Talk: The supersingular locus of a unitary Shimura variety in the ramified case

Motivation: In Kudla's talk he defined special cycles (KR-cycles) on Shimura variety for $GU(1,n-1)$. Important ingredient of his theorem was the structure of supersingular locus in reduction mod p, where p inert prime. Was consequence of structure of $\mathcal{N}_{red}$, where $\mathcal{N}$ = corresponding RZ-space.

Want to do same for p which ramify.

§1 Definition of $\mathcal{N}$

Notation: • $E/\mathbb{Q}_p$ ramified, with $p \neq 2$. Fix $\pi \in O_E$ s.t. $\pi_0 = \pi^2$ uniformizer of $\mathbb{Z}_p$.

• $\mathbb{F} = \overline{\mathbb{F}}_p$, $W = W(\mathbb{F})$, $W_{\mathbb{Q}}$, σ

• $\breve{E} = E \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}$, σ.

Let $Nilp = \{ O_{\breve{E}}\text{-schemes } S, \text{ s.t. } \pi \cdot O_S \text{ loc. nilp.} \}$

$S \in Nilp \mapsto \overline{S}$.

Fix $n \geq 1$. ~~Let~~ We will consider triples (X, ι, λ)

where X = p-divis. gp of dim n and height $2n$

$\iota : O_E \longrightarrow \mathrm{End}(X)$

$\lambda : X \xrightarrow{\sim} X^\vee$ principal quasi-polariz, comp w. ι.

Fix such a triple $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ over $\mathbb{F}$, s.t. $\mathbb{X}$ supersingular.
(unique up to isogeny).

Consider functor $\mathcal{N}$: Nilp $\longrightarrow$ (Sets), where

$\mathcal{N}(S) = \{$ iso-classes of $(X, \iota, \lambda, \varrho)$, where

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\,\mathbb{F}} \bar{S}$$

O_E-linear quasi-isog., s.t. $\varrho^*(\lambda_{\mathbb{X}}) \sim \lambda_X \}$

Impose two condition on action of O_E on Lie X:

- $\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T - a) \cdot (T - \bar{a})^{n-1}$, $a \in O_E$
- $\bigwedge^2 (\iota(\pi) + \pi \mid \mathrm{Lie}\, X) = 0$, $\bigwedge^n (\iota(\pi) - \pi \mid \mathrm{Lie}\, X) = 0$

  if $n \geq 3$.

Theorem: The functor is representable by a separated formal scheme $\mathcal{N}$ loc. formally of finite type over $O_{\breve{E}}$. The formal scheme is flat over $O_{\breve{E}}$, and formally smooth of relative dimension $n-1$ except at those isolated points of $\mathcal{N} \otimes_{O_{\breve{E}}} \mathbb{F}$ corresp. to $(X, \iota, \lambda, \varrho)$, where $\mathrm{Lie}(\iota(\pi)) = 0$. //

Let $\bar{\mathcal N}^0 = \mathcal N \otimes_{O_{\breve E}} \mathbb F \mid$ locus where $\mathrm{ht}(\varrho) = 0$.

Then for $S \in (\mathrm{Sch}/\mathbb F)$,

$$\bar{\mathcal N}^0(S) = \{ (X, \iota, \lambda, \varrho) \mid \wedge^2(\iota(\pi) \mid \mathrm{Lie}\, X) = 0 \text{ for } n \geq 3 \}.$$

(Note that $a \equiv \bar a \mod \pi$).

Let N = rational Dieudonné module of $\mathbb X$, with action by E, and skew-sym. form $\langle\, , \rangle$ w. values in $W_{\mathbb Q}$

Let

$$C = N^\tau, \quad \text{where } \tau = \iota(\pi) V^{-1}. \quad (\Pi = \iota(\pi))$$

Then $C \otimes_{\mathbb Q_p} W_{\mathbb Q} = N$. ($\mathbb X$ supersingular).

Fix $u \in \mathbb Z_{p^2}$ with $\dfrac{u}{u^\sigma} = -\dfrac{\pi_0}{p}$ and set

$$\langle\, , \rangle' = u^{-1} \cdot \langle\, , \rangle.$$

Then $\langle\, , \rangle' \mid C$ skew-sym. form w. values in $\mathbb Q_p$.

