Bonn, Dec. 05

Talk: On the geometry of the $\Gamma_1(p)$-moduli problem, 3.

Recall situation: $\pi : \mathcal{A}_1 \longrightarrow \mathcal{A}_0$ Galois cover, with Galois gp $T(\mathbb{F}_p) = (\mathbb{F}_p^\times)^d$, étale in generic fiber.

Stratification:
$$\mathcal{A}_0 \otimes \mathbb{F}_p = \bigcup \mathcal{A}_{0,S}$$
$$\mathcal{A}_1 \otimes \mathbb{F}_p = \bigcup \mathcal{A}_{1,S}$$

and $\pi_S : (\mathcal{A}_{1,S})_{red} \longrightarrow \mathcal{A}_{0,S}$ is étale Galois cover with gp of Deck transf. $T^S(\mathbb{F}_p)$ (quotient of $T(\mathbb{F}_p)$).

Furthermore, $\mathcal{A}_0 \otimes \mathbb{F}_p$ is a reduced DNC and the $\mathcal{A}_{0,S}$ are the corresp. "open" strata. And $\mathcal{A}_1 \otimes \mathbb{F}_p$ is a DNC, all divisors are of multiplicity $p-1$ (have equation).

Last time we had sketched the proof of

Theorem (monodromy theorem): For any $\dot{x} \in \mathcal{A}_{0,S}(\overline{\mathbb{F}}_p)$, the homo defined by π_S,
$$\pi_1(\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p, \dot{x}) \longrightarrow T^S(\mathbb{F}_p)$$
is surjective.

Last time, there were 2 gaps. One was filled right away by Viehmann. The other is the following lemma.

Lemma: Every connected component of $\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p$ has a point of $\mathcal{A}_{0,\mathbb{Z}/d} \otimes \overline{\mathbb{F}}_p$ in its closure.

Proof in the case when $\mathcal{A}$ is projective $/\mathbb{Z}_p$ (in the general case, there are some difficulties with references): By an obvious induction, and since $\overline{\mathcal{A}}_{0,S}$ is smooth, it suffices to show that $\mathcal{A}_{0,S}$ cannot be closed if $S \neq \mathbb{Z}/d$. Consider the morphism into the „absolute moduli scheme",

$$\mathcal{A}_0 \longrightarrow \mathcal{A} .$$

This morphism is finite and flat. Furthermore

$$\mathcal{A} \otimes \mathbb{F}_p = \mathcal{A}^{(0)} \dot{\cup} \mathcal{A}^{(1)} \cup \dots \cup \mathcal{A}^{(d-1)} .$$

It follows that the image of a connected component of $\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p$ is a connected component of $\mathcal{A}^{(h)}$, where $h = d - |S|$. However, by Ito, $\mathcal{A}^{(h-1)}$ defines a divisor in every connected component of $\mathcal{A}^{(h)}$, hence $\mathcal{A}^{(h)}$ is not closed, unless $h = 0$. Hence also the connected component of $\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p$ is not closed. //

Remark: This extends to the non-compact case, once one knows that the Satake compact. of $\mathcal{A}$ exists: this adds only a subscheme of codim ≥ 2 (unless $d = 2$, which is trivial anyway).

We now want to apply the monodromy theorem to calculate the vanishing cycle sheaves for $\mathcal{A}_0$ and $\mathcal{A}_1$.

Recollections from SGA 7, II.

~~Let Z be a henselian trait.~~

Let Y be a scheme over a field k, with separable closure $\bar{k}$, and let $\bar{Y} = Y \otimes_k \bar{k}$. Let G be a profinite group with a contin. homo

$$u : G \longrightarrow Gal(\bar{k}/k).$$

Then $Gal(\bar{k}/k)$ acts on $\bar{Y}$, and so does G. Let $\mathcal{F}$ be a sheaf on étale site of $\bar{Y}$. An action of G on $\mathcal{F}$ compat. with the action of $Gal(\bar{k}/k)$ $\overset{=}{\text{df}}$ action on $(\bar{Y}, \mathcal{F})$, it is continuous if $\forall\ U \longrightarrow Y$ étale, the induced action of G on $\mathcal{F}(\bar{U})$ is continuous (automatic if $G = Gal(\bar{k}/k)$ and $\mathcal{F}$ comes from $\mathcal{F}_0 / Y$; characterizes these sheaves).

