Bonn, Dec 09

Talk: On the geometry of the $\Gamma_1(p)$-moduli problem, 1+2

Want to give some details necessary to complete my write-up of joint paper w. Haines (talked about this on 6 Jan. 09).

(1.) Moduli problem: $F/\mathbb{Q}$ imag.-quadr. $(p) = \mathfrak{p}\cdot\bar{\mathfrak{p}}$

D/F simple alg. with center F, with

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p = D_{\mathfrak{p}} \times D_{\bar{\mathfrak{p}}}$$

where $D_{\bar{\mathfrak{p}}} \simeq M_d(\mathbb{Q}_p)$ (fix).

Fix $\mathbb{Z}_{(p)}$-order O_D s.t. under fixed iso have

$$O_D \otimes_{\mathbb{Z}_{(p)}} O_{F_{\bar{\mathfrak{p}}}} = M_d(\mathbb{Z}_p).$$

Let AV_{O_D} be the cat. fibered over $(Sch/\mathbb{Z}_p)$ of abelian var. with O_D-action, up to isogeny prime to p.

Def: $\Gamma_0(p)$-moduli pb. $\mathcal{M}_0$ over $\mathbb{Z}_p$: $S \mapsto \{$iso-classes of diagrams$\}$

a)
$$\begin{array}{ccccccc}
A_0 & \xrightarrow{\alpha} & A_1 & \xrightarrow{\alpha} & \cdots & \xrightarrow{\alpha} & A_d = A_0 \\
\lambda_0 \downarrow & & \downarrow & & & & \downarrow \\
\hat{A}_0 & \longrightarrow & \hat{A}_{d-1} & \longrightarrow & & & \hat{A}_0
\end{array}$$

Here $A_i \in AV_{O_D}$, α isogeny of degree p^{2d}, s.t. $\alpha^d = p\cdot id_{A_0}$

λ_i given up to common factor in $\mathbb{Q}^\times$.

λ_0 p-principal polariz.

b) Level structure outside p.

Conditions:

(i) tangent condition on each A_i,

$$\mathrm{char}(\iota(x) \mid \mathrm{Lie}\, A_i) = \mathrm{char}(x)^{d-1} \cdot \overline{\mathrm{char}(x)} \Rightarrow \dim A_i = d^2$$

(ii) Rosati$_{\lambda_0}$ preserves $\mathcal{O}_D$.

Consider p-div. gp of A_i, with its operation of

$$\mathcal{O}_D \otimes \mathbb{Z}_p = \mathcal{O}_{D_{\mathfrak{p}}} \oplus \mathcal{O}_{D_{\bar{\mathfrak{p}}}} = M_d(\mathbb{Z}_p)^{opp} \times M_d(\mathbb{Z}_p)$$

$$\Rightarrow \quad A_i(p^\infty) = A_i(\mathfrak{p}^\infty) \times A_i(\bar{\mathfrak{p}}^\infty)$$

and $A_i(\bar{\mathfrak{p}}^\infty) = X_i^d$ canonically, where $\begin{cases} \dim X_i = 1 \\ \mathrm{ht}\, X_i = d \end{cases}$

Hence obtain

$$X_0 \xrightarrow{\alpha} X_1 \longrightarrow \cdots \xrightarrow{\alpha} X_d = X_0$$

where $\mathrm{ht}(\alpha) = p$, $\alpha^p = p \cdot \mathrm{id}_{X_0}$.

Def: $\Gamma_1(p)$-moduli pb. $\mathcal{M}_1$ over $\mathcal{M}_1$.

Let $G_i = \ker(X_{i-1} \to X_i)$ finite flat gp scheme over S.

$\mathcal{M}_1(S)$: choose a generator of G_i, $\forall i$ (in the sense of course WS 08).

There is a variant of this construction, which also gives an interesting complement to the course on moduli of elliptic curves

Thm (Oort/Tate): Let OT be the stack over $\mathrm{Spec}\, \mathbb{Z}_p$ of finite flat group schemes of order p. Then

(i) OT is an Artin stack isomorphic to

$$\left[\operatorname{Spec} \mathbb{Z}_p[X,Y]/(XY - w_p) \,/\, \mathbb{G}_m\right],$$

where $\mathbb{G}_m$ acts via $\lambda \cdot (X,Y) = (\lambda^{p-1}X, \lambda^{1-p}Y)$.

Here $w_p \in p \cdot \mathbb{Z}_p^\times$ explicit element.

