O'fach, Feb. 08

PD + DL

$G_0 / \mathbb{F}_q$. Let $\mathbb{F} = \overline{\mathbb{F}}_q$, and $G = G_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

Two classes of varieties $/ \mathbb{F}$:

① DL-varieties (DL, 1976): X = variety of Borel's over $\mathbb{F}$

$w \in W \rightsquigarrow$

$$X(w) = X_{G_0}(w) = \{ x \in X \mid \operatorname{inv}(x, Fx) = w \}.$$

(F = Frobenius over $\mathbb{F}_q$). Then

$X(w)$ smooth quasi-projective var. of dimension $\ell(w)$,

$G_0(\mathbb{F}_q)$ acts. If $F^e(w) = w$, then $X(w)$ defined over $\mathbb{F}_{q^e}$.

② PD (Faltings, RZ, 1993):

$\mathcal{N}$ = conj.-class of 1-ps of G. Let $X(\mathcal{N}) = G/P_\nu$ smooth proj. var.

$X(\mathcal{N})^{ss} = X_{G_0}(\mathcal{N})^{ss}$ = open subset of semi-stable points:

explanation: $\nu \in \mathcal{N} \rightsquigarrow \mathcal{F}_\nu$ $\mathbb{Z}$-filtr. on $\operatorname{Lie}(G_0) \otimes_{\mathbb{F}_q} \mathbb{F}$.

$\mathcal{F}_\nu$ semi-stable $\underset{\text{Df.}}{=}$ $\forall\, U \subset \operatorname{Lie}(G_0)$ $\mathbb{F}_q$-subspace, have

$$\frac{\deg(\mathcal{F}_\nu | U)}{\dim U} \leq 0 = \frac{\deg(\mathcal{F}_\nu)}{\dim \operatorname{Lie} G_0}.$$

$G_0(\mathbb{F}_q)$ acts. If $F^e(\mathcal{N}) = \mathcal{N}$, then $X(\mathcal{N})^{ss}$ defined over $\mathbb{F}_{q^e}$.

Examples: a) Drinfeld space: $G_0 = GL_n$.

- $w = s_1 \dots s_{n-1} = (1 2 \dots n)$. Then

$$X(w) = \Omega^n_{\mathbb{F}_q} = \mathbb{P}^{n-1} \setminus \bigcup_{H/\mathbb{F}_q} H$$

- $N = (x, y^{(n-1)}) \in (\mathbb{Z}^n)_+$, where $x > y$. Then $X(N)^{ss} = \Omega^n_{\mathbb{F}_q}$.

  Similar for $(x^{(n-1)}, y)$: dual proj. space.

b) $G_0 = GL_3$ generic data.

- $w = w_0$. Then

$$X(w_0) = \{ \mathcal{U}_1 \subset \mathcal{U}_2 \subset V = \mathbb{F}^3 \mid F\mathcal{U}_1 + \mathcal{U}_2 = V,\ F\mathcal{U}_2 + \mathcal{U}_1 = V \}.$$

- $N = (x_1 > x_2 > x_3)$. Then

$$X(N)^{ss} = \left\{ \text{———} \;\middle|\; \begin{array}{ll} x_1 - x_2 > x_2 - x_3: & \mathcal{U}_1 + F\mathcal{U}_1 + F^2\mathcal{U}_1 = V \\ x_1 - x_2 < x_2 - x_3: & \mathcal{U}_2 \cap F\mathcal{U}_2 \cap F^2\mathcal{U}_2 = 0 \\ x_1 - x_2 = x_2 - x_3: & \mathcal{U}_1 \cap F\mathcal{U}_1 = 0,\ \mathcal{U}_2 + F\mathcal{U}_2 = V \end{array} \right\}$$

Motivation for these varieties:

- DL : construct $G_0(\mathbb{F}_q)$-equiv. Galois cov. $\dot{X}(w) \to X(w)$, with Galois group $T_w(\mathbb{F}_q)$. Consider $\theta: T_w(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times \rightsquigarrow \mathcal{F}_\theta$ on $X(w)$.

  Interesting repns. of $G_0(\mathbb{F}_q)$ on $H^*(X(w), \mathcal{F}_\theta)$.

- PD : toy models for their p-adic big cousins.

We know surprisingly little about these varieties (coho., π_1, etc.). My talk concerns the progress in the last year. May always assume G_0 adjoint, simple.

1. DL: <u>Proposition (Lusztig 77, Bonnafé/Rouquier 06)</u>: $X(w)$ connected iff w elliptic.
$w = 1 \Rightarrow X(w) = X(\mathbb{F}_q)$.

<u>Theorem (DL 76, Haastert 86)</u>: For simplicity assume even that G_0 is absolutely simple. a) If $q \geq h(-1)$, then $X(w)$ is affine.

b) $X(w)$ is always quasi-affine.

