SPECIAL CYCLES ON UNITARY SHIMURA VARIETIES II: GLOBAL THEORY

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Abstract

We introduce moduli spaces of abelian varieties which are arithmetic models of Shimura varieties attached to unitary groups of signature $(n-1,1)$. We define arithmetic cycles on these models and study their intersection behaviour. In particular, in the non-degenerate case, we prove a relation between their intersection numbers and Fourier coefficients of the derivative at $s=0$ of a certain incoherent Eisenstein series for the group $\mathrm{U}(n,n)$. This is done by relating the arithmetic cycles to their formal counterpart from part I via non-archimedean uniformization, and by relating the Fourier coefficients to the derivatives of representation densities of hermitian forms. The result then follows from the main theorem of [39] and a counting argument.

1. Introduction

This paper is the global counterpart to [39]. Our purpose here is to
• introduce moduli spaces of abelian varieties which are arithmetic models of Shimura varieties attached to unitary groups of signature $(n-1,1)$,
• define arithmetic cycles on these models and
• study the intersection pattern of these arithmetic cycles, and in particular prove in the *non-degenerate* case the relation between their arithmetic intersection numbers and Fourier coefficients of the derivative at $s=0$ of a certain incoherent Eisenstein series $E(z,s,\Phi)$ for the group $\mathrm{U}(n,n)$ that was predicted in §16 of [30].

We now describe our results in more detail. Let $\boldsymbol{k}$ be an imaginary quadratic field, with ring of integers $O_{\boldsymbol{k}}$. Let n be a positive integer and let r with $0\leq r\leq n$. We then consider the moduli space $\mathcal{M}(n-r,r)$ over $\operatorname{Spec} O_{\boldsymbol{k}}$ which parametrizes principally polarized abelian varieties (A,λ) of dimension n with an action $\iota: O_{\boldsymbol{k}}\to \operatorname{End}(A)$ of $O_{\boldsymbol{k}}$. We require that the Rosati involution corresponding to λ induces the non-trivial automorphism of $O_{\boldsymbol{k}}$ and that the representation of $O_{\boldsymbol{k}}$ on the Lie algebra of A is equivalent to the sum of $n-r$ copies of the natural representation and of r copies of the conjugate representation. Hence, for $n=1$ and $r=0$ we obtain the usual moduli space $\mathcal{M}_0$ of elliptic curves with complex multiplication by $O_{\boldsymbol{k}}$ (these moduli spaces are Deligne-Mumford stacks and not schemes, but we will neglect this fact in the introduction).

The special cycles of interest to us are defined as follows. To a pair (A,ι,λ), resp. (E,ι_0,λ_0) of objects of $\mathcal{M}(n-r,r)$ resp. $\mathcal{M}_0$ over a common connected base S, we associate the free $O_{\boldsymbol{k}}$-module $\operatorname{Hom}_{O_{\boldsymbol{k}}}(E,A)$ with the positive definite $O_{\boldsymbol{k}}$-valued hermitian form given by

$$h'(x,y)=\lambda_0^{-1}\circ y^\vee\circ\lambda\circ x\in \operatorname{End}_{O_{\boldsymbol{k}}}(E)=O_{\boldsymbol{k}}\ .$$

For $m > 0$, let $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})_{\geq 0}$ be a positive semi-definite hermitian matrix of size m. The special cycle $\mathcal{Z}(T)$ is the moduli space over $\mathcal{M}(n-r,r) \times \mathcal{M}_0$ which parametrizes m-tuples of homomorphism $\mathbf{x} = [x_1, \dots, x_m] \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(A, E)^m$ such that $h'(\mathbf{x}, \mathbf{x}) = (h(x_i, x_j)) = T$. We refer to T as the *fundamental matrix* of the collection $\mathbf{x}$. After extending scalars from $O_{\boldsymbol{k}}$ to $\mathbb{C}$, these cycles coincide with the cycles studied in [33, 35].

These cycles are most interesting in the case when $r = 1$. In this case, when $m = 1$ and $T = t \in \mathbb{Z}_{>0}$, they are divisors, which for $n = 2$ are essentially identical with the cycles considered by Gross and Zagier [14]. For $m \geq 1$ and $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})_{>0}$, the cycle $\mathcal{Z}(T)$ has codimension m in the generic fiber of $\mathcal{M}(n-1,1) \times \mathcal{M}_0$, which has dimension $n-1$, but may have lower codimension in $\mathcal{M}(n-1,1) \times \mathcal{M}_0$, which has dimension n. We call $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})_{>0}$ *non-degenerate* if $\mathcal{Z}(T)$ has codimension m in $\mathcal{M}(n-1,1) \times \mathcal{M}_0$. We are led to study the following intersection problem.

Let $n = \sum_{i=1}^r m_i$ and let $T_i \in \mathrm{Herm}_{m_i}(O_{\boldsymbol{k}})$. The intersection (fiber product) of the cycles $\mathcal{Z}(T_i)$ decomposes according to the fundamental matrices into the disjoint sum, comp. [30],

$$\mathcal{Z}(T_1) \times \cdots \times \mathcal{Z}(T_r) = \coprod_{T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})} \mathcal{Z}(T) ,$$

where the fiber product is taken over $\mathcal{M}(n-1,1) \times \mathcal{M}_0$. Here the matrices T have diagonal blocks $T_1, \dots, T_r$. The most tractable part of this intersection are the summands corresponding to *nonsingular* matrices $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$. In this case it is easy to see that the support of $\mathcal{Z}(T)$ is concentrated in finitely many fibers in positive characteristics. We define the contribution of such T to the arithmetic intersection number of *non-degenerate* $\mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r)$ as

$$\langle \mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r) \rangle_T = \sum\nolimits_p \chi\big(\mathcal{Z}(T)_p, \mathcal{O}_{\mathcal{Z}(T_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(T_r)}\big) \cdot \log p .$$

Here the derived tensor product appears because the cycles $\mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r)$ do not in general intersect properly, not even along the part $\mathcal{Z}(T)$ corresponding to a non-singular fundamental matrix T. Our main result concerns the case when T itself is *non-degenerate*, i.e. when $\mathcal{Z}(T)$ is a finite set of points. In this case, all diagonal blocks occurring in T are automatically non-degenerate. To formulate our main result we must introduce the *incoherent Eisenstein series.*

These series are the analogues for $\mathrm{U}(n,n)$ of the incoherent Eisenstein series for $\mathrm{Sp}(2n)$ defined and discussed in detail in [30]. To introduce them, we fix a character η of $\boldsymbol{k}_{\mathbb{A}}^\times / \boldsymbol{k}^\times$ whose restriction to $\mathbb{Q}_{\mathbb{A}}^\times$ is the quadratic character associated to $\boldsymbol{k}$. (Such a choice determines splittings of metaplectic covers of unitary groups and hence allows us to work on the linear groups.) If V is a hermitian space over $\boldsymbol{k}$ of dimension n and $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A})^n)$ is a factorizable Schwartz function on $V(\mathbb{A})^n$, with a suitable K_∞-finiteness condition on φ_∞, there is a corresponding standard section $\Phi(s) = \Phi_\varphi(s) = \otimes_v \Phi_{\varphi_v}(s)$ of the global degenerate principal series representation $I(s, \eta)$ of $\mathrm{U}(n,n)$, induced from the character $\eta(\det) \,|\det|^s$ of the maximal parabolic with Levi factor $\mathrm{GL}(n)/\boldsymbol{k}$ (Siegel parabolic). This section is *coherent* in the sense that it arises from the global hermitian space V. More precisely, at $s = 0$ and for a finite place v non-split in $\boldsymbol{k}$, the local degenerate principal series $I_v(0, \eta_v)$ is the direct sum of two irreducible representations $R_n(U_v^\pm)$ for local hermitian spaces $U_v^\pm$ of dimension n over $\boldsymbol{k}_v$, where $\chi_v(\det(U_v^\pm)) = \pm 1$, [43]. The space $R_n(U_v^\pm)$ is the image of the local Schwartz space $S((U_v^\pm)^n)$ under the Rallis coinvariant map. At a split finite place, the local degenerate principal series $I_v(0, \eta_v)$ is irreducible. At the archimedean place,

$$I_\infty(0, \eta_\infty) = \oplus_{0 \leq r \leq n} R_n(n-r, r),$$

where $R_n(n-r,r)$ is the image of the Schwartz space of a hermitian space of signature $(n-r,r)$, [44]. For a global hermitian space V the image $R_n(V)$ of $S(V(\mathbb{A})^n)$ in the global degenerate principal series $I(0,\eta)$ under the map $\varphi \to \Phi_\varphi(0)$ is the irreducible representation

$$R_n(V) = \otimes_v R_n(V_v).$$

This representation is then realized as an automorphic representation of $\mathrm{U}(n,n)$ by taking the value at $s=0$ of the Siegel-Eisenstein series $E(h,s,\Phi_\varphi)$ formed from a coherent section $\Phi_\varphi(s)$. This is a special case of the regularized Siegel-Weil formula, proved in the case of unitary groups by Ichino [22], [23]. There is a second type of irreducible constituent of $I(0,\eta)$ – note that this representation is unitarizable and hence completely reducible. These have the form

$$R_n(\mathcal{C}) = \otimes_v R_n(\mathcal{C}_v),$$

where $\{\mathcal{C}_v\}_v$ is a collection of local hermitian spaces of dimension n such that $\mathcal{C}_v = V_v$ for almost all v, and

$$\prod_v \chi_v(\det(\mathcal{C}_v)) = -1.$$

These constituents of $I(0,\eta)$ and the associated Siegel-Eisenstein series are *incoherent* in the sense that they do not arise from a global hermitian space. For any standard section $\Phi(s)$ with $\Phi(0) \in R_n(\mathcal{C})$, one has $E(h,0,\Phi) = 0$, and the kernel of the Eisenstein map $E(0)$ from $I(0,\eta)$ to the space of automorphic forms on $\mathrm{U}(n,n)$ is precisely the direct sum of the $R_n(\mathcal{C})$'s as $\mathcal{C}$ runs over the incoherent collections.

The incoherent Eisenstein series of interest to us are obtained by fixing a global hermitian space V of signature $(n-1,1)$ for which there exists a self-dual $O_{\boldsymbol{k}}$-lattice L. Such a space will be called a *relevant hermitian space*; the set $\mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ of isomorphism classes of such spaces is finite. Let $\varphi_f \in S(V(\mathbb{A}_f)^n)$ be the characteristic function of $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^n$, and let φ'_∞ be the Gaussian in $S((V'_\infty)^n)$, where V'_∞ has signature $(n,0)$. The resulting standard global section

$$\Phi(s,L) = \Phi_{\varphi'_\infty}(s) \otimes \Phi_{\varphi_f}(s,L)$$

is incoherent and the corresponding Siegel-Eisenstein series $E(h,s,L)$ depends only on the genus of L, i.e., the orbit of L under the action of $G_1^V(\mathbb{A}_f)$, where $G_1^V = U(V)$ is the isometry group of V. There are either one or two genera of self-dual lattices in V; we denote by $E(h,s,V)$ the sum of the $E(h,s,L)$ as L runs over representatives for the genera. Finally, we let

$$E(h,s) = \sum_{V \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})} E(h,s,V) \tag{1.1}$$

be the sum over the isomorphism classes of relevant hermitian spaces. Our main result expresses some of the non-singular Fourier coefficients $E'_T(h,0)$ of the derivative $E'(h,0)$ at $s=0$ in terms of arithmetic intersection numbers of special cycles. For this, it is more convenient to pass to the corresponding classical Eisenstein series $E(z,s)$, as in (9.1), where $z \in D_n$, the hermitian symmetric space for $\mathrm{U}(n,n)$. This Eisenstein series has the form, [5], [57], [45],

$$E(z,s) = \det v(z)^{\frac{s}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \det(cz+d)^{-n} |\det(cz+d)|^{-s}\, \Phi_f(\gamma,s),$$

where the series is convergent for $\mathrm{Re}(s) > n$. Here $\Gamma = U(n,n) \cap \mathrm{GL}_{2n}(O_{\boldsymbol{k}})$, Γ_∞ is the subgroup for which lower left $n \times n$ block is zero. The meromorphic analytic continuation and holomorphy on the line $\mathrm{Re}(s) = 0$ follow from Langlands' general results. By the incoherence condition,

$E(z,0)=0$, and, as explained in section 9, the Fourier expansion of the central derivative has the form

$$E'(z,0)=\sum_{T\in\mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}} a(T)\,q^T+\sum_{\substack{T\\ \text{other}}} a(T,v(z))\,q^T,$$

where $q^T=\exp(2\pi i\mathrm{tr}(Tz))$. In particular, the terms for positive definite T are holomorphic functions of z.

Theorem 1.1. *Let $T\in\mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ be nonsingular with diagonal blocks $T_1,\dots,T_r$. Let $\mathrm{Diff}_0(T)$ be the set of primes p that are inert in $\boldsymbol{k}$ for which $\mathrm{ord}_p(\det(T))$ is odd.*

(i) If $|\mathrm{Diff}_0(T)|\geq 2$, then $\mathcal{Z}(T)=\emptyset$ and $E'_T(z,0)=0$.

(ii) If $\mathrm{Diff}_0(T)=\{p\}$ with $p>2$, then T is non-degenerate if and only if it is $\mathrm{GL}_n(O_{\boldsymbol{k},p})$-equivalent to $\mathrm{diag}(1_{n-2},p^a,p^b)$ for some $0\leq a<b$ with $a+b$ odd. In this case $\mathcal{Z}(T)$ has support in the supersingular locus in characteristic p and

$$\langle\mathcal{Z}(T_1),\dots,\mathcal{Z}(T_r)\rangle_T=\mathrm{length}(\mathcal{Z}(T))\cdot\log p\ .$$

Furthermore, in this case

$$E'_T(z,0)=E'_T(z,0,V)=C_1\cdot\langle\mathcal{Z}(T_1),\dots,\mathcal{Z}(T_r)\rangle_T\cdot q^T\ .$$

for an explicit constant C_1, independent of T, where $V\in\mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ is the unique relevant hermitian space which differs from V_T only at ∞ and p.
Here V_T denotes the space $\boldsymbol{k}^n$ with hermitian form defined by T.

The strategy of the proof of this theorem is similar to that of the proof of the analogous theorem for Shimura curves [42]. We first prove that for nonsingular $T\in\mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ the cycle $\mathcal{Z}(T)$ is either empty or concentrated in the supersingular locus in finitely many characteristics p, where p is not split in $\boldsymbol{k}$, and in fact concentrated in characteristic p, if $\mathrm{Diff}_0(T)=\{p\}$. Then we use the theory of non-archimedean uniformization of [52], to reduce the calculation of the length of $\mathcal{Z}(T)$ to a combination of a local calculation on a formal moduli space of p-divisible groups and a point count. The first problem was solved in our previous paper [39]. The second problem is solved here in section 12. It then remains to calculate the Fourier coefficient corresponding to T of the derivative of the incoherent Eisenstein series. For this we use the Siegel-Weil formula established by Ichino [22], [23], in this case. We must ultimately calculate some representation densities for hermitian forms, which is in general a very difficult task, even though a general formula due to Hironaka [17], [18], exists. Fortunately in the non-degenerate case, these calculations are manageable and a direct comparison gives the formula in our main theorem.

