Madison, Sept. 04.

# Talk: Local models of Shimura varieties

Motivation at this conference (on Arakelov geometry): Let $E$ = number field, let $\mathcal{X}$ be arithmetic scheme over $\operatorname{Spec} O_E$, e.g. integral model of Shimura variety. Let $\mathcal{Z} \subset \mathcal{X}$ closed subscheme which is of finite length. The arithmetic degree of $\mathcal{Z}$ for Ar.-g. is

$$\widehat{\deg}\, \mathcal{Z} = \log |\mathcal{Z}| = \sum_{x \in \mathcal{Z}(\bar{\kappa}_p)} \lg O_{\mathcal{Z},x} \cdot \log p .$$

if $\mathcal{Z}$ is concentrated in char $p$.

What I am going to present is a method to treat another instance of counting points where not all points are counted w. multipl. 1, but are weighted: the semisimple zeta function of arithm. variety.

Let us localize at $p$: $E$ now local field, let $X/E$ smooth with model $\mathcal{X}$ over $O_E$. Then $\log Z_p^{ss} =$

$$\sum_{n=1}^{\infty} \sum_{x \in \mathcal{X}(\kappa_p^n)} \operatorname{Tr}^{ss}(\operatorname{Frob}_p^n ; R\Psi_x) \cdot \frac{T^n}{n} .$$

Here $\operatorname{Tr}^{ss} = \operatorname{Tr}(\operatorname{Frob}_p^n ; R\Psi_x^{I})$ if $I$ acts ~~through finite~~ trivially through finite grp. ~~Galois group~~

And $R\Psi$ = complex of nearby cycles in $\ell$-adic coho.

If $x$ smooth, then $R\Psi = \mathbb{Q}_\ell$ and $\mathrm{Tr} = 1$. Otherwise complicat

Example: If $\mathcal{X} \to \operatorname{Spec} O_E$ has strict semi-stable reduction, and $r$ divisors meet in $x \in \mathcal{X}(\kappa_E^n)$, then

$$R^i \Psi_{\mathbb{Q}_\ell, x} = \Lambda^i \left( \mathbb{Q}_\ell(-1)^{r-1} \right)$$

and hence $$\mathrm{Tr}^{ss}\left( \mathrm{Frob}_q^n ; R\Psi_x \right) = (1-q)^{r-1}.$$

<u>Want to calculate this in case of Sh.-variétés which are moduli spaces of abelian variétés</u>: (parahoric level str.) method of local models replaces these complicated schemes by algebraic variétés which can be defined by linear algebra, but have étale locally same singularities - hence can serve to determine $R\Psi$. <u>So far, this method does not apply to the Arakelov problem for special cycles on these Sh.-variétés!</u>

<u>Principle</u>: Start w. moduli pb. on $(Sch/E) \rightsquigarrow X = \mathcal{M}_E$. Then extend "in the obvious way" to $\mathcal{X} = \mathcal{M}$ on $(Sch/O_E)$. This will then be our model. <u>Naive</u> (Rapo-Zink): does not always give a good model.

## 1. The Ur-example: moduli of elliptic curves.

Fix $m \geq 3$, $(m, p) = 1$ (auxiliary). Put $E = \mathbb{Q}_p$. Define functor $\mathcal{M}_E$ on $(Sch/\mathbb{Q})$ by

$$\mathcal{M}_E(S) = \{(\underset{\text{elliptic curve}/S}{A}, \alpha),\ \alpha : A_m \simeq (\mathbb{Z}/m)^2\}/\simeq .$$

This is repr. by quasi-proj. scheme smooth of rel. dim 1 over $\operatorname{Spec} \mathbb{Q}_p$

Same formulation gives functor $\mathcal{M}$ on $(Sch/\mathbb{Z}_p)$, again repr. by quasi-proj. schem over $\operatorname{Spec} \mathbb{Z}_p$.

Thm: $\mathcal{M}$ is smooth of rel. dim 1 over $\operatorname{Spec} \mathbb{Z}_p$.

