Toronto, March 03

Talk : Introduction to local models of Shimura varieties

Try to give a feel through examples.

1. Motivation

For the purposes of this talk, a Shimura variety is a scheme that solves a moduli problem of abelian varieties with additional data (like polariz. + endomorphisms + level structures).

More precisely let B a semi-simple algebra / $\mathbb{Q}$ with positive involution $*$ and $\mathcal{O}_B$ a $*$-stable order.

Then one considers a functor

$$(\mathrm{Sch} / \mathrm{Spec}\, E_{global}) \longrightarrow (\mathrm{Sets})$$

which to S associates the tuples $(A, \imath, \lambda, \alpha)$

- where A / S abelian scheme of a fixed dimension,
- $\imath : \mathcal{O}_B \longrightarrow \mathrm{End}_S(A)$
- λ a polarization of A
- α a level structure on A

satisfying certain conditions, e.g.

- Rosati involution of λ induces $*$ on O_B
- rep'n of O_B on Lie A is given in advance.

Here $E_{global} \subset \bar{\mathbb{Q}}$ is a number field over which this moduli pb. makes sense: Assume repres. by smooth scheme $\mathcal{M}_{global}$

<u>Problem</u>: Fix p and a prime ideal $\mathfrak{p}$ of E_{global} over p. Let $E = E_{global, \mathfrak{p}}$ Construct a scheme $\mathcal{M}$ over Spec O_E

- with generic fiber

$$\mathcal{M} \otimes_{O_E} E \text{ isomorphic to } \mathcal{M}_{global} \otimes_{E_{global}} E$$

- flat over Spec O_E, with mild singularities along special fiber.

<u>We want this for 2 reasons</u>:

i) (topological): To calculate (semi-simple) zeta function of $\mathcal{M}_{global}$ at $\mathfrak{p}$, via

$$\log Z_{\mathfrak{p}}^{ss}(M_{global}, T) = \sum_{n=1}^{\infty} \sum_{x \in M(\kappa_{\mathfrak{p}^n})} \mathrm{Tr}^{ss}(\mathrm{Frob}_{\mathfrak{p}^n} ; R\Psi_x) \frac{T^n}{n}$$

To calculate the (semisimple) trace of the Frobenius on the sheaf of nearby cycles need a grip on singularities of M.

ii) (algebraic): To study congruence properties of modular forms need commutative algebra structure of M, e.g. M Cohen-Macaulay (q-expansion principle and refinements).

For the non-specialist: Flatness means continuity of fibers, e.g. equi-dimensional, generic fiber is dense (topological flatness)

A good theory seems to exist only when the level structure is parahoric at p.

Naive idea: Define M as solution to a moduli problem over $(Sch / \mathrm{Spec}\, O_E)$ which extends the restriction of M to $(Sch / \mathrm{Spec}\, E_{global, \mathfrak{p}})$. Works in many cases.

2. The Ur-example: moduli space of elliptic curves

Here $E_{global} = \mathbb{Q}$. Fix $m \geq 3$, $(m,p) = 1$.

Consider functor

$$(Sch/\mathbb{Q}) \to Sets, \quad S \mapsto (A, \alpha), \text{ where}$$

A elliptic curve over S, $\alpha : A_m \simeq (\mathbb{Z}/m)^2$.

This is representable by quasi-projective scheme $\mathcal{M}_{global}$ smooth of relative dimension 1 over $\operatorname{Spec} \mathbb{Q}$.

Now take same functor over $(Sch / \operatorname{Spec} \mathcal{O}_E)$ (here $\mathcal{O}_E = \mathbb{Z}_p$), get $\mathcal{M} / \operatorname{Spec} \mathcal{O}_E$.

<u>Thm:</u> $\mathcal{M}$ is smooth over $\operatorname{Spec} \mathcal{O}_E$.

