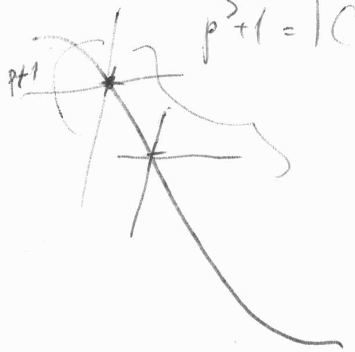
$n=3$

SPECIAL CYCLES ON UNITARY LOCAL SHIMURA VARIETIES

$p^3+1 = |C(\mathbb{F}_{p^2})|$



(Bonn, October 2007)

$p \neq 2$

1. Introduction

joint work with S. Kudla.

$0 = X_0^{p+1} + X_1^{p+1} + X_2^{p+1}$

In many cases, can construct in a modular fashion special cycles on *integral models* of Shimura varieties, can form generating series using them, and can compare them with specific modular forms (in fact, special values of derivatives of certain Eisenstein series)[Kudla's dream].

Previous results on special cycles on Shimura varieties attached to $\mathrm{SO}(2, n-2)$: existence of integral models is problematic for general n. For $n = 2, 3, 4$ can use exceptional isomorphisms with symplectic groups to deal with this problem (previous work with Kudla, Yang).

Recently we have shown that for the Shimura varieties attached to $\mathrm{GU}(1, n-1)$ this dream can be realized to some degree. Here:

Shimura variety: moduli scheme of abelian varieties of dimension n with CM by imaginary quadratic field $\boldsymbol{k}$, with k-linear polarization, such that the Lie algebra satisfies the signature condition $(1, n-1)$.

Special cycles: locus of points where the corresponding abelian variety splits off a copy of the elliptic curve with CM by $\boldsymbol{k}$.

Eisenstein series: Hermitian Siegel Eisenstein series on $\mathrm{U}(n, n)$.

Actually, the global theory has still to be worked out. But the local theory is in pretty good shape. Here replace everywhere

abelian varieties $\mapsto$ p-divisible groups,

Shimura variety $\mapsto$ local Shimura variety,

Eisenstein series $\mapsto$ representation densities of hermitian forms.

2. Statement of the main result

We write $\boldsymbol{k} = \mathbb{Q}_{p^2}$ for the unramified quadratic extension of $\mathbb{Q}_p$ and $O_{\boldsymbol{k}} = \mathbb{Z}_{p^2}$ for its ring of integers. We also write $\mathbb{F} = \bar{\mathbb{F}}_p$ and $W = W(\mathbb{F})$ for its ring of Witt vectors. There are two embeddings φ_0 and $\varphi_1 = \varphi_0 \circ \sigma$ of $\boldsymbol{k}$ into $W_{\mathbb{Q}} = W \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $(\mathbb{X}, \iota)$ be a fixed *supersingular* p-divisible group of dimension n and height $2n$ over $\mathbb{F}$ with an action $\iota : O_{\boldsymbol{k}} \to \operatorname{End}(\mathbb{X})$ satisfying the *signature condition* $(1, n-1)$,

$$\operatorname{char}(\iota(\alpha), \operatorname{Lie} \mathbb{X})(T) = (T - \varphi_0(\alpha))(T - \varphi_1(\alpha))^{n-1} \in \mathbb{F}[T]. \tag{2.1}$$

Let $\lambda_{\mathbb{X}}$ be a p-principal polarization of $\mathbb{X}$ for which the Rosati involution $*$ satisfies $\iota(\alpha)^* = \iota(\alpha^\sigma)$. The data $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ is unique up to isogeny. We denote by $\mathcal{N}_0$ the formal scheme over W which parametrizes the quadruples $(X, \iota, \lambda; \rho)$ where (X, ι, λ_X) is as above (in particular X satisfies the signature condition $(1, n-1)$), and where ρ is a quasi-isogeny of height 0 from (the reduction modulo p of) X to the constant p-divisible group $\mathbb{X}$ compatible with ι and with λ. Then $\mathcal{N}_0$ is formally smooth of relative dimension $n-1$ over W. The underlying reduced scheme $\mathcal{N}_{0,\mathrm{red}}$ is a singular scheme of dimension $[(n-1)/2]$ over $\mathbb{F}$.

In order to explain our results, we need to recall some of the results on the structure of $\mathcal{N}_{\mathrm{red}}$ due to Vollaard [thesis], as completed recently by Vollaard and Wedhorn [?].

Let N be the polarized isocrystal N of $\mathbb{X}$. Then $N = N_0 \oplus N_1$ (action of $\boldsymbol{k}$ on N). Then $p^{-1}F^2$ is a σ^2-linear automorphism of N_0 with all slopes zero. Hence get $\boldsymbol{k}$-form C of N_0 over $\boldsymbol{k}$. The polarization form on N defines a hermitian form on C. In this way we obtain a hermitian vector space $(C, \{\ ,\ \})$ of dimension n over $\boldsymbol{k}$ and with parity orddet$(C) \equiv n + 1 \bmod 2$ (this determines $(C, \{\ ,\ \})$ up to isomorphism).

