Non-archimedean period domains

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Talk: Non-arch. period domains

- Thank organizers
- 2 halfes:
  - explain what NPD are, even to classical Hodge theory people.
  - recent results on them.

<u>Classical case</u> (Griffiths / Schmid):

G = reductive alg. group / $\mathbb{R}$

D = $G(\mathbb{R})$-conjug-class of polariz. HS of type G, i.e.

$$h : \mathbb{C}^\times \longrightarrow G.$$

Open embedding

$$D \hookrightarrow \mathcal{F}$$

$$h \longmapsto \mathcal{F}_h^\bullet \text{ Hodge filtr.}$$

Examples: • $\mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1_{\mathbb{C}}$, $G = PGL_2$

• Siegel $\frac{1}{2}$-space $\subset$ sympl. Grassm, $G = PSp_{2n}$

• open unit disc $\subset \mathbb{P}^n$, $G = PU(1,n)$.

Deligne : HS $\Leftrightarrow$ $h : \mathbb{C}^\times \longrightarrow GL(V)$

Classical period domains $\simeq (G, h)$.

These examples all of type $(1,0)+(0,1)$. Also higher weights.

Interesting Diff-geo of D,

L^2-coho. $\Leftrightarrow$ rep'n thy of $G(\mathbb{R})$.

p-adic HS :

$F/\mathbb{Q}_p$ finite

$L = \widehat{F^{un}}$, $\sigma \in Gal(L/F)$.

Isocrystal $= (V, \Phi)$

form a semi-simple F-linear abelian category

slopes

$\{$iso-classes of (V, Φ) w. $\dim V = n\}$

$\hookrightarrow (\mathbb{Q}^n)_+$

$\mathbb{Z}$-filtr. on VS of dim. n of type μ,

$\mu \in (\mathbb{Z}^n)_+$: $\mathcal{F}^\bullet$ s.t.

$$\dim gr_i^{\mathcal{F}} = \#\{j \mid \mu_j = i\}.$$

like punctured unit disc

like universal covering

In other words, the entries μ_j are the jumps,

they appear w. mult. = size of jump.

Definition (Fontaine): Let (V, Φ).

Let $\mathcal{F}^\bullet$ on $V \otimes_L K$.

Then $(V, \Phi, \mathcal{F}^\bullet)$ weakly admissible

(i) $|\mu(\mathcal{F}^\bullet)| = |\nu(V, \Phi)|$

(ii) $|\mu(\mathcal{F}'^\bullet)| \leq |\nu(V', \Phi)|$,

$\forall \; (V', \Phi) \subset (V, \Phi)$.

Colmez-Fontaine: If $F = \mathbb{Q}_p$, $|K : L| < \infty$,

$$\left\{ \begin{array}{l} \text{crystalline Gal-reps} \\ \text{of } \mathrm{Gal}(\bar{K}/K) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{weakly admiss.} \\ \text{filtered iso-} \\ \text{crystals } / K \end{array} \right\}$$

explain | |

This will come up again later.

Period domains: Fix V_0 with $V = V_0 \otimes_F L$

Fix $\mu \in (\mathbb{Z}^n)_+$, $n = \dim V_0$.

$\mathcal{F}(\mu)$ = space of filtr. of type μ on V
(partial flag variety of V_0) $/ F$

$\mathcal{F}(b,\mu)^{wa}$ = space of w.a. filtr.
(here: $\Phi = b \cdot (id_{V_0} \otimes \sigma)$).

Facts: • $\mathcal{F}(b,\mu)^{wa}$ admissible open rigid-an. subset of $\mathcal{F}(\mu) \otimes_F L$.

• if $s > 0$ s.t. $s \cdot \nu_b \in \mathbb{Z}^n$, then $\mathcal{F}(b,\mu)^{wa}$ defined over F_s

• $J(F) = \mathrm{Aut}(V, \Phi)$ acts on $\mathcal{F}(b,\mu)^{wa}$.

explain F_{Δ}

Examples: (i) $b = 1$, $\mu = (n-1, -1^{(n-1)})$.

$$\mathcal{F}(b,\mu)^{wa} = \mathbb{P}^{n-1} \setminus \bigcup_{H/F} H \quad , \quad J(F) = GL_n(F)$$

Drinfeld space Ω^n_F

(ii) $b = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \pi & & & 0 \end{pmatrix}$, $\mu = (1, 0^{(n-1)})$.

$$\mathcal{F}(b,\mu)^{wa} = \mathbb{P}^{n-1} \quad , \quad J(F) = D^{\times}_{1/n} .$$

Gross-Hopkins ex.