Now let (Z, η, s) be a henselian triple, let $(\bar{Z}, \bar{\eta}, \bar{s})$ be deduced from a geometric point $\bar{\eta}$ over η. Let $f : X \longrightarrow Z$ morphism. Get diagram

$$\begin{array}{ccccc} X_{\bar{\eta}} & \hookrightarrow & \bar{X} & \hookleftarrow & X_{\bar{s}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\eta} & \hookrightarrow & \bar{Z} & \hookleftarrow & \bar{s} \end{array}$$

which is cartesian over

$$\begin{tikzcd} \bar\eta \arrow[r, hook] \arrow[d] & \bar Z \arrow[d] & \bar s \arrow[l, hook'] \arrow[d] \\ \eta \arrow[r] & Z & s \arrow[l, hook'] \end{tikzcd}.$$

Let $\mathcal F$ sheaf on X_η. Let

$$\Psi^0_\eta(\mathcal F) = \bar i^* \bar j_* \mathcal F_{\bar\eta}$$

By transport of structure, this is ~~indeed~~ a sheaf on $X_{\bar s}$, equipped with a continuous action of $Gal(\bar\eta/\eta)$, i.e. element of $Sh(X_s \times_s \eta)$.

If now $\Lambda = \mathbb{Z}/\ell^n$, with $(\ell, p) = 1$, then for a Λ-module $\mathcal F$, also $\Psi^0_\eta(\mathcal F)$ is a Λ-module; may derive this functor, get ~~object~~ complex in a suitable triang. category of Λ-sheaves,

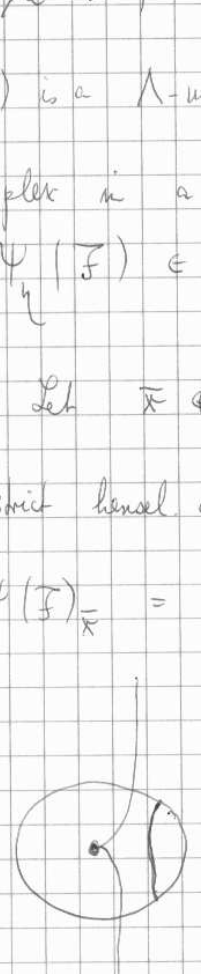
$$R\Psi_\eta(\mathcal F) \in D^+(X_s \times_s \eta, \Lambda).$$

<u>Explicit expression:</u> Let $\bar x$ ~~$\in X_{\bar s}$~~ geometric point of X_s. ~~and~~

Let Z_{hr} = strict hensel. of Z, with generic point η_{hr}. Then

$$R^i\Psi(\mathcal F)_{\bar x} = H^i(X_{(\bar x)} \times_{\eta_{hr}} \bar\eta, \mathcal F).$$

<u>Picture:</u>



Properties of $R\Psi$: Let $f : X \to X'$ morphism of Z-schemes.

a) $$R\Psi\, Rf_* \longrightarrow Rf_*\, R\Psi .$$

Isomorphism if f proper.

b) $$f^* R\Psi \longrightarrow R\Psi f^*$$

Isomorphism if f smooth.

Special case of a): $X' = Z$. Then

$$H^i(X_{\bar\eta}, \mathcal{F}) \simeq H^i(X_{\bar s}, R\Psi(\mathcal{F}))$$

as $Gal(\bar\eta/\eta)$-modules, if X/S proper.

c) Calculation of $R\Psi$ in the case of DNC, with multiplicities prime to p. Let $x \in X_{\bar s}$ and $\bar x$ a geometric point over x. Let $\{C_i\}$ be the ~~[illegible]~~ local divisors through x, and consider the homological complex conc. in degrees $0, -1$:

$$K : \mathbb{Z}^C \longrightarrow \mathbb{Z}$$
$$n_i \longmapsto \sum_i n_i d_i \qquad , \qquad d_i = \mathrm{mult}(C_i).$$

Then
- $R\Psi$ is tame, hence $Gal(\bar\eta/\eta)$-action factors through η_t, and
- $$R^*\Psi_{\bar x} = \Lambda^*(H_1(K) \otimes \Lambda(-1)) \otimes R^0\Psi_{\bar x}.$$
- $$R^0\Psi_{\bar x} = \Lambda^E ,$$

where E finite set of cardinality $|H_0(K)|$, on which inertia acts transitively

We now apply these general considerations to $\mathcal{A}_0$ and $\mathcal{A}_1$, over $Z = \operatorname{Spec} \mathbb{Z}_p$. For $\mathcal{A}_0$, this is easy when we restrict to a stratum $\mathcal{A}_{0,S}$:

$$i_{0,S}^* R\Psi_0 = \Lambda^*(\ker(\mathbb{Z}^S \to \mathbb{Z}) \otimes \overline{\mathbb{Q}}_\ell(-1)).$$

trivial local system, with triv. $Gal(\overline{\eta}/\eta)$-action (restriction to $\mathcal{A}_{0,S}$).

