How geometry meets arithmetic: global and local moduli spaces

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The three layers of Algebraic Geometry

Algebraic Geometry=mathematical discipline that studies the geometry of the zero loci of polynomials.

Examples:

$$X^n + Y^n - Z^n = 0$$

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$$

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$$

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = 0$$

$$X_0^2 + X_1^2 + X_3^2 + X_4^2 = 0, 2X_0^3 + X_1^3 + 5X_2^3 + 11X_3^3 + 7X_4^3 = 0$$

Layer 1: varieties defined by equations

Hierarchy by complication of equation, e.g. the degree of defining polynomials:

- *linear equations*: part of linear algebra
- *quadratic equations*: when only one equation, then still part of linear algebra; if more than one equation, becomes more complicated (*Note*: any algebraic variety can be described by quadratic equations!).
- *one cubic equation in 3 variables*: describes elliptic curves, geometry is understood, arithmetic is very mysterious.
- *one cubic equation in 4 variables*: much studied in the 19th century
- *one quartic equation in 4 variables*: have been intensely studied up to the present.

Layer 2: varieties defined by characterization

Study of properties of algebraic varieties (sometimes given by equations) leads to characterization by properties.

- *nonsingular algebraic curves of genus g*: over the ground field $\mathbb{C}$, these correspond to *compact Riemann surfaces* of genus g.
- *abelian varieties of dimension g*: g-dimensional projective group varieties.
- *K3-surfaces*: nonsingular projective surfaces which are simply connected, and with a nowhere vanishing holomorphic 2-form.

Layer 3: varieties defined as parameter spaces

Algebraic varieties of one characterization class form an algebraic variety by themselves.

- *parameter space $\mathcal{M}_g$ of curves of genus g*: a quasi-projective variety of dimension $3g - 3$ over the ground field.
- *parameter space of polarized abelian varieties of dimension g*: a quasi-projective algebraic variety of dimension $\frac{1}{2}g(g+1)$ over the ground field. Can in fact be defined as a scheme, even an arithmetic scheme.
- *parameter space of polarized K3-surfaces*: a quasi-projective algebraic variety of dimension 19 over the ground field.

Riemann's concept of moduli

Riemann (1857, in his first paper on *abelian functions*):

"Es hängt also [...] die [...] Klasse algebraischer Gleichungen von $3p-3$ *stetig veränderlichen Größen ab, welche die Moduln dieser Klasse genannt werden sollen."*

In today's language (analytic):

There exists a moduli space of compact Riemann surfaces of genus $g \geq 2$*, which is a complex-analytic variety of dimension* $3g-3$*.*

In today's algebraic language:

There exists a moduli space of smooth projective curves of genus $g \geq 2$, which is a quasi-projective algebraic variety $\mathcal{C}_g$ of dimension $3g-3$.

Explanation:

This means:

$\mathcal{C}_g$ is an algebraic variety with the following universal property: Any family of curves of genus g over a base S induces a morphism from S to $\mathcal{C}_g$ such that:

- the morphism only depends on the isomorphism class of the family of curves over S.
- this rule is functorial in S.
- when $S = \mathrm{Spec}\,\mathbb{C}$, the rule induces a bijection between the sets of isomorphism classes of curves of genus g over $\mathbb{C}$ and the set $\mathcal{C}_g(\mathbb{C})$.

Also known to Riemann:

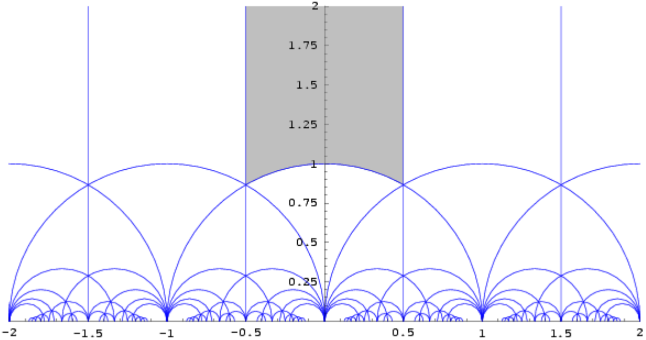
There exists a moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g, which is a quasi-projective algebraic variety of dimension $g(g+1)/2$.