Fancy proof: Have diagram

$$\begin{array}{ccccc} & & \mathcal{M}^{\natural} & & \\ & \swarrow^{\pi} & & \searrow^{\varphi} & \\ \mathcal{M} & & & & \mathbb{P}^1 \end{array}$$

Here in $\mathcal{M}^{\natural}$ add to $(A, \alpha)$ a basis of $H_1^{DR}(A)$,

$$\eta : \mathcal{O}_S^2 \xrightarrow{\sim} H_1^{DR}(A).$$

Hence $\pi$ is pbs under $GL_2$. Then $\varphi$ maps $(A, \alpha, \eta)$ to

$$[\mathcal{F} = \eta^{-1}(\omega_{A/S}) \subset \mathcal{O}_S^2] \in \mathbb{P}^1(S).$$

By Serre-Tate + Groth-Messing, $\varphi$ satisf. the infinit. criterion for smoothness. Hence $\mathcal{M}$ and $\mathbb{P}^1$ isom., loc. for étale topology.

Here $\mathbb{P}^1$ was the <u>local model for this moduli problem</u> (<u>very linear</u>)

<u>Next let us consider</u> $\Gamma_0(p)$-<u>moduli problem</u>. Again $E = \mathbb{Q}_p$, $n \geq 3$.

$$\mathcal{M}_E(S) = \{ (A_0 \xrightarrow{\psi} A_1, \alpha),\ \psi \text{ isogeny of degree } p \} / \simeq .$$

Can again extend this to quasi-proj. scheme over $\mathrm{Spec}\, \mathbb{Z}_p$.

<u>Thm</u>: $\mathcal{M}$ is flat of relative dimension 1 over $\mathrm{Spec}\, \mathbb{Z}_p$ and has semistable reduction.

<u>Proof</u>: As before get diagram with smooth morphisms ($\pi$ plus under smooth gp scheme).

$$\begin{tikzcd} & \mathcal{M}^{\psi} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & \mathbb{M} \end{tikzcd}$$

Here $\mathbb{M}$ is the <u>local model</u>, repr. following functor over $\mathrm{Spec}\, \mathbb{Z}_p$.

Let $V = \mathbb{Q}_p^2$ with std basis $e_1, e_2$. Let

$$\Lambda_0 = \mathrm{span}_{\mathbb{Z}_p}\{e_1, e_2\}, \quad \Lambda_1 = \mathrm{span}_{\mathbb{Z}_p}\{p^{-1}e_1, e_2\}$$

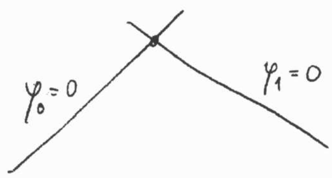
$$\Lambda_0 \longrightarrow \Lambda_1 \xrightarrow{p} \Lambda_0 .$$

Then $M(S) =$ ~~isom. classes of~~ $\{$ commut. diagrams

$$\begin{array}{ccccc} \Lambda_{0S} & \longrightarrow & \Lambda_{1S} & \longrightarrow & \Lambda_{0S} \\ \cup & & \cup & & \cup \\ \mathcal{F}_0 & \xrightarrow{\varphi_0} & \mathcal{F}_1 & \xrightarrow{\varphi_1} & \mathcal{F}_0 \end{array} \quad \}$$

where $\mathcal{F}_0, \mathcal{F}_1$ are loc. direct summands free of rank 1.

<u>Picture of</u> $M \otimes \mathbb{F}_p$ :




Choose coord. locally around worst point

$$(\mathcal{F}_0^0, \mathcal{F}_1^0) = (p\Lambda_1, \Lambda_0) \in M(\mathbb{F}_p).$$

Get equ.: $X_0 \cdot X_1 = p$.

Hence have expression for $R\Psi_x$, $\forall x$. : $\begin{cases} 1 & x \text{ smooth} \\ 1-q & \text{oth.} \end{cases}$

In the remainder I want to discuss an assortment of 2 examples.

For efficiency, I will not formulate precisely the moduli problem $\mathcal{M}$, but concentrate on local models.

$\boxed{\text{\underline{Naive idea} works in „unramified situation", but not in „ramified situation"}}$

## 2. The unramified unitary group

$\mathcal{M}_E$ Let $n = r+s$. Let $L =$ ~~imaginary~~ quadr. field $\subset \overline{\mathbb{Q}}_p$, consider moduli space of abelian var. $A$ of dim. $n$, with action of $O_L$, and $O_L$-linear principal polarization s.t. $\mathrm{tr}(\iota(b), \mathrm{Lie}\, A) = r\cdot b + s\cdot \bar{b}$, $\forall\, b \in O_L$.
+ level structure away from $p$ + parahoric level structure at $p$.
In this case $E = \begin{cases} L & r \neq s \\ \mathbb{Q}_p & r = s \end{cases}$.

