SOME QUESTIONS ABOUT $\mathcal{G}$-BUNDLES ON CURVES
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1. Introduction

Two remarks on Arnaud:

1) Fast talker: my terror in Raynaud's course on algebraic curves.

2) Excellent Expositions: Paper by Beauville-Laszlo is the origin and primary source for the theory of algebraic loop groups.

This theory was further developed by Beauville, Laszlo, Sorger, Faltings. The results today are inspired by the paper of Faltings.

The theory has applications in the theory of vector bundles on algebraic curves, in geometric Langlands theory, and in the theory of local models of Shimura varieties.

Joint work with G. Pappas.

2. Definitions

Let G be a linear algebraic group G over $k((t))$.

Associated **algebraic loop group**=the ind-group scheme LG over k, with points with values in a k-algebra R equal to $G(R((t)))$.

If P is a **parahoric subgroup** of $G(k((t)))$, Bruhat and Tits have associated to P a smooth group scheme with connected fibers over $\operatorname{Spec} k[[t]]$, with generic fiber G and with group of $k[[t]]$-rational points equal to P. Denoting by the same symbol P this group scheme, there is associated to it a group scheme L^+P over k, with points with values in a k-algebra R equal to $P(R[[t]])$.

The *fpqc*-quotient $\mathcal{F}_P = LG/L^+P$ is representable by an ind-scheme, and is called the **partial affine flag variety** associated to P.

Examples 2.1. a) Let $G = \mathrm{GL}_n$. Then $LG = L\mathrm{GL}_n$ with R-points $\mathrm{GL}_n(R((t))$. An example of a parahoric subgroup is $P = \mathrm{GL}_n(k[[t]])$, and then the corresponding group scheme over k is $\mathrm{Res}_{k[[t]]/k}(\mathrm{GL}_n)$. The corresponding partial affine flag variety is the **affine Grassmannian** $\mathcal{G}$ with $\mathcal{G}(R)$ equal to the set

$$\mathcal{G}(R) = \{R[[t]]\text{-lattices in } R((t))^n\} \ .$$

Another parahoric subgroup is the **Iwahori subgroup** $\mathcal{B} = \pi^{-1}(B)$, where $\pi : \mathrm{GL}_n(k[[t]]) \to \mathrm{GL}_n(k)$, and where B is the Borel group of upper triangular matrices. The corresponding partial affine flag variety is the **full affine flag variety** $\mathcal{F}$ with $\mathcal{F}(R)$ equal to the set

$$\mathcal{F}(R) = \{\text{complete periodic chains of } R[[t]]\text{-lattices in } R((t))^n\} \ .$$

(write down)

b) Let $G = \mathrm{SL}_n$. Then can consider the same kind of objects. For instance, the corresponding affine Grassmannian $\mathcal{G}$ parametrizes **normalized lattices**, i.e., $\bigwedge^n \mathcal{L} = R[[t]]$.

c) More generally, we can start with a reductive group G_0 over k and take $G = G_0 \otimes_k k((t))$. We call this the **classical case**.

d) Here is a non-classical case: Let $K = k((t))$ and let $K' = k((u))$ with $u^2 = t$. Let G be the special unitary group defined by a K'/K-hermitian vector space of dimension n. In this case the full affine flag variety $\mathcal{F}$ parametrizes normalized complete periodic lattice chains in $R((u))^n$ which are **selfdual**(*when n is odd*; otherwise $\mathcal{F}$ is one of two connected components of this functor).

3. Structural results

In [PR3] we studied these partial affine flag varieties. The main structural results are as follows. We assume that G is a connected reductive algebraic group and that k is algebraically closed.

1) $\pi_0(LG) = \pi_0(\mathcal{F}_P) = \pi_1(G)_I$. *Here $\pi_1(G)$ denotes the algebraic fundamental group of G in the sense of Borovoi, and $I = \mathrm{Gal}(\overline{k((t))}/k((t)))$ the inertia group.*

   In the classical case, this is due to Beauville-Laszlo-Sorger, and to Beilinson-Drinfeld.

2) *If G is semi-simple and splits over a tamely ramified extension of $k((t))$, and if $(\mathrm{char}(k), |\pi_1(G)|) = 1$, then LG and $\mathcal{F}_P$ are reduced ind-schemes.*

   In the classical case this is due to Beauville-Laszlo, Laszlo-Sorger in characteristic 0, in positive characteristic for $G = \mathrm{SL}_n$ to PR, and in the general case to Faltings. Note that $L\mathrm{PGL}_2$ is non-reduced in characteristic 2.