Hence C becomes n-dimensional E-vs, with hermitian form

$$(x, y) = \langle \Pi x, y \rangle' + \langle x, y \rangle' \pi$$

Now ~~we~~ extend $(\,,)$ to $C \otimes_E \breve E$, via

$$(x \otimes a, y \otimes a') = a \cdot a'^\sigma \cdot (x, y).$$

Dieudonné theory gives:

Proposition: Bijection between $\bar{\mathcal{N}}^\circ(\mathbb{F})$ and

$$\mathcal{V}(\mathbb{F}) = \{ M \subset C \otimes_E \breve{E} \mid M^\# = M, \quad \pi\tau(M) \subset M \subset \pi^{-1}\tau(M),\ M \overset{\leq 1}{\subset} M + \tau(M) \}$$

§ 2 Structure of $\bar{\mathcal{N}}^\circ_{red}$.

Definition: A *vertex lattice* in $C_{\bar{\mathbb{Q}}}$ a lattice Λ s.t.

$$\pi\Lambda \subset \Lambda^\# \subset \Lambda .$$

The dimension $\dim_{\mathbb{F}_p} (\Lambda / \Lambda^\#)$ is called the *type of* Λ.

Fact:
- the type is always an even integer $\leq n$
- there are vertex lattices of any type $\leq \begin{cases} n & \text{if } n \text{ even} \\ n-1 & \text{if } n \text{ odd} \end{cases}$

$\Rightarrow$ C split if n even (contrast to unramified case)

Theorem: To every vertex lattice Λ, there is associated a closed irreducible subscheme $\mathcal{N}_\Lambda$ of $\bar{\mathcal{N}}^\circ_{red}$, such that

$$\bar{\mathcal{N}}^\circ_{red} = \bigcup_\Lambda \mathcal{N}_\Lambda \text{, and}$$

$$\mathcal{N}_\Lambda(\mathbb{F}) = \{ M \in \mathcal{V}(\mathbb{F}) \mid M \subset \Lambda \otimes_{O_E} O_{\breve{E}} \}$$

Furthermore,

- $\mathcal{N}_\Lambda \subset \mathcal{N}_{\Lambda'} \iff \Lambda \subset \Lambda'$.

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \begin{cases} \mathcal{N}_{\Lambda \cap \Lambda'} & \text{if } \Lambda \cap \Lambda' \text{ vertex} \\ \emptyset & \text{oth.} \end{cases}$$

- $\dim \mathcal{N}_\Lambda = \frac{1}{2} t(\Lambda)$, and

$$\mathcal{N}_\Lambda^{sing} = \begin{cases} \emptyset & \text{if } n \leq 3 \\ \bigcup\limits_{\substack{\Lambda' \subset \Lambda, \\ t(\Lambda') = 0}} \mathcal{N}_{\Lambda'} & \text{if } n > 3. \end{cases}$$

isolated singularities

- Let $\mathcal{N}_\Lambda^0 = \mathcal{N}_\Lambda \setminus \bigcup\limits_{\Lambda' \subsetneq \Lambda} \mathcal{N}_{\Lambda'}$

Then $\mathcal{N}_\Lambda^0$ is isomorphic to DL-variety to a symplectic gp of size $t(\Lambda)$ over $\mathbb{F}_p$ and a Coxeter elt.

In particular, $\dim \mathcal{N}^0 = \left[\frac{n}{2}\right]$.

<u>Picture for $n=2$ and $n=3$</u>: $\bar{\mathcal{N}}^0_{red}$ a union of $\mathbb{P}^1$'s.

$\mathbb{P}^1$'s $\hat{=}$ $\mathcal{N}_\Lambda$, $t(\Lambda) = 2$

intersection points $\hat{=}$ ——— $t(\Lambda) = 0$.

Every line contains $p+1$ intersection points.