<u>Comments</u>: DL-crit.: Let $D(C,w) = \{x \in X_*(T)_{\mathbb{R}} \mid \alpha(x) > 0,\ \forall \alpha > 0 \text{ with } w(\alpha) < 0\}$. $\exists x \in D(C,w)$ s.t. $F^*(x) - w(x) \in C$.

<u>Theorem (OR, He, Bonnafé/Rouquier 07)</u>: Same assumptions. If w is of minimal length in its F-conjugacy class, then $X(w)$ is affine.

→ Comments: OR for split classical gps: list, check DL-crit., He for remaining cases (computer), BR: prove this for larger set, but to prove larger need computer, probably both sets the same

<u>Example</u>: Let G_0 be split of type G_2. Then $X(w)$ is always affine, except when $q = 2$ and $w = s_1 s_2 s_1$ or $w = s_2 s_1 s_2$.

I suspect that in the last two cases $X(w)$ <u>not</u> affine!

The previous theorem can be used to prove the following vanishing theorem.

<u>Theorem (OR 07)</u>: Let $\theta : T_w(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ with corresponding lisse sheaf $\mathcal{F}_\theta$ on $X(w)$. Then

$$H^i_c(X(w), \mathcal{F}_\theta) = 0 \quad \text{for} \quad 0 \leq i < \ell(w).$$

$\xrightarrow{\text{DL}}$ If furthermore θ is non-singular (i.e. $\mathrm{Stab}_W(\theta)$ trivial).

then $H^i_c(X(w), \mathcal{F}_\theta) = 0$ for $i \neq \ell(w)$. + Poincaré.

Comments: • For $q \geq h$, this is in original paper of DL.
• Last statement follows from first, because $H^i_c(\mathcal{F}_\theta) \simeq H^i(\mathcal{F}_\theta)$ if θ non-sing. (DL)

Last statement is due to Haastert (86). First statement for θ = trivial is due to Digne/Michel/Rouquier (07).

of Proof:

2. PD : Here the cohomology has been determined completely by Orlik. Complicated formulae (only involves $\mathrm{Ind}_P^G 1$.) An important consequence is

Theorem (Orlik 01): Let $r_0 = \mathrm{rk}_{\mathbb{F}_q}(G_0)$. If N is non-trivial (i.e. $X(N) \neq pt$), then

$$H^i_c(X(w)^{ss}, \mathbb{Q}_\ell) = \begin{cases} 0 & \text{for } 0 \leq i < r_0 \\ \mathrm{St}_{G_0} & \text{for } i = r_0. \end{cases}$$

Comparison between rk and dim G/P gives.

Corollary: Same assumptions. Let N be non-trivial. Then $X(N)^{ss}$ is never affine, unless $G_0 = PGL_n$ and $N = (x, y^{(n-1)})$ or $N = (x^{(n-1)}, y)$, in which case $X(N)^{ss} \cong \Omega^n_{\mathbb{F}_q}$.

Orlik has also recently „determined" π_1.

Theorem (Orlik 07): Same assumptions. We have

$$\pi_1(X(\mathcal{N})^{ss}) = \{1\},$$

unless $G_0 = PGL_n$, and $\mathcal{N} = (x_1, \dots, x_n) \in (\mathbb{Z}^n)_+$ with $\sum x_i = 0$

and either $x_2 < 0$ or $x_{n-1} > 0$. In this last case

$$\pi_1(X(\mathcal{N})^{ss}) = \pi_1(\Omega^n_{\mathbb{F}_q}).$$

In all other cases codim Complement ≥ 2.

Comments: What is RHS? Part of problem $\pi_1(DL) = ?$.

(3) DL $\Leftrightarrow$ PD Comparison of vanishing of DL and non-vanishing for PD gives.

Proposition (OR 07) Same assumptions. A DL-variety $X_{G_0}(w)$ is never homeomorphic to a PD $X_{G_0}(\mathcal{N})^{ss}$, unless $G_0 = PGL_n$, $w =$ Coxeter element, and $\mathcal{N} = (x, y^{(n-1)})$ or $(x^{(n-1)}, y)$, in which case both are univ. homeomorphic to $\Omega^n_{\mathbb{F}_q}$.

So, even though these varieties look quite similar at first glance, they are radically different!

Example: Let $x_2 < 0$. Then

$$X(\mathcal{N})^{ss} = \{ \mathcal{V}_\bullet \mid \mathcal{V}_1 \text{ not contained in } \mathbb{F}_q\text{-rat. hyperplane} \}$$

Hence $X(\mathcal{N}) \xrightarrow{\pi} \mathbb{P}^{n-1}$ induces iso

$$X(\mathcal{N})^{ss} = \pi^{-1}(\Omega^n_{\mathbb{F}_q}),$$ fibers are flag var. (simply connected)