3) *If G splits over a tamely ramified extension of $k((t))$, then all Schubert varieties in $\mathcal{F}_P$ are normal projective algebraic varieties, with only rational singularities.*

   In characteristic 0, this is due to Kumar, Littelmann, Mathieu (with an a priori different definition of loop groups). In arbitrary characteristic, this is due to Faltings in the classical case.

4) *Let G be semi-simple and simply connected and absolutely simple. If G splits over a tamely ramified extension of $k((t))$, then*

$$\mathrm{Pic}(\mathcal{F}_P) = \bigoplus \mathbf{Z} \cdot \epsilon_i \ ,$$

where the sum ranges over the affine roots corresponding to the walls not bounding the facet in the Bruhat-Tits building fixed by P.

(This identification is given by the *degree morphism* in all cases, except if G is a ramified special unitary group as in the previous section in an *odd* number of variables, in which case the surjectivity of the degree morphism has not been established.

Same references as for 3).

I gave a sketch of the proofs of these theorems in the fall of 2005 in Orsay.

4. The conjectures

Let k be an algebraically closed field, and let X be a smooth connected projective curve over k. Let $\mathcal{G}$ be a smooth affine group scheme over X. We assume that the generic fiber $\mathcal{G}_\eta$ is a connected reductive group scheme over $K = k(X)$, and that for every $x \in X(k)$, denoting by $\mathcal{O}_x$ the completion of the local ring at x and by K_x its fraction field, $\mathcal{G}_x = \mathcal{G} \times_X \operatorname{Spec}(\mathcal{O}_x)$ is a parahoric group scheme for $\mathcal{G}_{\eta_x} = \mathcal{G}_\eta \otimes_K K_x$.

Examples 4.1. a) A particular class of examples arises in the following way. Let G be connected reductive group scheme over k. Then $\mathcal{G} = G \times_{\operatorname{Spec} k} X$ is an example of the kind of group schemes we will consider (**constant group scheme**).

b) We may generalize this as follows. Let $x \in X(k)$. Then the parahoric subgroups in $G(K_x)$ contained in $G(\mathcal{O}_x)$ are in one-to-one correspondence with the parabolic subgroups of G. More precisely, if $P \subset G$ is a parabolic subgroup, then the corresponding parahoric subgroup $\mathcal{P}$ is equipped with a morphism of group schemes over $\operatorname{Spec} \mathcal{O}_x$,

$$\mathcal{P} \longrightarrow G \times_{\operatorname{Spec} k} \operatorname{Spec} \mathcal{O}_x \tag{4.1}$$

which in the generic fiber is the identity of $\mathcal{G}_{\eta_x}$ and which in the special fiber has image equal to P.

Suppose now that $\mathcal{G}$ is a group scheme equipped with a morphism $\mathcal{G} \to G \times_k X$ which, when localized at x is of the previous nature for all $x \in X(k)$. Hence there is a finite set of points $\{x_1, \dots, x_n\}$ such that this morphism is an isomorphism outside this finite set, and parabolic subgroups $P_1, \dots, P_n$ such that the localization of $\mathcal{G}$ at x_i corresponds to P_i in the sense explained above. Then there is an equivalence of categories:

$$\{\mathcal{G}\text{-torsors on } X\} \iff$$

$$\{G\text{-torsors on } X \text{ with quasi-parabolic structure of type } (P_1, \dots, P_n) \text{ with respect to } (x_1, \dots, x_n)\}$$

in the sense of [Laszlo-Sorger].

c) Another example is given by a special unitary group corresponding to a double cover X' of X. □

Let $\mathcal{M}_{\mathcal{G}/X}$ denote the **algebraic stack of $\mathcal{G}$-torsors on X**.

4 conjectures on the geometry of $\mathcal{M}_{\mathcal{G}/X}$

Conjecture 4.2. (I,Uniformization) *Let $x \in X(k)$. Let $\mathcal{P}$ be a $\mathcal{G}$-torsor over $X \times S$. If $\mathcal{G}_\eta$ is semi-simple, then after an $fppf$ base change $S' \to S$, the restriction of $\mathcal{P} \times_S S'$ to $(X \smallsetminus \{x\}) \times S'$ is trivial.*

From this conjecture we would obtain a uniformization of $\mathcal{M}_{\mathcal{G}/X}$. Namely, assuming $\mathcal{G}_\eta$ semi-simple, and choosing a uniformizer at x, we would have an isomorphism

$$\mathcal{M}_{\mathcal{G}/X} \;=\; \Gamma_{X\smallsetminus\{x\}}(\mathcal{G})\backslash L(\mathcal{G}_{\eta_x})/L^+(\mathcal{G}_x). \tag{4.2}$$