(iii) GL_3. : $b = 1$, $\mu = (x_1, x_2, x_3)$, $x_1 > x_2 > x_3$

$|\mu| = 0$

Proposition (Fontaine, R):

$$\mathcal{F}(b,\mu)^{wa} \neq \emptyset \iff \nu_b \leq \mu .$$

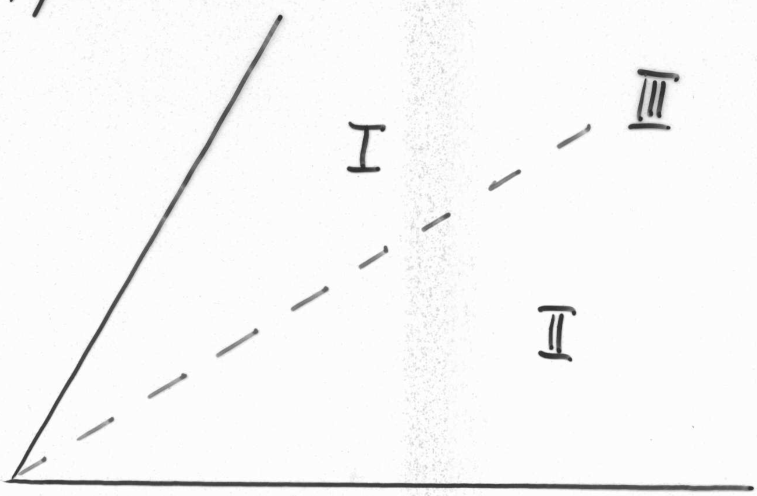
Drinfeld space = most famous, but highly non-typical

subdivision of positive Weyl chamber similar to wall crossing in GIT

On the RHS: dominance order.

<u>Example</u>: $G = GL_3$, μ regular $\Rightarrow$

$\mathfrak{F}(G,\{\mu\})$ = variety of all flags $\{(0) \subset V_1 \subset V_2 \subset V\}$



$$\mathfrak{F}(G,\{\mu\})^{wa} = \left\{ \begin{array}{ll} V_2 \text{ contains no } k\text{-rational line} & \mathrm{I} \\ V_1 \text{ not contained in } k\text{-rat. plane} & \mathrm{II} \\ V_1 \text{ and } V_2 \text{ not } k\text{-rational} & \mathrm{III} \end{array} \right.$$

Remarks: (i) Have $\mathcal{F}(b,\mu)^{wa}$ for triples (G, b, μ) where

G = reductive gp $/F$

$b \in G(L)$

μ = 1-PS of G / mod conjug.

(ii) $\mathcal{F}(b,\mu)^{wa}$ can be identified with semi-stable locus in $\mathcal{F}(\mu)$ (GIT) (van der Put / Voskuil, Totaro).

(iii) Inclusion $\mathcal{F}(b,\mu)^{wa} \subset \mathcal{F}(\mu)$ similar to (ss. VB / C) $\subset$ (all VB / C).

Behind this : (X) - Then of Faltings - Totaro.

Theorem (Orlik): Let G split, assume ν_b scalar. Then $H_c^*(\mathcal{F}(b,\mu)\otimes_{F'}^{wa}\hat{F}, \mathbb{Q}_\ell)$ is a smooth rep'n of $J(F)\times W_{F_s}$, with

$$H_c^* = \bigoplus_{\mu'=w\mu} v_{P_{\mu'}}^{J}(-\ell(\mu'))[-2\ell(\mu') - a_{P_{\mu'}}^J]$$

v_P^J = general. Steinberg rep'n.

Corollary: Same hypotheses. Then

$$H_c^i(\mathcal{F}(b,\mu)\otimes_{F'}^{wa}\hat{F}, \mathbb{Q}_\ell) = 0 \text{ for } 0 \leq i < F\text{-rk}(J$$

$$= v_{P_0}^J \text{ for } i = F\text{-rk}(J$$

- Drinfeld = the first to notice that $^{\alpha}\Omega_n$ has interesting ℓ-adic coho., his result on Ω_2 was generalized by Schneider - Stuhler to all Ω_n.
- Need to use <u>continuous</u> ℓ-adic coho. with compact support, to get smooth rep'ns of $J(F)$
- conjectured by Rapo 12 years ago
- EP-charact. calculated by Kottwitz + Rapo using Langlands - Lemma from theory of Eisenstein series. Inspired by Atiyah - Bott. ($\rightarrow$ Laumon - Rapo)
- Orlik constructs a resolution of any overconvergent sheaf on $\mathbb{F} \setminus \mathbb{F}^{wa}$. Get sepectral sequence. Collapses because few extensions of gener. Steinberg's. (Orlik, Dat)
- The quantities $\ell(\mu')$ and $a^J_{P_{\mu'}}$ are pulling to opposite sides: if $\ell(\mu')$ grows, $a^J_{P_{\mu'}}$ gets smaller $\rightarrow$ hard to get vanishing theorems
- No parity (for Ω_n have parity)

$$G = GL_3$$

i	I	III
0		
1		
2	ϑ_B^G	ϑ_B^G
3	$\vartheta_{P_2}^G(-1)$	$\vartheta_{P_1}^G(-1) + \vartheta_{P_2}^G(-1)$
4	$\vartheta_B^G(-1) + \vartheta_G^G(-2)$	$2\,\vartheta_G^G(-2)$
5	$\vartheta_{P_2}^G(-2)$	(0)
6	$\vartheta_G^G(-3)$	$\vartheta_G^G(-3)$

