SOME REMARKS ON SPECIAL CYCLES ON SHIMURA CURVES

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1. Without level structure

Definition 1.1. Let E be an elliptic curve over a base scheme S. An endomorphism $j \in \mathrm{End}_S(E)$ is called *special* if $\mathrm{tr}(j) = 0$

Let $V(E)$ denote the set of special endomorphisms of E. Then, if S is connected, $V(E)$ is a finitely generated free $\mathbb{Z}$-module equipped with a quadratic form given by

$$q(j) = \deg(j) = -j^2 \qquad (\text{multiple of } \mathrm{id}_E)\ . \tag{1.1}$$

Note that q is positive-definite.

Let $\mathcal{M}$ denote the moduli stack of elliptic curves. This is a smooth DM-stack of relative dimension one over $\mathrm{Spec}\,\mathbb{Z}$.

Let $t \in \mathbb{Z}_{>0}$. Let $\mathcal{Z}(t)$ be the stack of pairs (E, j) where $j \in V(E)$ with $q(j) = t$. The forgetful morphism

$$\mathcal{Z}(t) \longrightarrow \mathcal{M} \tag{1.2}$$

is finite and unramified. We call $\mathcal{Z}(t)$, or its image in $\mathcal{M}$, a *special cycle on* $\mathcal{M}$.

Proposition 1.2. *$\mathcal{Z}(t)$ is a relative divisor, i.e., a divisor flat over* $\mathrm{Spec}\,\mathbb{Z}$*. In other words, $\mathcal{Z}(t)$ is locally for the étale topology defined by one equation which is neither a unit nor divisible by any prime number p.*

Next we want to intersect two special cycles $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$. Let $T \in \mathrm{Sym}_2(\mathbb{Z})$. Let

$$\mathcal{Z}(T) = \{(E, j_1, j_2) \mid (j_1, j_2) \in V(E)^2,\ q(j_1, j_2) = T\}\ . \tag{1.3}$$

Here

$$q(j_1, j_2) = \begin{pmatrix} q(j_1) & \frac{1}{2}(j_1, j_2) \\ \frac{1}{2}(j_1, j_2) & q(j_2) \end{pmatrix} \tag{1.4}$$

where

$$(j_1, j_2) = q(j_1 + j_2) - q(j_1) - q(j_2) \tag{1.5}$$

is the bilinear form associated to the quadratic form q.

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We now obtain

$$\mathcal{Z}(t_1)\times_{\mathcal{M}}\mathcal{Z}(t_2)=\coprod_T \mathcal{Z}(T)\ . \tag{1.6}$$

Here the sum is over all $T\in \mathrm{Sym}_2(\mathbb{Z})$ with diagonal terms t_1 and t_2. Note that if t_1t_2 is not a perfect square, all T lie in $\mathrm{Sym}_2(\mathbb{Z})_{>0}$. For these T there is the following result.

Theorem 1.3. *Let $T\in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ with $\mathcal{Z}(T)\neq\emptyset$. Then $\mathcal{Z}(T)$ is a stack of finite length with support in the supersingular locus of a fiber $\mathcal{M}\otimes_{\mathbb{Z}}\mathbb{F}_p$, for a unique prime number p. Furthermore, $\mathcal{Z}(T)$ is the disjoint sum of local Artinian stacks, each of which has identical length given by the Gross-Keating formula,*

$$\lg(\mathcal{O}_{\mathcal{Z}(T),\xi})=\begin{cases}\sum_{j=0}^{\frac{a-1}{2}}(a+b-4j)p^j, & \text{if } a \text{ is odd}\\ \\ \sum_{j=0}^{\frac{a}{2}-1}(a+b-4j)p^j+\frac{1}{2}(b-a+1)p^{\frac{a}{2}}, & \text{if } a \text{ is even.}\end{cases}$$

Here $(0,a,b)$ are the Gross-Keating invariants of the matrix $\tilde{T}=\mathrm{diag}(1,T)\in\mathrm{Sym}_3(\mathbb{Z}_p)$. If $p\neq 2$, this means that T can be diagonalized to $\mathrm{diag}(up^a,u'p^b)$, with $u,u'\in\mathbb{Z}_p^\times$ and $0\leq a\leq b$. It turns out that $a+b$ is odd, so that the above formula makes sense.

