Some questions on $\mathcal{G}$-bundles

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Münster, June 06

Talk: Some questions on $\mathcal{G}$-bundles

j. with Pappas. Disclaimer: no specialist.

- $k = \bar{k}$. Let $G / k((t))$ linear alg. gp.

  $\rightsquigarrow$ LG, represent. by ind-scheme $/k$.

- Let $P / k[[t]]$ lin. affine gp. scheme

  $\rightsquigarrow$ L^+P repr. by scheme $/k$.

- Let $\mathcal{F}_P = LG / L^+P$: repr. by ind-scheme.

Apply this to P = unique connected ~~aff~~ smooth affine gp scheme $/ k[[t]]$

G reductive gp, s.t. $P(k[[t]])$ = parahoric gp. ~~scheme~~ of $G(k((t)))$.

Then $\mathcal{F}_P$ = partial affine flag var.

<u>Examples</u>: • G_0 / k, $G = G_0 \otimes_k k((t))$, $P = G_0 \otimes k[[t]]$.

Then $\mathcal{F}_P$ = affine Grassmannian.

- pf: In previous ex., parahoric subgps $\subset P$ $\hat{=}$ psgps "in" G_0.

In particular, taking Borel in G_0, get Iwahori.

<u>Results</u>: ~~Let~~ (i) $\pi_0(LG) = \pi_0(\mathcal{F}_P) = \pi_1(G)_I$.

(ii) If G semi-simple, splits over tame ext., and $(\mathrm{char}\, k, \pi_1(G)) = 1$

Then LG and $\mathcal{F}_P$ are reduced ind-schemes

(iii) If G splits over tame ext., then all Schubert var. in $\mathcal{F}_P$ are normal.

(iv) G semi-simple, simply conn., absolutely simple. G split over tame extension. Then

$$\underline{\deg} : \mathrm{Pic}(\mathcal{F}_p) \xrightarrow{\sim} \bigoplus \mathbb{Z}\, \varepsilon_i$$

Want to formulate conjectures in global case ($\hat{=}$ 2^{nd} part of Faltings's paper)

X = smooth conn. proj. curve $/k$.

$\mathcal{G}/X$ smooth affine gp scheme, s.t.

- $\mathcal{G}_\eta$ connected reductive gp scheme over $K = k(X)$
- $\mathcal{G}_x = \mathcal{G} \times_X \mathrm{Spec}(\mathcal{O}_x)$ is parahoric gp scheme for $\mathcal{G}_{\eta_x} = \mathcal{G}_\eta \times_K K_x$, $\forall x \in X(k)$.

~~Example (Laszlo, Sorger)~~

$\mathcal{M}_{\mathcal{G}/X}$ = algebraic stack of $\mathcal{G}$-torsors on X.

Example (Laszlo, Sorger): G_0 / k reductive gp. $\rightsquigarrow$ $\mathcal{G}_0 / X$ const.

$S \subset X(k)$ finite, fix $P_x \subset G_0$ psgrg, $\forall x \in S$

Then $\exists !$ $\mathcal{G}/X$ + $\mathcal{G} \longrightarrow \mathcal{G}_0$ iso ~~[illegible]~~ outside S and s.t. $\mathcal{G}_x$ parahoric $\hat{=}$ P_x.

Then $\mathcal{M}_{\mathcal{G}/X}$ = stack of ~~$\mathcal{G}_0$~~ G_0-bundles on X, with quasi-parabolic structure in S (in sense of [LS]).

Have 4 conjectures

Conjecture 1:

$$\pi_0(\mathcal{M}_{\mathcal{G}/X}) = \pi_1(\mathcal{G}_{\bar{\eta}})_\Gamma$$

(here $\Gamma = \pi_1(X, \bar{\eta})$)

First conjecture of Kottwitz style.

In partic., if $\mathcal{G}_\eta$ ss + siply conn., then $\mathcal{M}_{\mathcal{G}/X}$ should be connected.

If $\mathcal{G}/X$ is constant, then Γ-action is trivial. Over $\mathbb{C}$ Sorger

Conjecture 2: Let $x \in X(k)$.

Let $\mathcal{P}$ be a $\mathcal{G}$-torsor on $X \times S$.

