Bonn, July 09.

Talk: Occult period mappings

First recall 3 Torelli theorems for period morphisms:

P-Š 1) Let X, X' be smooth quartic surfaces in $\mathbb{P}^3$. Let

$$\varphi : H^2(X, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z})$$

be an isomorphism which respects the cup product pairings, the classes of hyperplane sections and the H.S. Then φ is induced by a unique isom. $X' \xrightarrow{\sim} X$.

C-G&C 2) Let X, X' be smooth cubic ~~3x~~ hypersurfaces in $\mathbb{P}^4$. Let

$$\varphi : H^3(X, \mathbb{Z}) \longrightarrow H^3(X', \mathbb{Z})$$

be isom. which ...

Voisin. 3) Let X, X' be smooth cubic hypersurfaces in $\mathbb{P}^5$. Let ...

In each of these cases leads to ~~open~~ period mappings which are ~~open~~ embeddings:

1) $\quad \wp : \text{Quartics}_2 \longrightarrow \{\text{pos. or. planes in } V(2,19)\}/\Gamma$

open embedding

2.) $\quad \wp : \underset{\dim 10}{\text{Cubics}_3} \longrightarrow \underset{\dim 15}{\mathfrak{h}_5/\Gamma}$ ~~$\mathcal{A}_5$~~ loc. closed, not open emb.

3.) $\quad \wp : \text{Cubics}_4 \longrightarrow \{\text{pos. or. plane in } V(2,20)\}/\Gamma$

open embedding.

First example of occult period maps concerns cubic surfaces in $\mathbb{P}^3$ (Allcock, Carlson, Toledo).

$$S = \{ F(X_0, \ldots, X_3) = 0 \} \subset \mathbb{P}^3$$

all coh.-gps are isomorphic (with all add. structure).

Let V = cyclic covering of $\mathbb{P}^3$ of degree 3, ramified along S.

$$V = \{ X_4^3 - F(X_0, \ldots, X_3) = 0 \} \subset \mathbb{P}^4 \quad \text{cubic 3-fold.}$$

V equipped w. action of μ_3.

Associate to V its intermediate Jacobian

$$A = A(V) = H^3(V, \mathbb{Z}) \backslash H^3(V, \mathbb{C}) / H^{2,1}(V).$$

This is an abelian var., of dimension 5, with action by $\mathcal{O}_k = \mathbb{Z}[\mu_3]$. It is principally polarized by its intersection form.

Such abelian varieties come in nice modular families, which I investigate in my joint work w. Kudla.

- $\mathcal{M}(k; n-r, r)$. Picard type over Spec k.

plus assumption: if n even, then 2 unramified.

• Specialize to $\mathcal{M}(k; n-1, 1)$ if k class no $= 1$., $\tau: k \to \mathbb{C}$

Then $\mathcal{M}(k, n-1, 1)_{\mathbb{C}} \simeq [\Gamma \backslash \mathcal{D}]$, orbifold

where L = perfect hermitian $\mathcal{O}_k$-module of signature $(n-1, 1)$

unique up to $\mathcal{U}(L \otimes \mathbb{Q})$.

$\mathcal{D}$ = space of negative lines in $L \otimes \mathbb{R} \simeq \mathbb{C}^n$

$\Gamma = U(L)$.

<u>Theorem</u> (ACT): (i) The object $(A(V), \iota, \lambda)$ lies in $\mathcal{M}(k; 4, 1)(\mathbb{C})$

(ii) This is compatible with families and defines a morphism of DM-stacks,

$$\varphi : \mathcal{C}ubics^{0}_{2, \mathbb{C}} \longrightarrow \mathcal{M}(k; 4, 1)_{\mathbb{C}}$$

(iii) This is an open embedding.

(iv) Its complement is the KM-divisor $\mathcal{Z}(1)$

(v) φ is defined over k.

<u>Explanation of</u> (iv): • Fix $(E, \iota, \lambda) \in \mathcal{M}_{(1,0)}(\mathbb{C})$, unique up to iso (class no $= 1$).

- $(V'(A,E), h')$ for (A,ι,λ) in $\mathcal{M}(k; n-1,1)_{\mathbb{C}}$
- For $t>0$ $\rightsquigarrow$

$$\mathcal{Z}(t) = \{ (A,\iota,\lambda; x) \mid h'(x,x) = t \}.$$

Then

$$\mathcal{Z}(t) = \Big[\Gamma \backslash \coprod_{(x,x)=t} \mathcal{D}_x \Big].$$ divisor on $\mathcal{M}(k; n-1,1)_{\mathbb{C}}$.

