Vortrag: Special cycles on Shimura varieties for unitary groups

0. Introduction and motivation.

A series of papers followed up on idea of Kudla: define special cycles on Shimura varieties for $SO(n,2)$ and relate their (arithmetic) intersection numbers to Fourier coeff of (derivatives of) Siegel-Eisenstein series.

Obstacle: To construct integral models of these Shimura varieties, use exceptional isom. for low n, $n = 0, 1, 2, 3$.

Idea (old idea of Kudla, but not so well-known (Annals paper)): Something similar should be doable for Shimura varieties assoc. to $GU(n-1,1)$: should be related to special value (derivative of) Eisenstein series on $U(r,r)$, $1 \leq r \leq n$, especially $r = n$ and special value at $s = 0$.

Advantage: all these Shimura varieties have integral models, at least for good primes.

Report on some joint preliminary experiments w. Kudla.

1. Moduli problem in char 0 and special cycles

<u>Data:</u> $k \hookrightarrow \mathbb{C}$ = imag.-quadr. field

$V, (,)$ = hermitian vector space of dim. n and signature $(n-1, 1)$

L = O_k- lattice in V where $(,)$ integral, i.e. $L \subset L^\vee$.

$G = GU(V, (,))$ alg. group $/\mathbb{Q}$

$K \subset G(\mathbb{A}_f)$ comp. open, suff. small.

<u>Moduli problem on (Sch/k):</u> $S \mapsto (A, \iota, \lambda, \eta)$ where

A = abelian scheme over S

$\iota : O_k \to \mathrm{End}(A)$ with $\mathrm{char}(\iota(\alpha) \mid \mathrm{Lie}\, A) = (T-\alpha)^{n-1} \cdot (T - \bar{\alpha})$
"$(n-1, 1)$-<u>special</u>"

$\lambda : A \to \hat{A}$ polariz. such that $\mathrm{Ker}\, \lambda \simeq L^\vee / L$ at all $\bar{s} \to S$

and $\iota(\alpha)^* = \iota(\bar{\alpha})$.

η level-K-structure (more complicated to state).

This functor is representable by a quasi-projective scheme $\mathcal{M}$ over $\mathrm{Spec}\, k$, of dimension $n-1$.

<u>Complex uniformization:</u> Let

$\mathcal{D} = \{$ space of negative lines in $V \otimes_k \mathbb{C} \}$

$I \mapsto (-1)$-eigenspace $\quad \hat{=} \{$ involutions $I \in G(\mathbb{R}) \mid (v, Iv) \geq 0, \forall v \}$.

To each I corresponds a change from the obvious complex structure J_0 on $V_{\mathbb{R}} = k_{\mathbb{R}}^n$ into new complex structure $J_I = I . J_0$.
Hence obtain on $\mathcal{D}$ a family of abelian varieties

$$A_I = (V_{\mathbb{R}}/L, J_I) \quad , \quad \text{obvious } O_k\text{-action}$$

with polarization λ_I with associated Riemann form

$$h_I(v,v') = \langle Jv, v'\rangle + i \cdot \langle v, v'\rangle ,$$

(where $\langle v, v'\rangle = tr\left(\frac{1}{\sqrt{d}} \cdot (v, v')\right)$.
Hence obtain $\mathcal{D} \longrightarrow \mathcal{M}_{\mathbb{C}}$,
is part of isomorphism

$$G(\mathbb{Q}) \backslash \left[\mathcal{D} \times G(\mathbb{A}_f)/K\right] \xrightarrow{\sim} \mathcal{M}_{\mathbb{C}} .$$

<u>Special cycles</u>: Roughly speaking the locus in $\mathcal{M}$, where (A, ι) splits off an elliptic curve w. CM by O_k.
<u>Definition</u>: A <u>special homomorphism of</u> (A, ι) is an element of

$$\mathrm{Hom}_{O_k}(E, A) ,$$

where (E, ι) is an elliptic curve w. CM by O_k (over S)
On the space of special homo's have <u>positive-definite hermitian form</u>

$$(f, g) = \hat{g} \circ \lambda \circ f \in \mathrm{End}_{O_k}(E) = O_k .$$

For $t > 0$, define relating η and f

$$Z(t) = \{ (A, \iota, \lambda, \eta ; f) \mid (f,f) = t \text{ ; integrality cond. } \}.$$

Then $Z(t) \to \mathcal{M}$ finite + unramified – do as if closed immersion

More generally, for $T \in \mathrm{Herm}_r(O_k)$, let

$$Z(T) = \{ (\text{——} ; \underline{f}) \mid \underline{f} = (f_1, \dots, f_r) \text{ } r\text{-tuple w. } (\underline{f}, \underline{f}) = T \}.$$

For $1 \leq r \leq n$, this cycle has codimension r.