If $p$ is unramified in $L$, the naive idea leads to following local model $M$. Let $V = \mathbb{Q}_p^n$ with basis $e_1, e_2, \dots, e_n$.
For $i = 0, \dots, n-1$, let $\Lambda_i = \mathrm{span}_{\mathbb{Z}_p}\{p^{-1}e_1, \dots, p^{-1}e_i, e_{i+1}, \dots, e_n\}$.
Get

$$\Lambda_0 \longrightarrow \Lambda_1 \longrightarrow \dots \longrightarrow \Lambda_{n-1} \xrightarrow{p} \Lambda_0$$

Choose $I = \{i_0 < i_1 < \dots < i_{m-1}\} \subset \{0, \dots, n-1\}$. The $M = M_I$ is described as $M(S) =$ ~~isom. classes of~~ commut. diagr.

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \dots & \longrightarrow & \Lambda_{i_{m-1},S} & \xrightarrow{p} & \Lambda_{i_0,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0 & \xrightarrow{\varphi_0} & \mathcal{F}_1 & \xrightarrow{\varphi_1} & \dots & \longrightarrow & \mathcal{F}_{m-1} & \longrightarrow & \mathcal{F}_0
\end{array}$$

Here $\mathcal{F}_i$ are loc. direct summands free of rank $r$.

For $|I| = 1$, $M_I$ is the Grassmannian $Gr_{r,n-r}$ over $\mathbb{Z}_p$.

Contrast to classical moduli pb's, like variety of complexes etc.: space fixed, maps vary

$\Updownarrow$ maps fixed, spaces vary.

In general have diagram

$$\begin{tikzcd} & M^{\triangledown} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ M & & N \end{tikzcd}$$

leave out.

where for $M^{\triangledown}$ we also fix a basis for $\mathcal{F}_i$, $\forall i \in I$.

Hence $\pi$ is phs. for $\prod_{i\in I} GL_r$. And

$$N_{r,m} = \{ (A_0, A_1, \dots, A_{m-1}) \in M_r^m \ ; \ A_0 \cdot A_1 \cdots A_{m-1} = A_1 A_2 \cdots A_{m-1} A_0 = \dots = p \cdot \mathbb{I} \}$$

The morphism $\varphi$ is smooth (Faltings, + many others).

In general, this does not seem to help much. But if $m = 2$, the special fiber of $N_{r,2}$ is

$$N \otimes \mathbb{F}_p = \{ (X, Y) \in M_r \times M_r \ ; \ XY = YX = 0 \}.$$

also appears in moduli space of bundles on semi-stable curves (Faltings)

Circular variety, studied by Strickland and later by Mehta + Trivedi, and by Faltings.

They in particular proved that $N \otimes \mathbb{F}_p$ is reduced.

This implies that $N_{r,2}$ is flat – and this is used in

$$\begin{tikzcd} & \mathcal{M}^{\triangledown} \arrow[dl, "\text{phs}"'] \arrow[dr, "\text{smooth}"] & \\ \mathcal{M} \otimes_{O_E} O_L & & M \otimes_{\mathbb{Z}_p} O_L \end{tikzcd}$$

Get as before after unramified base change

loc. isomorphic.

A lot of work on singularities: Faltings, Genestier, de Jong, ..

<u>Theorem</u> (Görtz): $M_I$ is flat. The special fiber is reduced, its irreducible components are normal with rational singularities.

For $n \leq 5$, $M_I$ is known to be Cohen-Macaulay.
For all $n$, $x \mapsto \mathrm{Tr}^{ss}(\mathrm{Frob}_{\mathbb{F}}^n; R\Psi_x)$ is known (Haines/Ngo)
Proof uses normality of Schubert varieties in affine flag variety. In fact, using this, one can circumvent the result of Strickland, etc.
use embedding $M \otimes \mathbb{F}_p \hookrightarrow \mathcal{F}$ affine flag variety for $SL_n(\mathbb{F}_p((t)))$ union of affine Schubert varieties

Work w. Pappas in progress

## 3. The ramified unitary group.

Now assume $p \neq 2$ ramified in $L$. ~~Change notation~~: Now $L/\mathbb{Q}_p$ ramified quadratic extension, fix $\varpi$ uniformizer with $\overline{\varpi} = -\varpi$.

transpose

Let $V$ = $L$-vector space of dim $n$, and

$\phi : V \times V \longrightarrow L$ non-deg. hermitian form.

