{O}_S$ loc. nilp.

For $S \in$ Nilp $\rightsquigarrow$ $\bar{S} = S \otimes_{\mathcal{O}_{\breve{F}}} \bar{k}$.

1.) Drinfeld: Let B/F quat.-division algebra. Consider $(X, \iota)/_S$

where X = p-div. gp of height $4 \cdot |F : \mathbb{Q}_p|$

$$\iota : \mathcal{O}_B \longrightarrow \operatorname{End}(X), \text{ s.t.}$$

$$\operatorname{char}(\iota(a) \mid \operatorname{Lie} X) = T^2 - \operatorname{Sprd}(a) \cdot T + \operatorname{Nrd}(a),$$

$$\forall a \in \mathcal{O}_B.$$

$$\left(\text{ right } \in \mathcal{O}_F[T] \longrightarrow \mathcal{O}_{\breve{F}}[T] \to \mathcal{O}_S[T]\right)$$

Lemma: Up to $\mathcal{O}_B$-linear isogeny ex! (X, ι) over $\bar{k}$.

Fix one $(\mathbb{X}, \iota_{\mathbb{X}})$. Drinfeld's functor:

$$S \longmapsto \left\{ (X, \iota, \varrho) \;\middle|\; \begin{array}{c} \varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S} \\ \mathcal{O}_B\text{-linear quasi., ht } \varrho = 0 \end{array} \right\}$$

Theorem: This functor is representable by $\hat{\Omega}^2_F \hat{\otimes}_{O_F} O_{\breve{F}}$, compatible with action of $SL_2(F)$ on both sides.

2) KR: Let E/F quadratic extension. Consider $(X, \iota, \lambda)_{/S}$ where

$$X = p\text{-div. gp of height } 4 \cdot |F:\mathbb{Q}_p|$$

$$\iota : O_E \longrightarrow \mathrm{End}(X), \quad \text{s.t.}$$

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = T^2 - \mathrm{Sp}(a)T + \mathrm{Nm}(a)$$

$$\lambda : X \longrightarrow X^\vee \quad \text{polarization s.t.}$$

$$\mathrm{Rosati}_\lambda \mid O_E = \text{non-trivial auto in } \mathrm{Gal}(E/F)$$

and

- if E/F unramified, then $\mathrm{Ker}\,\lambda \subset X[\iota(\pi)]$, of order q^2
- if E/F ramified, then $\mathrm{Ker}\,\lambda = (0)$.

Such triples $(X, \iota, \lambda)/\bar{k}$ not unique up to isogeny. However, uniqueness holds if:

- E/F unramified, X isoclinic. $p V^2$ all slopes 0
- E/F ramified, X isoclinic, $\epsilon(X, \iota, \lambda) = -1$.

[Explanation; if $F = \mathbb{Q}_p$: Let N = rational Dieudonné module, let Π uniformizer of E. Then want

a) $\tau = \Pi V^{-1}$ all slopes 0

b) Let $C = N^\tau$. Then C is hermitian E-VS of dimension 2

Then want C ~~non~~split, i.e. disc. $= -\det C$ $\notin \mathrm{Nm}$.

$\mathcal{O}_F$-Hodht $\mathrm{ht}_F(\mathcal{E}) = 2$, $\dim^F \mathcal{E} = 1$

This is equivalent to $\mathrm{inv}(X, \iota, \lambda) = -1$.

$\Longleftrightarrow$ $\mathbb{X} = \mathcal{E} \times \mathcal{E}$, $\iota = \mathrm{diag}$, $\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) / \bar{k}$. Functor:

$$S \longmapsto \{ (X, \iota, \lambda, \varrho) \mid \varrho : X \times_S \bar{S} \longrightarrow \mathbb{X}_{\bar{S}},$$

$$\mathcal{O}_E\text{-lin. qis of ht } 0, \text{ s.t. } \varrho^*(\lambda_{\mathbb{X}}) \sim \lambda_X \}$$

Theorem: This functor is repres. by $\hat{\Omega}^2_F \hat{\otimes}_{\mathcal{O}_F} \breve{\mathcal{O}}_F$, comp. with action of $SL_2(F) = SU_2(C)$ on both sides.

Caveat: If q even, assume E/F unramified.

(3) Global application: Restrict to KR case.

Fix hermit V of dim 2 over K, sgn $V = (1,1)$; fix $L \subset V$ selfdual lattice. $\varepsilon(V_p) = -1$, $C = C^p \cdot C_p$

Consider imag.-quadr. $K/\mathbb{Q}$, s.t. p not split in K. Consider foll. moduli pb. $\mathcal{M}$ over $(Sch / \mathbb{Z}_{(p)})$: $S \longmapsto \{$ isoclasses of $(A, \iota, \lambda, \eta^p) \}$, where

A = abelian scheme of dim 2 over S

$\iota : \mathcal{O}_K \longrightarrow \mathrm{End}(A)$ s.t.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, A) = T^2 - S(a) T + \mathrm{Nm}\, a.$$

λ polariz. of A "almost principal" s.t.

- p unramif. in K: $\mathrm{Ker}\,\lambda \subset A[p]$, $\mathrm{rk} = p^2$
- p ramified in K: $\mathrm{Ker}\,\lambda = 0$

$\eta^p : \hat{T}^p(A) \simeq L \otimes \hat{\mathbb{Z}}^p$ mod C^p level structure.

Theorem: Natural isomorphisms

- over $\cancel{\mathbb{Q}}$ $\mathbb{Q}$: $\mathcal{M} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ = canonical model of Sh-var. $Sh(GU(V), h)_C$,

  where $C = C^p \cdot C_p$, C_p = max. comp. in $GU(V_p)$.

- over $\cancel{\mathbb{Q}_{p}}$: $\breve{\mathbb{Z}}_p$

$$\mathcal{M} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p \simeq I(\mathbb{Q}) \backslash \left[\hat{\Omega}_{\mathbb{Q}_p} \times GU(V)(\mathbb{A}^\infty)/C \right],$$

where $I = \cancel{}\ GU(\bar{V})$ inner form of G.

RHS can be written as

$$\coprod_j \Gamma_j \backslash \hat{\Omega}_{\mathbb{Q}_p},$$

where Γ_j discrete cocomp. subgp of $\cancel{PGL}$ $SU(\bar{V}_p)$.