That $\mathcal{Z}(1) \subset$ complement comes from: intermediate jacobian of cubic 3-fold is <u>simple</u> as p.p.a.v. Not everything is surprising.

<u>explanation of (v)</u>: follows from Deligne's theorem (1972) that intermediate jacobian of complete inters. of Hodge level 1 is <u>algebraic construction</u>.

Before giving another example, we define a variant $\mathcal{M}(k;n-1,1)^{*}$, resp. $Z(t)^{*}$ of $\mathcal{M}(k;n-1,1)$, resp. $Z(t)$:

replace condition that λ be principal polariz. by the following: fix $d \mid \Delta$, d squarefree

$$(*) \qquad \mathrm{Ker}(\lambda : A \to A^{\vee}) \subseteq A[\iota\Delta], \quad \deg \lambda = \begin{cases} d^{n-1} & n \text{ odd} \\ d^{n-2} & n \text{ even} \end{cases}$$

$$2[\tfrac{n-1}{2}]$$

(loc. for fppf-topology isom. to $\prod_{p \mid d} \mathbb{F}_p$).

In our examples $\Delta = p$-power : then $d = p$.

<u>One more example</u>: Start with cubic 3-fold T

$\rightsquigarrow$ cubic 4-fold V, with μ_3-action.

Let $\qquad L = H^4_0(V, \mathbb{Z})$.

Have
$$\begin{cases} \text{Hodge structure} & L \otimes \mathbb{C} = \underset{\substack{1 \\ | \\ \zeta}}{L^{3,1}} \oplus \underset{\substack{20 \\ | \\ \text{half-half}}}{L^{2,2}} \oplus \underset{\substack{1 \\ | \\ \bar{\zeta}}}{L^{1,3}} \\ \text{eigendecomp.} & \end{cases}$$

Now define $\Lambda = \pi L^\vee \quad , \quad \pi = \sqrt{\Delta}$

$$A = \Lambda \backslash L \otimes \mathbb{C} / L^{3,1} + L^{2,2}_{\bar{\zeta}}$$

Then this is again of Picard type (except not princip.)!

(ACT, LS)

<u>Thm</u>: • Get open embedding

$$\varphi : \mathrm{Cubics}^{\circ}_{3,\mathbb{C}} \longrightarrow \mathcal{M}(k; 10, 1)^*_{\mathbb{C}} ,$$

• complement is KM-divisor $\mathcal{Z}(3)^*$.

• φ is defined over k.

better than <u>up-front</u> period map!
(which is <u>not</u> an open emb.)

<u>Comments</u>: (i) For rational HS with action by CM-field, this construction was defined by van Geemen: <u>half-twist</u>.

(ii) It is <u>not</u> clear, why $Z(3)^*$ is in the complement!

(iii) The last point is proved by imitating Deligne's proof.

<u>Lemma</u>: Let $\mathcal{C} = \mathrm{Cubics}_3^\circ$. Let $s \in \mathcal{C}(\mathbb{C})$.

(i) The monodromy rep'n $\rho : \pi_1(\mathcal{C}_{\mathbb{C}}, s) \to GL_k(\Lambda \otimes_{\mathcal{O}_K} k)$ is absol. irreduc.

(ii) ——— $\pi_1(\mathcal{C}_{\mathbb{C}}, s) \to GL_{K(\mathfrak{p})}(\Lambda/\mathfrak{p}\Lambda)$ ———, $\forall \mathfrak{p}$ prime to 3

(iii) ——— not abs. irred., but has a unique invar. subspace, namely the codim-1-image of $\pi\Lambda^\vee$ in $\Lambda/\pi\Lambda$

$$(\underbrace{\pi\Lambda \subset \pi\Lambda^\vee}_{10} \underbrace{\subset \Lambda}_{1}).$$

<u>Concluding remarks</u>: 2 more cases (Kondo):

- non-hyperelliptic curves of genus 3 $\longrightarrow \mathcal{M}(\mathbb{Q}(i); 6, 1)^*_{\mathbb{C}}$
- ——— 4 $\longrightarrow \mathcal{M}(\mathbb{Q}(\zeta_3), 9, 1)^*_{\mathbb{C}}$.

<u>But</u> no more hypersurface cases (but last one <u>not</u> a hypersurface case!)