Toronto, Oct. 2004

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Happy birthday, Jim !

1st in-depth explanation.

→ transp. 1

Similar counting formula was give by Kottwitz for the reduction mod p of Shimura variety.

Fix p. Let S_K = Shimura variety repr. moduli problem, with $K = K^p . K_p$ with $K_p = \mathcal{P}(\mathbb{Z}_p)$ parahoric.

The point in isogeny class should have the form

$$I(\mathbb{Q}) \backslash \left[G(\mathbb{A}_f^p)/K^p \times X_p \right] \circlearrowleft G(\mathbb{A}_f^p) \times \{ \Phi^{f_\mathfrak{p} \cdot \mathbb{Z}} \}$$

$$X_p \subset G(L)/\mathcal{P}(\mathcal{O}_L) \qquad f_\mathfrak{p} = [k_{E_\mathfrak{p}} : \mathbb{F}_p$$

<u>Kottwitz</u>:

$$\# \operatorname{Fix}_{g \cdot \Phi^j} = \sum_{\substack{(\gamma, \delta) \\ \in I(\mathbb{Q}) \times G(\mathbb{Q}_{p^j}) \\ \text{related}}} \mathrm{vol} \cdot O_\gamma(f_g) \cdot TO_\delta(\phi^{(j)})$$

char. fct. of $\{ M ; \mathrm{inv}(M, \Phi M) = \mu$

In general, S_K will have singularities, want to weight a point $x \in S_K(\mathbb{F}$

by

$$tr^{ss}(\Phi^j ; R\Psi_x)$$

(If Γ_0 acts through finite quotient, then $= tr(\Phi^j ; R\Psi_x^{\Gamma_0})$).

This should depend only on X_p-component of x, and be defined by

$$\psi^{(j)} : \big(\underbrace{G(L)/P(O_L)}_{\substack{\shortparallel \\ G(\mathbb{Q}_{p^j})/P(\mathbb{Z}_{p^j})}}\big)^{\Phi^j} \longrightarrow \bar{\mathbb{Q}}_\ell .$$

Approach through local models: Construct diagram of the form

$$S_K \otimes_{O_E} O_{E_\mathfrak{p}} \xleftarrow{\ \pi\ } \mathcal{M} \xrightarrow{\ \varphi\ } M ,$$

where π pHs under $P \otimes_{\mathbb{Z}_p} O_{E_\mathfrak{p}}$ and φ smooth of same relative dimension.

And M defined by linear algebra as a projective scheme / $\operatorname{Spec} O_{E_\mathfrak{p}}$.

In particular étale locally isomorphic.

Problem: Choice of model of S_K over $O_{E_\mathfrak{p}}$.

Naive idea: Extend moduli pb. over Sch / Spec E in „obvious way".

Turns out to work in unramified situation, but not in ramified situation.

Rest of the talk concentrate on unitary groups for $F/\mathbb{Q}$, $p \neq 2$.

S_K : Let $n = r+s$. Let F = imag.-quadr. field, $F \subset \mathbb{C}$

Moduli pb = (roughly) : abelian varieties A of dim. n, with O_F-action, with O_F-linear princ. polariz. s.t.

$$\mathrm{tr}(\iota(b); \mathrm{Lie}\, A) = r\, b + s \cdot \bar{b} \qquad , \quad b \in O_F .$$

+ level str. away from p + parahoric level str. at p.

$\rightsquigarrow$ representable by scheme over $\mathrm{Spec}\, E$, where

$$E = \begin{cases} F & r \neq s \\ \mathbb{Q} & r = s . \end{cases}$$

3 Cases: 1.) p splits in F

2.) p inert in F, unramified

3.) p ramified.

Ad 1.) transp. 2

Then M_I repr. by a projective scheme over Spec $\mathbb{Z}_p$, with generic fiber = Grass$(r, n-r)$.

<u>Singularities of</u> M: • $|I| = 1 \Rightarrow M =$ Grass smooth.

• $|I| = 2 \Rightarrow$ worst singularity of type

$$\{ (X, Y) \in M_r \times M_r \mid X \cdot Y = Y \cdot X = p \cdot E_r \}$$

(special fiber = circular variety (Strickland, Mehta-Trivedi, Faltings)

• $r = 1 \Rightarrow$ DNC

<u>Theorem 1</u> (Görtz): M_I is flat over Spec $\mathbb{Z}_p$. The special fiber is reduced, its irreducible components are normal w. rational singularities.

