Harvard Sept. 10

Talk ~~Arithmetic~~ Alg. cycles on Picard moduli spaces of abelian varieties ~~of Picard type~~

Joint with S. Kudla.

(1) <u>Moduli space</u>.

k = imag. quadr. field, O_k.

Fix $n \geq 1$ and r with $0 \leq r \leq n$.

$\mathcal{M}(n-r, r)$ = moduli space over O_k, parametrizing triples (A, λ, ι)

where (A, λ) = princ. polar. abelian var. of dim n

$\iota : O_k \to \mathrm{End}(A)$ action of O_k

s.t.

(i) $\mathrm{Rosati}_\lambda \mid \iota(O_k)$ = conj. on O_k

(ii) $\mathrm{char}(\iota(\alpha) \mid \mathrm{Lie}\, A) = (T-\alpha)^{n-r}(T-\bar{\alpha})^r$, $\alpha \in O_k$

Represented by ~~scheme~~ DM / stack, smooth of relative dimension $(n-r)r$ over Spec O_k ~~over Spec O_k~~, ~~smooth~~ over Spec $O_k[(2\Delta)^{-1}]$.

<u>Example</u> $n=1$, $r=0$ pos

$\mathcal{M}_0 = \{$ elliptic curves w. CM by O_k $\}$

Then $|\mathcal{M}_0|$ = Spec O_H.

We are especially interested in $\mathcal{M}(n-1,1)$.

<u>Complex uniformization of $\mathcal{M}(n-1,1)$:</u> Make simplifying assumption: $h_k = 1$ and if n even then $2 \nmid \Delta$.

Fix $\tau : k \hookrightarrow \mathbb{C}$. Then

orbifold.

$$\mathcal{M}(n-1,1)(\mathbb{C}) = [\Gamma(L) \backslash \mathcal{D}(L)],$$

where L = (unique up to iso) selfdual hermit. O_k - lattice of rank n, of sign. $(n-1,1)$.

$\Gamma(L) = \mathrm{Aut}(L)$

$\mathcal{D}(L)$ = space of negative lines in $L \otimes_{k,\tau} \mathbb{C}$

$\simeq$ complex unit ball of dim $n-1$.

$n=3$: Picard

<u>Special subvarieties of $[\Gamma(L)\backslash\mathcal{D}(L)]$:</u> Fix $x \in L$ with $(x,x)>0$

$$\rightsquigarrow \quad [\Gamma(L)_x \backslash \mathcal{D}(L)_x] \hookrightarrow [\Gamma \backslash \mathcal{D}] ,$$

where

$$\mathcal{D}(L)_x = \{ \ell \in \mathcal{D} \mid \ell \perp x \}$$

$$\Gamma(L)_x = \mathrm{Stab}_\Gamma(x).$$

Divisor, <u>roughly</u> like $\mathcal{M}(n-2,1)$

More convenient to fix $t \in \mathbb{Z}_{>0}$ and consider

$$\left[\Gamma(L) \backslash \coprod_{(x,x)=t} \mathcal{D}(L)_x\right] \xleftrightarrow[\text{unramified.}]{} [\Gamma \backslash \mathcal{D}].$$

More generally, let $m \leq n-1$ and $T \in \mathrm{Herm}_m(\mathcal{O}_k)_{>0}$

Then,
$$\left[\Gamma(L) \backslash \coprod_{(\underline{x},\underline{x})=T} \mathcal{D}(L)_{\mathrm{span}(\underline{x})}\right]$$

codimension m alg. cycle.

If $m = n-1$, then get points, and can form generating series with mult. of these points, etc. (Kudla-Millson).

<u>We are interested in arithmetic version of these alg. cycles.</u>

They will only exist after base change to HCF, so form

$$\mathcal{M} = \mathcal{M}(n-1,1) \times_{\mathrm{Spec}\, \mathcal{O}_k} \mathcal{M}_0.$$

Let $(A,\lambda,\iota), (E,\lambda_0,\iota_0) \in \mathcal{M}(S)$, S connected.

Free $\mathcal{O}_k$-module $\mathrm{Hom}_{\mathcal{O}_k}(E, A)$ with positive hermitian form $h(x,y) = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathcal{O}_k$.

For $t > 0$ $\rightsquigarrow$ $Z(t) = \{ (A, \lambda, \iota), (E, \lambda_0, \iota_0), x : E \longrightarrow A \mid$

$h(x, x) = t \}$.

<u>Prop.</u>: $Z(t)$ is a relative divisor in $\mathcal{M}$, at least over $O_k[(2\Delta)^{-1}]$.

<u>NB</u>: if $t = 1$, then x defines decomp. $A = E \times A'$,

hence roughly $Z(1)$ = locus in $\mathcal{M}$ where A splits off E.

More generally, for $T \in \mathrm{Herm}_m(O_k) \rightsquigarrow Z(T)$.

<u>Recursive property</u>: $Z(t_1) \cap \ldots \cap Z(t_m) = \coprod_{\mathrm{diag}(T) = (t_1, \ldots, t_m)} Z(T)$.

<u>Dimension</u>: $T \in \mathrm{Herm}_m(O)_{>0}$. Then $Z(T)_{\mathbb{C}}$

has pure codimension m in $\mathcal{M}_{\mathbb{C}}$.

Since $\dim(\mathcal{M}) = n$, would naively expect that for

$T \in \mathrm{Herm}_n(O_k)_{>0}$, the cycle $Z(T)$ is finite.

<u>Totally wrong!</u> for $n > 2$ depends on p-divis. of T.