Let $\langle v, w \rangle = \mathrm{Tr}_{L/\mathbb{Q}_p}(\varpi^{-1} \phi(v, w))$ alt. form.

Let $e_1, \dots, e_n$ basis of $V$ s.t. $\phi(e_i, e_{n+1-j}) = \delta_{ij}$.

For $i = 0, \dots, n-1$ put

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_L}\{\varpi^{-1} e_1, \dots, \varpi^{-1} e_i, e_{i+1}, \dots, e_n\}.$$

Extend to periodic lattice chain by $\Lambda_{i+kn} = \varpi^{-k} \cdot \Lambda_i$. Then

$$\hat{\Lambda}_j = \Lambda_{-j}$$ (for $\phi$ or $\langle\,,\,\rangle$ - the same)

Fix $I = \{ i_0 < i_1 < \dots < i_{m-1} \} \subset \{0, \dots, n-1\}$ with

$$i \in I,\ i \neq 0 \Rightarrow n-i \in I.$$

Also put $E = L$ if $r \neq s$, $E = \mathbb{Q}_p$ if $r = s$.

The <u>naive local model</u> is $O_E$-scheme repr. following moduli pb.

$M^{naive}_{-}(S) = M^{naive}_{I}(S) =$ ~~iso-classes of~~ diagrams

$$\begin{tikzcd}
\Lambda_{i_0,S} \arrow[r] & \Lambda_{i_1,S} \arrow[r] & \cdots \arrow[r] & \Lambda_{i_{m-1},S} \arrow[r, "\varpi"] & \Lambda_{i_0,S} \\
\mathcal{F}_{i_0} \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{F}_{i_1} \arrow[r] \arrow[u, phantom, "\cup"] & \arrow[r] & \mathcal{F}_{i_{m-1}} \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{F}_{i_0} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Here $\mathcal{F}_{i_j}$ are loc. direct summands free of rank $n$, stable under $O_L$.

Following conditions are imposed:

- $\forall i \in I \setminus 0$ the composition

$$\mathcal{F}_i \hookrightarrow \Lambda_{i,S} \simeq \underset{\text{via } \hat{\Lambda}_i = \Lambda_{-i}}{\Lambda^*_{n-i\,S}} \longrightarrow \mathcal{F}^*_{n-i} \quad \text{is zero}$$

If $0 \in I$, want $\mathcal{F}_0$ tot. isotropic (i.e. in middle $\Lambda_{0,S} = \Lambda^*_{0,S}$).

- $\forall i \in I$,

<u>Kottwitz condition.</u> $\quad \mathrm{char}_{\varpi | \mathcal{F}_i}(T) = (T-\varpi)^r \cdot (T+\varpi)^s \in O_E[T]$

Then: (i) (Pappas): If $|r-s| \geq 2$, then $M^{naive}_{\{0\}}$ not flat, because dim (special fiber) > dim (generic fiber). Probably if $|r-s| \leq 1$, then flat.

(ii) In general, it seems that (we can prove (?)): • have closed embedding

$$M^{naive} \otimes k_E \hookrightarrow \mathcal{F} = \text{affine flag variety for unitary gp rel. to } k_E((\sqrt{t}))/k_E((t)).$$

• affine Schubert varieties in $\mathcal{F}$ are normal.

? • Let $M$ = closure of $M_E$ in $M^{naive}$. Then special fiber is reduced, its irreducible comp. are normal w. rat. singularities.

But even if this can be proved, there is still the non-trivial problem of extending [HN] to this case, i.e. calculate $\mathrm{Tr}^{ss}(\mathrm{Frob}_x^r; R\Psi_x)$

Lot's of new phenomena: e.g. $M_{\{0\}}$ not irreducible

Notes for Toronto talk (refers to Madison talk)

(1) Motivation : • more on s.-s. L-fct., trace formula → Haines - Toronto

( ask Haines about unitary gp

• local models : in parahoric level str. want ~~flat~~ model $\mathcal{M}$ + diagram.

$$\mathcal{M} \xleftarrow{\pi} \tilde{\mathcal{M}} \xrightarrow{\varphi} \underline{M}$$

where $\pi$ = p.h.s. under smooth gp scheme
$\varphi$ = smooth morph. of same relative dim.

