Cambridge, May 2009

Talk: Algebraic cycles on moduli spaces of abelian varieties

(1) The 4 levels of algebraic geometry

- geo. of varieties defined by polynomial eqn.
- defined by geometric invariants, e.g.: curves of genus g, abelian varieties, K3-surfaces, CY-mfs. etc.
- moduli spaces of varieties with fixed geom. invariants, e.g.
  - moduli space M_g of curves of genus g (a stack, not a scheme)
  - moduli space A_g of abelian varieties of genus g + pric. polariz.
- study algebraic cycles / vector bdls on such moduli spaces

For M_g: Get tautological cycles from universal curve over M_g and its sheaf of relative differentials

Kontsevich / Witten: form a generating series from intersection no's of these tautological classes. Show that this is a „modular form" in some sense (fct'l eqns).

For A_g or rather variants: again, there are some kind of tautological cycles defined in terms of the moduli problem

Kudla: forms a generating series from intersection no's of these special cycles, predicts this is a modular form, in a very precise form.

The talk explains this program for the case of the Shimura variety ass. to $GU(n-1,1)$.

Joint work w. S. Kudla.

(2) The moduli space, the special cycles

k = imag.-quadr. field, $k \subset \mathbb{C}$

O_k

Fix r with $0 \le r \le n$.

$\mathcal{M}(n-r, r)$ = moduli space over $\operatorname{Spec} O_k$, parametriz. (A, λ, ι)

where (A, λ) abelian var. of dim. n, princip. polarized

$$\iota : O_k \longrightarrow \operatorname{End}(A)$$

s.t. (i) $\operatorname{Rosati}_\lambda \mid \iota(O_k)$ = cx. conj. $\curvearrowleft$ preserves $\iota(O_k)$

(ii) rep'n of O_k on $\operatorname{Lie}(A)$

$$= (n-r) \times \text{nat} + r \times \overline{\text{nat}}.$$

This moduli pb. is repres. by scheme/stack of relative dim. $(n-r) \cdot r$ over $\operatorname{Spec} O_k$.

<u>Example</u>: $n = 1$, $r = 0$. $\rightsquigarrow$

$$\mathcal{M}_0 = \{ \text{ell. curves, with CM by } O_k \}$$

Then

$$|\mathcal{M}_0| = \operatorname{Spec} O_H.$$

We are especially interested in $\mathcal{M}(n-1,1)$:

write $\mathcal{M}$, when n is understood.

Picard surfaces

<u>Note</u>: $\mathcal{M}^{\circ}_{\mathbb{C}}$ is alg. variety uniformized by $\mathbb{B}_{n-1}$.

in $\mathcal{M}_{\mathbb{C}}$

<u>Simplest tautological cycle</u>: locus where universal abelian

A E

variety splits off elliptic curve with CM by $\mathcal{O}_k$.

This is a divisor $\mathcal{Z}(1)_{\mathbb{C}}$ on $\mathcal{M}_{\mathbb{C}}$.

<u>Example</u>: $k = \mathbb{Q}(\sqrt{-3})$, $n = 5$. Then

$$\mathcal{M}_{\mathbb{C}} \setminus \mathcal{Z}(1)_{\mathbb{C}} = \text{moduli spaces of non-sing. cubic surfaces}$$

(Allcock, Carlson, Toledo).

<u>Precise definition of special cycles</u>: Let $(A,\lambda,\iota), (E,\lambda_0,\iota_0) \in \mathcal{M} \times \mathcal{M}_0$.

Consider free $\mathcal{O}_k$-module

$$\mathrm{Hom}_{\mathcal{O}_k}(E, A)$$

with positive-def hermitian form

$$h(x,y) = \lambda_0^{-1} \circ \hat{y} \circ \lambda \circ x \in \mathrm{End}_{\mathcal{O}_k}(E) = \mathcal{O}_k.$$

For $t \in \mathbb{Z}_{>0}$, let $Z(t) \subset \mathcal{M} \times_{O_k} \mathcal{M}_0$ be defined by

$$Z(t) = \{ (A, \lambda, \iota), (E, \lambda_0, \iota_0), \quad x : E \longrightarrow A, \quad h(x,x) = t \}$$

Then $Z(t)$ is a relative divisor in $\mathcal{M} \times_{O_k} \mathcal{M}_0$.

More generally, for $T \in \mathrm{Herm}_m(O_k)$, let

$$Z(T) = \{ \quad \ldots , \quad \underline{x} : E^m \longrightarrow A, \quad h(\underline{x},\underline{x}) = T \}.$$

<u>Good recursive properties</u>:

$$Z(t_1) \cap \ldots \cap Z(t_m) = \coprod_{\substack{T \in \mathrm{Herm}_m(O_k) \\ \mathrm{diag}(T) = (t_1, \ldots, t_m)}} Z(T).$$

<u>Prop</u>: Let $m \leq n$. Let $T \in \mathrm{Herm}_m(O_k)_{>0}$. Then $Z(T)_{\mathbb{C}}$ ~~purely~~ of codimension m in $(\mathcal{M} \times_{O_k} \mathcal{M}_0)_{\mathbb{C}}$ //

<u>Now</u> $\mathcal{M} \times_{O_k} \mathcal{M}_0$ has arithm. dimension n: would expect $Z(T)$ to ~~be~~ have finite support when $T \in \mathrm{Herm}_n(O_k)_{>0}$.

This turns out to be totally wrong.