Let us denote by RHS the formula of the right hand side of the Gross-Keating formula. It can be expressed in terms of derivatives of p-adic representation densities. Let $S_0=\mathrm{M}_2(\mathbb{Z}_p)^{\mathrm{tr}=0}$, with quadratic form given by the determinant. Let $\alpha_p'(T,S_0)$ denote the derivative at $X=1$ of the representation density polynomial $A(T,S_0)(X)$ of T by S_0 . Then

$$\mathrm{RHS}=-\frac{p^2}{p^2-1}\cdot\alpha_p'(T,S_0)\ . \tag{1.7}$$

Principle of the calculation: The calculation of $\lg(\mathcal{O}_{\mathcal{Z}(T),\xi})$ reduces by the Serre-Tate theorem to the following problem on p-divisible groups.

Let G be the p-divisible formal group of dimension one and height two over $\overline{\mathbb{F}}_p$. Then

$$\mathrm{End}(G)=O_D\ , \tag{1.8}$$

where O_D is the maximal order in the quaternion division algebra over $\mathbb{Q}_p$. Let

$$V(G)=\{x\in\mathrm{End}(G)\mid \mathrm{tr}(x)=0\}\ , \tag{1.9}$$

with quadratic form given by $q(x)=\mathrm{Nm}(x)$. We change our notation slightly and write

$$\mathcal{M}=\mathrm{Spec}\,W(\overline{\mathbb{F}}_p)[[t]] \tag{1.10}$$

for the universal deformation space of G. For $x\in V(G)$ with $q(x)\neq 0$, let $\mathcal{Z}(x)\subset\mathcal{M}$ be the locus (closed formal subscheme) where the endomorphism x deforms. This is a relative divisor over $\mathrm{Spf}\,\mathbb{Z}_p$, comp. Proposition **??** above.

Let now $(x,y)\in V(G)^2$ and $T=q(x,y)\in\mathrm{Sym}_2(\mathbb{Z}_p)$. Let $\mathcal{Z}(x,y)$ be the locus inside $\mathcal{M}$ where (x,y) deforms.

Theorem 1.4. *If* $\det(T) \neq 0$*, then*

$$\lg(\mathcal{Z}(x,y)) = \text{RHS} \ .$$

Proof. (Sketch, see [A] and [KRY] for details.) For simplicity assume $p \neq 2$. We then may assume that $T = \mathrm{diag}(up^a, u'p^b)$ with $u, u' \in \mathbb{Z}_p^\times$ and $0 \leq a \leq b$. By Gross's theory of quasi-canonical liftings we have an equality of divisors

$$\mathcal{Z}(x) = \sum\nolimits_{s=0}^{A} \mathcal{W}_s(k) \ . \tag{1.11}$$

Here $k = \mathbb{Q}_p(x)$ is the quadratic extension of $\mathbb{Q}_p$ generated by x inside D, and $A = [\frac{a}{2}]$. Furthermore

$$\mathcal{W}_s(k) = \mathrm{Spf}\, W_s(k) \ , \tag{1.12}$$

where $W_s(k)$ denotes the ring of integers in the ring class field of k with norm group equal to $\mathcal{O}_{k,s}^\times$, where

$$\mathcal{O}_s = \mathcal{O}_{k,s} = \mathbb{Z}_p + p^s \cdot \mathcal{O}_k \tag{1.13}$$

is the order of conductor s in k. The quasi-canonical divisiors are regular and prime to each other, and defined over $\mathrm{Spf}\, W_0(k)$.

The decomposition (**??**) comes about as follows. Over $\mathrm{Spec}\, W_s(k)$ there is the quasi-canonical lifting Γ_s of level s with respect to k. This is a p-divisible group which lifts G and with $\mathrm{End}_{W_s(k)}(\Gamma_s) = \mathcal{O}_s$. Note that $\mathbb{Z}_p[x] = \mathcal{O}_{k,A}$, which explains the range of the summation in (**??**).

All quasi-canonical liftings with respect to k are isogenous to each other, in a natural way. Fix a basis of the rational p-adic Tate module of Γ_0. Then the p-adic Tate module $T_p(\Gamma_s)$ becomes a lattice in $\mathbb{Q}_p^2$, and hence defines a vertex in the Bruhat-Tits building of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The fixed points of $k^\times$ are either the midpoint of an edge (if $k/\mathbb{Q}_p$ is ramified) or a vertex (if $k/\mathbb{Q}_p$ is unramified). The vertex corresponding to $T_p(\Gamma_s)$ has distance $s + \frac{1}{2}$ resp. s to this fixed point locus, and conversely any such vertex appears in this way (they are all conjugate under $\mathrm{Gal}(\overline{W_0(k)}/W_0(k))$. The pictures are, depending on whether k is ramified or unramified over $\mathbb{Q}_p$:

