Global and local moduli spaces, II

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Recollections from the end of last time

Had

- $W = W(\bar{\mathbb{F}}_p)$, the ring of Witt vectors
- $\mathrm{Nilp} = \mathrm{Nilp}_W$, the category of W-schemes S such that $p\mathcal{O}_S$ is a locally nilpotent ideal sheaf.

For a p-divisible group $\mathbb{X}$ over $\bar{\mathbb{F}}_p$, have functor on Nilp,

$$\mathcal{M}(S) = \big\{ \text{ iso-classes of } (X, \rho) \big\},$$

where

- X a p-divisible group over S
- $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{S}$ a quasi-isogeny.

This functor is representable by a formal scheme $\mathcal{M}$ over $\mathrm{Spf}\, W$.

General RZ-data

- **Case EL**: Let B be a finite-dimensional semi-simple $\mathbb{Q}_p$-algebra and let V be a finite left B-module. Let

  $$G = \mathrm{GL}_B(V).$$

  Furthermore, let $b \in G(W_{\mathbb{Q}})$, and μ a minuscule cocharacter of G, such that $[b] \in B(G, \mu)$. Finally, add some *integral* data.

  Example: Take $\mathbb{X}$ above. Let $B = \mathbb{Q}_p$. Write the *rational Dieudonné module* as $M_{\mathbb{Q}}(\mathbb{X}) = V \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}$, and the Frobenius endomorphism as $F = b \cdot (\mathrm{id}_V \otimes \sigma)$. And $\mu = (1^{(d)}, 0^{(h-d)})$, where $d = \dim \mathrm{Lie}\, \mathbb{X}$ and $h = \mathrm{height}\, \mathbb{X} = \dim V$.
  Integral data: $\mathcal{O}_B = \mathbb{Z}_p$, $V_{\mathbb{Z}_p} = M(\mathbb{X})$.

- **Case PEL:** Add a *polarization*.

Example: Variant of local Picard type, of last time: Let $E/\mathbb{Q}_p$ an unramified quadratic extension. Consider functor on Nilp

$$\widetilde{\mathcal{M}}(S) = \{ \text{ iso-classes of } (X, \iota, \lambda, \rho)\},$$

where

- X is a p-divisible group of height 4 over S
- $\iota : \mathcal{O}_E \to \mathrm{End}(X)$ an action of $\mathcal{O}_E$, of signature $(1, 1)$
- λ a *principal* polarization compatible with ι.
- ρ as before, compatible with ι, λ.

In this case $B = E$, and V is equipped with a hermitian form. The group G is the group of hermitian similitudes $\mathrm{GU}(V)$.

Construction: Associate to these data

- the reflex field $E = E_\mu = E(G, \mu)$, a finite extension of $\mathbb{Q}_p$.
- an algebraic group J over $\mathbb{Q}_p$ with

$$J(\mathbb{Q}_p) = \{g \in G(W_\mathbb{Q}) \mid gb\sigma(g)^{-1} = b\}.$$

- a formal scheme $\mathcal{M}$ over $\mathcal{O}_{\breve{E}}$, with an action of $J(\mathbb{Q}_p)$. Here $\breve{E} = E.W_\mathbb{Q}$.
- an unramified inner form G' of G, and an open compact subgroup K'_0 of $G'(\mathbb{Q}_p)$. [If G is connected, then G_{der} is simply connected, and $G' = G$.]
- a Weil descent datum on $\mathcal{M}$ from $\mathcal{O}_{\breve{E}}$ down to $\mathcal{O}_E$.

Different aspects of the theory

- Study of the singularities of the formal scheme $\mathcal{M}$, leads to theory of local models (see survey by Pappas, Rapo, Smithling, important recent work of Xinwen Zhu, Brian Smithling on conjectures of Pappas and Rapo).
- Period map, a rigid-analytic morphism from $\mathcal{M}^{\mathrm{rig}}$ to the Grassmannian attached to (G, μ) (important recent work of Faltings, Hartl, Kedlaya on the image of this map (conjecture of Hartl and Rapo/Zink)).
- Period domains, open subspaces of partial flag varieties described by semi-stability conditions and containing the image of period morphisms (recent monograph by Dat, Orlik, Rapo).

- Non-archimedean uniformization relates global moduli spaces and local moduli spaces (Rapo/Zink).
- ℓ-adic cohomology of the tower of rigid-analytic coverings of $\mathcal{M}^{\mathrm{rig}}$ (conjectures of Kottwitz, Harris, recent work of Fargues, Mantovan, Shin, Viehmann)
- Special divisors on $\mathcal{M}$, for RZ-spaces of local Picard type (recent work of Kudla/ Rapo, Terstiege, Howard, conjecture of Kudla/Rapo)
- Relations between different RZ-spaces (important not so recent work of Faltings, Fargues; new point of view of Scholze, using his theory of perfectoid spaces)
- Theory of local Shimura varieties (ideas of Scholze, using perfectoid spaces and a relative version of the Fargues-Fontaine curve).