We now put our main result in perspective and indicate possible directions of further research. Of course, the model for the results above and in fact the origin of the whole program lies in the theory of special cycles on Shimura varieties attached to orthogonal groups of signature $(n-1,2)$, and considered for low values of n in our papers [30], [36], [38], [41]. However, as explained in the introduction of [39], there are serious problems with the construction of arithmetic models of these Shimura varieties as soon as $n\geq 6$. By contrast, in the case considered here, which is related to Shimura varieties attached to unitary groups of signature $(n-1,1)$, manageable arithmetic models exist and can be studied for arbitrary n. In fact, we define such models for unitary groups of arbitrary signature $(n-r,r)$ – but with no level structure. In the case of deeper level structure such arithmetic models can surely also be defined, although results on arithmetic intersection numbers of special cycles will then be harder to come by.

In the case of signature $(n-1,1)$, the most promising next steps to be taken seem to be the following:

• Let $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with diagonal blocks $T_1, \ldots, T_r$, which are assumed to be non-degenerate. Assuming $\mathrm{Diff}_0(T) = \{p\}$ with $p > 2$ inert in $\boldsymbol{k}$, prove that $\langle \mathcal{Z}(T_1), \ldots, \mathcal{Z}(T_r)\rangle_T$ is given by the same formula as in the Main Theorem above. To prove this in general[1], one will have to deal with degenerate intersections, which most probably also requires a better knowledge of the structure of the special cycles outside the supersingular locus.

• Investigate in detail the reduction modulo a ramified prime p of the moduli space $\mathcal{M}(n-1,1)$ and its supersingular locus and use this to establish results on the arithmetic intersection numbers $\langle \mathcal{Z}(T_1), \ldots, \mathcal{Z}(T_r)\rangle_T$ in the case when $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ has $\mathrm{Diff}_0(T) = \emptyset$.

• One can define variants of our arithmetic models which involve some parahoric level structure. It should be possible to exploit our considerable knowledge on such models obtained in recent years, [46], [47], [48], [49], to investigate special cycles in this context.

The situation becomes more speculative when $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$ is singular. In this case one would like to equip the special cycles with appropriate Green forms and define classes in certain arithmetic Chow groups. Then various cup products of these arithmetic cycles should be related to various special values of derivatives of Eisenstein series on unitary groups of type (n,n). Here the fact that the moduli space $\mathcal{M}(n-1,1)$ is not proper will play a crucial role. The Eisenstein series we consider in this paper are conjecturally related to the cup product with values in $\widehat{\mathrm{CH}}^n\big(\mathcal{M}(n-1,1)\big)$ – but the non-compactness of these moduli spaces prevents us from making this more precise. No doubt the extended versions of arithmetic Chow groups defined by Burgos, Kramer, Kühn [7] will have an impact on these questions.

One may also try to generalize our results in other directions. One may expect similar results when $\boldsymbol{k}$ is replaced by an arbitrary CM-field. Also, our main results in this paper concern special cycles which lie above

$$\mathcal{M} = \mathcal{M}(n-1,1) \times \mathcal{M}(1,0)\ .$$

However, we define $\mathcal{M}(n-r,r)$ for arbitrary r and one can similarly define more general cycles over more general products. It is a challenge then to determine arithmetic intersection numbers and to form a sensible generating function using them, which can be compared with some automorphic counterpart.

As should be clear from the above description, the investigation of special cycles on unitary Shimura varieties is largely uncharted territory. This also explains why we have made an effort to explain the relation to previous work; we explain in section 3 the precise relation to the cycles in [28, 33, 35] (*KM-cycles*), and in section 14 the relation to the cycles (*Heegner points*) considered by Gross and Zagier. In a sequel, [40], we explain how KM-cycles arise in the theory of *occult period mappings*.

We now explain the lay-out of the paper. The paper has five parts.

In Part I, we give the definition of the moduli stack $\mathcal{M}(n-r,r)$ and define the special cycles $\mathcal{Z}(T)$ on them. We also show how to uniformize the orbifold of complex points of these stacks in terms of the space of negative r-planes in $\mathbb{C}^n$, and make the connection between the moduli stacks $\mathcal{M}(n-r,r)$ and certain Shimura varieties associated to unitary groups. In particular, this allows us to describe the set of connected components of $\mathcal{M}(n-r,r)_{\mathbb{C}}$, which is used later

[1] In this context, we refer to Terstiege's forthcoming paper [62], in which he deals with this problem in the case $n = 3$.

in the examples.

In Part II, we describe the completion of $\mathcal{M}(n-r,r)$ along the supersingular locus in the fiber of $\mathcal{M}(n-r,r)$ at a prime p which is inert in $\boldsymbol{k}$, in terms of the formal moduli space of p-divisible groups that was studied extensively in [65]. This p-adic uniformization is essentially just spelling out the general method of [52] in the special case at hand. We also exhibit an analogous p-adic uniformization of the completion of the special cycles along their supersingular locus.

Part III is devoted to the computation of the nonsingular Fourier coefficients of the central derivative of an incoherent Eisenstein series. First, we review the theory of theta integrals for unitary groups and the regularized Siegel-Weil formula, due in this case to Ichino [22], that relates these to special values of certain Eisenstein series for unitary groups. Then, in the incoherent case, the Tth Fourier coefficient of the central derivative for a positive definite T is expressed as a product of a representation number of T by a genus of definite lattices and a derivative of a local Whittaker function. The values and derivatives of such Whittaker functions are given in terms of representation densities for hermitian forms and their derivatives.

In Part IV we prove our main results, by determining explicitly the arithmetic intersection numbers in the non-degenerate case, and by comparing the result with the special values of the derivatives at 0 of the relevant Eisenstein series.

Finally, Part V is devoted to examples and variants of our main result. In particular, we give a more detailed description of the case $n=2$ and explain the precise relation of our special cycles for $\mathcal{M}(1,1)$ to those introduced by Gross and Zagier [14].

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Notation and conventions

We fix an imaginary quadratic field $\boldsymbol{k}=\mathbb{Q}(\sqrt{\Delta})$ with discriminant Δ, ring of integers $O_{\boldsymbol{k}}$, and nontrivial Galois automorphism $a\mapsto a^\sigma$, $a\in\boldsymbol{k}$. As usual, we denote by $h_{\boldsymbol{k}}$ the class number and by $w_{\boldsymbol{k}}=|O_{\boldsymbol{k}}^\times|$ the number of units. We view $\boldsymbol{k}$ as a subfield of $\mathbb{C}$ via an embedding τ and require that $\tau(\sqrt{\Delta})$ have positive imaginary part. For a rational prime p, we write $O_{\boldsymbol{k},p}=O_{\boldsymbol{k}}\otimes_{\mathbb{Z}}\mathbb{Z}_p$ and $O_{\boldsymbol{k},(p)}=O_{\boldsymbol{k}}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

For a hermitian space V, $(\ ,\ )$ of dimension n over $\boldsymbol{k}$, let $\det(V)\in\mathbb{Q}^\times/N(\boldsymbol{k}^\times)$ be the determinant of the matrix $((v_i,v_j))$ where $\{v_i\}$ is a $\boldsymbol{k}$-basis for V. Note that we take $(\ ,\ )$ to be conjugate linear in the second argument. Define invariants $\mathrm{sig}(V)=(r,s)$, $r+s=n$, and $\mathrm{inv}_p(V)=\chi_p(\det(V))$, where $\chi_p(a)=(a,\Delta)_p$. Here $(\ ,\ )_p$ is the quadratic Hilbert symbol for $\mathbb{Q}_p$. Note that $\mathrm{inv}_\infty(V)=(-1)^s$ and that $\mathrm{inv}_p(V)=1$ for all split primes p. For a fixed dimension n, the isometry class of the hermitian space V_p over $\boldsymbol{k}_p$ (resp. V_∞ over $\mathbb{C}$) is determined by $\mathrm{inv}_p(V_p)$ (resp. $\mathrm{sig}(V_\infty)$). Moreover, the isometry class of V over $\boldsymbol{k}$ is determined by the collection of its local invariants (Hasse principle) and, for any collection of local hermitian spaces $\{V_p\}$ satisfying the product formula

$$\prod_{p\leq\infty}\mathrm{inv}_p(V_p)=1,\tag{1.2}$$

there is a unique global hermitian space with the V_p's as its local completions (Landherr's Theorem).

For a hermitian space V over $\boldsymbol{k}$, there is an associated alternating form defined by $\langle x, y \rangle = \mathrm{tr}((x,y)/\sqrt{\Delta})$. Note that, for $a \in \boldsymbol{k}$, $\langle ax, y \rangle = \langle x, a^\sigma y \rangle$ and that the hermitian form is given by

$$2(x,y) = \langle \sqrt{\Delta}\, x, y \rangle + \langle x, y \rangle \sqrt{\Delta}. \tag{1.3}$$

Conversely, if V is a $\boldsymbol{k}$-vector space with a $\mathbb{Q}$-bilinear alternating form $\langle\ ,\ \rangle$ satisfying $\langle ax, y \rangle = \langle x, a^\sigma y \rangle$, then (1.3) defines a hermitian form on V. An $O_{\boldsymbol{k}}$-lattice in V is self-dual for $(\ ,\ )$ if and only if it is self-dual for $\langle\ ,\ \rangle$.

Contents

Part I: The global moduli problem and special cycles

2. The global moduli problem

2.1. For an integer $n \geq 1$ and a decomposition $n = (n-r) + r$ with $0 \leq r \leq n$, we define a groupoid

$$\mathcal{M}(n-r, r)^{\mathrm{naive}} = \mathcal{M}(\boldsymbol{k}, n-r, r)^{\mathrm{naive}}$$

fibered over $(\mathrm{Sch}/\mathrm{Spec}\, O_{\boldsymbol{k}})$ by associating to a locally noetherian $O_{\boldsymbol{k}}$-scheme S the groupoid of triples (A, ι, λ). Here A is an abelian scheme over S, $\iota : O_{\boldsymbol{k}} \to \mathrm{End}_S(A)$ is an action of $O_{\boldsymbol{k}}$ on A, and $\lambda : A \to A^\vee$ is a principal polarization such that

$$\iota(a)^* = \iota(a^\sigma)$$

for the corresponding Rosati involution $*$. In addition, the following signature condition is imposed:

$$\mathrm{char}(T, \iota(a) \mid \mathrm{Lie}\, A) = (T - \varphi(a))^{n-r}(T - \varphi(a^\sigma))^r, \qquad a \in O_{\boldsymbol{k}}. \tag{2.1}$$

where $\varphi : O_{\boldsymbol{k}} \to \mathcal{O}_S$ is the structure homomorphism. Here the left side is the characteristic polynomial in $\mathcal{O}_S[T]$ of the $\mathcal{O}_S$-module endomorphism of $\mathrm{Lie}\, A$ induced by $\iota(a)$. In particular, A is of relative dimension n over S.

A morphism in $\mathcal{M}(n-r, r)^{\mathrm{naive}}(S)$ from (A, ι, λ) to (A', ι', λ') is an $O_{\boldsymbol{k}}$-linear isomorphism $\alpha : A \to A'$ such that $\alpha^*(\lambda') = \lambda$.

Proposition 2.1. *$\mathcal{M}(n-r, r)^{\mathrm{naive}}$ is a Deligne-Mumford stack over $\mathrm{Spec}\, O_{\boldsymbol{k}}$. Furthermore, $\mathcal{M}(n-r, r)^{\mathrm{naive}} \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, O_{\boldsymbol{k}}[\Delta^{-1}]$ is smooth of relative dimension $(n-r)r$ over $\mathrm{Spec}\, O_{\boldsymbol{k}}[\Delta^{-1}]$.*

Proof. The representability by a DM-stack follows from the representability by a DM-stack of the stack of principally polarized abelian varieties and the relative representability of the forgetful map which forgets the $O_{\boldsymbol{k}}$-action. This relative representability follows from the theory of Hilbert schemes. The smoothness assertion is checked by the infinitesimal criterion for smoothness and Grothendieck-Messing theory [52], [46]. $\square$

Example 2.2. Let $n = 1, r = 0$. Then $\mathcal{M}(1, 0)^{\mathrm{naive}}$ parametrizes triples (E, ι_0, λ_0) where (E, ι) is an elliptic curve with complex multiplication by $O_{\boldsymbol{k}}$ such that the action of $O_{\boldsymbol{k}}$ on $\mathrm{Lie}\, E$ is the natural one. In this case, the polarization λ_0 is uniquely determined. The coarse moduli space of $\mathcal{M}(1, 0)^{\mathrm{naive}}$ is $\mathrm{Spec}\, O_H$ where H is the Hilbert class field of $\boldsymbol{k}$. Note that $\mathrm{End}_{O_{\boldsymbol{k}}}(E/S, \iota_0) = O_{\boldsymbol{k}}$ for any $(E, \iota_0, \lambda_0) \in \mathcal{M}(1, 0)^{\mathrm{naive}}(S)$. This example is discussed in [41]. We will abbreviate $\mathcal{M}(1, 0)^{\mathrm{naive}}$ to $\mathcal{M}_0$.

We note that there is a natural isomorphism between the moduli stacks $\mathcal{M}(n-r, r)^{\mathrm{naive}}$ and $\mathcal{M}(r, n-r)^{\mathrm{naive}}$ which associates to (A, ι, λ) its conjugate $(A, \bar\iota, \lambda)$, where the $O_{\boldsymbol{k}}$-action on A has been changed to its conjugate, i.e., $\bar\iota(a) = \iota(a^\sigma)$.

Remark 2.3. As was first pointed out by Pappas, [46], $\mathcal{M}(n-r, r)^{\mathrm{naive}}$ is not flat over $\mathrm{Spec}\, O_{\boldsymbol{k}}$ for $n \geq 3$. Pappas defines a closed substack of $\mathcal{M}(n-r, r)^{\mathrm{naive}}$ by imposing an additional condition as in the following definition.

Definition 2.4. Let $\mathcal{M}(n-r, r)$ be the closed substack of $\mathcal{M}(n-r, r)^{\mathrm{naive}}$ consisting of those triples (A, ι, λ) for which the action of $O_{\boldsymbol{k}}$ on $\mathrm{Lie}\, A$ satisfies the *wedge condition*:

$$\wedge^{r+1}(\iota(a) - a) = 0, \qquad \wedge^{n-r+1}(\iota(a) - a^\sigma) = 0. \tag{2.2}$$

For $n \leq 2$, this condition follows from the signature condition. Furthermore, over $\mathrm{Spec}\, O_{\boldsymbol{k}}[\Delta^{-1}]$,

$$\mathcal{M}(n-r, r)[\Delta^{-1}] = \mathcal{M}(n-r, r)^{\mathrm{naive}}[\Delta^{-1}]. \tag{2.3}$$

Theorem 2.5. (Pappas) *Let $r = 1$, and assume $2 \nmid \Delta$. Then the stack $\mathcal{M}(n-1, 1)$ is flat over $\mathrm{Spec}\, O_{\boldsymbol{k}}$.*

Proof. This is [46], Theorem 4.5, a). $\square$

Finally, the following stack will play a fundamental role, so that we introduce a symbol for it.

Notation 2.6. For a given fixed collection $\boldsymbol{k}, n, r$, we let

$$\mathcal{M} = \mathcal{M}(n-r, r) \times_{\operatorname{Spec} O_{\boldsymbol{k}}} \mathcal{M}_0$$

be the base change of $\mathcal{M}(\boldsymbol{k}; n-r, r)$ to $\mathcal{M}_0$.

