Strasbourg, March 2011

Talk: A new look at the Drinfeld p-adic halfplane.

Kudla

F = p-adic field, i.e. $|F:\mathbb{Q}_p| < \infty$. Have $(\mathcal{O}_F, \pi, k)$.

e.g. $F = \mathbb{Q}_p$.

Drinfeld half-plane assoc. to F:

$$\Omega_F = \mathbb{P}^1_F \setminus \mathbb{P}^1(F).$$

is rigid-analytic variety $/F$.

Formal model $\hat{\Omega}_F$ of Ω_F: Formal scheme over $\mathrm{Spf}\, \mathcal{O}_F$, defined by Deligne. Is obtained by glueing open formal subschemes $\hat{\Omega}_\Delta$, with Δ running through the simplices of $\mathcal{B}(PGL_2(F))$, and

$$\hat{\Omega}_\Delta \cap \hat{\Omega}_{\Delta'} = \begin{cases} \hat{\Omega}_{\Delta \cap \Delta'} & \text{if } \Delta \cap \Delta' \text{ simplex} \\ \emptyset & \text{if not.} \end{cases}$$

Furthermore, $\exists$ action of $PGL_2(F)$ on $\hat{\Omega}$ s.t.

$$g \cdot \hat{\Omega}_\Delta = \hat{\Omega}_{g\Delta}.$$

If $\Delta = [\Lambda_0]$, with $\Lambda_0 = [e_1, e_2]$, then

$$\hat{\Omega}_\Delta = \left(\mathbb{P}(\Lambda_0) \setminus \mathbb{P}(\Lambda_0)(k)\right)^\wedge$$

$$= \mathrm{Spf}\, \mathcal{O}_F [T, (T^q - T)^{-1}]^\wedge \qquad q = \# k.$$

If $\Delta = ([\Lambda_0], [\Lambda_1])$ with $\Lambda_1 = [\pi e_1, e_2]$, then

$$\hat{\Omega}_\Delta = \mathrm{Spf}\, \mathcal{O}_F [T_0, T_1, (1 - T_0^{q-1})^{-1}, (1 - T_1^{q-1})^{-1}]^\wedge / (T_0 T_1 - \pi)$$

and open immersion $\Omega_{[\Lambda_0]} \hookrightarrow \Omega_\Delta$ ~~given~~ induced by open immersion

$$\begin{tikzcd}
\mathrm{Spf}\, O_F[T, T^{-1}]^\wedge \arrow[rr, hook, "{T_1 \mapsto T^{-1},\ T_0 \mapsto \pi T}"] & & \mathrm{Spf}\left(O_F[T_0, T_1]/(T_0 T_1 - \pi)\right)^\wedge \\
\hat{\Omega}_{[\Lambda_0]} \arrow[u, hook] \arrow[rr, dashed] & & \hat{\Omega}_\Delta \arrow[u, hook]
\end{tikzcd}$$

The special fiber of $\hat{\Omega}$ is a union of $\mathbb{P}^1$'s parametrized by vertices of $\mathcal{B}$. More precisely can identify

$$\left(\text{dual graph of } \hat{\Omega} \otimes k\right) \longrightarrow \mathcal{B},$$

equiv. w.r.t. action of $PGL_2(F)$, and

$$\hat{\Omega}_{[\Lambda]} \otimes k = \mathbb{P}_{[\Lambda]} \otimes k \setminus \mathbb{P}_{[\Lambda]}(k).$$

$$\Omega_{[\Lambda, \Lambda']} \otimes k = \mathbb{P}_{[\Lambda]} \otimes k \cap \mathbb{P}_{[\Lambda']} \otimes k = pt.$$

$\hat{\Omega}$ has semi-stable reduction.

<u>Drinfeld moduli problem</u>: Let $\breve{F}$ = completion of max. unr. ext.

~~Let B = quat. division algebra over F.~~ Consider following moduli pb. on $Nilp = (O_{\breve{F}}\text{-schemes } S \text{ s.t. } \pi \text{ loc. nilp. in } O_S)$

For $S \in Nilp$, let $\bar{S} = S \otimes_{O_{\breve{F}}} \bar{k}$.

Let B = quat. division algebra over F. Consider (X, ι_B), where X/S formal O_F-p-divisible gp

$$\iota_B : O_B \longrightarrow \mathrm{End}(X)$$

extending action of O_F and satisfying <u>special condition</u> for action on Lie X (which is a module over $O_B \otimes_{O_F} O_S$).

$$\mathrm{char}(\iota(\alpha) \mid \mathrm{Lie}\, X) = (T-\alpha)\,(T-\alpha') \;,\quad \forall \alpha \in O_B.$$

For $F = \mathbb{Q}_p$, X is a p-div gp of ht 4 and dim 2.

Fix $(\mathbb{X}, \iota_B)$ over $\bar{k}$ (unique up to isomorphism).

$$\mathcal{M}(S) = \{ \text{iso-classes of } (X, \iota_B, \varrho),\ \text{where}$$

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \otimes_{\mathrm{Spec}\, \bar{k}} \bar{S} \quad O_B\text{-lin.}$$

$$\text{quasi-iso. of ht zero} \}.$$

<u>Theorem (Drinfeld → BC)</u>: $\mathcal{M}$ is representable by the formal scheme $\hat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{F}}$, compatible with action of $SL_2(F)$ on both sides [for $\mathcal{M}$ post-composing ϱ with g].

<u>Our new result is as follows</u>.

Let E/F quadratic extension. Consider triples (X, ι, λ) where

- X/S is formal O_F- p-divisible group
- $\iota : O_E \longrightarrow \mathrm{End}(X)$, ext. O_F-action and satisfying Kottwitz condition on Lie X,

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T-a)\cdot(T-\bar{a}) \;,\quad \forall a \in O_E.$$

- $\lambda : X \longrightarrow \check{X}$ ~~O_E~~ polariz s.t.