<u>Qu.</u>: Is M_I C-M. ? ($\to$ arithm. modular forms, Kisin.)

<u>Theorem 2</u> (Haines, Ngo): (i) The function $\varphi^{(i)} : G(\mathbb{Q}_{p^i}) / B(\mathbb{Z}_{p^i}) \longrightarrow \overline{\mathbb{Q}}$ lies in the center of the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_{p^i}) // B(\mathbb{Z}_{p^i}))$, and can be uniquely characterized. (<u>Kottwitz conjecture</u>).

(ii) The action of Γ_0 on $R\Psi$ is unipotent.

<u>Qu</u>: Is Γ_0-action on $R^i \Psi_x$ trivial ?

<u>Proofs of both theorems</u> use embedding of

$$M \otimes_{\mathbb{Z}} \mathbb{F}_p \hookrightarrow \mathcal{F} = \text{affine flag variety for } SL_n(\mathbb{F}_p((t))),$$

as union of affine Schubert varieties.

- for Thm 1: use normality + F-splitting of Schubert varieties
- for Thm 2: use deformation of $\mathcal{F}$ to affine Grassmannian, à la Gaitsgory.

Ad 2: In this case, local model is isom. to M after $\otimes_{O_{E_\mathfrak{p}}} O_L$.

Thm 1': the same (geometric properties)

Thm 2': more problematic. Seems O.K. for Iwahori subgp

$(I = \{0, \dots, n-1\})$, more generally standard parahoric (cont. i hs.)

Ad 3: Here the situation changes drastically (Pappas)!

(Work in progress with Pappas.).

Change notation: F resp. E become F_p resp. E_p.

To avoid technical problems, assume n odd.

Then $\mathrm{Stab}_{G(\mathbb{Q}_p)}(\Lambda_I)$ parahoric subgp., hence can formulate

moduli pb. (even over $\mathrm{Spec}\, E$, this is a problem!).

transp. 3 + 4

- Pappas: Let $I = \{0\}$. Then $M^{naive}_{\{0\}}$ not flat, unless $|r-s| \le 1$.

    (dimension).

Hence forced to take M_I = flat closure of generic fiber

(we have conjectural functor description of M_I in many cases).

Theorem (Pappas, R): . Then special fiber of M_I reduced

and ... – provided either $\begin{cases} n = 3 & \text{or} \\ I = \{0\} \text{ or } I = \{k, k+1\}, & \text{if } n = 2k+1. \end{cases}$

Proof is again based on embedding into affine flag variety, this time for unitary group, splitting over ramified extension.

This leads to geometric version of Bruhat-Tits theory: Let $K = k((t))$, let G/K reductive group. Associate $\mathcal{G}/\operatorname{Spec} k$ with

$$\mathcal{G}(R) = G(R((t))).$$

Then $\mathcal{G}$ is an ind-group scheme over $\operatorname{Spec} k$.

Theorem: Let $k = \bar{k}$

(i) $\mathcal{G}$ reduced iff $\operatorname{Hom}_K(G, G_m) = 0$.

(ii) $\pi_0(\mathcal{G}) = \pi_1(G)_{\Gamma_0}$

(iii) Let $P \subset G(K)$ parahoric subgp. Then $\exists!$ connected affine group scheme $\mathcal{P} \subset \mathcal{G}$ with $\mathcal{P}(k) = P$. The fpqc-quotient $\mathcal{G}/\mathcal{P}$ exists as ind-scheme, and quotient map loc. trivial for Z-topology.

(iv) Let G = unitary gp., char $k = p > 2$. Then Schubert varieties in $\mathcal{F} = \mathcal{G}/\mathcal{B}$ (= closures of $\mathcal{B}$-orbits) are normal and their $\cup$, $\cap$ are reduced and compat. F-split (gen. of Faltings's thm).

Remarks

Haines/Ngo not proved. We know the following: $(r,s) = (n-1, 1)$.

Then $\overline{M}_{\{0\}}$ has 2 $\mathcal{P}$-orbits, a unique singular point. Have

$$\psi^{(t)} = \begin{cases} 1 & \text{on these 2 orbits} \\ 0 & \text{else} \end{cases} \qquad \text{(Krämer)}$$

Very striking! Perhaps Γ_0 acts non-trivially on $R^i\psi_x$ for $i > 0$

(iii) Contrast to n even. If $n = 2$, $r = 1 = s$, have picture for $\overline{M}_I^{\text{naive}}$: