Rom, Oct. 09

Talk: Arithmetic cycles on moduli spaces of abelian varieties of Picard type

Joint with S. Kudla.

(1) Moduli space.

k = imag.-quadr. field , O_k.

Fix $n \geq 1$ and r with $0 \leq r \leq n$.

$\mathcal{M}(n-r, r)$ = moduli space over O_k, parametrizing triples (A, λ, ι)

where (A, λ) = princ. polar. abelian var. of dim n

$\iota : O_k \longrightarrow \mathrm{End}(A)$ action of O_k.

s.t.

(i) $\mathrm{Rosati}_\lambda \mid \iota(O_k)$ = conj. on O_k

(ii) $\mathrm{char}(\iota(\alpha) \mid \mathrm{Lie}\, A) = (T-\alpha)^{n-r} \cdot (T-\alpha^\sigma)^r$, $\alpha \in O_k$.

Represented by scheme / stack of relative dimension $(n-r) \cdot r$ ~~over Spec O_k~~, smooth over $\mathrm{Spec}\, O_k[(2\Delta)^{-1}]$.

Example: $n = 1$, $r = 0$ $\rightsquigarrow$

$\mathcal{M}_0$ = { elliptic curves w. CM by O_k }

Then $|\mathcal{M}_0| = \mathrm{Spec}\, O_H$.

We are especially interested in $\mathcal{M}(n-1,1)$.

<u>Complex uniformization of</u> $\mathcal{M}(n-1,1)$: Fix $\tau : k \to \mathbb{C}$

Let $\mathcal{L}_{(n-1,1)}$ = set of iso-classes of self-dual hermitian $\mathcal{O}_k$-lattices of signature $(n-1,1)$.

$\# = h_k$?

Then

$$\mathcal{M}(n-1,1)(\mathbb{C}) \simeq \coprod_{L \in \mathcal{L}_{(n-1,1)}} [\Gamma(L) \backslash D(L)],$$

where

$$D(L) = \text{space of negative lines in } L \otimes_{\mathcal{O}_k} \mathbb{C}$$

$$\simeq \text{cx. unit ball of dimension } n-1.$$

Inside $[\Gamma(L) \backslash D(L)]$ have special divisors defined by $x \in L$ with $(x,x) > 0$, via

$$\Gamma(L)_x \backslash D(L)_x$$

where $D(L)_x = \{ \ell \in D(L) \mid \ell \perp x \}$.

roughly like $D(L)$ but for signature $(n-2,1)$.

<u>Need algebraic definition of special cycles</u>: they will only exist after base to the Hilbert class field, more precisely on $\mathcal{M} = \mathcal{M}_{(n-1,1)} \times_{\mathcal{O}_k} \mathcal{M}_0$.

We are especially interested in $\mathcal{M}(n-1,1)$:

write $\mathcal{M}$, when n is understood.

Picard surfaces

<u>Note</u>: $\mathcal{M}^{\circ}_{\mathbb{C}}$ is alg. variety uniformized by $\mathbb{B}_{n-1}$.

in $\mathcal{M}_{\mathbb{C}}$

<u>Simplest tautological cycle</u>: locus where universal abelian variety A splits off elliptic curve E with CM by $\mathcal{O}_k$.

This is a divisor $\mathcal{Z}(1)_{\mathbb{C}}$ on $\mathcal{M}_{\mathbb{C}}$.

<u>Example</u>: $k = \mathbb{Q}(\sqrt{-3})$, $n = 5$. Then

$$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{Z}(1)_{\mathbb{C}} = \text{moduli spaces of non-sing. cubic surfaces}$$

(Allcock, Carlson, Toledo).

<u>Precise definition of special cycles</u>: Let $(A,\lambda,\iota), (E,\lambda_0,\iota_0) \in \mathcal{M}(S)$,

Consider free $\mathcal{O}_k$-module

$$\mathrm{Hom}_{\mathcal{O}_k}(E, A)$$

with positive-def. hermitian form

S connected

$$h(x,y) = \lambda_0^{-1} \circ \hat{y} \circ \lambda \circ x \in \mathrm{End}_{\mathcal{O}_k}(E) = \mathcal{O}_k .$$

For $t \in \mathbb{Z}_{>0}$, let $Z(t) \subset \mathcal{M}$ be defined by

$$Z(t) = \{ (A,\lambda,\iota),(E,\lambda_0,\iota_0),\ x : E \longrightarrow A,\ h(x,x) = t \}$$

Then $Z(t)$ is a relative divisor in $\mathcal{M}$.

More generally, for $T \in \mathrm{Herm}_m(O_k)$, let

$$Z(T) = \{ \quad \ldots \quad \underline{x} : E^m \longrightarrow A,\ h(\underline{x},\underline{x}) = T \}$$

<u>Good recursive properties</u>:

$$Z(t_1) \cap \ldots \cap Z(t_m) = \coprod_{\substack{T \in \mathrm{Herm}_m(O_k) \\ \mathrm{diag}(T) = (t_1,\ldots,t_m)}} Z(T).$$

<u>Prop</u>: Let $m \le n$. Let $T \in \mathrm{Herm}_m(O_k)_{>0}$. Then $Z(T)_{\mathbb{C}}$ purely of codimension m in $(\mathcal{M})_{\mathbb{C}}$ //

<u>Now</u> $\mathcal{M}$ <u>has arithm. dimension</u> n: would expect $Z(T)$ to ~~be~~ have finite support when $T \in \mathrm{Herm}_n(O_k)_{>0}$.