2.2. Suppose that $(E, \iota_0, \lambda_0) \in \mathcal{M}_0(S)$ and $(A, \iota, \lambda) \in \mathcal{M}(n-r, r)(S)$, i.e., for an element of $\mathcal{M}(S)$, are given. When S is connected, we consider the free $O_{\boldsymbol{k}}$-module of finite rank

$$V'(A, E) = \operatorname{Hom}_{O_{\boldsymbol{k}}}(E, A).$$

On this $O_{\boldsymbol{k}}$-module there is a $O_{\boldsymbol{k}}$-valued hermitian form given by

$$h'(x, y) = \iota_0^{-1}(\lambda_0^{-1} \circ y^\vee \circ \lambda \circ x) \in O_{\boldsymbol{k}}. \tag{2.4}$$

where $y^\vee : A^\vee \to E^\vee$ denotes the dual homomorphism.

Lemma 2.7. *The hermitian form h' on $V'(A, E)$ is positive-definite.*

Proof. Consider the endomorphism $\alpha \in \operatorname{End}(E \times A)$ given by

$$\alpha = \begin{pmatrix} 0 & \lambda_0^{-1} x^\vee \lambda \\ x & 0 \end{pmatrix}.$$

The adjoint α^* with respect to the polarization (λ_0, λ) of $E \times A$ is $\alpha^* = \alpha$. Hence

$$\alpha\alpha^* = \operatorname{diag}(\lambda_0^{-1} x^\vee \lambda x, x \lambda_0^{-1} x^\vee \lambda) \in \operatorname{End}(E) \times \operatorname{End}(A).$$

The positivity of the Rosati involution implies that the first entry of this diagonal matrix is positive, as had to be shown. □

Definition 2.8. Let $T \in \operatorname{Herm}_m(O_{\boldsymbol{k}})$ be an $m \times m$ hermitian matrix, $m \geq 1$, with coefficients in $O_{\boldsymbol{k}}$. The *special cycle* $\mathcal{Z}(T)$ attached to T is the stack of collections $(A, \iota, \lambda, E, \iota_0, \lambda_0; \mathbf{x})$ where $(A, \iota, \lambda) \in \mathcal{M}(n-r, r)(S)$, $(E, \iota_0, \lambda_0) \in \mathcal{M}_0(S)$, and $\mathbf{x} = [x_1, \ldots, x_m] \in \operatorname{Hom}_{O_{\boldsymbol{k}}}(E, A)^m$ is an m-tuple of homomorphisms such that

$$h'(\mathbf{x}, \mathbf{x}) = (h'(x_i, x_j)) = T. \tag{2.5}$$

Proposition 2.9. *$\mathcal{Z}(T)$ is representable by a DM-stack. The natural morphism $\mathcal{Z}(T) \to \mathcal{M}$ is finite and unramified.*

Proof. Given an S-valued point $(A, \iota, \lambda, E, \iota_0, \lambda_0)$ of $\mathcal{M}$, the functor $\underline{\operatorname{Hom}}_{O_{\boldsymbol{k}}}(E, A)$ on (Sch/S) defined by

$$S' \rightsquigarrow \operatorname{Hom}_{O_{\boldsymbol{k}}}(E \times_S S', A \times_S S')$$

is representable by a scheme which is unramified over S (by rigidity) and satisfies the valuative criterion for properness (by the Néron property of abelian schemes). The finiteness now follows, since, by the positive definiteness of h', the set

$$\{\ \mathbf{x} \in \operatorname{Hom}_{O_{\boldsymbol{k}}}(E, A)^m \mid h'(\mathbf{x}, \mathbf{x}) = T\ \}$$

is finite. □

2.3. There is an obvious Tate module variant of the hermitian space $V'(A,E)$. Let F be an algebraically closed field of characteristic p and let

$$(A,\iota,\lambda;E,\iota_0,\lambda_0)\in\mathcal{M}(F).$$

Denoting by $T^p(A)^0$, resp. $T^p(E)^0$ the rational Tate modules prime to p of A resp. E, let

$$V'_{\mathbb{A}_f^p}=\mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(T^p(E)^0,T^p(A)^0),$$

with hermitian form

$$h'(x,y)=\iota_0^{-1}(\lambda_0^{-1}\circ y^\vee\circ\lambda\circ x)\in\boldsymbol{k}\otimes\mathbb{A}_f^p.$$

Then the natural embedding $V'(A,E)\to V'_{\mathbb{A}_f^p}$ is isometric. On the other hand, the hermitian form $h'(\ ,\ )$ on $V'_{\mathbb{A}_f^p}$ is related to the Weil pairing as follows. We fix an isomorphism $\mathbb{A}_f^p(1)\simeq\mathbb{A}_f^p$. Then the natural pairing e_A takes values in $\mathbb{A}_f^p$,

$$e_A: T^p(A)^0\times T^p(A^\vee)^0\longrightarrow\mathbb{A}_f^p,$$

and there is an hermitian form $h=h_\lambda$ on $T^p(A)^0$ given by

$$2\,h_\lambda(x,y)=e_A(\delta x,\lambda(y))+\delta e_A(x,\lambda(y)),\tag{2.6}$$

where $\delta=\sqrt{\Delta}$.
The same construction can be made with E in place of A. The hermitian forms h_λ and h_{λ_0} define a hermitian structure $h(\ ,\ )$ on $\mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(T^p(E)^0,T^p(A)^0)$, which is independent of the trivialization of $\mathbb{A}_f^p(1)$. Hence we may replace the base scheme $\operatorname{Spec}F$ by any connected $O_{\boldsymbol{k},(p)}$-scheme S.

Lemma 2.10. *The two hermitian forms $h'(\ ,\ )$ and $h(\ ,\ )$ on*

$$\mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(T^p(E)^0,T^p(A)^0)$$

are identical.

Proof. We choose an identification of $T^p(E)^0$ with $\boldsymbol{k}\otimes\mathbb{A}_f^p$, i.e., a basis vector 1 in $T^p(E)^0$. We calculate for $x,y\in\mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(T^p(E)^0,T^p(A)^0)$

$$\begin{aligned}2\,h_\lambda(x(1),y(1))&=e_A(\delta x(1),\lambda(y(1)))+\delta\,e_A(x(1),\lambda(y(1)))\\&=e_E(\delta,x^\vee\lambda\,y(1))+\delta\,e_E(1,x^\vee\lambda\,y(1))\\&=2\,h_{\lambda_0}(1,\lambda_0^{-1}\,x^\vee\lambda\,y(1))\\&=2\,h'(x,y)\,h_{\lambda_0}(1,1).\end{aligned}$$

This proves the claim. □

2.4. We define the set $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ of *relevant hermitian spaces* of dimension n over $\boldsymbol{k}$ as the set of isomorphism classes of hermitian spaces V with $\mathrm{sig}(V)=(n-r,r)$ and which contain a self-dual $O_{\boldsymbol{k}}$-lattice.

Lemma 2.11. *(i) The cardinality of $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ is*

$$|\mathcal{R}_{(n-r,r)}(\boldsymbol{k})|=2^{\delta-1},$$

where δ is the number of primes that ramify in $\boldsymbol{k}$, i.e., the number of distinct prime divisors of Δ.
(ii) The number of strict *similarity classes of relevant hermitian spaces is*

$$|\mathcal{R}_{(n-r,r)}(\boldsymbol{k})/\mathrm{str.sim.}| = \begin{cases} 2^{\delta-1} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Here by a strict similarity we mean a similarity such that the scale factor is positive.

Proof. By the Hasse principle, the isomorphism class of a hermitian space is determined by its localizations. The existence of a self-dual lattice is equivalent to the condition that the determinant $\det(V)$ lies in $\mathbb{Z}_p^\times N(\boldsymbol{k}_p^\times)$ for all finite primes p. For split and ramified primes, this local condition is automatic, while, for inert primes it is equivalent to $\mathrm{inv}_p(V) = (\det V, \Delta)_p = 1$. Since the signature is also fixed, the relevant spaces are determined by the collection of signs $\mathrm{inv}_p(V)$ for $p \mid \Delta$, and any collection of signs is realized, subject to the condition that

$$\mathrm{inv}_\infty(V) = \prod_{p|\Delta} \mathrm{inv}_p(V).$$

Here note that $\mathrm{inv}_\infty(V) = (-1)^r$. This proves (i).
To prove (ii), note that the determinants of similar spaces differ by the nth power of the scale factor. Thus, for n even, two hermitian space are similar if and only if they are isomorphic, while, for n odd, two relevant hermitian spaces are locally similar by a unit at each ramified prime and hence are globally similar, by the Hasse principle for similitudes. $\square$

Proposition 2.12. *(i) There is a natural disjoint decomposition of algebraic stacks*

$$\mathcal{M}(n-r,r) = \coprod_{V \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})/\mathrm{str.sim.}} \mathcal{M}(n-r,r)^V.$$

(ii) There is a natural disjoint decomposition of algebraic stacks

$$\mathcal{M} = \mathcal{M}(n-r,r) \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathcal{M}_0 = \coprod_{V \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})} \mathcal{M}^V.$$

For n even, this decomposition is obtained by base change from that in (i).

Remark 2.13. Of course, for n odd, part (i) is trivial, since there is only one strict similarity class of relevant V's.

Proof. Let (A, ι, λ) be in $\mathcal{M}(n-r,r)(S)$ for a connected base S. Let $s : \mathrm{Spec}\, F \to S$ be a geometric point of S. First suppose that F has characteristic zero, and choose an isomorphism $\hat{\mathbb{Z}}(1) \xrightarrow{\sim} \hat{\mathbb{Z}}$ over F. We obtain from the pull-back $\xi = \xi(s)$ to F of (A, ι, λ) the rational Tate module $T(A)^0$ with its Riemann form $\langle\ ,\ \rangle_\lambda$ associated to the polarization λ. This satisfies $\langle ax, y \rangle_\lambda = \langle x, a^\sigma y \rangle_\lambda$ and hence determines a hermitian form $(\ ,\ )_\lambda$ by (1.3). Hence we obtain a hermitian space $\mathcal{V}$ over $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{A}_f$ with a self-dual lattice given by the Tate module of (A, ι, λ). We claim that there is a unique element $V(\xi)$ in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ which after tensoring with $\mathbb{A}_f$ gives the hermitian space $\mathcal{V}$. The uniqueness is clear by the Hasse principle and the product formula. For the existence, note that the point ξ arises, via base change, from a point $\xi_0 \in \mathcal{M}(n-r,r)(F_0)$ for a subfield $F_0 \subset F$ which is finitely generated over the prime field and hence has a complex embedding. If F has a complex embedding $F \hookrightarrow \mathbb{C}$, let $V(\xi) = H_1(A_{\mathbb{C}}, \mathbb{Q})$ be the rational homology of the corresponding complex abelian variety, which, by the same argument as above, is a hermitian vector space over $\boldsymbol{k}$. By the signature condition, the space $V(\xi)$ has signature $(n-r,r)$, and by the compatibility between singular homology and Tate module, $V(\xi) \otimes \mathbb{A}_f = \mathcal{V}$.

Thus $V(\xi) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$.
Next, suppose that F has characteristic $p > 0$, and choose an isomorphism $\hat{\mathbb{Z}}^p(1) \xrightarrow{\sim} \hat{\mathbb{Z}}^p$ over F. We obtain a hermitian space $\mathcal{V}^p$ over $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ with a self-dual lattice. There exists a unique hermitian space $V(\xi)$ with $V(\xi) \otimes \mathbb{A}_f^p \simeq \mathcal{V}^p$ and with $\mathrm{sig}(V(\xi)) = (n-r, r)$. We claim that $V(\xi) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$. If p is ramified or split in $\boldsymbol{k}$, the space $V(\xi)_p$ always has a self-dual $O_{\boldsymbol{k}} \otimes \mathbb{Z}_p$-lattice, so that $V(\xi) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$. If p is inert in $\boldsymbol{k}$, we use the fact that $\mathcal{M}(n-r,r)$ is smooth at p. Hence there exists a point $\tilde{\xi} = (\tilde{A}, \tilde{\iota}, \tilde{\lambda}) \in \mathcal{M}(n-r,r)(W(F))$ lifting ξ, where $W(F)$ is the ring of Witt vectors. Then

$$T^p(A)^0 = T^p(\tilde{A})^0 = V(\tilde{\xi}) \otimes_{\mathbb{Q}} \mathbb{A}_f^p,$$

as hermitian spaces over $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Also, by the previous argument, $V(\tilde{\xi})$ has signature $(n-r, r)$, so that $V(\xi)$ and $V(\tilde{\xi})$ are locally isomorphic at all places other than p. Hence $V(\xi)_p \simeq V(\tilde{\xi})_p$ as well and this space has a self-dual $\mathbb{Z}_p \otimes O_{\boldsymbol{k}}$-lattice. Again, we conclude that $V(\xi) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$.

The space $V(\xi)$ attached above to (A, ι, λ) depends on the choice of the geometric point s of S, on the complex embedding, and on the trivialization of the group of roots of unity. A change in these choices changes the space $V(\xi)$ within a strict similarity class, as we check below. However, if (A, ι, λ) and (E, ι_0, λ_0) are points of $\mathcal{M}(n-r,r)$ and $\mathcal{M}_0$ over S, then attaching as above the hermitian spaces $V(\xi)$ to (A, ι, λ) and $V(\xi_0)$ to (E, ι_0, λ_0), the hermitian space $V = \mathrm{Hom}_{\boldsymbol{k}}(V(\xi_0), V(\xi)) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ is independent of all choices. Indeed, if the trivialization $\widehat{\mathbb{Z}}^p(1) \simeq \widehat{\mathbb{Z}}^p$ of the prime-to-p roots of unity is changed by a scalar $c \in \big(\widehat{\mathbb{Z}}^p\big)^\times$, then $V(\xi) \otimes \mathbb{A}_f^p$ and $V(\xi_0) \otimes \mathbb{A}_f^p$ are both scaled by the same scalar c, hence $V \otimes \mathbb{A}_f^p = \mathrm{Hom}_{\boldsymbol{k}}(V(\xi_0), V(\xi)) \otimes \mathbb{A}_f^p$ is unchanged. Since the archimedean localization of V is determined by the signature condition, the product formula and the Hasse principle imply that V is unchanged in its isometry class. Similarly, if two geometric points s and s' have the same image in S, then $V(\xi) \otimes \mathbb{A}_f^p$ and $V(\xi') \otimes \mathbb{A}_f^p$ are scales of one another by the scalar $c \in \big(\widehat{\mathbb{Z}}^p\big)^\times$ which compares the corresponding trivializations of the prime-to-p roots of unity; the same applies to $V(\xi_0) \otimes \mathbb{A}_f^p$ and $V(\xi_0') \otimes \mathbb{A}_f^p$, hence $\mathrm{Hom}_{\boldsymbol{k}}(V(\xi_0), V(\xi)) \otimes \mathbb{A}_f^p \simeq \mathrm{Hom}_{\boldsymbol{k}}(V(\xi_0'), V(\xi')) \otimes \mathbb{A}_f^p$, and we conclude as before by the product formula and the Hasse principle. The same argument takes care of the change of the complex embedding. The independence of the image point in S of the chosen geometric point is proved by using the specialization homomorphism from a generic geometric point.

This defines the claimed disjoint decomposition in (ii), and also proves (i). □

Over $\mathrm{Spec}\, O_{\boldsymbol{k}}[\frac{1}{2}]$, a slight refinement of the decomposition of Proposition 2.12 will be useful. For a relevant hermitian space V, let $G_1^V = \mathrm{U}(V)$ be the isometry group. Then $G_1^V(\mathbb{A}_f)$ acts on the set of self dual lattices in V by $g : L \mapsto V \cap \big(g(L \otimes \widehat{\mathbb{Z}})\big)$. The orbit of a lattice L is the G_1^V-genus of L; we denote it by $[[L]]$. The following result, due to Jacobowitz [24], sections 9 and 10, describes the orbits.