(ii) The universal group scheme $\mathcal{G}$ over OT is isomorphic to

$$\mathcal{G} = \left[\operatorname{Spec} \mathcal{O}_{OT}[Z]/(Z^p - XZ) \,/\, \mathbb{G}_m\right] \quad \lambda(X,Y,Z) = (\lambda^{p-1}X, \lambda^{1-p}Y, \lambda Z)$$

with zero section $Z = 0$. (As scheme! - the gp structure is another matter.) $\longrightarrow$ Recto

(iii) Cartier duality interchanges X and Y.

Let $\mathcal{G}^\times$ = closed relative subscheme defined by $Z^{p-1} = X$.

Hence $\mathcal{G}^\times \longrightarrow OT$ is finite + flat of degree $p-1$, étale after base change $\times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{Q}_p$.

<u>Back to</u> $\mathcal{M}_0$: for $i = 1, \dots, d$, get

$$\varphi_i : \mathcal{M}_0 \longrightarrow OT$$

Hence

$$\begin{array}{ccc} & & \mathcal{G}^\times \times \dots \times \mathcal{G}^\times \\ & & \downarrow \\ \mathcal{M}_0 & \longrightarrow & OT \times_{\mathbb{Z}_p} \dots \times OT \end{array}$$

Then

$\mathcal{M}_1$ = 2-fiber product:

Unsymmetric description of $\mathcal{M}_0$ resp. $\mathcal{M}_1$.

$\mathcal{M}_0(S) =$ $(A,\iota) \in AV_{\mathcal{O}_D}$ plus $\mathbb{Q}$-class of principal polariz. plus

level-K^p-structure $\bar{\eta}^p$ $\mapsto$ X ht d, dim 1.

PLUS filtration $0 = H_0 \subset H_1 \subset \dots \subset H_d = X[p]$

by finite flat gp schemes s.t. $G_i = H_i/H_{i-1}$ finite flat of order p

$\mathcal{M}_1(S) =$ in addition a generator of G_i, $i = 1, \dots, d$.

Proposition 1 : a) $\mathcal{M}_0$ is a regular scheme of dimension d, with strict semi-stable reduction over $\operatorname{Spec} \mathbb{Z}_p$. More precisely, let

$\mathcal{M}_{0,i}$ = locus where $\operatorname{Lie} X_{i-1} \to \operatorname{Lie} X_i$ vanishes.

Then each $\overline{\mathcal{M}_{0,i}}$ is a smooth divisor and

$S(\dot{x}) = \{ i \mid a_i \text{ infinit.} \}$
$$\mathcal{M}_0 \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \underset{\text{strict}}{\bigcup} \overline{\mathcal{M}_{0,i}} . \quad \rightsquigarrow \underset{\text{smooth of dim } n-|S|.}{\mathcal{M}_{0,S}}$$

b) $\mathcal{M}_1$ is a regular scheme with tame semi-stable reduction, with all divisors of multiplicity $p-1$.

Etale locally on the base, around $\dot{x} \in \mathcal{M}_{0,S}(\overline{\mathbb{F}}_p)$, the morphism $\mathcal{M}_1 \to \mathcal{M}_0$ has the form

$$\operatorname{Spec} W(\overline{\mathbb{F}}_p)[T_1', \dots, T_d'] / \Big(\prod_{j \in S} T_j'\Big)^{p-1} - \varpi_p \longrightarrow \operatorname{Spec} W(\overline{\mathbb{F}}_p)[T_1, \dots, T_d] / \prod_{j \in S} T_j - \varpi_p$$

$$T_i'^{\,p-1} \longleftarrow\!\shortmid T_i$$

//

c) $\mathcal{M}_1 \to \mathcal{M}_2$ is a Galois covering with Galois group $T(\mathbb{F}_p) = \prod_{i \in \mathbb{Z}/d} \mathbb{F}_p^\times$. The Galois covering is étale over the generic fiber but ramified in the special fiber. More precisely, let

$$\mathcal{M}_{1,S} = \pi^{-1}(\mathcal{M}_{0,S}) , \quad T^S = T / T_S .$$

Then $\pi \mid (\mathcal{M}_{1,S})_{red}$ is an étale Galois covering of $\mathcal{M}_{0,S}$ with Galois group $T^S(\mathbb{F}_p)$.

② The monodromy theorem.

We denote by a $\overline{}$ extension of scalars from $\mathbb{F}_p$ to $\overline{\mathbb{F}}_p$.

Theorem: Let $S \subset \mathbb{Z}/d$, $S \neq \emptyset$.