<u>This turns out to be totally wrong.</u>

(3.) To formulate the result, define for $T \in \mathrm{Herm}_n(O_k)_{>0}$,

$$\mathrm{Diff}_0(T) = \{ p \mid p \text{ inert in } k,\ \mathrm{ord}_p(\det T) \text{ odd} \}$$

<u>finite set</u>

Theorem: (i) Let $T \in \mathrm{Herm}_n(O_k)_{>0}$.

a) If $|\mathrm{Diff}_0(T)| > 1$, then $Z(T) = \emptyset$.

b) If $\mathrm{Diff}_0(T) = \{p\}$, then

supersingular locus

$$Z(T) \subset (\mathcal{M})_p^{ss}$$

$\longrightarrow$ picture of s.s. locus.

c) If $\mathrm{Diff}_0(T) = \emptyset$, then

$$Z(T) \subset \bigcup_{p \text{ ram.}} (\mathcal{M})_p^{ss}.$$

(ii) Assume we are in case b) and $p > 2$. Then, if

(Jordan dec.)

T is $GL_n(O_{k_p})$-equiv. to $1_{n_0} + p \cdot 1_{n_1} + \dots + p^k \cdot 1_{n_k}$,

then $Z(T)$ is equidimensional of dimension $[(n-n_0-1)/2]$.

(the supersingular locus has dimension $[(n-1)/2]$.

Furthermore, $Z(T)$ is zero-dimensional (T non-degenerate)

iff

$$T \sim \mathrm{diag}(1_{n-2}, p^a, p^b) \qquad 0 \le a < b,\ a+b \text{ odd}.$$

(iii) Let T be non-degenerate, then $Z(T)$ decomposes into a disjoint sum of local Artin schemes

$$Z(T) = \coprod_{y \in Z(T)(\bar{\mathbb{F}}_p)} Z(T)_y.$$

Each $\lg(Z(T)_y) = \frac{1}{2} \cdot \sum_{i=0}^{a} p^i(a+b-2i+1)$.

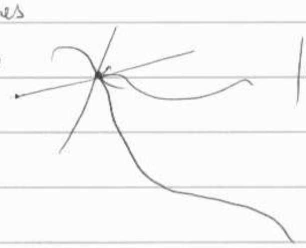
<u>Supersingular locus in</u> $\mathcal{M}_p \otimes \overline{\mathbb{F}}_p$:

$$\left(\mathcal{M}_p \otimes \overline{\mathbb{F}}_p\right)^{ss} = \widetilde{\mathcal{M}}_p^{ss} \,/\, \Gamma$$

Γ = discrete p-adic gp, acts properly disc. for Z-topology.

$n = 3$:

p+1 branches

$\widetilde{\mathcal{M}}_p^{ss} =$ 

$|C(\mathbb{F}_{p^2})| = p^3 + 1$

$C : X_0^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$

(4.) Form generating series:

$T \in \mathrm{Herm}_n(\mathcal{O}_k)_{>0}$ with $\mathrm{diag}(T) = (t_1, \dots, t_n)$,

let

$$\widehat{\deg}\, Z(T) = \chi\big(Z(T), \mathcal{O}_{Z(t_1)} \overset{\mathbb{L}}{\otimes} \cdots \overset{\mathbb{L}}{\otimes} \mathcal{O}_{Z(t_n)}\big) \cdot \log p.$$

This occurs then in the q-expansion of a modular form, as follows:

Form Eisenstein series for quasi-split unitary group rel. to $k/\mathbb{Q}$

$$U(n,n) = U(W, \langle\,,\rangle), \quad \langle\,,\rangle = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Then Siegel psgp has unipotent radical $\mathrm{Herm}_n(k)$.

Form

$\in \mathbb{C}$ $\quad C(s)\det(v(z))^{\frac{s}{2}}$ $\quad v(z) = $ "$\mathrm{Im}(z)$"

$$E(z, s, \Phi) = * \sum_{\Gamma_\infty \backslash \Gamma} \det(cz+d)^{-n} \cdot |\det(cz+d)|^{-s} \cdot \Phi(\gamma, s),$$

$\in$ tube domain, $z = u(z) + i\cdot v(z)$

where Φ: carefully chosen Siegel-Weil section of induced rep'n corresp. to $s \in \mathbb{C}$, depends on choice of T

(there are $2^{\delta-1}$ of them.

center of symmetry for fctl. equ.

<u>Theorem</u>: (i) $E(z, 0, \Phi) = 0$.

(ii) Consider the Fourier expansion of $E'(z, 0, \Phi)$,

$$E'(z, 0, \Phi) = \sum_{T \in \mathrm{Herm}_n(\mathcal{O}_k)} a(T, v(z)) \cdot q^T ,$$

where

$$q^T = \exp\left(2\pi i \,\mathrm{Tr}(Tz)\right)$$

$$v(z) = \text{"}\,\mathrm{Im}(z)\,\text{"}, \text{ as before}$$

If $T \in \mathrm{Herm}_n(\mathcal{O}_k)_{>0}$ ~~is such that $\mathrm{Diff}_0(T)=\{p\}$, $p>2$,~~

<u>then</u> $a(T, v(z)) = a(T)$ <u>is indep't of</u> z. If furthermore $\mathrm{Diff}_0(T) = \{p\}$, with $p > 2$, and T is non-degen. with $\mathrm{diag}(T) = (t_1, \dots, t_n)$, then

$$a(T) = C \cdot \widehat{\deg}\, Z(T) ,$$

for some constant C indep't of T, p, etc.

<u>Hope</u>: Last relation holds for any $T \in \mathrm{Herm}_n(\mathcal{O}_k)_{>0}$.

The other coeff. are a mystery ($\longrightarrow$ KRY).