Proposition 2.14. ([24]) *Suppose that V_p is a non-degenerate hermitian space of dimension n over $\boldsymbol{k}_p/\mathbb{Q}_p$ and that V_p contains a self-dual lattice. Then the unitary group $\mathrm{U}(V_p)$ acts transitively on the set of self-dual lattices in V_p except in the following cases:*
(a) $p = 2$, $\boldsymbol{k}/\mathbb{Q}_p$ is ramified, $n = \dim V_p$ is even and V_p is a split space.
(b) $p = 2$, $\boldsymbol{k} = \mathbb{Q}_p(\sqrt{\Delta})$ where $\mathrm{ord}_2(\Delta) = 3$, $n = \dim V_p$ is even and V_p is the sum of a 2-dimensional anisotropic space and a split space of dimension $n-2$.
Then there are two $\mathrm{U}(V_p)$-orbits of self-dual lattices in V_p.

Remark 2.15. (i) Explicit representatives for the two orbits can be given as follows. Let $H(0)$ be the hyperbolic plane, i.e., 2-dimensional space with $O_{\boldsymbol{k},2}$-basis e, f and $(e,f)=1$, $(e,e)=(f,f)=0$. Note that the $\mathbb{Z}_2$-ideal generated by the set of values (x,x), $x\in H(0)$ is then $\mathrm{tr}(O_{\boldsymbol{k},2})=2\mathbb{Z}_2$. Let $\dim V_2=n=2k$. In case (a), the orbit representatives are $H(0)^k$ and $\mathrm{diag}(1,-1)\oplus H(0)^{k-1}$. In case (b), take $\kappa\in\mathbb{Z}_2^\times$ such that κ is not a norm from $\boldsymbol{k}_2$. There is then a unit $\lambda\in O_{\boldsymbol{k},2}^\times$ with $N(\lambda)-\kappa\in 4\mathbb{Z}_2$. This element is unique modulo $2O_{\boldsymbol{k},2}$. Write $\Delta=4d$ with $d\in 2\mathbb{Z}_2$. Then the $O_{\boldsymbol{k},2}$-lattice $D(0)$ of rank 2 with hermitian form defined by

$$\begin{pmatrix} -d & \lambda^\sigma \\ \lambda & \frac{\kappa-N(\lambda)}{d} \end{pmatrix}$$

is unimodular, anisotropic, and has $(x,x)\in 2\mathbb{Z}_2$ for all $x\in D(0)$. The orbit representatives in case (b) are $D(0)\oplus H(0)^{k-1}$ and $\mathrm{diag}(1,-\kappa)\oplus H(0)^{k-1}$.
(ii) By analogy with the case of symmetric bilinear forms, [54], we will call a self-dual (unimodular) hermitian lattice type II if the values (x,x) for $x\in L$ are all even and type I otherwise.
(iii) By Proposition 10.4 of [24], for a quadratic extension E/F of local fields of characteristic zero and residue characteristic 2, the isometry class of a unimodular hermitian lattice L is determined by the rank, the determinant $\det(L)\in F^\times/N(E^\times)$ and the O_F-ideal $\mu(L)$ generated by the values (x,x) for $x\in L$. This ideal has the form $\mathcal{P}_F^r$ with $r\geq 0$ and contains the ideal $\mathrm{tr}(O_E)\supset 2O_F$. Thus, the number of isometry classes can grow with $\mathrm{ord}_F(2)$.

Corollary 2.16. *For a relevant hermitian space V in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$, the number of G_1^V-genera of self-dual lattices in V is 2 or 1 depending on whether or not one of the exceptional cases (a) and (b) occurs at the prime $p=2$.*

The following fact will also useful.

Lemma 2.17. *Let*

$$G^V(\mathbb{A}_f)^0=\{\ g\in G^V(\mathbb{A}_f)\mid \nu(g)\in\widehat{\mathbb{Z}}^\times\ \}.$$

Then the orbit of a self-dual lattice under the action of $G^V(\mathbb{A}_f)^0$ is the same as the orbit under $G_1^V(\mathbb{A}_f)$.

Proof. The only issue is to show that, in the case where there are two $G_1^V(\mathbb{Q}_2)$-orbits of self-dual lattices in V_2, the action of $G^V(\mathbb{Q}_2)^0$ preserves these orbits. But if $g\in G^V(\mathbb{Q}_2)^0$, and L is a self-dual lattice in V_2, then the ideals $\mu(L)$ and $\mu(gL)$ in $\mathbb{Z}_2$ generated by the values (x,x) for $x\in L$ (resp. gL) are the same. By Proposition 10.4 of [24], cf. Remark 2.15, L and gL are isometric. □

Definition 2.18. Let $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ be the set of isomorphism classes of pairs $V^\sharp:=(V,[[L]])$ where V is a relevant hermitian space and $[[L]]$ is a G_1^V-genus of self-dual hermitian lattices in V.

The G_1^V-genus is determined by its type, as defined in (ii) of the preceding remark. Of course, if n is odd or if n is even and 2 is unramified in $\boldsymbol{k}$, all self-dual lattices are of type I and the natural map from $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ to $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ is a bijection.

We write

$$\mathcal{M}(n-r,r)[\frac{1}{2}]=\mathcal{M}(n-r,r)\times_{\mathrm{Spec}\,O_{\boldsymbol{k}}}\mathrm{Spec}\,O_{\boldsymbol{k}}[\frac{1}{2}]$$

and

$$\mathcal{M}[\frac{1}{2}] = \mathcal{M} \times_{\operatorname{Spec} O_{\boldsymbol{k}}} \operatorname{Spec} O_{\boldsymbol{k}}[\frac{1}{2}].$$

Proposition 2.19. *(i) There is a natural disjoint decomposition of algebraic stacks*

$$\mathcal{M}(n-r,r)[\frac{1}{2}] = \coprod_{V^\sharp \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp/\mathrm{str.sim.}} \mathcal{M}(n-r,r)[\frac{1}{2}]^{V^\sharp}.$$

(ii) There is a natural disjoint decomposition of algebraic stacks

$$\mathcal{M}[\frac{1}{2}] = \coprod_{V^\sharp \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp} \mathcal{M}[\frac{1}{2}]^{V^\sharp}.$$

For n even, this decomposition is obtained by base change from that in (i).

Proof. Away from characteristic 2, the Tate module $T_2(A)$ is a unimodular $O_{\boldsymbol{k},2}$-lattice where, as explained in the proof of Proposition 2.12, the hermitian form is well defined up to scaling by an element of $\mathbb{Z}_2^\times$. The norm, $\mu(T_2(A))$, of this lattice is thus well defined and hence so is the type of the genus of self-dual lattices in V determined by $T^p(A)$. $\square$

We will frequently abuse notation and write $\mathcal{M}(n-r,r)^{V^\sharp}$ and $\mathcal{M}^{V^\sharp}$ instead of $\mathcal{M}(n-r,r)[\frac{1}{2}]^{V^\sharp}$ and $\mathcal{M}[\frac{1}{2}]^{V^\sharp}$ when working away from characteristic 2.

2.5. We next obtain some information about the support of the cycle $\mathcal{Z}(T)$.

Lemma 2.20. *If $T \in \operatorname{Herm}_m(O_{\boldsymbol{k}})_{>0}$ for $m > n-r$, then $\mathcal{Z}(T)_{\mathbb{Q}} = \emptyset$.*

Proof. Obviously, it suffices to prove that $\mathcal{Z}(T)(\mathbb{C}) = \emptyset$, and this is part of Corollary 3.6. $\square$

We now assume that the matrix T is nonsingular of rank n. In this case more can be said.

Lemma 2.21. *Let $0 < r < n$. Let $T \in \operatorname{Herm}_n(O_{\boldsymbol{k}})_{>0}$. Then $\operatorname{supp}(\mathcal{Z}(T))$ is contained in the union over finitely many inert or ramified p of the supersingular locus of $\mathcal{M}_p$.*

Proof. Since T is nonsingular, any point $(A, \iota, \lambda; E, \iota_0, \lambda_0; \mathbf{x})$ of $\mathcal{Z}(T)$ defines an $O_{\boldsymbol{k}}$-isogeny $E^n \to A$. Over $\mathbb{C}$ such an isogeny cannot exist since the representations of $\boldsymbol{k} \otimes \mathbb{C}$ on $(\operatorname{Lie} E)^n$ and on $\operatorname{Lie} A$ are non-isomorphic by the determinant condition. Hence $\mathcal{Z}(T)$ is concentrated in the fibers at p for finitely many p. Furthermore, if p is inert or ramified, then E is supersingular and hence so is A.
It remains to exclude the case of split primes p. Let M, resp. M_0, be the Dieudonné module of A, resp. E^n, and let N, resp. N_0 be the corresponding rational Dieudonné module. The action of $O_{\boldsymbol{k}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ decomposes these Dieudonné modules into the direct sum of Dieudonné modules $M = M^1 \oplus M^2$, and $M_0 = M_0^1 \oplus M_0^2$, and similarly for the rational Dieudonné modules and the Lie algebras. An $O_{\boldsymbol{k}}$-linear isogeny $E^n \to A$ induces isomorphisms of rational Dieudonné modules $N \simeq N_0$, and $N^i \simeq N_0^i$ for $i = 1, 2$. Now $\operatorname{ord}\det(V|N^i) = \dim M^i/VM^i = \dim(\operatorname{Lie} A)^i$, and $\operatorname{ord}\det(V|N_0^i) = \dim M_0^i/VM_0^i = \dim(\operatorname{Lie} E^n)^i$. Hence $\dim(\operatorname{Lie} A)^i = \dim(\operatorname{Lie} E^n)^i$ for $i = 1, 2$. Since $(\operatorname{Lie} E^n)^i = (0)$ for one i, this contradicts the nontriviality hypothesis made on the signature. $\square$

In the case of signature $(n-1, 1)$ one can go a bit further.

Proposition 2.22. *Let $r=1$ and suppose that $T\in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$. Let V_T denote the positive-definite hermitian space $V_T=\boldsymbol{k}^n$ with hermitian form given by T. Let $\mathrm{Diff}_0(T)$ be the set of primes p that are inert in $\boldsymbol{k}$ for which $\mathrm{ord}_p(\det(T))$ is odd.*
(i) If $|\mathrm{Diff}_0(T)|>1$, then $\mathcal{Z}(T)$ is empty.
(ii) If $\mathrm{Diff}_0(T)=\{p\}$, then

$$\mathrm{supp}(\mathcal{Z}(T))\subset \mathcal{M}_p^{V,\mathrm{ss}},$$

where $V\in\mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ is the unique relevant hermitian space with $\mathrm{inv}_\ell(V)=\mathrm{inv}_\ell(V_T)$ for all finite primes $\ell\neq p$.
Here $\mathcal{M}_p^{V,\mathrm{ss}}$ denotes the supersingular locus in the fiber of $\mathcal{M}^V$ at p.
(iii) If $\mathrm{Diff}_0(T)$ is empty, then, for each $p\mid\Delta$, there is a unique relevant hermitian space $V^{(p)}\in\mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ for which $\mathrm{inv}_\ell(V^{(p)})=\mathrm{inv}_\ell(V_T)$ for all $\ell\neq p$. Then

$$\mathrm{supp}(\mathcal{Z}(T))\ \subset\ \bigcup_{p\mid\Delta}\mathcal{M}_p^{V^{(p)},\mathrm{ss}}.$$

Proof. Let $p\in\mathrm{Diff}_0(T)$, and let $(A,\iota,\lambda;E,\iota_0,\lambda_0;\mathbf{x})\in\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$. Let $V'=V'(A,E)$. Then V' can be identified with V_T, and the natural map

$$V'\otimes\mathbb{A}_f^p\to V'_{\mathbb{A}_f^p}$$

is an isomorphism. On the other hand, $V'_{\mathbb{A}_f^p}$ can be identified with $\mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(T^p(E)^0,T^p(A)^0)$ (cf. section 2.3), and therefore admits a self-dual $\hat{\mathbb{Z}}^p$-lattice. Hence $\mathrm{ord}_\ell(\det(T))$ is even for all inert $\ell\neq p$. This proves (i), and (ii) and (iii) follow from this and the previous lemma. □

3. Uniformization of the complex points

In this section, we fix an embedding τ of $\boldsymbol{k}$ into $\mathbb{C}$ and discuss the complex points of our moduli spaces $\mathcal{M}(n-r,r)$ and of the special cycles.

3.1. Let $V,(\ ,\ )$ be a hermitian vector space over $\boldsymbol{k}$ of signature $(n-r,r)$. Let $G=G^V=\mathrm{GU}(V)$ be the similitude group of V, viewed as a reductive algebraic group over $\mathbb{Q}$. Thus, for any $\mathbb{Q}$-algebra R,

$$G(R)=\{\ g\in\mathrm{End}_{\boldsymbol{k}}(V)\otimes_{\mathbb{Q}}R\mid gg^*=\nu(g)\in R^\times\ \},$$

where $*$ is the involution on $\mathrm{End}(V)$ determined by $(\ ,\ )$. Let $G_1=\mathrm{U}(V)$.

We note that

$$\nu(G(\mathbb{R}))=\begin{cases}\mathbb{R}^\times & \text{for signature } (r,r),\\ \mathbb{R}^\times_+ & \text{for signature } (n-r,r), \text{ with } r\neq n-r.\end{cases}$$

This fact distinguishes the case $n-r\neq r$ from the case $n-r=r$.

Let us first assume that $r\neq n-r$. Let $\mathtt{h}:\mathbb{C}\to\mathrm{End}_{\boldsymbol{k}}(V)\otimes\mathbb{R}$ be an $\mathbb{R}$-algebra homomorphism such that $\mathtt{h}(z)^*=\mathtt{h}(\bar z)$, and such that the form $\langle\,\mathtt{h}(i)x,y\,\rangle$ is symmetric and positive definite on $V\otimes_{\mathbb{Q}}\mathbb{R}$. Note that $\mathtt{h}(z)\mathtt{h}(z)^*=\nu(\mathtt{h}(z))=|z|^2$, so that $\mathtt{h}:\mathbb{C}^\times\to G(\mathbb{R})$. Let J_0 be the complex structure on $V_{\mathbb{R}}=V\otimes_{\mathbb{Q}}\mathbb{R}$ given by multiplication by $\sqrt{\Delta}\otimes|\Delta|^{-\frac12}$. Here $\sqrt{\Delta}\in\boldsymbol{k}$ is the element for which $\tau(\sqrt{\Delta})$ has positive imaginary part. For a homomorphism $\mathtt{h}$ as above, let U be the subspace of $V_{\mathbb{R}}$ on which $\mathtt{h}(i)=-J_0$. Then U is a complex r-plane in $(V_{\mathbb{R}},J_0)$ on which the hermitian form $(\ ,\ )$ is negative definite, and every such r-plane arises in this way. Note that $\mathtt{h}(i)=J_0$ on the positive $n-r$-plane $U^\perp$. Thus, we can identify the space of homomorphisms

$\mathtt{h} : \mathbb{S} \to G(\mathbb{R})$ of the above type with the space $D = D(V)$ of negative r-planes in $(V_{\mathbb{R}}, J_0)$. The subspace of $V_{\mathbb{C}}$ on which $\mathtt{h}(z)$ acts by z is isomorphic, as a representation of $\boldsymbol{k}$, to $(n-r)\cdot\tau + r\cdot\bar{\tau}$.