<u>Fancy proof:</u> Consider the following diagram

$$\begin{tikzcd} & \tilde{\mathcal{M}} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & \mathbb{P}^1 \end{tikzcd}$$

Here $\tilde{\mathcal{M}} = \{(A, \alpha, \eta \ \mathcal{O}_S\text{-basis of } H_{1DR}(A))\}$,

hence π is phs under GL_2. Furthermore, φ

maps (A, α, η) to $\eta^{-1}(\omega_A)$,

$$\begin{array}{ccc} \eta : \mathcal{O}_S^2 & \xrightarrow{\sim} & H_{1,DR}(A) \\ \cup & & \cup \\ \eta^{-1}(\omega_A) & & \omega_A \end{array}$$

By the theorems of Grothendieck-Messing + Serre-Tate, φ satisfies the infinites. criterion for smoothness. Hence φ is smooth; whence the smoothness of $\mathcal{M}$ //.

Now let us consider the $\Gamma_0(p)$-moduli problem.

Consider functor

$$S \mapsto (A_0 \to A_1, \alpha)$$

where $A_0 \to A_1$ is isogeny of degree p between elliptic curves, α a level structure as before (of either A_0 or A_1).

If we take this functor on $(Sch / \mathbb{Z}_p)$, this is represented by a quasi-projective scheme.

<u>Thm</u>: $\mathcal{M}$ is flat of relative dimension 1 over $\operatorname{Spec} \mathbb{Z}_p$ and has semi-stable reduction

<u>Proof:</u> As before get diagram with smooth morphisms

$$\begin{tikzcd} & \widetilde{\mathcal{M}} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & M \end{tikzcd}.$$

Here M is the projective scheme repr. following functor over $\mathbb{Z}_p$: Let $V = \mathbb{Q}_p^2$ with std. basis e_0, e_1.

Let $\Lambda_0 = \mathrm{span}_{\mathbb{Z}_p}\{e_0, e_1\}$, $\Lambda_1 = \mathrm{span}\{p^{-1}e_0, e_1\}$.

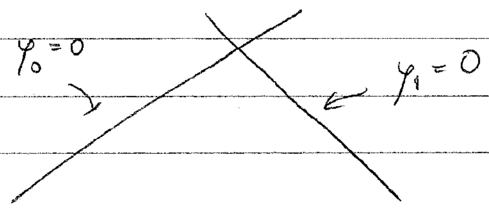
Hence $\Lambda_0 \longrightarrow \Lambda_1 \xrightarrow{p} \Lambda_0$.

Now $M(S)$ = isom.-classes of commut. diagrams of $\mathcal{O}_S$-homo

$$\begin{tikzcd} \Lambda_0 \otimes \mathcal{O}_S \arrow[r] & \Lambda_1 \otimes \mathcal{O}_S \arrow[r] & \Lambda_0 \otimes \mathcal{O}_S \\ \mathcal{F}_0 \arrow[u, phantom, "\cup"] \arrow[r, "\varphi_0"] & \mathcal{F}_1 \arrow[u, phantom, "\cup"] \arrow[r, "\varphi_1"] & \mathcal{F}_0 \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $\mathcal{F}_0, \mathcal{F}_1$ are loc. direct sums free of rank 1.

<u>Picture for $M \otimes \mathbb{F}_p$:</u>



Choose coordinates locally around the worst point

$$(\mathcal{F}_0^0, \mathcal{F}_1^0) = (p\Lambda_1, \Lambda_0) \in M(\mathbb{F}_p)$$

eqn.: $X_0 X_1 = p$.

3. The Hilbert - Blumenthal case

Let F_{global} be a totally real extension of degree g of $\mathbb{Q}$. Fix $m \geq 3$, $(m,p)=1$. Consider the following functor

$$S \mapsto (A, \iota, \kappa, \alpha)$$

where : A abelian variety of dimension g over S

$\iota : \mathcal{O}_{F_{global}} \longrightarrow \mathrm{End}_S(A)$ homo

$\kappa : \mathfrak{D}^{-1} \longrightarrow \underline{\mathrm{Hom}}_{sym, \mathcal{O}_{F_{global}}}(A, A^\vee)$

positive isom. of $\mathcal{O}_F$ - modules (sheaves of)

$\alpha : A_m \simeq (\mathcal{O}_F / m\mathcal{O}_F)^2$ $\mathcal{O}_F$ - linear iso

This is represented by smooth quasi-projective scheme of dimension g over $\mathbb{Q}$, M_{global}.