HERE THE PICTURE SHOULD GO

Let

$$n(k; b, s) = \lg(W_s(k)/I(b)) \ , \tag{1.14}$$

where $I(b)$ is the ideal defining the locus where any $y \in V(G)$ deforms which anti-commutes with k and with $q(y) = up^b$ for some $u \in \mathbb{Z}_p^\times$. This number is explicitly known, cf. [A]. From

(**??**) we obtain

$$\lg(\mathcal{Z}(x,y)) = \sum\nolimits_{s=0}^{A} n(k;b,s)\ . \tag{1.15}$$

Inserting the values for $n(k;b,s)$ then leads to Theorem **??**. □

Remark 1.5. Similar results hold for Shimura curves corresponding to an indefinite quaternion algebra B over $\mathbb{Q}$ and special cycles on them, as long as the prime number p does not divide the discriminant of B. In fact, by the Serre-Tate theorem this extension is immediate.

2. $\Gamma_0(p^n)$-level structure

Let $n \geq 1$. Let $\tilde{\mathcal{M}} = \mathcal{M}_{\Gamma_0(p^n)}$ be the stack of elliptic curves with $\Gamma_0(p^n)$-level structure. Hence $\tilde{\mathcal{M}}$ parametrizes cyclic isogenies of degree p^n,

$$\lambda : E \longrightarrow E'\ . \tag{2.1}$$

In joint work with S. Kudla (in progress) we are trying to generalize the results of section 1. Define

$$V(E \xrightarrow{\lambda} E') = \{(j,j') \in \mathrm{End}(E) \times \mathrm{End}(E') \mid \lambda \circ j = j' \circ \lambda\}\ . \tag{2.2}$$

For a connected base scheme S, $V(E \xrightarrow{\lambda} E')$ is again a quadratic space with quadratic form $q((j,j')) = q(j) = q(j')$. Hence we can again define $\tilde{\mathcal{Z}}(t)$ for $t > 0$ and $\tilde{\mathcal{Z}}(T)$ for $T \in \mathrm{Sym}_2(\mathbb{Z})$.

Warning 2.1. *$\tilde{\mathcal{Z}}(t)$ has embedded components and hence is not a divisor.*

Let $\tilde{\mathcal{Z}}(t)^\flat$ be the associated divisor.

The local version of $\tilde{\mathcal{Z}}(t)$ resp. of $\tilde{\mathcal{Z}}(t)^\flat$ can be defined as well. Up to isomorphism there is a unique cyclic isogeny of degree p^n,

$$\lambda : G \longrightarrow G\ . \tag{2.3}$$

Let $\mathbf{x} = (x,x') \in O_D^2$ with $\lambda \circ x = x' \circ \lambda$. Let us again slightly modify the notation and denote now by $\tilde{\mathcal{M}}$ the universal deformation space of $\lambda : G \longrightarrow G$. Then to $\mathbf{x} = (x,x')$ there is associated the closed locus $\tilde{\mathcal{Z}}(\mathbf{x}) \subset \tilde{\mathcal{M}}$ where $\mathbf{x}$ deforms. Again let $\tilde{\mathcal{Z}}(\mathbf{x})^\flat$ be the associated divisor.

Our first result is the decomposition of $\tilde{\mathcal{Z}}(\mathbf{x})^\flat$ into quasi-canonical divisors. Let $\delta = 0$ if k is unramified, and $\delta = 1$ if k is ramified. For any pair (c,d) of non-negative integers with $c + d = n$ and any s with $c \leq s + \delta$, we associate a regular irreducible divisor $\tilde{\mathcal{W}}_s(k)^{(c,d)}$ of $\tilde{\mathcal{M}}$, as follows.

Let $s^\circ = \max\{s, s+\delta-c+d\}$. Then as $W_0(k)$-scheme $\tilde{\mathcal{W}}_s(k)^{(c,d)}$ is equal to $\mathrm{Spf}\, W_{s^\circ}(k)$. The closed immersion $\tilde{\mathcal{W}}_s(k)^{(c,d)} \longrightarrow \tilde{\mathcal{M}}$ is obtained by noting that over $\mathrm{Spec}\, W_{s^\circ}(k)$ we have the cyclic isogeny of degree p^n, given by the composition of isogenies of degree p,

$$\Gamma_s \longrightarrow \Gamma_{s-1} \longrightarrow \ldots \longrightarrow \Gamma_{s-c} \longrightarrow \Gamma_{s-c+1} \longrightarrow \ldots \longrightarrow \Gamma_{s-c+d}\ , \tag{2.4}$$

provided $c \leq s$. The cyclicity of this isogeny translates into the condition that the distance between the vertices in the Bruhat-Tits building corresponding to Γ_s and Γ_{s-c+d} is equal to n (no backtracking in the path). The first c isogenies are canonical, and the last d isogenies are non-canonical. If $\delta = 1$ and $c = s+1$, then the isogeny of degree p^n is the composition of c canonical isogenies and d non-canonical isogenies, as follows,

$$\Gamma_s \longrightarrow \Gamma_{s-1} \longrightarrow \ldots \longrightarrow \Gamma_0 \longrightarrow \Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \ldots \longrightarrow \Gamma_d \ . \tag{2.5}$$

All the closed immersions thus obtained are conjugate over $W_0(k)$.