If $\mathcal{G}_\eta$ semi-simple, then

$$\exists\ S' \longrightarrow S \text{ fppf}, \text{ s.t.}$$

$$\mathcal{P} \times_S S' \mid (X \setminus \{x\}) \times S' \text{ trivial.}$$

If true, then $\forall\ x$

$$\mathcal{M}_{\mathcal{G}/X} = \Gamma_{X \setminus \{x\}}(\mathcal{G}) \setminus \mathcal{F}_x,$$

$$\mathcal{F}_x = L\mathcal{G}_{\eta_x} / L^+\mathcal{G}_x.$$

If $\mathcal{G}$ constant, then Drinfeld - Simpson (Harder).

Let $p_x : \mathcal{F}_x \longrightarrow \mathcal{M}_{\mathcal{G}}/X$.

Before next slide, recall $\overset{\text{local theory:}}{\text{central charge}}$

$$c_x : \mathrm{Pic}(\mathcal{F}_x) \longrightarrow \mathbb{Z}.$$

Get exact sequence

$$0 \to X^*(\mathcal{G}(x)) \to \mathrm{Pic}(\mathcal{F}_x) \xrightarrow{c_x} \mathbb{Z} \to 0$$

$\uparrow$

trivial if $x \notin \mathrm{Bad}(\mathcal{G}) = \{x \mid \mathcal{G}_x \text{ not special}\}$

Conjecture 3: Let $\mathcal{G}_\eta$ s-s, simply connected and absolutely simple.

(i) For $x \in X(k)$, let $\tilde{c} = c_x \circ \rho_x^*$,

$$\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X}) \longrightarrow \mathrm{Pic}(\mathcal{F}_x) \longrightarrow \mathbb{Z}.$$

This homo is surjective, indept of x.

(ii) Let $\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})^\circ = \ker \tilde{c}_{\mathcal{G}/X}$.

Then

$$\mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})^\circ \simeq \bigoplus_{x \in X(k)} X^*(\mathcal{G}(x)).$$

(iii) $\exists!$ functorial section of $\tilde{c}_{\mathcal{G}/X}$ (on category of all $\mathcal{G}$ w. same generic fiber).

Conj. is true in [LS]-setting, w. Faltings for (i) in char $p > 0$.

Conjecture 4: Let $\operatorname{char} k = 0$, $\mathcal{G}$ as in Conj. 3. Let $S \subset X(k)$ finite with $\mathrm{Bad}(\mathcal{G}) \subset S$. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{M}_{\mathcal{G}/X})$ dominant.

Then $H^0(\mathcal{M}_{\mathcal{G}/X}, \mathcal{L})$ is a f.-d. VS canonically isomorphic to

$$\left[\bigotimes_{\substack{\cancel{x \in X(k)} \\ x \in S}} H^0(\mathcal{F}_x, p_x^* \mathcal{L})\right]^{H^0(X \setminus S,\, \mathrm{Lie}\, \mathcal{G})}.$$

$\mathcal{L}$ dominant $\overset{=}{\text{Df}}$ $\forall x$, $\underline{\deg}\, p_x^*(\mathcal{L}) \geq\geq 0$.

Then, $\forall x$

$$H^0(\mathcal{F}_x, p_x^*(\mathcal{L})) = V^*_{\underline{\deg}(p_x^*\mathcal{L})}.$$

dual of integr. highest wt rep'ns of $\mathrm{Lie}\, \widetilde{L}\mathcal{G}_{\eta_x}$.

action of $H^0(X \setminus S, \mathrm{Lie}\, \mathcal{G})$ via lifting of $H^0(X\setminus S, \mathrm{Lie}\,\mathcal{G}) \to \bigoplus_{x \in S} \mathrm{Lie}\, L\mathcal{G}_{\eta_x}$ to $\bigoplus_{x\in S} \mathrm{Lie}\, \widetilde{L}\mathcal{G}_{\eta_x} /$ centers

In [LS]-setting, this is proved by Laszlo / Sorger

If $\mathcal{G}$ constant, then have a formula for dimension on RHS if G has type A, B, C, D, G_2 (Faltings) (Verlinde formula).