Every intersection point lies on $\begin{cases} 2 \ \mathbb{P}^1\text{'s} & n=2 \\ p+1 \ \mathbb{P}^1\text{'s} & n=3. \end{cases}$

Combinatorics of strata given by $\mathcal{B} = \mathcal{B}(SU(C), \mathbb{Q}_p)$:

$n=2$: $\quad \mathrm{Vert}(\mathcal{B}) \triangleq \{ \Lambda \mid t(\Lambda) = 2 \}$

$\quad \mathrm{Edges}(\mathcal{B}) \triangleq \{ \Lambda \mid t(\Lambda) = 0 \}$. $\quad \triangleq$ dual graph of $\mathcal{N}^0_{red}$

$n=3$: $\quad \mathrm{Vert}_2(\mathcal{B}) \triangleq \{ \Lambda \mid t(\Lambda) = 2 \}$

$\quad \mathrm{Vert}_0(\mathcal{B}) \triangleq \{ \Lambda \mid t(\Lambda) = 0 \}$ $\quad \triangleq$ incidence graph of $\mathcal{N}^0_{red}$

Remark: For $n=2$, have isomorphism, cf. [KR],

$$\mathcal{M}^0 \simeq \hat{\Omega}^2_{\mathbb{Q}_p}$$

Corollary: $\mathcal{N}^0_{red}$ is connected. $\longrightarrow$ Miaofen

Explanation of DL-variety: Let $(V, \langle\, ,\rangle)$ symplectic vs of dimension $2m$ over $\mathbb{F}_p$. Then

$$X(w) = \{ \text{isotropic flags } (0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m \text{ of } V \otimes \mathbb{F} \mid \mathcal{F}_{i-1} = \mathcal{F}_i \cap \Phi(\mathcal{F}_i),\ i = 1, \dots, m \}.$$

This is smooth irreduc. affine variety of dimension m.

Let

$$S_V = \{ U \text{ Lagrange of } U \otimes \mathbb{F} \mid \dim U \cap \Phi(U) \geq m-1 \}.$$

One proves in fact: Let Λ be vertex lattice of type $t(\Lambda) = 2m$.

Let $V = \Lambda / \Lambda^\#$, with symplectic form induced by $p \cdot \langle\, ,\rangle'$.

Proposition: Then ex commutative, cartesian diagram

$$\begin{array}{ccc} \mathcal{N}_\Lambda & \xrightarrow{\sim} & S_\Lambda \\ \cup & & \cup \quad \Leftarrow \text{ via } \mathcal{F}_0 \mapsto U = \mathcal{F}_m. \\ \mathcal{N}^\circ_\Lambda & \xrightarrow{\sim} & X(w) \end{array}$$

//

Now S_V is singular for $m \geq 2$, namely $S_V^{sing} = S_V(\mathbb{F}_p)$.

Proposition: Let $\overline{X(w)}$ be the closure of $X(w)$ in G/B. Then $\overline{X(w)}$ is smooth, and $\partial \overline{X(w)}$ is a DNC. Furthermore, the morphism $X(w) \hookrightarrow S_V$ extends to

$$\pi: \overline{X(w)} \longrightarrow S_V$$

which is a resolution of singularities in the weak sense.

Remark: The trace along S_V of the BT-stratific. of $\mathcal{N}_\Lambda$ can be described explicitly in terms of linear algebra.

§ 3 Conclusion / Outlook.

For $x \in C \rightsquigarrow Z(x) \subset \mathcal{N}$, with

$$Z(x)(\mathbb{F}) = \{ M \in \mathcal{V}(\mathbb{F}) \mid x \in M^\# \}.$$

Again, for $\underline{x} = [x_1, \ldots, x_m] \in C$ have

$$Z(\underline{x}) = Z(x_1) \cap \ldots \cap Z(x_m) \quad , \quad \text{and}$$

<u>Proposition:</u> $$\mathcal{N}^{0}_{red} \cap \mathcal{Z}(\underline{x}) = \bigcup_{\underline{x} \in \Lambda^{\#}} \mathcal{N}_{\Lambda} .$$

Want calculation of

$$(\mathcal{Z}(x_1), \ldots, \mathcal{Z}(x_n)).$$

<u>Expect from Terstiege:</u>

- if $\dim(LHS) = 0$ (generaliz. of [KR])
- if $n \leq 3$ (—— [T])

But, contrary to unramified case, no KR-conjecture!