Hence $\mathcal{M}$ is loc. around each pt isom. to $M$.

<u>Naive idea:</u> take „same moduli pb" over $O_E$ as over $E$.

Turns out : works "$\Longleftrightarrow$" unramified situation.

Want to discuss 2 cases of unitary gp : unramified, ramified

(2)

(1) The unramified unitary group

Here $\mathcal{M} \leftarrow \mathcal{M}^{\#} \longrightarrow M \otimes_{\mathbb{Z}_p} O_L$, where $M$ following projective scheme. Transparency Proof uses Groth-Messing

<u>Singularities:</u> $|I| = 1 \Rightarrow$ Grassm. smooth.

$|I| = 2 \rightarrow$ worst Sing. $\{(X, Y) \in M_r \times M_r \; ; \; XY = YX = 0\}$

Strickland, Mehta + Trivedi, Faltings

$r = 1 \Rightarrow$ DNC.

<u>Görtz:</u> statement.

Uses embedding $M \otimes \mathbb{F}_p \hookrightarrow \mathcal{F}_I$ affine flag var. for $SL_n(\mathbb{F}_p((t)))$

- union of affine Schubert var., $M \otimes \mathbb{F}_p = \bigcup_{w \in Adm(\mu)} S_w$
- all affine Schubert var. are normal (P+R, Faltings for all split ss. grps).

<u>Haines - Ngo</u> (confirms Kottwitz conj. in this case): $x \mapsto Tr^{ss}(Frob_q^n; R\Psi_x)$

also unitary gp $\xrightarrow{?}$ is constant along $S_w$, call it $\varphi(w)$. Then $w \mapsto \varphi(w)$ in center of $\mathcal{H}(G, K_I)$, is charact. through Satake isom.

~~Furthermore, I acts trivially on $R\Psi_x$, $\forall x$~~

② Ramified unitary group.

<u>change not.</u> : $L$ = local. at $p$ of previous $L$.

Pappas, Rapo: Let $G$ any ~~[illegible]~~ reductive alg. gp over $K = k((t))$. The assoc. loop gp $\mathcal{G}$ : ind-gp scheme over Spec $k$ with

$$\mathcal{G}(R) = G(R((t))) \quad , \forall \ k\text{-alg. } R.$$

Here want $G$ = quasi-split unitary gp. rel. to $\bar{\mathbb{F}}_p((\sqrt{t}))/\bar{\mathbb{F}}_p((t))$.

Let $\mathcal{F} = \mathcal{G}/P_I$ fpqc-quotient by parahoric type $I$. (alg. gp. / $\bar{\mathbb{F}}_p$).

<u>Then have</u>: $M^{naive} \otimes_{O_E} \bar{\mathbb{F}}_p \hookrightarrow \mathcal{F}$ closed subscheme, union of affine Schubert var.

Need to know more about $\mathcal{F}$

<u>Thm</u>: (i) $\mathcal{F}$ is reduced, $\pi_0(\mathcal{F}) = \pi_1(G)$.

(ii) All affine Schubert varieties are normal.

imitates Faltings's proof.

Let now $M$ = closure of $M^{naive}$. ~~[illegible]~~ Have a conj. description, but too hard to prove!

<u>Hope</u>: can deduce statement in le fiante. This is still in

look at yellow pages

the works. ~~These~~ There are new phenomena, e.g.

- even if $I$ minimal, i.e. $P_I$ maximal, $M \otimes \overline{\mathbb{F}}_p$ not irreducible

This makes it diffic. to apply Hironaka's Lemma: stated it.

Other method is due to Faltings - seems to apply in those cases when reducible special fib

Other Open questions: calculate $x \mapsto Tr^{ss}$, is action of $I$ through finite quotient, or even trivial?

Mention $R_{F/\mathbb{Q}_p}(\quad)$ : paper w. Pappas.

$GU(3)$ complete ? → Haines.

→ Krämer