In the case of signature (r, r), the form $\langle\, \mathtt{h}(i)x, y\,\rangle$ is only required to be symmetric and definite on $V \otimes_{\mathbb{Q}} \mathbb{R}$, and we define $\mathrm{sgn}(\mathtt{h}) = \pm 1$ so that $\mathrm{sgn}(\mathtt{h})\,\langle\, \mathtt{h}(i)x, x\,\rangle \geq 0$. On the complex r-plane U in $V(\mathbb{R})$ on which $\mathtt{h}(i) = -J_0$, the hermitian form $\mathrm{sgn}(\mathtt{h})\,(\ ,\ )$ is negative definite, and we may identify the space of homomorphisms $\mathtt{h} : \mathbb{S} \to G(\mathbb{R})$ of the above type with $D = D^+ \cup D^-$, where D^ϵ is the space of r-planes that are negative for $\epsilon\,(\ ,\ )$.

3.2. Next, we describe the complex points of $\mathcal{M}(n-r, r)$. Let $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$ be the set of isomorphism classes of self-dual hermitian $O_{\boldsymbol{k}}$-lattices of signature $(n-r, r)$. There is a natural surjective map $L \mapsto L \otimes \mathbb{Q}$ from $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$ to $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$. We fix representatives V for $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ and a compatible set of representatives L for $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$. We will write $D(L) = D(L \otimes \mathbb{Q})$ for the corresponding space of negative r-planes. We also fix the trivialization $\mathbb{A}_f(1) \xrightarrow{\sim} \mathbb{A}_f$ given by the inverse of the exponential map $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mu_n(\mathbb{C})$, $a \mapsto e(a/n)$, for $e(x) = \exp(2\pi i x)$.

Suppose that (A, ι, λ) is an element of $\mathcal{M}(n-r, r)(\mathbb{C})$, and let $H = H_1(A, \mathbb{Z})$. Then, via ι, H is an $O_{\boldsymbol{k}}$-lattice of rank n with an alternating form $\langle\, ,\, \rangle_\lambda$ determined by the polarization λ. Since the adjoint with respect to $\langle\, ,\, \rangle_\lambda$ is given by $\iota(a)^* = \iota(a^\sigma)$, there is a $O_{\boldsymbol{k}}$-valued hermitian form $(\ ,\ ) = (\ ,\ )_{\lambda,\boldsymbol{k}}$ on H such that $\langle\, x, y\,\rangle_\lambda = \mathrm{tr}((x,y)/\sqrt{\Delta})$, as in (1.3). Since the polarization λ is principal, the $O_{\boldsymbol{k}}$-lattice H is self dual with respect to $(\ ,\ )_{\lambda,\boldsymbol{k}}$. By the signature condition and the canonical isomorphism $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Lie}(A)$, the hermitian lattice H has signature $(n-r, r)$. Choose an isomorphism $j : H \to L$, where L is one of our fixed representatives. Via j the complex structure on $H_{\mathbb{R}}$ corresponds to $\mathtt{h}_z(i)$ for some $z \in D(L)$. Now we eliminate the choice of j; since any two choices of j differ by an element of Γ_L, the group of isometries of L, we obtain the following result.

Proposition 3.1. *There is an isomorphism of orbifolds*

$$\mathcal{M}(n-r, r)(\mathbb{C}) \xrightarrow{\sim} \coprod_L \left[\, \Gamma_L \backslash D(L)\, \right],$$

where L runs over $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$.

Consider the special case $n = 1$ and $r = 0$. A fractional ideal $\mathfrak{a}$ defines a self-dual hermitian lattice $L_{0,\mathfrak{a}}$ in the space $V_{0,\mathfrak{a}} = \boldsymbol{k}$ with hermitian form $(x, y) = N(\mathfrak{a})^{-1}\, x y^\sigma$. This gives an isomorphism

$$C(\boldsymbol{k}) \xrightarrow{\sim} \mathcal{L}_{(1,0)}(\boldsymbol{k}), \qquad [\mathfrak{a}] \mapsto [L_{0,\mathfrak{a}}], \tag{3.1}$$

where $C(\boldsymbol{k})$ is the ideal class group of $\boldsymbol{k}$ and $[L_{0,\mathfrak{a}}]$ denotes the isomorphism class of $L_{0,\mathfrak{a}}$. Since $D(L_{0,\mathfrak{a}})$ consists of a single point and $\Gamma_{L_{0,\mathfrak{a}}} = O_{\boldsymbol{k}}^\times$, we obtain

$$\mathcal{M}_0(\mathbb{C}) \xrightarrow{\sim} \coprod_{[\mathfrak{a}] \in C(\boldsymbol{k})} \left[\, O_{\boldsymbol{k}}^\times \backslash D(L_{0,\mathfrak{a}})\, \right] \simeq \left[\, O_{\boldsymbol{k}}^\times \backslash C(\boldsymbol{k})\, \right], \tag{3.2}$$

where $O_{\boldsymbol{k}}^\times$ acts trivially on $D(L_{0,\mathfrak{a}})$ and $C(\boldsymbol{k})$.

There is a slight variant of Proposition 3.1. Consider the map $\mathcal{L}_{(n-r,r)}(\boldsymbol{k}) \longrightarrow \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ given by $L \mapsto (L \otimes_{\mathbb{Z}} \mathbb{Q}, [[L]])$. By construction, the fiber in $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$ over $(V, [[L]]) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ is in bijection with

$$G_1^V(\mathbb{Q}) \backslash G_1^V(\mathbb{A}_f) / K_1^V,$$

where K_1^V is the stabilizer in $G_1^V(\mathbb{A}_f)$ of a self-dual lattice in $[[L]]$.

Corollary 3.2. *There is an isomorphism of orbifolds*

$$\mathcal{M}(n-r,r)(\mathbb{C}) \xrightarrow{\sim} \coprod_{V^\sharp} \big[\, G_1^V(\mathbb{Q})\backslash\big(\, D(V) \times G_1^V(\mathbb{A}_f)/K_1^{V^\sharp}\,\big)\,\big],$$

where $V^\sharp$ runs over $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$. $\square$

When n is odd, we will usually drop the $\sharp$ here. On the other hand, when there are two G_1^V-genera, the group $K_1^{V^\sharp}$ depends on $V^\sharp$, not just on V.

Remark 3.3. In the special case $n=1$ and $r=0$, the decomposition

$$\mathcal{M}_0(\mathbb{C}) \xrightarrow{\sim} \coprod_{V_0} \big[\, G_1^{V_0}(\mathbb{Q})\backslash\big(\, D(V_0) \times G_1^{V_0}(\mathbb{A}_f)/K_1^{V_0}\,\big)\,\big]$$

in Corollary 3.2 corresponds to the decomposition of $C(\boldsymbol{k})$ according to genera. More precisely, the isomorphism class of the hermitian space $V_{0,\mathfrak{a}} = L_{0,\mathfrak{a}} \otimes \mathbb{Q}$ is determined by the values $\chi_p(\det V_{0,\mathfrak{a}}) = \chi_p(N(\mathfrak{a})) = (N(\mathfrak{a}),\Delta)_p$ as p runs over the primes dividing Δ. But these are just the values of the genus characters $\xi_p([\mathfrak{a}]) = (N(\mathfrak{a}),\Delta)_p$. Thus the fibers of the map $C(\boldsymbol{k}) \simeq \mathcal{L}_{(1,0)}(\boldsymbol{k}) \to \mathcal{R}_{(1,0)}(\boldsymbol{k})$ are the genera, i.e., the cosets of the subgroup $C(\boldsymbol{k})^2$. On the other hand, the inclusion of $\boldsymbol{k}^1\backslash \boldsymbol{k}^1_{\mathbb{A}_f}/\widehat{O}^1_{\boldsymbol{k}}$ into $\boldsymbol{k}^\times\backslash \boldsymbol{k}^\times_{\mathbb{A}_f}/\widehat{O}^\times_{\boldsymbol{k}} = C(\boldsymbol{k})$ identifies

$$G_1^{V_0}(\mathbb{Q})\backslash G_1^{V_0}(\mathbb{A}_f)/K_1^{V_0} = \boldsymbol{k}^1\backslash \boldsymbol{k}^1_{\mathbb{A}_f}/\widehat{O}^1_{\boldsymbol{k}}$$

with $C(\boldsymbol{k})^2$.

Remark 3.4. For a relevant hermitian space $\tilde{V}$ in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$, the set of complex points $\mathcal{M}^{\tilde{V}}(\mathbb{C})$ of the summand $\mathcal{M}^{\tilde{V}}$ of $\mathcal{M}$, in the sense of the decomposition in Proposition 2.12, corresponds to the subset of the product $\mathcal{M}(n-r,r)(\mathbb{C}) \times \mathcal{M}_0(\mathbb{C})$ indexed by pairs $(V^\sharp, V_0)$ for which $\mathrm{Hom}_{\boldsymbol{k}}(V_0, V) \simeq \tilde{V}$.

3.3. Next we consider the complex points of the special cycles. Suppose that $(A,\iota,\lambda,E,\iota_0,\lambda_0;\mathbf{x})$ is a point of $\mathcal{Z}(T)(\mathbb{C})$, and let $H = H_1(A(\mathbb{C}),\mathbb{Z})$ and $H_0 = H_1(E(\mathbb{C}),\mathbb{Z})$. There are isomorphisms $j : H \xrightarrow{\sim} L$ and $j_0 : H_0 \xrightarrow{\sim} L_0$, for relevant hermitian lattices L and L_0 in our fixed sets of representatives, and we obtain data $z \in D(L)$ and, setting $\widetilde{L} := \mathrm{Hom}_{O_{\boldsymbol{k}}}(L_0, L)$,

$$\tilde{\mathbf{x}} = j \circ \mathbf{x} \circ j_0^{-1} \in \widetilde{L}^m.$$

Here we slightly abuse notation and write $\mathbf{x}$ for the collection of homomorphisms from H_0 to H induced by $\mathbf{x}$. Note that the lattice $\widetilde{L}$ has a hermitian form $\tilde{h}$ coming from the hermitian forms h_λ and h_{λ_0} on H and H_0 arising from the polarizations, as above. The pair $(z,\tilde{\mathbf{x}})$ satisfies the following incidence relations

(1) $\tilde{h}(\tilde{\mathbf{x}},\tilde{\mathbf{x}}) = T$.
(2) $z \in D(L)_{\tilde{\mathbf{x}}}$, where, for $\tilde{\mathbf{x}} = [\tilde{x}_1,\dots,\tilde{x}_m]$,

$$D(L)_{\tilde{\mathbf{x}}} = \{\, z \in D(L) \mid z \perp \tilde{x}_i(L_0) \ \text{ for all } i \,\}.$$

Note that condition (2) corresponds to the fact that the $\boldsymbol{k}$-linear maps $x_i : H_{0,\mathbb{R}} = \mathrm{Lie}\, E \to H_{\mathbb{R}} = \mathrm{Lie}\, A$ are holomorphic.

Let

$$\mathrm{Inc}_\infty(T; L, L_0) \quad \subset \quad D(L) \times \widetilde{L}^m$$

be the set of pairs satisfying conditions (1) and (2). The complex uniformization of the special cycles is now given by the following proposition.

Proposition 3.5. *Let $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$. Then there is an isomorphism of orbifolds*

$$\mathcal{Z}(T)(\mathbb{C}) \simeq \coprod_{[L]\in\mathcal{L}_{(n-r,r)}} \coprod_{[L_0]\in\mathcal{L}_{(1,0)}} \big[\, (O_{\boldsymbol{k}}^\times \times \Gamma_L)\backslash \mathrm{Inc}_\infty(T; L, L_0)\,\big].$$

Here $O_{\boldsymbol{k}}^\times$ acts by scalar multiplication on the $\tilde{\mathbf{x}}$-component of an element of $\mathrm{Inc}_\infty(T; L, L_0)$.

This can be written more explicitly. For a fractional ideal $\mathfrak{a}$ and a lattice L in $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$, let $L_{\mathfrak{a}} = \mathfrak{a} \otimes_{O_{\boldsymbol{k}}} L$, with hermitian form

$$(x, x)_{\mathfrak{a}} = \frac{(x, x)}{N(\mathfrak{a})}. \tag{3.3}$$

Then, $L_{\mathfrak{a}}$ is again a self-dual lattice,

$$L_{\mathfrak{a}} \simeq \mathrm{Hom}_{O_{\boldsymbol{k}}}(L_{0,\mathfrak{a}^{-1}}, L) \tag{3.4}$$

as hermitian lattices, and we have

$$\mathcal{Z}(T)(\mathbb{C}) \simeq \coprod_{[\mathfrak{a}]\in C(\boldsymbol{k})} \coprod_{[L]\in\mathcal{L}_{(n-r,r)}} \Bigg[(O_{\boldsymbol{k}}^\times \times \Gamma_L)\backslash \coprod_{\substack{\mathbf{x}\in L_{\mathfrak{a}}^m \\ (\mathbf{x},\mathbf{x})_{\mathfrak{a}}=T}} D(L)_{\mathbf{x}} \Bigg]. \tag{3.5}$$

If $\tilde{\mathbf{x}} \in (\tilde{L} \otimes \mathbb{R})^m$ with $\tilde{h}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) = T > 0$, then $m \le n - r$ and $D(L)_{\tilde{\mathbf{x}}}$ has codimension mr in $D(L)$.

Corollary 3.6. *For $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})_{>0}$, and $m \le n-r$, the cycle $\mathcal{Z}(T)(\mathbb{C})$ has codimension mr. If $m > n - r$, then $\mathcal{Z}(T)(\mathbb{C})$ is empty.*

Remark 3.7. The cycles occurring here are essentially those studied in the joint work of the first author with John Millson. More precisely, let $\mathrm{pr} : D(L) \to \Gamma_L\backslash D(L)$ be the projection. Then the cycles

$$Z(T; L_{\mathfrak{a}}, \Gamma_L) = \coprod_{\substack{\mathbf{x}\in L_{\mathfrak{a}}^m \\ (\mathbf{x},\mathbf{x})_{\mathfrak{a}}=T \\ \mathrm{mod}\ \Gamma_L}} \mathrm{pr}(D(L)_{\mathbf{x}})$$

of codimension mr in $\Gamma_L\backslash D(L)$ are those introduced in [33], [34], [35], and, in the case of signature $(2, 1)$, in [28].

There is an alternative version, where, with the notation as above, we fix isomorphisms $j_{\mathbb{Q}} : H_{\mathbb{Q}} \xrightarrow{\sim} V$ and $j_{0,\mathbb{Q}} : H_{0,\mathbb{Q}} \xrightarrow{\sim} V_0$. The lattice $j_{\mathbb{Q}}(H)$ determines a G_1^V-genus $[[L]]$ in V, i.e., an element $V^\sharp$ in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$. The lattices $j_{\mathbb{Q}}(H)$ and $j_{0,\mathbb{Q}}(H_0)$ correspond to cosets $gK_1^{V^\sharp}$ in $G_1^V(\mathbb{A}_f)$ and $g_0K_1^{V_0}$ in $G_1^{V_0}(\mathbb{A}_f)$, respectively, and so, we obtain a collection

$$(z, gK_1^{V^\sharp}, g_0K_1^{V_0}, \tilde{\mathbf{x}}) \in D(V) \times G_1^V(\mathbb{A}_f)/K_1^{V^\sharp} \times G_1^{V_0}(\mathbb{A}_f)/K_1^{V_0} \times \widetilde{V}(\mathbb{Q})^m,$$

where

$$\widetilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0, V), \qquad \tilde{\mathbf{x}} = j \circ \mathbf{x} \circ j_0^{-1}.$$

The data $(z, gK_1^{V^\sharp}, g_0K_1^{V_0}, \tilde{\mathbf{x}})$ satisfies the following incidence relations.