Here $\Gamma_{X\smallsetminus\{x\}}(\mathcal{G})$ denotes the ind-group scheme with k-rational points equal to

$$\Gamma_{X\smallsetminus\{x\}}(\mathcal{G})(k) \;=\; \Gamma(X \smallsetminus \{x\}, \mathcal{G})\ .$$

More precisely, (4.2) presents the affine partial flag variety $\mathcal{F}_x = L(\mathcal{G}_{\eta_x})/L^+(\mathcal{G}_x)$ as a $\Gamma_{X\smallsetminus\{x\}}(\mathcal{G})$-torsor over $\mathcal{M}_{\mathcal{G}/X}$. We will denote by p_x the *uniformization morphism*,

$$p_x : \mathcal{F}_x \to \mathcal{M}_{\mathcal{G}/X}\ . \tag{4.3}$$

Remarks 4.3. In the constant case $\mathcal{G} = G \times_{\mathrm{Spec}\, k} X$, this is the theorem of Drinfeld and Simpson [DS]. In the case $S = \mathrm{Spec}\, k$, this statement in this special case was proved much earlier by Harder [H]. Even for the Examples 4.1b) the conjecture is not trivial. □

Heinloth(in a recent letter): *True if $\mathcal{G}$ is either simply connected or quasisplit.*

The second conjecture concerns the set of connected components, and is of Kottwitz style.

Conjecture 4.4. (II, Connected components) *Denote by $\pi_1(\mathcal{G}_{\bar\eta})$ the algebraic fundamental group of $\mathcal{G}_{\bar\eta}$ in the sense of Borovoi. Then*

$$\pi_0(\mathcal{M}_{\mathcal{G}/X}) \;=\; \pi_1(\mathcal{G}_{\bar\eta})_\Gamma.$$

Here on the right hand side are the co-invariants under $\Gamma = \mathrm{Gal}(\bar\eta/\eta)$.

Remarks 4.5. a) In particular, if $\mathcal{G}_{\bar\eta}$ is semi-simple and simply connected, then $\mathcal{M}_{\mathcal{G}/X}$ should be connected. This would follow from the uniformization conjecture and the fact that LG is connected for any semi-simple simply connected group over $k((t))$, cf. result 1) above. Hence the conjecture II holds in this case by Heinloth.

b) If $\mathcal{G}$ is constant, i.e comes by extension of scalars from a group scheme G over k, then the action of Γ on $\pi_1(\mathcal{G}_{\bar\eta})$ is trivial. Over $\mathbf{C}$ Conjecture II follows from the topological uniformization theorem, [Sorger-Triest], Cor. 4.1.2.

c)**Heinloth**(same letter): *Conjecture II holds if $\mathcal{G}$ is quasisplit.*

The third conjecture concerns the Picard group of $\mathcal{M}_{\mathcal{G}/X}$. For this we assume that $\mathcal{G}_\eta$ is semi-simple, simply connected, and absolutely simple. Let us also assume that $\mathcal{G}_{\eta_x}$ splits over a tamely ramified extension of $K_x = k((t))$. In [PR3] we construct an exact sequence

$$0 \longrightarrow X^*(\mathcal{G}(x)) \longrightarrow \mathrm{Pic}(\mathcal{F}_x) \;\longrightarrow\; \mathbf{Z} \longrightarrow 0\ . \tag{4.4}$$

Here $X^*(\mathcal{G}(x))$ is the character group of the fiber of $\mathcal{G}$ in x, and $c_x : \mathrm{Pic}(\mathcal{F}_x) \;\to\; \mathbf{Z}$ is the *central charge homomorphism.*

Note that if $\mathcal{G}_x$ is a special maximal parahoric group, then $X^*(\mathcal{G}(x))$ is trivial; and this applies to all but finitely many points $x \in X$. Let us denote by $\mathrm{Bad}(\mathcal{G})$ the set of points x where $\mathcal{G}_x$ is not special.

Conjecture 4.6. (III, on Pic($\mathcal{M}$)) *Let $\mathcal{G}_\eta$ be semi-simple, simply connected and absolutely simple. We also assume that $\mathcal{G}_{\eta_x}$ splits over a tamely ramified extension of K_x, for all $x \in X(k)$.*

(i) For any $x \in X(k)$, consider the homomorphism

$$p_x^* : \mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X}) \longrightarrow \mathrm{Pic}(\mathcal{F}_x)$$

induced by the uniformization morphism. Composing with the homomorphism c_x, we obtain a homomorphism $\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X}) \longrightarrow \mathbf{Z}$. This homomorphism is surjective and independent of x. Let us denote this homomorphism by c or $c_{\mathcal{G}/X}$.