Define for $T \in \mathrm{Herm}_n(O_k)_{>0}$

$\mathrm{Diff}_0(T) = \{ p \mid p \text{ inert in } k, \ \mathrm{ord}_p(\det T) \text{ odd} \}$ <u>finite set</u>

Thm 1: (i) Let $T \in \mathrm{Herm}_n(O_k)_{>0}$.

a) If $|\mathrm{Diff}_0(T)| > 1$, then $Z(T) = \emptyset$

b) If $\mathrm{Diff}_0(T) = \{p\}$, then

$$Z(T) \subset \left(\mathcal{M} \otimes_O \mathbb{F}_{p^2}\right)^{ss}$$

dim $= \left[\frac{n-1}{2}\right]$, hence weak upper bd

$n=2$: good, $n>2$: not good.

c) If $\mathrm{Diff}_0(T) = \emptyset$, then

$$Z(T) \subset \bigcup_{p \text{ ramif.}} \left(\mathcal{M} \otimes_O \mathbb{F}_p\right)^{ss}.$$

(ii) Assume we are in case b) and $p > 2$. Then $Z(T)$ is zero-dim. (: T non-degenerate) if and only if

$$T \sim \mathrm{diag}(1_{n-2}, p^a, p^b) \quad 0 \le a < b, \ a+b \text{ odd}$$

(in general $Z(T)$ equi-dim., have formula for dimension)

(iii) Let T be non-degenerate. Then

$$Z(T) = \coprod_{\xi \in Z(T)(\bar{\mathbb{F}}_p)} Z(T)_\xi \quad ,$$

where $Z(T)_\xi$ local Artin scheme of length

$$\ell(Z(T)_\xi) = \sum_{i=0}^{a} p^i (a+b+1-2i), \quad \forall \xi.$$

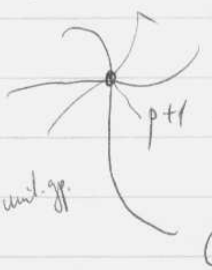
<u>Explanations:</u> a) <u>Supersingular locus</u> for $n=3$ is a curve of the form $\mathcal{M}_p^{ss} = \tilde{\mathcal{M}}_p^{ss} / \Gamma'$

(Γ' = discrete p-adic gp, acts prop. discont. for Zariski topol.)

and

Vollaard, Wedhorn

$\tilde{\mathcal{M}}_p^{ss}$ = building of unit. gp.



$|C(\mathbb{F}_{p^2})| = p^3+1$

$C: X_0^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$

gen. DL-variety

Similar picture for $n=4$. General case

b) <u>Diff.-set</u>: make 2 observations,

For $(A,\_;E,\_) \in Z(T)(k)$ get $E^n \longrightarrow A$, <u>isogeny</u>.

1. Compare Lie algebras as O-modules $\Rightarrow$ char $K = p$, and p inert or ramified, and A supersingular

2. Note that $\mathrm{Hom}_{O_k}(E,A)_{\mathbb{Q}} \simeq V_T$. OTOH, $V_T \otimes \mathbb{A}_f^p = \mathrm{Hom}(T^p(E), T^p(A)) \otimes \mathbb{A}_f^p$ contains a selfdual lattice. $\Longrightarrow$ (i)

c) (ii) + (iii) are reduced via RZ-uniformization to statement on formal groups.

Then one crucial ingredient is Zink's theory of displays + Gross-Keating theory of q.-c. liftings

Finally, relation to Eisenstein series

For selfdual L of sign. $(n-1,1)$ have incoherent Eisenstein series

$$E_L(z,s) \quad , \quad \text{where } z \in \mathcal{D}_{n,n} = \text{hermit. space for } U(n,n)$$
$$= \{ z \in M_n(\mathbb{C}) \mid v(z) > 0 \}$$

only dep. on iso-class of L

$$v(z) = \frac{1}{2i} \cdot (z - {}^t\bar{z}).$$

In particular, $E_L(z,0) \equiv 0$.

Let

$$E(z,s) = \sum_{L/\simeq} E_L(z,s)$$

Fourier expansion

$$E(z,s) = \sum_{T \in \mathrm{Herm}_n(\mathcal{O}_k)^\vee_{>0}} a(T)\, q^T + \sum_{T \text{ other}} a(T, v(z)) \cdot q^T$$

where $q^T = \exp(2\pi i \operatorname{tr}(Tz))$

<u>Theorem 2</u>: Let $\mathrm{Diff}_0(T) = \{p\}$, $p > 2$, and assume T non-degenerate. Then

$$a(T) = C \cdot \mathrm{lg}(Z(T)) \cdot \log p$$

where C const. indep't of T. In fact, if $t_1, \dots, t_n$ are the diagonal entries of T, then

$$\mathrm{lg}(Z(T)) = (Z(t_1), \dots, Z(t_n))_T \, . \quad \text{part along } Z(T)$$

(inters.-product of divisors, no higher tor-terms).

OTOH, if $|\mathrm{Diff}_0(T)| > 1$, then $a(T) = 0$ – checks aga

<u>Conjecture</u>: Let $T \in \mathrm{Herm}_n(O_k)_{>0}$ with $\mathrm{Diff}_0(T) = \{p\}$, $p >$
Let $t_1, \dots, t_n$ diag entries. Then

$$a(T) = C \cdot (Z(t_1), \dots, Z(t_n))_T \cdot \log p, \text{ same constant}$$

<u>Terstiege</u>: proved this for $n = 3$ (first non-trivial case).

<u>Seems</u>: need to understand inductive structure of cycles, as n varies (ongoing work with Kudla).