Passage to the generic fiber

Start with RZ-data, and associated RZ-space.
Let $\mathcal{M}^{\mathrm{rig}}$ be its generic fiber. Then get a tower

$$(\mathbb{M}_{K'})_{K'}; K' \subset G'(\mathbb{Q}_p) \text{ open compact subgroup}$$

with the following properties.

- $\mathbb{M}_{K'_0} = \mathcal{M}^{\mathrm{rig}}$. Here K'_0 is determined by the *integral data*.
- each member $\mathbb{M}_{K'}$ is equipped with an action of $J(\mathbb{Q}_p)$
- the tower admits an action of $G'(\mathbb{Q}_p)$.
- the tower is equipped with a Weil descent datum from $\breve{E}$ down to E.

Construction of local Langlands correspondences

- Let $\ell \neq p$. Let

$$H^*(\mathbb{M}_{K'}) = H_c^*(\mathbb{M}_{K'} \times_{\breve{E}} \hat{\overline{E}}, \hat{\overline{\mathbb{Q}}}_\ell(d)),$$

where $d = \dim M_{K'}$.

- $J(\mathbb{Q}_p) \times G'(\mathbb{Q}_p) \times W_E$ acts.

- For $\rho \in \mathrm{Groth}\big(J(\mathbb{Q}_p)\big)$, let

$$\begin{aligned} H^{i,j}(\mathbb{M}^\infty)_\rho = & \varinjlim_{K'} \mathrm{Ext}^j_{J(\mathbb{Q}_p)}(H^i(\mathbb{M}_{K'}), \rho) \\ & \in \mathrm{Groth}\big(G'(\mathbb{Q}_p) \times W_E\big). \end{aligned}$$

- Put

$$H^\bullet(\mathbb{M}^\infty)_\rho = \sum_{i,j} (-1)^{i+j} H^{i,j}(\mathbb{M}^\infty)_\rho$$

Conjecture of Kottwitz

Assumptions:

- b is basic, i.e., J is an inner form of G.
- G' is a B-strong inner form.
- the representation ρ is attached to a discrete Langlands parameter φ.

Conjecture (Kottwitz)

$$H^{\bullet}(\mathbb{M}^{\infty})_{\rho} = \sum_{\pi \in \Pi_{\varphi}} \check{\pi} \boxtimes \mathrm{Hom}_{S_{\varphi}}(\tau_{\pi} \otimes \tau_{\rho}, r_{\mu} \circ \varphi).$$

Theorem (Fargues)

The above conjecture holds true when $G = G' = \mathrm{GL}_n$, and when J anisotropic.

Local Shimura varieties

Let (G, b, μ) such that

- μ is minuscule
- $[b] \in B(G, \mu)$.

As for RZ data have $E = E_\mu$, $J = J_b$, $G' = G'_{(b,\mu)}$.

Conjecture

Can associate a tower of rigid-analytic spaces $(\mathbb{M}_{K'})_{K'} = (\mathbb{M}(G, b, \mu)_{K'})_{K'}$ over $\breve{E}$, with similar structure as in RZ case.

Scholze has ideas how to define this tower, using his theory of perfectoid spaces.

Conjecture of Harris/Viehmann

- Start with (G, b, μ) as above, where G is quasi-split. Then J_b is an inner form of a Levi subgroup L_b of G. Assume that $b \in L_b(W_{\mathbb{Q}})$.
- Let $L \supset L_b$ be another Levi subgroup. Let

$$I_{b,\mu,L} = \{\mu' \,|\text{cocharacter of } L \text{ conjugate in } G \text{ to } \mu \\ \text{with } [b]_L \in B(L, \mu')\}/\text{modulo } L\text{-conjugacy}.$$

Then $|I_{b,\mu,L}| = 1$ if G is split; in general finite set.

Conjecture (Harris/Viehmann)

$$H^\bullet(\mathbb{M}(G, b, \mu)^\infty)_\rho = \sum_{\mu' \in I_{b,\mu,L}} \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\big(H^\bullet(\mathbb{M}(L, b, \mu')^\infty)_\rho\big).$$

- Here P denotes a definite psgp with Levi component L determined by μ'.
- Implicitly there appears the inclusion $E(L, \mu') \subset E(G, \mu)$.

Corollary (of Conjecture)

Assume the H/V-conjecture. If b is not basic, and ρ has a discrete Langlands parameter, then LHS vanishes.

Theorem (Fargues)

The above corollary to the conjecture holds true when $G = G' = \mathrm{GL}_n$, and when J anisotropic.

Theorem

Assume (G, b, μ) is attached to unramified RZ data.
(i) (Mantovan) The conjecture holds true, if L is the Levi subgroup associated to a Hodge-Newton decomposition.
(ii) (Mantovan, Viehmann) The conjecture holds true at level K_0.

Theorem (Boyer, Harris/Taylor)

The conjecture holds in the Lubin-Tate case, i.e., when (G, b, μ) is associated to the RZ data $\big(B = \mathbb{Q}_p, V = \mathbb{Q}_p^n, \mu = (1, 0, \ldots, 0)\big)$, for any b.