(ii) Denote the kernel of $c_{\mathcal{G}/X}$ by $\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})^0$. There is a natural isomorphism

$$\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})^0 \;\simeq\; \bigoplus_{x \in X(k)} X^*(\mathcal{G}(x)).$$

(iii) Consider all group schemes $\mathcal{G}$ as above on X with a common generic fiber. Then there is a unique section of $c_{\mathcal{G}/X}$ which is functorial in homomorphisms $\mathcal{G} \to \mathcal{G}'$ which induce the identity homomorphism in the generic fiber.

Remarks 4.7. a) In the case of a constant group scheme, (i) and (ii) are due to Sorger [S] for $k = \mathbf{C}$ and to Faltings [F1] for arbitrary k, and (iii) is trivial.

b) In the case that $\mathcal{G}$ is of the type described in Examples 4.1 b), the point (ii) is proved by Laszlo and Sorger in [LS]. In this case, $\mathcal{G}$ comes with a morphism to $G \times_{\mathrm{Spec}\, k} X$. Therefore the positive answer to (i) for $G \times_{\mathrm{Spec}\, k} X$ implies that (iii) is true as well, via the splitting $\mathbf{Z} = \mathrm{Pic}(\mathcal{M}_{G\times X/X}) \to \mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})$. $\square$

We now come to the conformal blocks. Before this, we recall some facts from [PR3] about the Picard group of a partial affine flag variety $\mathcal{F} = LG/L^+P$. Here we are assuming that $\mathrm{char}(k) = 0$ and that the group G over $k((t))$ is semi-simple, simply connected and absolutely simple. A line bundle $\mathcal{L}$ on $\mathcal{F}$ is called *dominant* if its image $\deg(\mathcal{L}) \in \bigoplus \mathbf{Z} \cdot \epsilon_i$ has all coefficients ≥ 0. Then the Lie algebra of the universal extension $\tilde{L}G$ acts on the space of global sections $H^0(\mathcal{F}, \mathcal{L})$, and if $\mathcal{L}$ is dominant, this representation is the dual of the integrable highest weight representation corresponding to the element $\deg(\mathcal{L})$.

We now return to the global situation and a general group scheme $\mathcal{G}$. A line bundle $\mathcal{L}$ on $\mathcal{M}_{\mathcal{G}/X}$ is called *dominant* if $p_x^*(\mathcal{L})$ is a dominant line bundle on $\mathcal{F}_x$ for every x.

Conjecture 4.8. (IV, Conformal blocks) *Let* char $k = 0$, *and assume as before that $\mathcal{G}_\eta$ is semi-simple, simply connected and absolutely simple. Let S be a non-empty finite* *subset of* $X(k)$ *containing* $\mathrm{Bad}(\mathcal{G})$*. Let* $\mathcal{L}$ *be a dominant line bundle on* $\mathcal{M}_{\mathcal{G}/X}$*. There is a canonical isomorphism of finite-dimensional vector spaces*

$$H^0(\mathcal{M}_{\mathcal{G}/X}, \mathcal{L}) \simeq \Big[\bigotimes\nolimits_{x\in S} H^0(\mathcal{F}_x, p_x^*(\mathcal{L}))\Big]^{H^0(X\setminus S, \mathrm{Lie}(\mathcal{G}))} .$$

It is known (Faltings) that if S is enlarged to $S' \supset S$, the RHS does not change.

Remarks 4.9. In the 'classical' theory where $\mathcal{G} = G \times_{\mathrm{Spec}\, k} X$, one considers data which formally look very similar to the data above. One also fixes a finite set S of points, and dominant integral weights, one for each point $x_i \in S$. These are written traditionally as above in the form $\lambda_i = (\lambda_i^{(0)}, \ell)$, where $\lambda_i^{(0)}$ is a dominant weight for G and $\ell \in \mathbf{Z}$ is the central charge with $\langle \theta^\vee, \lambda_i^{(0)} \rangle \leq \ell$. These additional points and dominant integral weights are introduced to formulate and prove the fusion rules, which ultimately lead to an explicit determination of the dimension of the vector spaces in Conjecture 4.8.

On the other hand, in [Laszlo-Sorger] the set S and the dominant integral weights λ_i appear for essentially the same reason as here, except that here the situation is more general. □

In the classical case, when $\mathcal{G} = G \times_{\mathrm{Spec}\, k} X$, the dimension of the RHS in Conjecture 4.8 has been calculated by Faltings [Fa2] by using the factorization rules and the fusion algebra, at least when G is a classical group or of type G_2. We have not thought about how to generalize these further developments.

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