<u>Pullback under</u> $\mathcal{D} \to \mathcal{M}_{\mathbb{C}}$ (<u>complex uniformization</u>):

To $x \in V \mapsto \mathfrak{a}_x = \{ \alpha \in k \mid \alpha x \in L \}$ fractional ideal

$\mapsto$ elliptic curve E_x w. CM by O_k.

Get map of real tori $f_x : k_{\mathbb{R}} / \mathfrak{a}_x = E_x \longrightarrow V_{\mathbb{R}} / L$

$$\alpha \longmapsto \alpha x.$$

<u>Observation</u>: For $z \in \mathcal{D}$, with assoc. complex structure J_z,

this homo f_x is complex-analytic $\Longleftrightarrow z \in \mathcal{D}_x$, where

$$\mathcal{D}_x = \{ z \in \mathcal{D} \mid z \perp x \}.$$

(hence $\mathcal{D}_x \neq \emptyset \Longleftrightarrow (x,x) > 0$).

<u>Now inverse image of</u> $Z(t)$ <u>in</u> $\mathcal{D}$ <u>is equal to</u>

$$\bigcup_{x \in L(t)} \mathcal{D}_x$$

$$L(t) = \{ x \in L \mid (x,x) = t \} \quad \text{(integrality condition)}.$$

<u>Check</u>: $(f_x, f_y) = (x, y)$, $x, y \in V$

2. Arithmetic theory

level structure

Same functor definition yields quasi-projective $\mathcal{M}$ over $\operatorname{Spec} O_k[\frac{1}{N}]$,

and also $Z(T)$, for $T \in \operatorname{Herm}_r(O_k)$.

Go to extreme case with empty generic fiber:

<u>Lemma</u>: Let $T \in \operatorname{Herm}_n(O_k)_{>0}$. If $Z(T) \neq \emptyset$, then

$\exists!\ p$ such that $Z(T)_{red} \subset \mathcal{M}_p$. In this case

p is not split and $Z(T)_{red} \subset (\mathcal{M} \otimes \kappa_p)^{ss}$. //

Now fix p <u>good</u>, i.e.

- p inert in k
- $p \nmid N$, $p \nmid |L^\vee : L|$

In this case $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ smooth. Need description of

supersingular locus.

Review of results of Vollaard.

Fix alg. closure $\mathbb{F}$ of $\mathbb{F}_{p^2}$. Note $O_k \otimes \hat{\mathbb{Z}}_p = \mathbb{Z}_{p^2}$.

Fix $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ = supersingular p-div. gp of ht $2n$ over $\mathbb{F}$, with action of $\mathbb{Z}_{p^2}$ making it $(n-1,1)$-special, and p-principal polariz. comp. with ι.

Moduli functor on $(Sch/\mathbb{F})$: $S \mapsto (X, \iota, \lambda, \varrho)$, where

X = p-div. gp. / S w. action of $\mathbb{Z}_{p^2}$ making it $(n-1,1)$-special

$\lambda : X \longrightarrow X^\vee$ p-principal

$\varrho : X \longrightarrow \mathbb{X} \times S$ $\mathbb{Z}_{p^2}$-quasi-isogeny of height zero

$\hat{\varrho} \circ \lambda_{\mathbb{X}} \circ \varrho = \mathbb{Z}_p^\times$-multiple of λ.

This is representable by a formal scheme $\mathcal{N}$ over $\mathbb{F}$.

p-adic uniformization of $\mathcal{M}_p^{ss}$:

$$\mathcal{M}_p^{ss} \otimes_{\mathbb{F}_{p^2}} \mathbb{F} = G'(\mathbb{Q}) \backslash \left[\mathcal{N}_{red} \times \mathbb{Z} \times G'(\mathbb{A}_f^p)/K'^p \right].$$

Here G' = twisted form of G. $V' = \mathrm{Hom}_k^\circ(E, E^n)$

Description of $\mathcal{N}$: Let $N = N_0 \oplus N_1$ rational Dieudonné module of $\mathbb{X}$. ($\mathbb{Z}_{p^2} \otimes W = W \oplus W$) Let $\tau = p^{\mu} V^{-2} : N_0 \longrightarrow N_0$ σ^2-linear endo, all slopes are zero (since $\mathbb{X}$ is supersingular). Let

$(*) \Longrightarrow$

$$\{\ ,\ \} : N_0 \times N_0 \longrightarrow W_{\mathbb{Q}} \quad \text{via}$$

$$\{x, y\} = \delta^{-1} \cdot \langle x, {}^{\tau}F y \rangle$$

bad normaliz.: better would be $\delta^{-1} \cdot \langle x, V^{-1} y \rangle$

where $\delta \in \mathbb{Z}_{p^2}^{\times}$, $\bar{\delta} = -\delta$.