Explanation for $g = 1$: Abelian varieties of dimension 1 are equivalently

- *elliptic curves*
- non-singular cubics in $\mathbb{P}^2$ with a distinguished point (the *neutral element* for the group operation)
- over $\mathbb{C}$: $E = \mathbb{C}/\Lambda$, for a lattice Λ in $\mathbb{C}$.

Elliptic curves have an essentially unique principal polarization. Hence

$$\mathcal{A}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}.$$



Example

The various "triangles" are fundamental domains for the action of $\mathrm{SL}_2(\mathbb{Z})$.

The moduli scheme of abelian varieties à la Grothendieck-Mumford

Let $g \geq 1$. Fix an integer $N \geq 3$. Consider the functor $\mathcal{A} = \mathcal{A}_{g,N}$ on the category of schemes over $\mathrm{Spec}\, \mathbb{Z}[1/N]$,

$$\mathcal{A}(S) = \{ \text{iso-classes of triples } (A, \lambda, \eta) \},$$

where

- A an abelian scheme of relative dimension g over S
- λ a principal polarization
- η a level-N-structure.

Theorem

The functor $\mathcal{A}_{g,N}$ is representable by a smooth quasi-projective scheme of relative dimension $g(g+1)/2$ over $\mathrm{Spec}\, \mathbb{Z}[1/N]$.

Application: Construction of interesting Galois representations ($g = 1$)

Form the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module

$$H_N^* = H^*(\mathcal{A}_{1,N} \times_{\mathrm{Spec}\,\mathbb{Z}[1/N]} \mathrm{Spec}\,\bar{\mathbb{Q}}, \bar{\mathbb{Q}}_\ell).$$

Let $G = \mathrm{GL}_2$. Let π_f be an irreducible admissible ∞-dimensional representation of $G(\mathbb{A}_f)$. Then the π_f-isotypic component in H_N^1 is a multiple of a two-dimensional Galois representation $\rho(\pi_f)$.

Theorem (Eichler, Shimura, Ihara, Deligne, Langlands, Carayol, Scholze)

Let ρ_p be the restriction of $\rho(\pi_f)$ to a decomposition group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then

$$\rho_p \text{ and } \pi_p$$

are in Langlands correspondence (up to a Tate twist).

Why is this arithmetic?

Application: Consider the Ramanujam function $n \mapsto \tau(n)$, appearing in the q-development of the Δ-function

$$\Delta(z) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}, \quad q = e^{2\pi i z}, \, \mathrm{Im}(z) > 0,$$

i.e.,

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Theorem (Serre)

The set of prime numbers p with $\tau(p) = 0$ has density zero.

Conjecture (Lehmer 1947)

The above set of p is empty.

Variant: Picard moduli scheme

Let E be an imaginary quadratic field.
Let $n \geq 1$. Fix an integer $1 \leq r \leq n-1$. Fix an integer $N \geq 3$.
Consider the functor $\mathcal{A} = \mathcal{A}_{E,n,r,N}$ on the category of schemes over $\mathrm{Spec}\ O_E[1/N]$,

$$\mathcal{A}(S) = \{ \text{iso-classes of quadruples } (A, \iota, \lambda, \eta) \},$$

where

- A an abelian scheme of relative dimension n over S.
- $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ an action of $\mathcal{O}_E$, of signature $(r, n-r)$.
- λ a principal polarization compatible with ι,
- η a level-N-structure.

Theorem

The functor $\mathcal{A}$ is representable by a smooth quasi-projective scheme of relative dimension $r(n-r)$ over $\mathrm{Spec}\,\mathcal{O}_E[1/2N]$.

Construction of Galois representation

By a similar procedure as for $\mathcal{A}_{1,N}$, get ℓ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, of dimension $\binom{n}{r}$.

Using this construction for $r = 1$, get n-dimensional Galois representations.

Theorem (Harris/Taylor, Henniart, Scholze)(local Langlands correspondence)

There exists a one-to-one correspondence between the following two sets, with all sorts of good properties,

$$\{\textit{iso-classes of n-dim. } \ell\textit{-adic representations of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\}$$

and

$$\{\textit{iso-classes of irreducible admissible representations of } \mathrm{GL}_n(\mathbb{Q}_p)\}.$$

Localization of abelian varieties

- *archimedean localization:* associate to an abelian variety over $\mathbb{C}$ its underlying real Lie group, i.e., $A = \mathbb{C}^g/\Lambda \rightsquigarrow \mathbb{R}^{2g}/\Lambda$.