Proposition 2.2. *There is the following equality of divisors on $\tilde{\mathcal{M}}$,*

$$\tilde{\mathcal{Z}}(\mathbf{x})^\flat = \sum_{s-c+d \leq A} \tilde{\mathcal{W}}_s(k)^{(c,d)} \ .$$

Here again $A = [\frac{a}{2}]$ when $q(\mathbf{x}) = up^a$, and the sum runs over triples of non-negative integers (c,d,s) with $c+d = n$, $c \leq s+\delta$, $s-c+d \leq A$.

We now specialize to the case $n = 1$.

Conjecture 2.3. *Let $n = 1$. Then, at least if $p \neq 2$,*

$$\lg(\ker(\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})} \longrightarrow \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})^\flat})) = 1 \ .$$

This is true if $a \leq 1$. If this conjecture holds true, then one can show that for any prime divisor D on $\tilde{\mathcal{M}}$ relatively prime to $\tilde{\mathcal{Z}}(\mathbf{x})^\flat$,

$$(\tilde{\mathcal{Z}}(\mathbf{x})^\flat . D) = \lg(\tilde{\mathcal{Z}}(\mathbf{x}) \cap D) - 1 \ . \tag{2.6}$$

Let us assume this. Let $\mathbf{y} = (y, y') \in V(G \xrightarrow{\lambda} G)$ such that $T = q(\mathbf{x}, \mathbf{y}) = \mathrm{diag}(up^a, u'p^b)$ with $u, u' \in \mathbb{Z}_p^\times$ and $0 \leq a \leq b$. Then applying (**??**) to $\tilde{\mathcal{Z}}(\mathbf{y})^\flat$ instead of $\tilde{\mathcal{Z}}(\mathbf{x})^\flat$, and writing $\tilde{\mathcal{Z}}(\mathbf{x})^\flat$ as a sum of quasi-canonical divisors according to Proposition **??** we obtain

$$\begin{aligned}(\tilde{\mathcal{Z}}(\mathbf{x})^\flat, \tilde{\mathcal{Z}}(\mathbf{y})^\flat) &= \sum_{s-c+d \leq A} [\lg(\tilde{\mathcal{W}}_s(k)^{(c,d)} \cap \tilde{\mathcal{Z}}(\mathbf{y})) - 1] \\ &= \sum\nolimits_{s=1}^{A-1} [(n(k;b,s) - 1) + (n(k;b,s+1) - 1)] \\ &\quad + [(n(k;b,1) - 1) + n(k;b,A) - 1] \\ &\quad + \delta \cdot (n(k;b,0) - 1) \ .\end{aligned} \tag{2.7}$$

Here we used the equality

$$\lg(\tilde{\mathcal{W}}_s(k)^{(c,d)} \cap \tilde{\mathcal{Z}}(\mathbf{y})) = n(k;b,s^\circ) \ . \tag{2.8}$$

Inserting the known expression for the quantities $n(k;b,s)$ we can calculate the expression on the right hand side. On the other hand consider the derivative of the representation density $\alpha_p'(T, S_0')$, where S_0' denotes the quadratic space given by the intersection of $\mathrm{M}_2(\mathbb{Z}_p)^{\mathrm{tr}=0}$ with the Eichler order for $\Gamma(p)$. Comparing now with Yang's formulae [Y] for representation densities, we obtain

Projected Theorem 2.4. *Let* $n = 1$. *Let* $(\mathbf{x}, \mathbf{y}) \in V(G \xrightarrow{\lambda} G)^2$ *with* $T = q(\mathbf{x}, \mathbf{y}) \in \mathrm{Sym}_2(\mathbb{Z}_p)$ *non-singular. Then*

$$(\tilde{\mathcal{Z}}(\mathbf{x})^\flat . \tilde{\mathcal{Z}}(\mathbf{y})^\flat) = -\frac{1}{p-1} \cdot \alpha_p'(T, S_0') - 1 \ .$$