In terms of $(N_0, \{\ ,\ \})$ have

$$\mathcal{N}(\mathbb{F}) = \{ A \text{ lattice in } N_0 \mid pA^{\vee} \overset{1}{\subset} A \subset A^{\vee} \}.$$

$(*) \Rightarrow$ Let $C = N_0^{\tau}$, with hermit. form $\{\ ,\ \}$. Can describe $\mathcal{N}(\mathbb{F})$ in terms of the building of G'_{der}, where $G' = GU(C, \{\ ,\ \})$.

Vertex lattice = $\mathbb{Z}_{p^2}$-lattice Λ in C with $p\Lambda^{\vee} \subset \Lambda \subset \Lambda^{\vee}$.

type of Λ = $t(\Lambda) = [\Lambda : p\Lambda^{\vee}]$ any odd integer between 1 and n.

Closed DL-variety assoc. to Λ: smooth irreduc. alg. variety $Y(\Lambda)$ of dimension $\frac{1}{2}(t-1)$ over $\mathbb{F}$.

And

$$(*) \qquad \mathcal{N}_{red}(\mathbb{F}) = \bigcup_{\Lambda} Y(\Lambda) \qquad \text{via incidence rel'n}$$

where $Y(\Lambda) \supset Y(\Lambda') \Leftrightarrow \Lambda \supset \Lambda'$

$$Y(\Lambda) \cap Y(\Lambda') = \begin{cases} Y(\Lambda \cap \Lambda') & \text{if } \Lambda \cap \Lambda' \text{ vertex lattice} \\ \emptyset & \text{oth.} \end{cases}$$

e.g., if $t(\Lambda) = 3$, then $Y(\Lambda)$ Fermat curve $x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0$.

If $n = 3$, then $(*)$ comes from isom. of schemes $/\mathbb{F}$.

big gap!

<u>Back to special cycles.</u> Fix $T \in \mathrm{Herm}_n(O_k)_{>0}$. Then $Z(T)$ is a finite disjoint sum of schemes of the following kind: let $x_1, \dots, x_n \in C$ with $-p^{-1} \cdot \{\underline{x}, \underline{x}\} = T$.

To $\underline{x}$ is associated a closed formal subscheme $\mathcal{N}_{\underline{x}}$ of $\mathcal{N}$, with

$$\mathcal{N}_{\underline{x}}(\mathbb{F}) = \{ A \in \mathcal{N}(\mathbb{F}) \mid \mathrm{span}_W(\underline{x}) \subset A \}.$$

<u>So far, we know this about</u> $\mathcal{N}_{\underline{x}}$:

a) $\mathcal{N}_{\underline{x}}(\mathbb{F}) = \bigcup_{\underline{x} \subset p\Lambda^\vee} Y(\Lambda)$, and in fact, it suffices to take union over all Λ with $t(\Lambda) = t(T)$.

Here $t(T) =$ greatest odd integer $\leq n - \mathrm{rk}(\overline{T})$

b) $\mathcal{N}_{\underline{x}}$ is irreducible $(= Y(\Lambda)) \iff$ in Jordan decomp. of $T \sim 1_{n_0} + p\, 1_{n_1} + p^2 \cdot 1_{n_2} + \dots$ have $n_0 + n_1 \geq n-2$

c) $\mathcal{N}_{\underline{x}}$ is a point $\iff \mathrm{rk}(\overline{T}) \geq n-2$, i.e. $n_0 \geq n-2$

<u>Conjecture</u>: Let $T \sim 1_{n-2} \oplus \begin{pmatrix} p^a & \\ & p^b \end{pmatrix}$, with $0 \leq a \leq b$. Then

$$\lg(\mathcal{N}_{\underline{x}}) = \frac{1}{2} \sum_{l=0}^{a} p^l (a + b - 2l + 1).$$

True, if $a = 0$. (theory of quasi-canonical liftings of Gross)

Would check w. Fourier coeffic. of Eisenstein series.