(0) $\tilde{\mathbf{x}} \in (g(\tilde{L} \otimes \widehat{\mathbb{Z}})g_0^{-1})^m$, where $\tilde{L} = \mathrm{Hom}_{O_{\boldsymbol{k}}}(L_0, L)$.
(1) $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) = T$.
(2) $z \in D(V)_{\tilde{\mathbf{x}}}$, where

$$D(V)_{\tilde{\mathbf{x}}} = \{\, z \in D(V) \mid z \perp \tilde{x}_i(V_0) \ \text{ for all } i \,\}.$$

For given relevant spaces $V^\sharp \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ and $V_0 \in \mathcal{R}_{(1,0)}(\boldsymbol{k})$, let

$$\mathrm{Inc}_\infty(T; V^\sharp, V_0) \;\subset\; D(V) \times G_1^V(\mathbb{A}_f)/K_1^{V^\sharp} \times G_1^{V_0}(\mathbb{A}_f)/K_1^{V_0} \times \widetilde{V}(\mathbb{Q})^m,$$

be the subset of collections satisfying conditions (0), (1) and (2).

Proposition 3.8. *Let $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$. Then there is an isomorphism of orbifolds*

$$\mathcal{Z}(T)(\mathbb{C}) \simeq \coprod_{V^\sharp \in \mathcal{R}^\sharp_{(n-r,r)}} \coprod_{V_0 \in \mathcal{R}_{(1,0)}} \big[\, (G_1^V(\mathbb{Q}) \times G_1^{V_0}(\mathbb{Q}))\backslash \mathrm{Inc}_\infty(T; V^\sharp, V_0)\,\big].$$

□

4. Relation to Shimura varieties

In this section, we discuss the relation between our moduli space $\mathcal{M}(n-r,r)$ and Shimura varieties for unitary similitude groups. When $n > 1$, we assume that $r(n-r) > 0$, and we fix an embedding τ of $\boldsymbol{k}$ into $\mathbb{C}$.

4.1. We fix a hermitian vector space V over $\boldsymbol{k}$ of signature $(n-r, r)$ and write $G = G^V = \mathrm{GU}(V)$. For an open compact subgroup $K \subset G(\mathbb{A}_f)$, there is a Shimura variety Sh_K^V over[2] $\boldsymbol{k}$ with

$$\mathrm{Sh}_K^V(\mathbb{C}) \simeq G(\mathbb{Q})\backslash\big(D \times G(\mathbb{A}_f)/K\big). \tag{4.1}$$

Remark 4.1. We note that two hermitian spaces $V, (\ ,\ )$ and $V', (\ ,\ )'$ which are strictly similar, i.e., $V \simeq V'$ with $(\ ,\ )' = c\ (\ ,\ )$ for $c \in \mathbb{Q}_+^\times$, define the same Shimura varieties.

4.2. These Shimura varieties are related to the moduli stacks of an 'up-to-isogeny' moduli problem [8, 27]. For a hermitian space V over $\boldsymbol{k}$ with signature $(n-r, r)$ and a compact open subgroup K in $G^V(\mathbb{A}_f)$, we define a groupoid $\underline{\mathrm{Sh}}_K^V$ fibered over the category of locally noetherian $\boldsymbol{k}$-schemes. For a locally noetherian $\boldsymbol{k}$-scheme S, the objects of $\underline{\mathrm{Sh}}_K^V(S)$ are collections $(A, \iota, \lambda, \bar\eta)$, where

(1) A is an abelian scheme over S, up to isogeny, with an action of $\boldsymbol{k}$, $\iota : \boldsymbol{k} \to \mathrm{End}^0(A)$,
(2) λ is a polarization[3],
(3) $\bar\eta$ is a K-level structure, i.e., a K-orbit of $\boldsymbol{k} \otimes \mathbb{A}_f$-linear isomorphisms

$$\eta : T(A)^0 \to V(\mathbb{A}_f)$$

such that the polarization form on $T(A)^0$ and the symplectic form $\langle\, ,\, \rangle$ on $V(\mathbb{A}_f)$ coincide up to a scalar in $\mathbb{A}_f^\times$.

In addition, the action ι is supposed to be compatible with the polarization, i.e., for $a \in \boldsymbol{k}$, $\iota(a)^* = \iota(a^\sigma)$, where $*$ is the Rosati involution on $\mathrm{End}^0(A)$ determined by λ. Finally, the action of $\boldsymbol{k}$ is supposed to satisfy the determinant condition of type $(n-r, r)$ (Kottwitz condition):

$$\det(T - \iota(a) \mid \mathrm{Lie}\,(A)) = (T - \varphi(a))^{n-r}(T - \varphi(a^\sigma))^r \in \mathcal{O}_S[T], \tag{4.2}$$

where $\varphi : \boldsymbol{k} \to \mathcal{O}_S$ is the structure homomorphism.

---

[2]In the case where n is even and $r = n-r$, the reflex field is $\mathbb{Q}$, and we take Sh_K^V to be the base change to $\boldsymbol{k}$ of the usual canonical model.

[3] By this we mean that λ is an isomorphism in the isogeny category, and that for some $a \in \mathbb{Q}_+^\times$, $a\lambda$ is a polarization in the sense of Deligne, [8], p.145. Specifically, for an abelian variety A up to isogeny over a field k, a polarization is an element of $\mathrm{NS}(A) \otimes \mathbb{Q}$ such that a positive multiple is defined by a projective embedding. Note that $\mathrm{NS}(A) \otimes \mathbb{Q}$ can be identified with the symmetric elements in $\mathrm{Hom}(A, A^\vee) \otimes \mathbb{Q}$. This identification is being made here.

The morphisms between two objects $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$ in $\underline{\mathrm{Sh}}_K^V(S)$ are given by $\boldsymbol{k}$-linear isogenies $\mu : A \to A'$ carrying $\bar{\eta}$ to $\bar{\eta}'$ and carrying λ to a $\mathbb{Q}_+^\times$- multiple of λ'. All such morphisms are isomorphisms.

Remark 4.2. Let us be more precise about (3). For this we recall the discussion on p.390-91 of [27]. Suppose that (A, ι, λ) is an abelian scheme over a connected base S, up to isogeny, together with a polarization λ and a compatible $\boldsymbol{k}$-action. The rational Tate module of A is then a smooth $\mathbb{A}_f$-sheaf on S. For a fixed geometric point s of S, the rational Tate module of A is determined by the rational Tate module $H_1(A_s, \mathbb{A}_f)$ viewed as a $\pi_1(S, s)$-module. The polarization induces an alternating form on $H_1(A_s, \mathbb{A}_f)$ valued in $\mathbb{A}_f(1) = \widehat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathbb{Q}$, where

$$\widehat{\mathbb{Z}}(1) = \varprojlim_n \mu_n(k(s)), \tag{4.3}$$

for $k(s)$ the residue field of s. If we fix an isomorphism

$$\widehat{\mathbb{Z}}(1) \xrightarrow{\sim} \widehat{\mathbb{Z}}, \tag{4.4}$$

we obtain an $\mathbb{A}_f$-valued alternating form on $H_1(A_s, \mathbb{A}_f)$ and a corresponding $\boldsymbol{k} \otimes \mathbb{A}_f$-valued hermitian form, since the polarization is compatible with the $\boldsymbol{k}$-action. A change in the isomorphism (4.4) results in a scaling of these forms by an element of $\widehat{\mathbb{Z}}^\times$. A K-level structure is a K-orbit $\bar{\eta}$ in the set of $\boldsymbol{k} \otimes \mathbb{A}_f$-linear isomorphisms

$$\eta : H_1(A_s, \mathbb{A}_f) \xrightarrow{\sim} V(\mathbb{A}_f)$$

which preserve the hermitian forms up to a scalar in $\mathbb{A}_f^\times$. Finally, the group $\pi_1(S, s)$ acts on $H_1(A_s, \mathbb{A}_f)$ by similitudes of the alternating form, and it is required that the orbit $\bar{\eta}$ be fixed by this action. As a result, the notion of K-level structure is independent of the choice of the geometric point s and of the trivialization (4.4).

A 'p-integral version' of the following proposition is proved in [27], and the proof transposes easily to the situation considered here, comp. also [8].

Proposition 4.3. *$\underline{\mathrm{Sh}}_K^V$ is a smooth Deligne-Mumford stack over* $\operatorname{Spec} \boldsymbol{k}$*. For K sufficiently small, $\underline{\mathrm{Sh}}_K^V$ is a quasi-projective scheme over $\boldsymbol{k}$, naturally isomorphic to the canonical model of the Shimura variety Sh_K^V.*
In particular, if K is sufficiently small, the set of $\mathbb{C}$-points $\underline{\mathrm{Sh}}_K^V(\mathbb{C})$ of $\underline{\mathrm{Sh}}_K^V$, via the fixed embedding τ of $\boldsymbol{k}$ into $\mathbb{C}$, is canonically identified with the double coset space $\mathrm{Sh}_K^V(\mathbb{C})$ of (4.1). When K is not sufficiently small, then $\underline{\mathrm{Sh}}_K^V(\mathbb{C})$ is canonically identified with the space

$$\left[\, G(\mathbb{Q}) \backslash D(V) \times G(\mathbb{A}_f)/K \,\right]$$

viewed as an orbifold.

4.3. The relation between the Shimura variety $\underline{\mathrm{Sh}}_K^V$ and our moduli stack $\mathcal{M}(n-r, r)$ is now given by the following result.

Proposition 4.4. *Let $V^\sharp \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ be a relevant hermitian space. Then there is an isomorphism of stacks over $\boldsymbol{k}$,*

$$\mathcal{M}(n-r, r)^{V^\sharp} \times_{\operatorname{Spec} O_{\boldsymbol{k}}} \operatorname{Spec} \boldsymbol{k} \simeq \underline{\mathrm{Sh}}_K^V,$$

where $\mathcal{M}(n-r, r)^{V^\sharp}$ is the component of $\mathcal{M}(n-r, r)$ associated to the strict similarity class of $V^\sharp$ in (i) of Proposition 2.19 and K is the stabilizer in $G^V(\mathbb{A}_f)$ of a self-dual lattice in the G_1^V-genus given by $V^\sharp$.

Remark 4.5. Note the stacks on both sides of this isomorphism depend only on the strict similitude class of V and recall that, when n is odd, there is only one such class.

Proof. For a connected locally noetherian scheme S over $\boldsymbol{k}$, let $\xi = (A, \iota, \lambda)$ be an object of $\mathcal{M}(n-r,r)^{V^\sharp}(S)$. We view A as an abelian scheme up to isogeny with polarization given by λ and we extend ι to an action of $\boldsymbol{k}$. In order to complete the definition of the corresponding object of $\underline{\mathrm{Sh}}_K^V(S)$, we need to define the level structure $\bar{\eta}$. Since ξ lies in $\mathcal{M}(n-r,r)^{V^\sharp}(S)$, and after choosing a trivialization (4.4) of the roots of unity, we have an $\boldsymbol{k} \otimes \mathbb{A}_f$-linear similitude

$$j_{\mathbb{A}_f} : T(A)^0 \xrightarrow{\sim} V(\mathbb{A}_f),$$

unique up to an element of $G^V(\mathbb{A}_f)^0$. The image $j_{\mathbb{A}_f}(T(A))$ is a self-dual lattice in V in the genus $[[L]]$ given by $V^\sharp$, and so, adjusting by an element of $G(\mathbb{A}_f)^0$ if necessary, we can assume that $j_{\mathbb{A}_f}(T(A)) = L \otimes \widehat{\mathbb{Z}}$. Let $\eta = j_{\mathbb{A}_f}$; the K-orbit $\bar{\eta}$ of η is then uniquely determined, and the collection $(A, \iota, \lambda, \bar{\eta})$ is an object in $\underline{\mathrm{Sh}}_K^V(S)$.

Conversely, if an object $(B, \iota, \lambda, \bar{\eta})$ of $\underline{\mathrm{Sh}}_K^V(S)$ is given, the $O_{\boldsymbol{k}} \otimes \widehat{\mathbb{Z}}$-lattice

$$(\eta)^{-1}(L \otimes \widehat{\mathbb{Z}}) \subset T(B)^0$$

is independent of the choice of η in the K-orbit. There is a unique abelian scheme A over S, equipped with a quasi-isogeny with B, such that

$$T(A) = (\eta)^{-1}(L \otimes \widehat{\mathbb{Z}}).$$

Moreover, there is a unique $a \in \mathbb{Q}_+^\times$ such that $a\lambda$ is a principal polarization of A. To see this, note that, under η, the $O_{\boldsymbol{k}} \otimes \mathbb{A}_f$-valued hermitian forms on $T(B)$ and $V(\mathbb{A}_f)$ coincide up to a scalar in $(\mathbb{A}_f)^\times$. Of course, this scalar is only well defined up to an element of $\widehat{\mathbb{Z}}^\times$. In any case, by passing to $a\lambda$, we can arrange it so that this scalar lies in $\widehat{\mathbb{Z}}^\times$ and hence $a\lambda$ defines a principal polarization of A, since $T(A)$ correponds to $L \otimes \widehat{\mathbb{Z}}$ and hence is now self-dual. The given action of $\boldsymbol{k}$ on B defines an action of $O_{\boldsymbol{k}}$ on A, since $T(A)$ is an $O_{\boldsymbol{k}}$-lattice. The Kottwitz condition on $(B, \iota, \lambda, \bar{\eta})$ implies the signature condition on (A, ι, λ). Furthermore, by construction $T(A)^0$ is similar to $V \otimes \mathbb{A}_f$, so that $V(\xi)$ is strictly similar to V, again by the signature condition. Thus, we obtain a collection $(A, \iota, a\lambda)$ of $\mathcal{M}(n-r,r)^{V^\sharp}(S)$. ∎

4.4. In this section, we review the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\boldsymbol{k})$ on the connected components of Sh_K^V and of $\mathcal{M}(n-r,r)$. Here we will work with the canonical model over the reflex field E, where $E = \mathbb{Q}$, if $r = n-r$, and $E = \boldsymbol{k}$ otherwise.

Let T be the torus over $\mathbb{Q}$ given by

$$T = \begin{cases} T^1 \times \mathbb{G}_m & \text{if } n \text{ is even,} \\ \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}(\mathbb{G}_{m,\boldsymbol{k}}) & \text{if } n \text{ is odd,} \end{cases}$$

where $T^1 = \ker(N : \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}(\mathbb{G}_{m,\boldsymbol{k}}) \to \mathbb{G}_m)$. For a fixed hermitian space V of signature $(n-r,r)$ with $G = G^V = \mathrm{GU}(V)$, we define a surjective homomorphism $\nu^o : G \to T$ by

$$\nu^o(g) = \begin{cases} (\frac{\det(g)}{\nu(g)^k}, \nu(g)) & \text{if } n = 2k \text{ is even,} \\ \frac{\det(g)}{\nu(g)^k} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Note that, in the odd case $N(\nu^o(g)) = \nu(g)$. In particular, $\ker(\nu^o) = \mathrm{SU}(V)$ in both cases. Then we have

$$\pi_0(\mathrm{Sh}_K^V) \simeq T(\mathbb{Q}) \backslash T(\mathbb{A}) / \nu^o(K_\infty \times K) = T(\mathbb{Q})^0 \backslash T(\mathbb{A}_f) / \nu^o(K), \tag{4.5}$$

where $T(\mathbb{Q})^0 = T(\mathbb{Q}) \cap T(\mathbb{R})^0 T(\mathbb{A}_f)$ and K_∞ is the centralizer of $\mathtt{h}$ in $G(\mathbb{R})$. Note that $\nu^o(K_\infty) = T(\mathbb{R})^0$, the identity component of $T(\mathbb{R})$.