For $\mathcal{A}_1$, we first define the $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-sheaf on $\operatorname{Spec} \overline{\mathbb{F}}_p$ by

$$\mathcal{L} = R\Psi_\eta(\beta_* \overline{\mathbb{Q}}_\ell), \quad \beta : Z_{p-1} \to Z.$$

Here Z_{p-1} is obtained from $\operatorname{Spec} \mathbb{Z}_p$ by extracting a $(p-1)$-st root of ϖ_p. Hence $\mathcal{L}$ corresponds to the induced representation $\operatorname{Ind}_{Gal(\eta_{p-1})}^{Gal(\eta)}(\overline{\mathbb{Q}}_\ell)$. Its restriction to the inertia group I_p is via the regular representation of the tame quotient μ_{p-1} of I_p. Using now the description of the morphism π locally around $\mathcal{A}_{0,S}$ and the SGA7-formulas, we obtain:

$$i_{1,S}^*(R\Psi_1) = i_S^*(\mathcal{L}_{\mathcal{A}_1} \otimes \pi^*(R\Psi_0)).$$

Here $\mathcal{L}_{\mathcal{A}_1}$ denotes the object in $Sh(\mathcal{A}_1 \times_S \eta, \overline{\mathbb{Q}}_\ell)$ obtained from $\mathcal{L}$ by pullback by the structure morphism.

We now deduce from this a useful expression for $R\Psi_1$. We have, by proper push-forward,

$$\pi_*(R\Psi_1) = R\Psi_0(\pi_{\eta *}\overline{\mathbb{Q}}_\ell)$$
$$= \bigoplus R\Psi_\chi \quad ,$$

where for $\chi : T(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times$, we denote $R\Psi_\chi = R\Psi(\overline{\mathbb{Q}}_{\ell,\chi})$ under the decomposition $\pi_{\eta *}\overline{\mathbb{Q}}_\ell = \bigoplus_\chi \overline{\mathbb{Q}}_{\ell,\chi}$.

The restriction of $\pi_*(R\Psi_1)$ to $\mathcal{A}_{0,S}$ is given by the projection formula,

$$i_S^*(\pi_*(R\Psi_1)) = i_S^*(\pi_*(\pi^*(\mathcal{L}_{\mathcal{A}_0} \otimes R\Psi_0)))$$
$$= \mathcal{L}_{\mathcal{A}_{0,S}} \otimes i_S^*(R\Psi_0) \otimes \pi_{S*}(\overline{\mathbb{Q}}_\ell).$$

<u>Theorem</u>: If χ does not factor through $T^S(\mathbb{F}_p)$, the restriction $i_S^*(R\Psi_\chi)$ is zero. Assume now that χ is induced by a character $\overline{\chi} : T^S(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times$.

Then

$$i_S^*(R\Psi_\chi) = (\mathcal{L}_{\mathcal{A}_{0,S}} \otimes \overline{\mathbb{Q}}_{\ell\,\overline{\chi}}) \otimes i_S^*(R\Psi_0) \quad ,$$

where $\overline{\mathbb{Q}}_{\ell\,\overline{\chi}}$ is the local system of rank 1 on $\mathcal{A}_{0,S}$ defined on each connected component of $\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p$ by the composition $\pi_1(\mathcal{A}_{0,S} \otimes \overline{\mathbb{F}}_p, \bar{x}) \to T^S(\mathbb{F}_p) \xrightarrow{\overline{\chi}} \overline{\mathbb{Q}}_\ell$

$\to$ Since π_S def. over $\mathbb{F}_p$, $\overline{\mathbb{Q}}_{\ell\,\overline{\chi}}$ is $\in Sh(\mathcal{A}_{0,S} \times_s \eta)$.

The action of the inertia group on $\mathcal{L}_{\mathcal{A}_{0,S}}$ is through a finite group, and

$$i_S^*(R^*\Psi_\chi)^{I_p} = \overline{\mathbb{Q}}_{\ell,\overline{\chi}} \otimes i_S^*(R^*\Psi_0).$$