A *vertex lattice* is a $O_{\boldsymbol{k}}$-lattice Λ in C with

$$p\Lambda^\vee \subset \Lambda \subset \Lambda^\vee.$$

The *type* of a vertex lattice Λ is the index $t(\Lambda)$ of $p\Lambda^\vee$ in Λ (which is always an odd integer between 1 and n). To every vertex lattice Λ, there is associated a locally closed irreducible subset $\mathcal{V}(\Lambda)^o$ of dimension $\frac{1}{2}(t(\Lambda) - 1)$ of $\mathcal{N}_{0,\mathrm{red}}$ such that

a) the closure $\mathcal{V}(\Lambda)$ of $\mathcal{V}(\Lambda)^o$ is the finite disjoint union

$$\mathcal{V}(\Lambda) = \bigcup_{\Lambda' \subset \Lambda} \mathcal{V}(\Lambda')^o,$$

where Λ' runs over all vertex lattices contained in Λ (and of type $t(\Lambda') \leq t(\Lambda)$).

b) the union of $\mathcal{V}(\Lambda)^o$, as Λ ranges over all vertices, is equal to $\mathcal{N}_{0,\mathrm{red}}$.

c) $\mathcal{N}_{0,\mathrm{red}}$ is connected.

We next define special cycles on $\mathcal{N}_0$. Let $\overline{\mathbb{Y}}$ denote the supersingular p-divisible group of dimension 1 with $O_{\boldsymbol{k}}$-action which satisfies the signature condition $(0,1)$ (with its natural polarization $\lambda_{\overline{\mathbb{Y}}}$) (can be obtained from $\mathbb{Y}$ by changing the $O_{\boldsymbol{k}}$-action). Note that $\overline{\mathbb{Y}}$ is rigid, i.e., has *canonical lifting* over W (this corresponds to the case $n = 1$, hence the relative dimension is 0).

The *space of special homomorphisms* is the $\boldsymbol{k}$-vector space

$$\mathbb{V} := \mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We have a hermitian form on $\mathbb{V}$ given by

$$h(x,y) = \lambda_{\overline{\mathbb{Y}}}^{-1} \circ \hat{y} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}) \otimes \mathbb{Q} \overset{\iota^{-1}}{\underset{\sim}{\longrightarrow}} \boldsymbol{k}.$$

(In fact, $(\mathbb{V}, h)$ can be identified with $(C, p^{-1}\{\ ,\ \})$.) For a special homomorphism $x \in \mathbb{V}$, define the *special cycle* $\mathcal{Z}(x)$ *associated to* x *in* $\mathcal{N}_0$ as the locus of points (X, ι, λ) in $\mathcal{N}_0$, where the quasi-homomorphism $x : \overline{\mathbb{Y}} \longrightarrow \mathbb{X}$ extends to a homomorphism from the canonical lift of $\overline{\mathbb{Y}}$ to X.

Proposition 2.1. *Let $x \neq 0$. Then $\mathcal{Z}(x)$ is a relative (formal) divisor in $\mathcal{N}_0$.*

Proof. Analogous to the proof of Terstiege [Diplomarbeit, appears in Manus. math.] of a similar statement regarding the HB-case (case $\mathrm{SO}(2,3)$). □

More generally, for an m-tuple $\mathbf{x} = (x_1, \ldots .x_m)$ of special homomorphisms $x_i \in \mathbb{V}$, the *associated special cycle* $\mathcal{Z}(\mathbf{x})$ is the intersection of the special cycles associated to the components of $\mathbf{x}$.

Naively, one might think that intersecting n special divisors $\mathcal{Z}(x_i)$ on the (formal) scheme $\mathcal{N}_0$ of dimension n in a *non-degenerate way* yields a scheme of dimension 0. Non-degeneracy is measured by the 'fundamental matrix' $T = h(\mathbf{x}, \mathbf{x}) \in \mathrm{Herm}_n(\boldsymbol{k})$. It turns out that the situation is more involved, and is very similar to the situation encountered for $\mathrm{SO}(2, n-2)$ for $n = 3, 4$.

We may now formulate our main theorem.

Theorem 2.2. *Let* $\mathbf{x} \in \mathbb{V}^n$. *If* $\mathcal{Z}(\mathbf{x})$ *is non-empty, then the corresponding fundamental matrix* T *lies in* $\mathrm{Herm}_n(O_{\mathbf{k}})$. *Let us assume this.*

(i) $\mathcal{Z}(\mathbf{x})$ *is a union of strata* $\mathcal{V}(\Lambda)^o$.

Suppose furthermore that T *is non-singular.*

(ii) Let $\mathrm{red}(T)$ *denote the image of* T *in* $\mathrm{Herm}_n(\mathbb{F}_{p^2})$, *and let* $t_0(T)$ *be the largest odd integer less than or equal to* $n - \mathrm{rank}(\mathrm{red}(T))$. *Then* $\mathcal{Z}(\mathbf{x})$ *is purely of dimension* $\frac{1}{2}(t_0(T)-1)$.