(i) The lisse $\mathbb{F}_p$-sheaf $\prod_{i \notin S} \mathcal{G}_i$ on $\overline{\mathcal{M}}_{0,S}^{\circ}$ defines for any $\dot{x} \in \overline{\mathcal{M}}_{0,S}(\overline{\mathbb{F}}_p)$ a homomorphism

$$\pi_1(\overline{\mathcal{M}}_{0,S}^{\circ}, \dot{x}) \longrightarrow T^S(\mathbb{F}_p).$$

This homomorphism is surjective.

(ii) ~~Let~~ The natural map

$$\pi_0(\overline{\mathcal{M}}_{1,S}) \longrightarrow \pi_0(\overline{\mathcal{M}}_{0,S})$$

is bijective.

(iii) Note that these ~~three~~ statements are in fact equivalent (in addition to being true): We know that $T^S(\mathbb{F}_p)$ is the group of deck transformations for the covering $\overline{\mathcal{M}}_{1,S}^{\circ} \to \overline{\mathcal{M}}_{0,S}$. Hence if the inverse image of a connected component of $\overline{\mathcal{M}}_{0,S}^{\circ}$ is connected, π_1 maps surjectively to the group of deck transformations. Conversely, if the inverse image is not connected, then the map from π_1 to the group of deck transformations would factor ~~over~~ a through a proper subgroup.

That (iii) is equivalent is also clear.

(iii) The ~~representation~~ rep'n of the fundamental group $\pi_1(\bar{\mathcal{M}}^{\#}_{0,S}, \dot{x})$ ~~acts transitively~~ on $\pi_{S*}(\mathbb{Q}_\ell)_{\dot{x}}$ is via the regular rep'n of $T^S(\mathbb{F}_p)$.

We will prove the theorem in the form (i). For simplicity assume $S = \{1, \dots, l\}$ (one can reduce to this case).

We will show something stronger. Let $y \in \mathcal{M}_{0,\mathbb{Z}/d}(\mathbb{F}_p)$ be a point in the closure of the connected component of $\overline{\mathcal{M}}^{0}_{0,S}$ containing the base point $\dot{x}$. Let

$$R := \quad \simeq \quad W[[u_1, \dots, u_{d-1}]]$$

be the ~~completed local ring of y in~~ universal deformation ring of the formal group X_0 ~~also~~ defined by y. Similarly, let us consider the finite ring homomorphisms

$$R \subset R^0 \subset R^1 \subset \widetilde{R},$$

$$\begin{array}{c} \widetilde{R}_1 \\ \uparrow \\ R_1 \\ \uparrow \\ R_0 \\ \uparrow \\ R \end{array}$$

where R^0 = univ. def. of X_0 + chain ~~comes~~ $H_\bullet = H_\bullet(y)$

R^1 = ——〃—— , + generators of $G_i(y) = H_i(y)/H_{i-1}(y)$

$\widetilde{R}$ = univ. def. of X_0 + Drinfeld-level structure
$$\varphi : \mathbb{F}_p^d \longrightarrow X_0[p].$$

Let $\mathfrak{p}^0_h \subset R^0$ be the prime ideal defined by $\overline{\mathcal{M}}_{0,S}$.

~~Then we obtain~~ Let $K^0_h = \mathrm{Frac}(R^0/\mathfrak{p}^0_h)$.

Then we obtain a commutative diagram,

$$\begin{tikzcd} & \mathrm{Gal}(\eta) \arrow[d] & \\ \mathrm{Gal}(x_h^0) \arrow[ur] \arrow[r] \arrow[dr] & \pi_1(\overline{\mathcal{M}}^{\#}_{0,S}, \bar{x}) \arrow[d] & \\ & T^S(\mathbb{F}_p). & \end{tikzcd}$$

η = generic point of connected comp. of $\overline{\mathcal{M}}^{\#}_{0,S}$.

Hence it suffices to prove the surjectivity of the oblique arrow.

Note that the morphism $R^0 \hookrightarrow \widetilde{R}$ is induced on the moduli side by

$$(X_0, \varphi) \mapsto X_0, \; H_\cdot(\varphi), \quad \text{where}$$

$$[H_i] = \sum_{x \in \mathrm{span}(e_1, \dots, e_i)} [\varphi(x)].$$

Similarly, $R^1 \hookrightarrow \widehat{R}$ is induced by

$$(X_0, \varphi) \longmapsto (X_0, \; H_\cdot(\varphi), \; \text{generator } \varphi(e_i) \text{ of } H_i/H_{i-1}).$$

Let

$$\begin{aligned} G &= GL_d(\mathbb{F}_p) \\ &\cup \\ P_h &= \mathrm{Stab}(\mathrm{span}(e_1, \dots, e_h)) \\ &\cup \\ Q_h &= \text{subgp which acts trivially on } \mathbb{F}_p^n / \mathrm{span}(e_1, \dots, e_h)) \\ &= \begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array} \right) \\ h \end{array}. \end{aligned}$$

Recall (Drinfeld): $\tilde{R}$ regular local ring, with system of regular parameters $\varphi(e_1), \dots, \varphi(e_d)$.