  $\mathrm{Rosati}_\lambda \mid O_E$ = non-triv. auto in $\mathrm{Gal}(E/F)$

<u>and</u>:

- if E/F <u>unramified</u>, then $\mathrm{Ker}\,\lambda = O_E/(\pi)$ - group scheme of height 1 in sense of Raynaud, i.e. of order q^2.
- if E/F <u>ramified</u>, then λ principal.

Fix $(\mathbb{X}, \iota, \lambda_{\mathbb{X}}) / \bar{k}$ (again unique up to iso.) $\rightsquigarrow \mathcal{N}_E$:

~~Theorem~~: $(X, \iota, \lambda, \varrho)$, s.t. $\varrho^*(\lambda_{\mathbb{X}})$ differs from $\lambda \times_S \bar{S}$ locally by a unit in $O_F^\times$.

<u>Theorem</u>: $\mathcal{N}_E$ is representable by $\hat{\Omega}_F \times_{\mathrm{Spf}\, O_F} \mathrm{Spf}\, O_{\breve{F}}$, compatible with action of $SL_2(F) \simeq SU(F)$, where SU/F = auto of split hermitian space of di 2 rel. E/F.
$= \mathrm{Aut}^0(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$.

<u>Proof is by comparison with $\mathcal{M}$.</u> Uses following proposition, implicit in Drinfeld, BC:

<u>Proposition</u>: Let $\Pi \in O_B$ be uniformizer with $\Pi^2 \in O_F$ and consider involution

$$b \longmapsto b^* = \Pi\, b' \,\Pi^{-1},$$

where $b \mapsto b'$ main involution.

a) On $\mathbb{X}$ ex. principal polariz. $\lambda^0_{\mathbb{X}}$ with assoc. Rosati $b \mapsto b^*$.

Furthermore, $\mathring{\lambda}_{\mathbb{X}}$ is unique up to $O_F^\times$.

b) Now fix $\mathring{\lambda}_{\mathbb{X}}$: Let $(X, \iota_B, \varrho) \in \mathcal{M}(S)$. On X $\exists!$ principal polariz $\mathring{\lambda}_X$ s.t. $\mathring{\lambda}_X \mid X \times_S \bar{S} = \varrho^*(\mathring{\lambda}_{\mathbb{X}})$.

Sketch of proof of theorem for E/F unramified: It is based on following prop. Embed $E \subset B$ s.t. $O_B = O_E[\Pi]/\Pi^2 - \pi$, $\Pi a = \sigma(a)\Pi$

As framing object, start with Drinfeld $(\mathbb{X}, \iota_B)$ – then take $(\mathbb{X}, \iota_B | O_E, \mathring{\lambda}_{\mathbb{X}} \circ \iota_B(\Pi\delta))$ Here $\delta \in O_E^\times$ s.t. $\delta^2 \in O_F^\times \setminus O_F^{\times,2}$. (rel. to Π)

Proposition: Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$. There exists a unique principal O_F-polariz. $\mathring{\lambda}_X$ on X with Rosati inv. inducing on O_E the trivial auto and with

$$\lambda_X \mid X \times_S \bar{S} = (\mathring{\lambda}_X \mid X \times_S \bar{S}) \circ \varrho^*(\iota_B(\Pi))$$

~~Here $\mathring{\lambda}_{\mathbb{X}}$ is relative to the choice Π of uniformizer, and we took $(\mathbb{X}, \iota_B | O_E, \lambda_{\mathbb{X}} = \mathring{\lambda}_{\mathbb{X}} \circ \iota_B(\Pi\delta))$ as framing object. Here $O_E = O_F[\delta]/(\delta^2)$.~~

The theorem follows easily from the proposition: namely,

$$\mathcal{M} \longrightarrow \mathcal{N}_E : (X, \iota_B, \varrho) \mapsto (X, \iota_B | O_E, \mathring{\lambda}_X \circ \iota_B(\Pi\delta), \varrho).$$

Inverse given by $(X, \iota, \lambda, \varrho) \longmapsto (X, \iota_B, \varrho)$, where

$$\iota_B(\Pi) = (\mathring{\lambda}_X)^{-1} \circ \lambda.$$

The proof of the proposition imitates Drinfeld's proof: On $X^\vee$ ex! polarization $\lambda_{X^\vee}$ s.t. $\lambda_{X^\vee} \circ \lambda = [\pi]$.
Hence $(X^\vee, \iota_{X^\vee}, \lambda_{X^\vee})$ is an object of $\mathcal{N}_E(S)$, if we define quasi-isogeny $\varrho_{X^\vee}$ correctly. Hence we obtain an automorphism of formal scheme

$$\alpha: \mathcal{N}_E \longrightarrow \mathcal{N}_E \quad : (X, \iota, \lambda, \varrho) \mapsto (X^\vee, \iota_{X^\vee}, \lambda_{X^\vee}, \varrho_{X^\vee})$$

In fact, α is an involution and it commutes with action of

$$SU(F) = \{ g \in End(\mathbb{X}) \mid g^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}, \ det(g) = 1 \}.$$

Now one shows that such an automorphism has to be the identity. The functorial iso $X \to X^\vee$ is the searched-for polarization $\mathring{\lambda}_X$.

<u>Concluding remark</u>: Have considered

E/F unramified, $|Ker\,\lambda| = q^2$ } <u>semi-stable red.</u>
E/F ramified, $Ker\,\lambda$ triv.

What about other 2 cases?

E/F unramified, $Ker\,\lambda$ triv. <u>form. smooth</u>

E/F ramified, $|Ker\,\lambda| = q^2$ <u>set of isolated points</u> → spin condition.