- *p-adic localization:* associate to an abelian variety A its p-divisible group $X = A(p^\infty)$, where

$$A(p^\infty) = \varinjlim A[p^n].$$

Big difference:

- all archimedean localizations of elements of $\mathcal{A}_g(\mathbb{C})$ are isomorphic.

- not all p-divisible groups of elements of $\mathcal{A}_1(\overline{\mathbb{F}}_p)$ are isomorphic, not even isogenous (ordinary elliptic curves versus supersingular elliptic curves).

Local moduli spaces

Let $W = W(\bar{\mathbb{F}}_p)$ be the ring of Witt vectors. Let $\mathrm{Nilp} = \mathrm{Nilp}_W$ be the category of W-schemes S such that $p\mathcal{O}_S$ is a locally nilpotent ideal sheaf.

Let $\mathbb{X}$ be a p-divisible group over $\bar{\mathbb{F}}_p$. Functor on Nilp,

$$\mathcal{M}(S) = \{ \text{ iso-classes of } (X, \rho) \},$$

where

- X a p-divisible group over S
- $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{\mathbb{F}}_p} \bar{S}$ a quasi-isogeny.

Theorem (Rapoport/Zink)

The functor $\mathcal{M}$ is representable by a formal scheme locally formally of finite type over $\mathrm{Spf}\, W$.

(locally of the form

$$\mathrm{Spf}\, W[[x_1, \ldots, x_n]]\langle y_1, \ldots, y_m \rangle / \mathit{ideal})$$

Examples and variants

1. Let $X = A(p^\infty)$, where $A \in \mathcal{A}_1(\overline{\mathbb{F}}_p)$ (hence $\mathrm{height}(X) = 2$). Then
$$\mathcal{M} = \coprod \mathrm{Spf}\, W[[T]],$$
with index set equal to $\mathbb{Z}^2$ when A is ordinary, and equal to $\mathbb{Z}$ when A is supersingular.
2. (local analogue of Picard moduli problem) Let $E/\mathbb{Q}_p$ quadratic extension. Let $\widetilde{W} = W.\mathcal{O}_E$. Consider functor on $\mathrm{Nilp}_{\widetilde{W}}$
$$\widetilde{\mathcal{M}}(S) = \{ \text{ iso-classes of } (X, \iota, \lambda, \rho)\},$$
where
   - X is a p-divisible group of height 4 over S
   - $\iota : \mathcal{O}_E \to \mathrm{End}(X)$ an action of $\mathcal{O}_E$, of signature $(1,1)$
   - λ a polarization compatible with ι, which is *principal* when $E/\mathbb{Q}_p$ is ramified, and has kernel of order p^2, when $E/\mathbb{Q}_p$ is unramified.
   - ρ as before, compatible with ι, λ.

Theorem (Kudla/Rapoport)

$\widetilde{\mathcal{M}}$ is representable by the formal scheme

$$\coprod \widehat{\Omega}_{\mathbb{Q}_p} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\widetilde{W},$$

with index set equal to $\mathbb{Z}$.

Explanation: Here $\widehat{\Omega}_{\mathbb{Q}_p}$ is Deligne's formal model of Drinfeld's p-adic halfplane. Properties of $\widehat{\Omega}_{\mathbb{Q}_p}$:

- it is a p-adic formal scheme with semi-stable reduction over $\mathrm{Spf}\,\mathbb{Z}_p$
- its *generic fiber* is Drinfeld's rigid-analytic space

$$\Omega_{\mathbb{Q}_p} = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$$

- its *special fiber* is a union of $\mathbb{P}^1$'s, organized according to the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Picture of the special fiber of $\widehat{\Omega}_{\mathbb{Q}_2}$

Various aspects of the theory

- the theory of local models (Görtz, Pappas, Rapo, Smithling, X. Zhu).
- the theory of the period map (Gross/Hopkins, Faltings, Hartl, Rapo/Zink).
- the theory of non-archimedean period domains (Dat, Kottwitz, Orlik, Rapo).
- the non-archimedean uniformization (Drinfeld, Rapo/Zink).
- the ℓ-adic cohomology of RZ-spaces (Fargues, Harris, Mantovan, Kottwitz, Shin, Viehmann).
- the theory of special divisors on RZ-spaces of local Picard type (Howard, Kudla/Rapo, Terstiege).
- the relations between different RZ-spaces (Fargues, Faltings, Scholze).
- the speculative theory of local Shimura varieties (Scholze).

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