Let $\tilde{\mathfrak{p}}_h$ = prime ideal gener. by $\varphi(e_1), \dots, \varphi(e_h)$.

$$\tilde{R}_h = \tilde{R}/\tilde{\mathfrak{p}}_h$$

$$\tilde{K}_h = \mathrm{Frac}(\tilde{R}_h).$$

Similarly, let

$$\left.\begin{array}{lll} \mathfrak{p}_h = \tilde{\mathfrak{p}}_h \cap R, & R_h = R/\mathfrak{p}_h, & K_h = \mathrm{Frac}(R_h) \\ \mathfrak{p}_h^1 = \tilde{\mathfrak{p}}_h \cap R^1, & R_h^1 = R^1/\mathfrak{p}_h^1, & K_h^1 = \mathrm{Frac}(R_h^1) \\ \mathfrak{p}_h^0 = \tilde{\mathfrak{p}}_h \cap R^0, & R_h^0 = R^0/\mathfrak{p}_h^0, & K_h^0 = \mathrm{Frac}(R_h^0) \end{array}\right\} (*)$$

(see below for this seeming notational conflict)

For $h = 0$, set $\tilde{R}_0 = \tilde{R}$, $\tilde{K}_{\text{[illegible]}} = \mathrm{Frac}(\tilde{R}_0)$.

Similarly, $K_{\text{[illegible]}} = \mathrm{Frac}(R)$, $K^0 = \mathrm{Frac}(R^0_{\text{[illegible]}})$, $K^1_{\text{[illegible]}} = \mathrm{Frac}(R^1_{\text{[illegible]}})$

<u>Theorem</u> (Strauch): (i) The extension $\tilde{K}/K$ is Galois with Galois group G.

(ii) The ~~prime ideal $\tilde{\mathfrak{p}}_h$ lies over $\mathfrak{p}_{\text{[illegible]}h}$ and the~~ extension $\tilde{K}_h / K_{\text{[illegible]}h}$ is normal.

(iii) The ~~[illegible]~~ decomposition subgroup of $\tilde{\mathfrak{p}}_h$ is equal to P_h and the inertia subgroup is equal to Q_h. The canonical map

$$P_h / Q_h \longrightarrow \mathrm{Aut}(\tilde{K}_h / K_{\text{[illegible]}h})$$

is an isomorphism. Hence

$$\mathrm{Aut}(\tilde{K}_h / K_h) \simeq GL_{n-h}(\mathbb{F}_p). \quad //$$

(*)

~~Corollary~~: Let

$$PB_h = \left(\begin{array}{c|c} * & * \\ \hline 0 & \begin{matrix} \ddots & * \\ 0 & \ddots \end{matrix} \end{array}\right)$$

half-Borel inside P_h,

$$PU_h = \left(\begin{array}{c|c} * & * \\ \hline 0 & \begin{matrix} 1 & * \\ 0 & 1 \end{matrix} \end{array}\right)$$

half-unipotent radical inside PU_h.

(i) $\mathfrak{p}^0_h$ coincides with the prime ideal of R^0 introduced earlier.

Corollary: Then (ii) Under the Strauch identifications,

$$\mathrm{Aut}(\tilde{K}_h / K^0_h) = PB_h / Q_h$$

$$\mathrm{Aut}(\tilde{K}_h / K^1_h) = PU_h / Q_h$$

$$\mathrm{Aut}(K^1_h / K^0_h) = PB_h / PU_h = (\mathbb{F}_p^\times)^{n-h}$$

Hence $\mathrm{Aut}(K^1_h / K^0_h)$ maps surjectively to $T^S(\mathbb{F}_p)$, and so does $\mathrm{Gal}(K^0_h)$.

Proof: (ii) This is clear from the above description, and the fact that K^1_h / K^0_h is separable since of degree prime to p so that

$$\mathrm{Aut}(K^1_h / K^0_h) = \mathrm{Gal}(K^1_h / K^0_h) \quad //$$

(i) It is clear that $\tilde{\mathfrak{p}}_h \cap R^0 \subset \mathfrak{p}^0_h$. Identical heights //