(iii) Let

$$T = 1_{n_0} + p1_{n_1} + \cdots + p^k 1_{n_k}$$

be a Jordan decomposition of T *and let*

$$n^+_{\text{even}} = \sum_{\substack{i\geq 2\\ \text{even}}} n_i \qquad \textit{and} \qquad n^+_{\text{odd}} = \sum_{\substack{i\geq 3\\ \text{odd}}} n_i .$$

Then $\mathcal{Z}(\mathbf{x})$ *is irreducible if and only if*

$$\max(n^+_{\text{even}}, n^+_{\text{odd}}) \leq 1.$$

(iv) $\mathcal{Z}(\mathbf{x})$ *is of dimension zero if and only if* T *is* $O_{\mathbf{k}}$*-equivalent to* $\mathrm{diag}(1_{n-2}, p^a, p^b)$, *where* $a+b$ *is odd. In this case,* $\mathcal{Z}(\mathbf{x})$ *consists of a single point* ξ, *and the length of the local ring* $\mathcal{O}_{\mathcal{Z}(\mathbf{x}),\xi}$ *is equal to*

$$\mathrm{length}_W(\mathcal{O}_{\mathcal{Z}(\mathbf{x}),\xi}) = \frac{\alpha'_p(S,T)}{\alpha_p(S,S)} .$$

[Here $S = 1_n$ and $\alpha_p(S,S)$ denotes a hermitian representation density, and $\alpha'_p(S,T)$ the derivative of a representation density (defined as usual through interpolation).]

We note that the RHS of the last identity can be explicitly computed; the result is

$$\frac{\alpha'_p(S,T)}{\alpha_p(S,S)} = \frac{1}{2}\cdot \sum\nolimits_{i=0}^{a} p^i(a+b-2i+1), \qquad \text{if } 0 \leq a \leq b\,. \tag{2.2}$$

In the form given above it should hold for any nonsingular fundamental matrix, as follows.

Conjecture 2.3. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{V}^n$ *be such that* $\mathcal{Z}(\mathbf{x}) \neq \emptyset$ *and such that the fundamental matrix* T *is nonsingular. Then*

$$\chi(\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) = \frac{\alpha'_p(S,T)}{\alpha_p(S,S)} .$$

The Euler-Poincaré characteristic appearing here is indeed finite, since it can be shown that $\mathcal{O}_{\mathcal{Z}(\mathbf{x})}$ is annihilated by a power of p. In the case that $\mathcal{Z}(\mathbf{x})$ is of dimension zero, Terstiege [poster session] has shown that there are no higher Tor-terms on the LHS of the above identity, so that indeed the statement (iv) of the above theorem confirms the conjecture in this case.

3. The structure of $\mathcal{N}_0$

We continue to denote by N be the isocrystal of $\mathbb{X}$. Then N has an action of $\boldsymbol{k}$ and a skew-symmetric $W_{\mathbb{Q}}$-bilinear form $\langle\,,\,\rangle$ satisfying

$$\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}, \tag{3.1}$$

and with $\langle \alpha x, y\rangle = \langle x, \alpha^{\sigma} y\rangle$, for $\alpha \in \boldsymbol{k}$.

Proposition 3.1. ([?], Lemmas 1.4, 1.6) *There is a bijection between $\mathcal{N}_0(\mathbb{F})$ and the set of W-lattices M in N such that M is stable under F, V, and $O_{\boldsymbol{k}}$, and with the following properties.*

$$\mathrm{char}(\alpha, M/VM)(T) = (T-\varphi_0(\alpha))(T-\varphi_1(\alpha))^{n-1} \in \mathbb{F}[T], \tag{3.2}$$

and

$$M = M^{\perp} \tag{3.3}$$

where

$$M^{\perp} = \{\, x \in N \mid \langle x, M\rangle \subset W\,\}. \tag{3.4}$$

The isomorphism $O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} W \oplus W$ yields a decomposition $M = M_0 \oplus M_1$ into rank n submodules, and F and V have degree 1 with respect to this decomposition. Also M_0 and M_1 are isotropic with respect to $\langle\,,\,\rangle$. Moreover, the determinant condition (3.2) is equivalent to the chain condition

$$pM_0 \overset{n-1}{\subset} FM_1 \overset{1}{\subset} M_0, \tag{3.5}$$

or, equivalently,

$$pM_1 \overset{1}{\subset} FM_0 \overset{n-1}{\subset} M_1.$$

(Here the numbers above the incusion signs indicate the lengths of the respective cokernels.) Note that $M_0 = M \cap N_0$ and $M_1 = M \cap N_1$ for the analogous decomposition $N = N_0 \oplus N_1$.

Since the isocrystal N is supersingular, the operator $\tau = V^{-1}F = pV^{-2}$ is a σ^2-linear automorphism of degree 0 and has all slopes 0. Let

$$C = N_0^{\tau=1}$$