Let M be defined by same moduli problem $/ \operatorname{Spec} \mathbb{Z}_p$.

This is not flat!

Explanation: Let $(A, \iota, \kappa, \alpha) / k = \bar{k}$. If char $k = 0$, then

$\mathrm{Lie}\, A$ free module over $\mathcal{O}_{F_{global}} \otimes k = F_{global} \otimes k$, hence (*) $\mathrm{char}(b;\, \mathrm{Lie}\, A) = \prod_{i=1}^{g} (T - \varphi_i(b))$, $\forall\, b \in O_F$.

If char $k = p$, where p unramified in F_{global}, have

$$O_F \otimes k = \bigoplus_{i=1}^{g} k, \qquad \mathrm{Lie}\, A = \bigoplus_i \mathrm{Lie}^i(A).$$

If $g \geq 2$, it is easy to construct examples where $\mathrm{Lie}^i(A) = (0)$ for some i. Then $(A, \iota, \kappa, \alpha)$ cannot be lifted to char 0, hence $\mathcal{M}$ not flat.

Impose condition (*) (<u>Kottwitz condition</u>), have

<u>Theorem</u>: The new functor is representable by a quasi-pr. scheme which is flat over Spec $\mathbb{Z}_p$ and smooth if p unramified in F_{global}.

Put $F = F_{global} \otimes \mathbb{Q}_p$. Again get diagram as before, where M projective scheme, repr. following functor on $(Sch/\mathbb{Z}_p)$: Let $\Lambda = O_F^2$.

Then $M(S) = \{ \mathcal{F} \subset \Lambda \otimes O_S ;\ \mathcal{F}$ loc. direct summand of rank g, O_F-stable, $\mathrm{char}(b; \mathcal{F}) = \prod (T - \varphi_i(b)) \}$.

2 <u>extreme cases</u>:

p <u>unramified</u> in F_{global}: Then $M \otimes W(\mathbb{F}_p) = \prod_{i=1}^{g} \mathbb{P}^1$.

p *totally ramified in* F_{global}: Then can relate M to space of matrices, via 2 smooth morphisms,

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "\tau"'] \arrow[dr, "\psi"] & \\ M & & N \end{tikzcd}$$

Here $\tilde{M} = \{ \mathcal{F} \subset \Lambda \otimes \mathcal{O}_S,\ \eta = \mathcal{O}_S\text{-basis of } \mathcal{F} \}$

$$N = \{ X \in M_g \ ;\ \det(T \cdot I - X) = \prod_{i=1}^{g} (T - \varphi_i(\pi)),\ Q(X) = 0 \}.$$

Here π denotes a uniformizer in $\mathcal{O}_F = \mathcal{O}_{F_{global}} \otimes \mathbb{Z}_p$ satisfying an Eisenstein equation $Q(\pi) = 0$. Hence special fiber is

$$N \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \{ X \in M_g ;\ \det(T \cdot I - X) = T^g \}$$

$\Rightarrow X^g = 0$ Cayley-Hamilton

Hence $N \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is the "true" nilpotent variety, reduced normal irreduc. CM - variety. Hence the same follows for M.

Again can soup this up to include $\Gamma_0(p)$ - moduli pb.

4. The Hilbert - Siegel case

Let F_{global} as before, and $m \geq 3$, $(m,p)=1$. Consider the following functor for fixed $r \geq 1$,

$$S \longmapsto (A, \iota, \lambda, \alpha)$$

where A abelian variety of dimension $g \cdot r$ over S

$$\iota : \mathcal{O}_F \longrightarrow \mathrm{End}_S(A)$$

λ $\mathcal{O}_{F_{global}}$ - linear polarization

α level m-structure $A_m \simeq (\mathcal{O}_F / m\mathcal{O}_F)^{2r}$

s.t.

$$\mathrm{char}(b; \mathrm{Lie}\, A) = \left(\prod_{i=1}^{g} (T - \varphi_i(b))\right)^r, \quad b \in \mathcal{O}_F.$$

Represented by quasi-proj. scheme $\mathcal{M}$ over $\mathrm{Spec}\, \mathbb{Z}_p$.