Next, we recall the standard description of the action of the Galois group on this set, cf. [49], for example. For any $\mathtt{h} : \mathbb{S} \to G(\mathbb{R})$, a simple calculation shows that

$$\nu^o \circ \mathtt{h}(z) = \begin{cases} \big(\big(\frac{z}{\bar{z}}\big)^{k-r}, z\bar{z}\big) & \text{if } n = 2k \text{ is even,} \\ \big(\frac{z}{\bar{z}}\big)^{k-r} \ z & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Moreover, writing $\mathbb{S} = \mathbb{S}_{\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{R}$ where $\mathbb{S}_{\mathbb{Q}} = \mathrm{R}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$, there is a homomorphism of the form (cf. [49], 1.c),

$$\rho = N_{E/\mathbb{Q}} \circ \nu^o \circ \mu_h : \mathbb{S}_{\mathbb{Q}} \longrightarrow T,$$

given by[4]

$$\rho(a) = \begin{cases} \big(\big(\frac{a}{\bar{a}}\big)^{k-r}, a\bar{a}\big) & \text{if } n = 2k \text{ is even and } r \neq n-r, \\ (1, a) & \text{if } n = 2k \text{ is even and } r = n-r, \\ \big(\frac{a}{\bar{a}}\big)^{k-r} \ a & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Finally, the action of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ on $\pi_0(\mathrm{Sh}_K^V)$ is given, on the right side of (4.5), by multiplication by $\rho(x_\sigma)$, where $x_\sigma \in \mathbb{A}_E^\times$ is an element whose image under the Artin reciprocity map is $\sigma | E^{\mathrm{ab}}$. As in [49], we normalize this map so that a local uniformizer corresponds to the inverse of the Frobenius.

Part II: The supersingular locus

5. Uniformization of the supersingular locus

Let p be a prime inert in $\boldsymbol{k}$. In this section we review the p-adic uniformization of $\mathcal{M}(n-r,r)$ along the supersingular locus of its reduction, following the procedure of Chapter 6 of [52].

Let $\mathbb{F} = \bar{\mathbb{F}}_p$ and let $W = W(\mathbb{F})$ with a fixed embedding τ of $\boldsymbol{k}_p$.

As in [63] and [39], we fix a supersingular p-divisible formal group $\mathbb{X}$ over $\mathbb{F}$ of dimension n and height $2n$ with an action ι of $O_{\boldsymbol{k}}$ satisfying the determinant condition of type $(n-r,r)$ and a p-principal polarization $\lambda_{\mathbb{X}}$ for which the Rosati involution $*$ satisfies $\iota(a)^* = \iota(a^\sigma)$. The collection $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ is unique up to isomorphism.

Let $\mathcal{N} = \mathcal{N}(n-r,r)$ be the functor on $\mathrm{Nilp} = \mathrm{Nilp}_W$ whose value on $S \in \mathrm{Nilp}$ is the set of isomorphism classes of collections $\xi = (X, \iota, \lambda_X, \rho_X)$ where X is a p-divisible group over S with an $O_{\boldsymbol{k}}$-action satisfying the determinant condition of type $(n-r,r)$ and λ_X is a p-principal polarization compatible with the $O_{\boldsymbol{k}}$-action. Finally,

$$\rho_X : X \times_W \mathbb{F} \longrightarrow \mathbb{X} \times_{\mathbb{F}} \bar{S}$$

is an $O_{\boldsymbol{k}}$-equivariant quasi-isogeny of height 0 such that $\rho_X^\vee \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Z}_p^\times$-multiple of λ_X in $\mathrm{Hom}_{O_{\boldsymbol{k}}}(X, X^\vee) \otimes \mathbb{Q}$. Here $\bar{S} = S \times_W \mathbb{F}$. An isomorphism between two collections $\xi = (X, \iota, \lambda_X, \rho_X)$ and $\xi' = (X', \iota', \lambda_{X'}, \rho_{X'})$ is an $O_{\boldsymbol{k}}$-linear isomorphism $\alpha : X \to X'$, compatible with ρ_X and $\rho_{X'}$ over $\bar{S}$, such that $\alpha^*(\lambda_{X'}) \in \mathbb{Z}_p^\times \lambda_X$.

[4] In [49], eq. (1.15), the exponent -1 in the first factor should be eliminated.

In fact, for our global construction, it will be convenient to choose $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ as follows. Suppose that $\xi^o = (A^o, \iota^o, \lambda^o)$ of $\mathcal{M}(n-r, r)(\mathbb{F})$ lies in the supersingular locus, and let $(\mathbb{X}, \iota, \lambda_{\mathbb{X}}) = (X(A^o), \iota^o, \lambda_{X(A^o)})$ be the corresponding p-divisible group with its additional structure.

We also fix a trivialization of the prime-to-p roots of unity over $\mathbb{F}$,

$$\widehat{\mathbb{Z}}^p(1) \simeq \widehat{\mathbb{Z}}^p. \tag{5.1}$$

Then the Weil pairing on $T^p(A^o)^0$ takes values in $\mathbb{A}_f^p(1) \simeq \mathbb{A}_f^p$, and there is a unique relevant space $V \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ such that $T^p(A^o)^0$ is isometric to $V \otimes \mathbb{A}_f^p$. If $p \neq 2$, there is a unique $V^\sharp = (V, [[L]]) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$ such that $T^p(A^o)^0$ is isometric to $V \otimes \mathbb{A}_f^p$, and the Tate module $T_2(A^o)$ determines the type of the genus $[[L]]$ of self-dual lattices, cf. the proof of Proposition 2.19.

For the rest of this section, we will discuss only the case $p \neq 2$ and leave it to the reader to make the slight modifications needed in the $p = 2$ case.

Lemma 5.1. *For each $V^\sharp = (V, [[L]]) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$, the supersingular locus in $\mathcal{M}(n-r, r)^{V^\sharp}(\mathbb{F})$ is non-empty.*

Proof. Fix a supersingular elliptic curve E over $\mathbb{F}$ and an embedding $O_{\boldsymbol{k}} \hookrightarrow \mathrm{End}(E) = O_B$, where B is the quaternion algebra over $\mathbb{Q}$ ramified at ∞ and p. We assume that $O_{\boldsymbol{k}}$ acts on $\mathrm{Lie}\,(E)$ via the standard map of $O_{\boldsymbol{k}}$ to $\mathbb{F}$ and we give E its canonical principal polarization. The corresponding Rosati involution on B is the main involution $b \mapsto b'$. Let $A = E^n$ so that $\mathrm{End}(A) = M_n(O_B)$ and let λ_0 be the product polarization. The Rosati involution is then $b \mapsto {}^t b'$. Define $\iota : O_{\boldsymbol{k}} \longrightarrow \mathrm{End}(A)$ by

$$\iota(a) = \mathrm{diag}(\underbrace{a, \dots, a}_{n-r}, \underbrace{\bar{a}, \dots, \bar{a}}_{r}).$$

Then (A, ι, λ_0) gives a point in the supersingular locus of $\mathcal{M}(n-r, r)(\mathbb{F})$. For our fixed trivialization (5.1), the procedure above yields a relevant hermitian space $V(\lambda_0) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})$. We next show that, by modifying the choice of the polarization and performing an isogeny, we can obtain any given $V^\sharp = (V, [[L]])$. Suppose that a relevant space $V^\sharp = (V, [[L]])$ is given. Any polarization λ on A can be written as $\lambda = \lambda_0 \circ \beta$, for $\beta \in M_n(O_B) \cap \mathrm{GL}_n(B)$ with ${}^t\beta' = \beta > 0$, and the corresponding Rosati involution is given by $b \mapsto \beta {}^t b' \beta^{-1}$. This will have the required compatibility with ι precisely when β centralizes $\iota(O_{\boldsymbol{k}})$. By (2.6), the hermitian forms h_λ and h_{λ_0} defined on $T^p(A)^0$ by λ and λ_0 are related by

$$h_\lambda(x, y) = h_{\lambda_0}(x, \beta y). \tag{5.2}$$

Choose a prime $q \nmid \Delta$ such that $(\Delta, q)_\ell = \mathrm{inv}_\ell(V)\,\mathrm{inv}_\ell(V(\lambda_0))$ for all $\ell \mid \Delta$ and such that $q \equiv 1 \mod 8$, if $2 \nmid \Delta$. In particular, since V and $V(\lambda_0)$ both have signature $(n-r, r)$ and both contain a unimodular lattice, their invariants agree at infinity and at all unramified finite primes, and hence

$$\prod_{\ell \mid \Delta} (\Delta, q)_\ell = 1. \tag{5.3}$$

It follows that $(\Delta, q)_q = 1$ as well so that q is split in $\boldsymbol{k}$. Let $i(q) = \mathrm{diag}(q, 1, \dots, 1) \in \mathrm{End}(A)$, and let $\lambda = \lambda_0 \circ i(q)$ be the resulting (non-principal) polarization. Let $V(\lambda)$ be the hermitian space of signature $(n-r, r)$ with an isometry $V(\lambda) \otimes_{\mathbb{Q}} \mathbb{A}_f^p \simeq T^p(A)^0$ where the hermitian form on $T^p(A)^0$ is defined by λ. By (5.2), the invariants of $V(\lambda)$ satisfy $\mathrm{inv}_\ell(V(\lambda)) = (\Delta, q)_\ell\,\mathrm{inv}_\ell(V(\lambda_0)) = \mathrm{inv}_\ell(V)$ for all $\ell \neq p$ and hence $\mathrm{inv}_p(V(\lambda)) = \mathrm{inv}_p(V)$ as well. Fixing an isometry $V \simeq V(\lambda)$, we obtain an isometry $\psi : V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \simeq T^p(A)^0$. We can choose an abelian variety B over $\mathbb{F}$, isogenous to A by a prime to p isogeny such that $T^p(B) \subset T^p(A)^0$ is the image of $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$ under ψ. The collection (B, ι, λ) then defines an element of the supersingular locus of $\mathcal{M}(n-r,r)^{V^\sharp}(\mathbb{F})$. $\square$

For a given $V^\sharp = (V, [[L]]) \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$, we fix a base point $\xi^o = (A^o, \iota^o, \lambda^o)$ lying in the supersingular locus of $\mathcal{M}(n-r,r)^{V^\sharp}(\mathbb{F})$.

We fix a self-dual lattice $L \in V$ in the given G_1^V-genus and an isomorphism

$$\eta^o : T^p(A^o)^0 \xrightarrow{\sim} V(\mathbb{A}_f^p) \tag{5.4}$$

such that (i) η^o is an isometry and (ii) $\eta^o(T^p(A^o)) = L \otimes \widehat{\mathbb{Z}}^p$. Let K^p be the stabilizer of L in $G(\mathbb{A}_f^p)$, and note that K^p is a subgroup of

$$G(\mathbb{A}_f^p)^0 = \{\ g \in G(\mathbb{A}_f^p) \mid \nu(g) \in (\widehat{\mathbb{Z}}^p)^\times\ \}.$$

We choose a lift $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ to W and let $\widetilde{A}^o$ be the corresponding lift of A^o. Then there is a canonical isomorphism

$$\tilde{\eta}^o : T^p(\widetilde{A}^o)^0 \longrightarrow T^p(A^o)^0 \xrightarrow{\eta^o} V(\mathbb{A}_f^p). \tag{5.5}$$

Using these objects, we can define a morphism of functors on Nilp_W

$$\Theta : \mathcal{N} \times G(\mathbb{A}_f^p)^0 \ \longrightarrow\ \mathcal{M}(n-r,r) \tag{5.6}$$

as follows. For $S \in \mathrm{Nilp}_W$, let $\widetilde{A}_S^o = \widetilde{A}^o \times_W S$. Note that, over the special fiber $\bar{S} = S \times_W \mathbb{F}$, there is a canonical isomorphism

$$\widetilde{A}_S^o \times_W \mathbb{F} = A^o \times_{\mathbb{F}} \bar{S}.$$

Thus there is an $O_{\boldsymbol{k}}$-action

$$\iota_{\bar{S}}^o = \iota^o : O_{\boldsymbol{k}} \longrightarrow \mathrm{End}_{\bar{S}}(A^o \times \bar{S}) = \mathrm{End}(A^o),$$

and a polarization

$$\lambda_{\bar{S}}^o = \lambda^o \times 1_{\bar{S}} : A^o \times \bar{S} \longrightarrow A^{o\vee} \times \bar{S}.$$

By 'rigidity', there are unique extensions of these to $\widetilde{A}_S^o$, i.e., there is an $O_{\boldsymbol{k}}$-action by quasi-isogenies,

$$\tilde{\iota}_S^o : O_{\boldsymbol{k}} \longrightarrow \mathrm{End}_S^0(\widetilde{A}_S^o) = \mathrm{End}^0(A^o),$$

and a (quasi-) polarization

$$\widetilde{\lambda}_S^o : \widetilde{A}_S^o \longrightarrow (\widetilde{A}_S^o)^\vee.$$

Finally, there is an isomorphism

$$\tilde{\eta}_S^o : T^p(\widetilde{A}_S^o)^0 \longrightarrow T^p(A^o)^0 \xrightarrow{\eta^o} V(\mathbb{A}_f^p)$$

derived from (5.5). Note that $(\widetilde{A}_S^o, \tilde{\iota}_S^o, \widetilde{\lambda}_S^o)$ need not be an element of $\mathcal{M}(n-r,r)(S)$, since $\tilde{\iota}_S^o$ (resp. $\widetilde{\lambda}_S^o$) is only a quasi-action (resp. quasi-polarization).

Remark 5.2. The choice of trivialization (5.1) of the prime-to-p roots of unity over $\mathbb{F}$ made above gives a canonical choice of such a trivialization at all geometric points of any scheme $S \in \mathrm{Nilp}_W$. The isomorphism $\tilde{\eta}_S^o$ is always an isometry for this choice.

Note that the restriction to $\bar{S}$ of the p-divisible group of $(\widetilde{A}_S^o, \tilde{\iota}_S^o, \widetilde{\lambda}_S^o)$ is canonically identified with $(\mathbb{X}_{\bar{S}}, \iota, \lambda_{\mathbb{X},\bar{S}})$. Also note that all of these constructions are functorial in S.