3. To formulate the result, define for $T \in \mathrm{Herm}_n(O_k)_{>0}$,

$$\mathrm{Diff}_0(T) = \{ p \mid p \text{ inert in } k, \ \mathrm{ord}_p(\det T) \text{ odd} \}$$

finite set

Theorem: (i) Let $T \in \mathrm{Herm}_n(O_k)_{>0}$.

a) If $|\mathrm{Diff}_0(T)| > 1$, then $Z(T) = \emptyset$.

b) If $\mathrm{Diff}_0(T) = \{p\}$, then

supersingular locus

$$Z(T) \subset \left(\mathcal{M} \times_{O_k} \mathcal{M}_0\right)_p^{ss}$$

c) If $\mathrm{Diff}_0(T) = \emptyset$, then

$$Z(T) \subset \bigcup_{p \text{ ram.}} \left(\mathcal{M} \times_{O_k} \mathcal{M}_0\right)_p^{ss}.$$

(ii) Assume we are in case b) and $p > 2$. Then, if

(Jordan dec.)

T is $GL_n(O_k)$-equiv. to $1_{n_0} + p \cdot 1_{n_1} + \dots + p^k \cdot 1_{n_k}$,

then $Z(T)$ is equidimensional of dimension $\left[(n-n_0-1)/2\right]$.

[the supersingular locus has dimension $\left[(n-1)/2\right]$.

Furthermore, $Z(T)$ is zero-dimensional (T non-degenerate)

iff

$$T \sim \mathrm{diag}(1_{n-2}, p^a, p^b) \qquad 0 \le a < b, \ a+b \text{ odd}.$$

(iii) Let T be non-degenerate, then $Z(T)$ decomposes into a disjoint sum of local Artin schemes

$$Z(T) = \coprod_{\xi \in Z(T)(\mathbb{F}_p)} Z(T)_\xi .$$

Each $\lg(Z(T)_\xi) = \frac{1}{2} \cdot \sum_{i=0}^{a} p^i (a+b-2i+1)$.

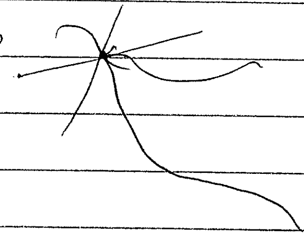
Supersingular locus in $\mathcal{M}_p \otimes \overline{\mathbb{F}}_p$:

$$\left(\mathcal{M}_p \otimes \overline{\mathbb{F}}_p\right)^{ss} = \widetilde{\mathcal{M}}_p^{ss} / \Gamma$$

Γ = discrete p-adic gp, acts properly disc. for Z-topology

$n = 3$:

p+1 branches



$|C(\mathbb{F}_{p^2})| = p^3 + 1$

$\widetilde{\mathcal{M}}_p^{ss} =$

$C : X_0^{p+1} + X_1^{p+1} + X_2^{p+1} = 0$

4. Form generating series: for $t_1, \dots, t_n \in \mathbb{Z}_{>0}$,

and $T \in \mathrm{Herm}_n(O_k)_{>0}$ with $\mathrm{diag}(T) = (t_1, \dots, t_n)$ and

for $\mathrm{Diff}_0(T) = \{p\}$, let

$$\langle Z(t_1), \dots, Z(t_n) \rangle_T = \chi\big(Z(T), O_{Z(t_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{Z(t_n)}\big) \cdot \log p.$$

This occurs then in the q-expansion of a modular form, as follows:

Form Eisenstein series for quasi-split unitary group rel. to $k/\mathbb{Q}$

$$U(n,n) = U(W, \langle\, , \rangle), \qquad \langle\, , \rangle = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Then Siegel psgp has unipotent radical $\mathrm{Herm}_n(k)$.

Form

$\in \mathbb{C}$ $\qquad C(s)\det(z))^{\frac{s}{2}}$ $\qquad v(z) =$ "$\mathrm{Im}(z)$"

$$E(z, s, \Phi) = * \sum_{\Gamma_\infty \backslash \Gamma} \det(cz+d)^{-n} \cdot |\det(cz+d)|^{-s} \cdot \Phi(\gamma, s),$$

$\in$ tube domain

where Φ: carefully chosen Siegel-Weil section of induced rep'n

corresp. to $s \in \mathbb{C}$.

center of symmetry for fctl. equ.

Theorem: (i) $E(z, 0, \Phi) = 0$

(ii) Consider the Fourier expansion of $E'(z, 0, \Phi)$,

$$E'(z, 0, \Phi) = \sum_{T \in \mathrm{Herm}_n(\mathcal{O}_k)} a(T, v(z)) \cdot q^T ,$$

where

$$q^T = \exp(2\pi i \, \mathrm{Tr}(Tz))$$

$$v(z) = \text{"}\mathrm{Im}(z)\text{"}.$$

If $T \in \mathrm{Herm}_n(\mathcal{O}_k)_{>0}$ ~~is such that $\mathrm{Diff}_0(T) = \{p\}, p > 2$,~~

then $a(T, v(z)) = a(T)$ is indep't of z. If furthermore $\mathrm{Diff}_0(T) = \{p\}$, with $p > 2$, and T is non-degen. with $\mathrm{diag}(T) = (t_1, \dots, t_n)$, then

$$a(T) = C \cdot \langle Z(t_1), \dots, Z(t_n) \rangle_T ,$$

for some constant C indep't of T, p, etc.

Hope: Last relation holds for any $T \in \mathrm{Herm}_n(\mathcal{O}_k)_{>0}$.

The other coeff. are a mystery ($\longrightarrow$ KRY).