Again have diagram of smooth morphisms,

$$\begin{tikzcd} & \widehat{\mathcal{M}} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & M \end{tikzcd}$$

Let Λ free $\mathcal{O}_F$-module of rank $2r$ with symplectic form $\langle \, , \, \rangle$ s.t. $\langle bv, w \rangle = \langle v, bw \rangle$

Then $M(S) = \{ \mathcal{F} \subset \Lambda \otimes \mathcal{O}_S$; loc. direct summand free of rank gr totally isotropic for $\langle \, , \, \rangle$, $\mathcal{O}_F$ - stable with $\mathrm{char}(b; \mathcal{F}) = \left(\prod (T - \varphi_i(b))\right)^r \}$.

Hence if p unramified in F, then

$M \otimes W(\bar{\mathbb{F}}_p)$ = product of g copies of Grassm. of Lagrangian subspaces in sympl. vs. of dim $2r$

Hence, Theorem: If p unramified in F_{global}, then $\mathcal{M}$ is smooth.

Other extreme: p totally ramified in F

Theorem: Let $\mathcal{M}^{can}$ be the scheme-theoretic closure (hence $\mathcal{M}^{ca}$ flat) of the generic fiber in $\mathcal{M}$. Then $\mathcal{M}^{can} = \mathcal{M}_{red}$ and $\mathcal{M}^{ca}$ has a reduced irreducible special fiber which is normal, CM, with only rational singularities.

Conjecture: $\mathcal{M}^{can} = \mathcal{M}$, i.e. $\mathcal{M}$ is flat. (seems difficult).

Principle of proof: Suffices to consider local model M. Let K be the Galois closure of F. Define a scheme $\tilde{M}$ as repres. the following functor on $(Sch/\mathrm{Spec}\, O_K)$

Number the embeddings $\varphi_i : F \to K$, $i = 1, \dots, g$.

splitting model

$$\tilde{M}(S) = \{ (0) = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \dots \subset \mathcal{F}^g \subset \Lambda \otimes \mathcal{O}_S \ ;$$

$\mathcal{F}^i$ loc. dir. summands of rk ir, stable $\mathcal{O}_F$

$$(b \otimes 1 - 1 \otimes \varphi_i(a))(\mathcal{F}^i) \subset \mathcal{F}^{i-1}$$

$\mathcal{F}^i$ totally isotropic and

$$\prod_{i+1 \leq q \leq g} (b \otimes 1 - 1 \otimes \varphi_q(b))(\mathcal{F}^i)^{\perp} \subset \mathcal{F}^i,$$

$\forall i. \}$

<u>One now checks:</u> $\tilde{M}$ is smooth over $\operatorname{Spec} \mathcal{O}_K$ with generic fiber $M \otimes_{\mathbb{Z}_p} \mathcal{O}_K$, get a morphism $(\mathcal{F}^\bullet \mapsto \mathcal{F}^g)$.

$$\tilde{M} \longrightarrow M \otimes_{\mathbb{Z}_p} \mathcal{O}_K \longrightarrow M$$

, projective, iso on generic fibers.

Hence M^{can} = image of $\tilde{M}$ in M.

Now relate special fibers to <u>affine Grassmannian</u>, apply Thm of Faltings-Görtz on normality of Schubert varieties.

Can soup this up to include other parahoric level structures:

- if p <u>unramified</u>, then naive method gives a flat model with reduced fiber whose irreducible components are normal, CM, rational singularities (Görtz)

- if p ramified, can again define splitting model and take its image in naive local model. This has all good properties.