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
P_1 \arrow[dr, dash] & & P_2 \arrow[dl, dash] \\
 & B &
\end{tikzcd}$$

$$G \text{ split} \Longrightarrow \quad H_c^i = \begin{cases} (0) & i < a_B^G \\ \vartheta_B^G & i = a_B^G \end{cases}.$$

Drinfeld case: $G = J = GL_n$

$$H_c^{n-1+i}(\Omega^n_F \hat\otimes \hat{\bar{F}}, \mathbb{Q}_\ell) = v^G_{P_{(i,1^{(n-i)})}}(-i)$$

$$R\Gamma_c(\Omega^n_F \hat\otimes \hat{\bar{F}}, \overline{\mathbb{Q}}_\ell) \in D^b(\overline{\mathbb{Q}}_\ell, PGL_n(F))$$

Theorem (Dat): $\forall$ std psgp P_I

$$H^*\left(R\mathrm{Hom}_{D^b(\overline{\mathbb{Q}}_\ell, PGL_n(F))}\left(R\Gamma_c(\Omega^n_{\hat{\bar{F}}}, \mathbb{Q}_\ell), v^G_{P_I}\right)\right)$$

$$\simeq \sigma(v^G_{P_I})\left(\frac{n-1}{2}\right).$$

$\sigma(v^G_P)$ = Langlands corresp. of v^G_P,

indecomp. rep of W_F of degree n.

Spell out result in Drinfeld case.

Observations: lack of symmetry, not all std reps appear.

Idea of Def: Consider derived cat. of complexes of smooth rep'ns of $PGL_n(F)$ on $\overline{\mathbb{Q}}_\ell$-VS, with action of W_F.

Makes completely explicit the relation to Langlands correspondence.

Theorem (Dat): $\forall \pi$ irr. of $GL_n(F)$, ρ irr. D

$$H^*\left(R\mathrm{Hom}_{D^b(GD)}\left(R\Gamma_c(\tilde{\Omega}^n_F \otimes_F \hat{F}, \mathbb{Q}_\ell), \pi \otimes \rho\right)\right)$$

$$\underset{W_K}{\cong} \begin{cases} \sigma(\pi)\left(\frac{n-1}{2}\right) & \text{if } \rho = LJ(\check{\pi}) \\ 0 & \text{oth.} \end{cases}$$

$$GL_n(F) \times D^\times_{1/n} \times W_F.$$

Ω^n_F has a system of étale Galois coverings, $\mathcal{O}_D^\times$ acts.

On $\Omega^n_F \otimes_F \hat{\bar{F}}$ have action of $GL_n(F) \times D^\times \times W_F$.

$$GD = GL_n(F) \times D^\times / F^\times$$

Proof: involves at the same time Gross-Hopkins case and is the final stone i edifice of many people:

- Boyer
- Harris / Taylor
- Faltings / Fargues

Conjecture (R., Zink): $\exists$ étale

$$\breve{\mathcal{F}}(b,\mu)^{a} \longrightarrow \breve{\mathcal{F}}(b,\mu)^{wa}$$

which is bijective on rig.-anal. points + $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathbb{Q}_p\text{-}\mathcal{L}oc_{\breve{\mathcal{F}}(b,\mu)^{a}}$$

s.t. $\forall \ \mathcal{F}^{\bullet} \in \breve{\mathcal{F}}(b,\mu)^{a}(K)$, $\forall \ \varrho \in \mathrm{Rep}_{\mathbb{Q}_p} G$,

$\mathcal{L}_{\varrho, \mathcal{F}^{\bullet}}$ = crystalline Galois rep'n for

$$(V_{\varrho}, \varrho(b), \varrho(\mathcal{F}^{\bullet})).$$

(Colmez-Fontaine)

Hartl: If μ minuscule, constructs open L-analy[tic] subset of $\breve{\mathcal{F}}(b,\mu)^{wa}$. The assoc. rigid space should be $\breve{\mathcal{F}}(b,\mu)^{a}$.

<u>Generaliz. to other period domains:</u>

To get more interesting rep'ns than Steinberg's, need étale coverings. This is difficult.

One might try to calculate the coho of Hartl's space (??)

Left out: periods in classical case,
in p-adic case works so far only in minuscule case