Proposition 5.3. *For a given $S \in \mathrm{Nilp}_W$, let $(\widetilde{A}^o_S, \widetilde{\iota}^o_S, \widetilde{\lambda}^o_S, \tilde{\eta}^o_S)$ be the collection of objects just defined.*
For each object $\xi = (X, \iota, \lambda_X, \rho_X)$ in $\mathcal{N}(S)$ and coset $gK^p \in G(\mathbb{A}^p_f)^0/K^p$, there is an object $\Theta(\xi, gK^p) = (A, \iota, \lambda)$ of $\mathcal{M}(n-r, r)(S)$ and a $O_{\boldsymbol{k}}$-linear quasi-isogeny $\phi : A \to \widetilde{A}^o_S$ uniquely characterized by the following properties:
(i) The polarization λ agrees with $\phi^\vee \circ \widetilde{\lambda}^o_S \circ \phi$.
(ii) Let
$$\eta = \tilde{\eta}^o_S \circ \phi_* : T^p(A)^0 \longrightarrow V(\mathbb{A}^p_f).$$
(Note that the map η is a symplectic isomorphism, by (i) and Remark 5.2.) Then
$$\eta(T^p(A)) = g \cdot (L \otimes \widehat{\mathbb{Z}}^p).$$

(iii) Let $(X(A), \iota)$ be the p-divisible group of (A, ι) with $O_{\boldsymbol{k}}$-action. Then there is an isomorphism
$$i : (X(A), \iota) \xrightarrow{\sim} (X, \iota)$$
such that the quasi-isogeny ϕ induces ρ_X over $\bar{S}$, i.e., the diagram
$$\begin{array}{ccc} (X(A), \iota) \times_W \mathbb{F} & \xrightarrow{\sim} & (X, \iota) \times_W \mathbb{F} \\ & {\scriptstyle \phi_*}\searrow & \downarrow \rho_X \\ & & (\mathbb{X}_{\bar{S}}, \iota) \end{array}$$
commutes. The maps $\lambda_{X(A)}$ and $i^*(\lambda_X)$ agree up to a factor in $\mathbb{Z}_p^\times$.

The construction of A, ϕ is functorial in S in the obvious sense.

Proof. A detailed proof can be found in [52], §6. The point is that we have the lattice
$$(\tilde{\eta}^o_S)^{-1}(gL \otimes \widehat{\mathbb{Z}}^p) \quad \subset T^p(\widetilde{A}^o_S)^0,$$
while ρ_X determines a p-divisible group in the isogeny class of $X(\widetilde{A}^o_S)$, the p-divisible group of $\widetilde{A}^o_S$. Together these two determine a unique abelian scheme A over S with a quasi-isogeny to $\widetilde{A}^o_S$. The quasi-polarization and quasi-$O_{\boldsymbol{k}}$-action on $\widetilde{A}^o_S$ determine a principal polarization and $O_{\boldsymbol{k}}$-action on A, as required. □

Let $I(\mathbb{Q}) = I^V(\mathbb{Q})$ be the group of quasi-isogenies in $\mathrm{End}^0_{O_{\boldsymbol{k}}}(A^o)$ that preserve the polarization λ^o. Note that $I(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of an algebraic group defined over $\mathbb{Q}$. Any $\gamma \in I(\mathbb{Q})$ induces a quasi-isogeny $\alpha_p(\gamma)$ of height 0 of the p-divisible group $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ of (A^o, ι, λ^o) and hence acts on $\mathcal{N}$ by sending $\xi = (X, \iota, \lambda_X, \rho_X)$ to $\alpha_p(\gamma)\xi = (X, \iota, \lambda_X, \alpha_p(\gamma) \circ \rho_X)$. The element γ also induces an automorphism γ_* of $T^p(A^o)^0$ and hence defines an element
$$\alpha^p(\gamma) = \eta^{p,o} \circ \gamma_* \circ (\eta^{p,o})^{-1} \in G(\mathbb{A}^p_f),$$
of scale factor 1.

Lemma 5.4. *Any $\gamma \in I(\mathbb{Q})$ induces an isomorphism*
$$\Theta(\alpha_p(\gamma)\xi, \alpha^p(\gamma) gK^p) \simeq \Theta(\xi, gK^p),$$
as points in $\mathcal{M}(n-r, r)(S)$. Conversely, any isomorphism
$$\Theta(\xi, gK^p) \simeq \Theta(\xi', g'K^p)$$
is induced by a unique $\gamma \in I(\mathbb{Q})$.

Proof. The quasi-isogeny $\gamma : A^o \to A^o$ lifts uniquely to a quasi-isogeny $\widetilde{\gamma}_S : \widetilde{A}^o_S \to \widetilde{A}^o_S$ which commutes with the $O_{\boldsymbol{k}}$-action and preserves the polarization $\widetilde{\lambda}^o_S$. Moreover, the diagram

$$\begin{array}{cccccc} \tilde{\eta}^{p,o}_S : & T^p(\widetilde{A}^o_S) & \longrightarrow & T^p(A^o) & \xrightarrow{\eta^{p,o}} & V(\mathbb{A}^p_f) \\ & {\scriptstyle (\widetilde{\gamma}_S)_*}\downarrow & & {\scriptstyle \gamma_*}\downarrow & & {\scriptstyle \alpha^p(\gamma)}\downarrow \\ \tilde{\eta}^{p,o}_S : & T^p(\widetilde{A}^o_S) & \longrightarrow & T^p(A^o) & \xrightarrow{\eta^{p,o}} & V(\mathbb{A}^p_f) \end{array} \tag{5.7}$$

commutes. Suppose that $\Theta(\xi, gK^p) = (A, \iota, \lambda)$ with quasi-isogeny $\phi : A \to \widetilde{A}^o_S$. Then the same collection (A, ι, λ) with quasi-isogeny $\widetilde{\gamma}_S \circ \phi$ satisfies the conditions of Proposition 5.3 for the pair $(\xi', g'K^p) = (\alpha_p(\gamma)\xi, \alpha^p(\gamma)gK^p)$. Hence γ induces an isomorphism $\Theta(\xi, gK^p) \simeq \Theta(\xi', g'K^p)$. Conversely, an isomorphism $\beta : \Theta(\xi, gK^p) \simeq \Theta(\xi', g'K^p)$ defines a quasi-isogeny $\phi^{-1} \circ \beta \circ \phi' : \widetilde{A}^o_S \to \widetilde{A}^o_S$. This then defines an element $\gamma \in I(\mathbb{Q})$ which induces the isomorphism β. $\square$

In the present situation, the uniformization theorem, Theorem 6.30 of [52], amounts to the following. It reflects the process of forgetting the quasi-isogeny ϕ in the construction of Proposition 5.3. Recall that we fixed a trivialization (5.1) of the prime-to-p roots of unity over $\mathbb{F}$.

Theorem 5.5. *Let $\widehat{\mathcal{M}}(n-r,r)^{\rm ss}$ denote the formal completion of $\mathcal{M}(n-r,r) \times_{\operatorname{Spec} O_{\boldsymbol{k}}} \operatorname{Spec} W(\mathbb{F})$ along its supersingular locus. For a relevant space $V^\sharp = (V, [[L]])$ in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$, let $\widehat{\mathcal{M}}(n-r,r)^{V^\sharp,\rm ss}$ be the open and closed sublocus where the rational Tate module $T^p(A)^0$ is isomorphic to $V \otimes \mathbb{A}^p_f$ and the type of the hermitian lattice $T_2(A)$ coincides with the type of the G^V_1-genus $[[L]]$. Then the map Θ induces an isomorphism*

$$\Theta : \big[\, I^V(\mathbb{Q}) \backslash (\mathcal{N} \times G^V(\mathbb{A}^p_f)^0/K^p) \,\big] \ \xrightarrow{\ \sim\ } \widehat{\mathcal{M}}(n-r,r)^{V^\sharp,\rm ss}$$

of formal algebraic stacks over W, where K^p is the stabilizer of L in $G^V(\mathbb{A}^p_f)$. $\square$

Note that our space $\mathcal{N}$ is (a slight variant of) the space $\breve{\mathcal{M}}$ in [52] and that Theorem 6.30 of loc. cit. actually gives a stronger result, not needed here, involving the descent of both sides to $\operatorname{Spf}(O_{\boldsymbol{k}} \otimes \mathbb{Z}_p)$.

Remark 5.6. Recall from Proposition 2.19 the disjoint decomposition of $\mathcal{M}(n-r,r)[\frac{1}{2}]$ according to strict similarity classes of relevant hermitian spaces in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$. Then, it is clear that by taking the disjoint sum of the spaces $\widehat{\mathcal{M}}(n-r,r)^{V^\sharp,\rm ss}$, as V runs through the elements in the fixed strict similarity class $[V]$, we obtain the formal completion of $\mathcal{M}(n-r,r)^{[V]^\sharp} \times \operatorname{Spec} W(\mathbb{F})$ along its supersingular locus.

A special case of the previous result occurs for $\mathcal{M}_0$. In this case the formal scheme $\mathcal{N}_0 = \mathcal{N}(1,0)$ is trivial, i.e., is equal to Spf W (*canonical lifting*).

Let us now combine the uniformization theorems for $\mathcal{M}(n-r,r)$ and for $\mathcal{M}_0$ to obtain a uniformization theorem for $\mathcal{M} = \mathcal{M}(n-r,r) \times_{\operatorname{Spec} O_{\boldsymbol{k}}} \mathcal{M}_0$. Recall the disjoint sum decomposition of $\mathcal{M}[\frac{1}{2}]$ according to relevant spaces $\tilde{V}^\sharp \in \mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp$, cf. Proposition 2.19. We denote by $\widehat{\mathcal{M}}^{\tilde{V}^\sharp,\rm ss}$ the formal completion of $\mathcal{M}^{\tilde{V}^\sharp} \times_{\operatorname{Spec} O_{\boldsymbol{k}}} W(\mathbb{F})$ along its supersingular locus. Now there is a decomposition

$$\widehat{\mathcal{M}}^{\tilde{V}^\sharp,\rm ss} = \coprod_{(V^\sharp, V_0)} \widehat{\mathcal{M}}^{(V^\sharp,V_0),\rm ss}, \tag{5.8}$$

where

$$\widehat{\mathcal{M}}^{(V^\sharp,V_0),\rm ss} = \widehat{\mathcal{M}(n-r,r)}^{V^\sharp,\rm ss} \times \widehat{\mathcal{M}_0}^{V_0,\rm ss}, \tag{5.9}$$

and $(V^\sharp, V_0)$ runs over pairs in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})^\sharp \times \mathcal{R}_{(1,0)}(\boldsymbol{k})$ with $\mathrm{Hom}_{\boldsymbol{k}}(V_0, V) \simeq \tilde{V}$. Note that the indexing on the right hand side of (5.8) is determined by our choice of trivialization (5.1).

Corollary 5.7. *For a pair $(V^\sharp, V_0)$, with $\big(\mathcal{M}(n-r,r)^{V^\sharp} \times \mathcal{M}_0^{V_0}\big)^{\mathrm{ss}}(\mathbb{F})$ non-empty, there is an isomorphism of formal stacks over* $\mathrm{Spf}\, W$

$$\widehat{\mathcal{M}}^{(V^\sharp,V_0),\mathrm{ss}} \simeq \big[\,(I^V(\mathbb{Q}) \times I^{V_0}(\mathbb{Q}))\backslash\big(\mathcal{N} \times \mathcal{N}_0 \times G^V(\mathbb{A}_f^p)^0/K^{V^\sharp,p} \times G_0^{V_0}(\mathbb{A}_f^p)^0/K_0^p\,\big)\big].$$

□

6. Special cycles in the supersingular locus

In this section, we utilize the uniformization described in the previous section to study the intersection of our special cycles with the supersingular locus. To lighten notation, we write $\mathcal{M}(n-r,r)$ for $\mathcal{M}(n-r,r) \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, W$, $\mathcal{M}_0$ for $\mathcal{M}_0 \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, W$, and $\mathcal{M}$ for $\mathcal{M} \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, W$. We denote the supersingular locus by $\mathcal{M}(n-r,r)^{\mathrm{ss}}$, $\mathcal{M}_0^{\mathrm{ss}}$, and $\mathcal{M}^{\mathrm{ss}}$, respectively. We continue to treat the case $p \neq 2$, where it is (sometimes) necessary to keep track of the type of the G_1^V-genera. When $p = 2$ is inert, the G_1^V-genera do not play a role.

For $T \in \mathrm{Herm}_m(\boldsymbol{k})$, we write $\mathcal{Z}(T)$ for $\mathcal{Z}(T) \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, W$, and introduce the fiber product:

$$\begin{array}{ccc} \widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T) & \longrightarrow & \widehat{\mathcal{M}}^{(V^\sharp,V_0),\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathcal{Z}(T) & \longrightarrow & \mathcal{M}. \end{array} \tag{6.1}$$

Note that the formal stack $\widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T)$ is the formal completion of $\mathcal{Z}(T)$ along $\mathcal{Z}^{(V^\sharp,V_0),\mathrm{ss}}(T) = \mathcal{Z}(T) \times_{\mathcal{M}} \mathcal{M}^{(V^\sharp,V_0),\mathrm{ss}}$.

Having fixed a base point $(A^o, \iota^o, \lambda^o; E^o, \iota_0^o, \lambda_0^o)$ in $\mathcal{M}^{(V^\sharp,V_0),\mathrm{ss}}(\mathbb{F})$ and lifts $\widetilde{A}^o$ and $\widetilde{E}^o$ to W, etc., as in the previous section, we have the uniformization of $\widehat{\mathcal{M}}^{(V^\sharp,V_0),\mathrm{ss}}$ given by Corollary 5.7. Our goal is to give a similar uniformization of the formal stack $\widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T)$.

To the base point $(A^o, \iota^o, \lambda^o; E^o, \iota_0^o, \lambda_0^o)$ in $\big(\mathcal{M} \times \mathcal{M}_0\big)^{\mathrm{ss}}(\mathbb{F})$ there are associated two hermitian spaces, $\tilde{V}$ and $\tilde{V}'$. Here $\tilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0, V)$; it may also be characterized as the unique relevant space in $\mathcal{R}_{(n-r,r)}(\boldsymbol{k})$ such that

$$\tilde{V} \otimes \mathbb{A}_f^p \simeq \mathrm{Hom}_{\boldsymbol{k} \otimes \mathbb{A}_f^p}\big(T^p(E^o)^0, T^p(A^o)^0\big), \tag{6.2}$$

i.e., $\tilde{V}$ is uniquely defined by the condition that $(A^o, \iota^o, \lambda^o; E^o, \iota_0^o, \lambda_0^o) \in \mathcal{M}^{\tilde{V},\mathrm{ss}}(\mathbb{F})$. Note that the isomorphism (6.2) is determined by our choices of η and η_0 as in (5.4).
The space $\tilde{V}'$ is given as

$$\tilde{V}' = \mathrm{Hom}^0_{O_{\boldsymbol{k}}}(E^o, A^o) \tag{6.3}$$

with hermitian form defined by

$$h'(x, y) = (\lambda_0^o)^{-1} \circ y^\vee \circ \lambda^o \circ x. \tag{6.4}$$

Note that the natural action of the group $I^V(\mathbb{Q})$ (resp. $I^{V_0}(\mathbb{Q})$) on V' by post-multiplication (resp. by pre-multiplication) preserves this hermitian form. For $x \in \tilde{V}'$, let

$$\underline{x} = \eta^o \circ x \circ (\eta_0^o)^{-1} \in \mathrm{Hom}_{\boldsymbol{k} \otimes \mathbb{A}_f^p}(V_0(\mathbb{A}_f^p), V(\mathbb{A}_f^p)), \tag{6.5}$$

and let

$$\underline{\underline{x}} \in \mathrm{Hom}_{O_{\boldsymbol{k}} \otimes \mathbb{Z}_p}(\mathbb{X}_0, \mathbb{X}) = \mathbb{V} \tag{6.6}$$

be the corresponding homomorphisms. Note that there is a natural action of the group $I^V(\mathbb{A}_f^p)\times I^{V_0}(\mathbb{A}_f^p)$ (resp. $I^V(\mathbb{Q}_p)\times I^{V_0}(\mathbb{Q}_p)$) on the space $\mathrm{Hom}(V_0(\mathbb{A}_f^p), V(\mathbb{A}_f^p))$ (resp. $\mathbb{V}$), and the maps $x\mapsto \underline{x}$ and $x\mapsto \underline{\underline{x}}$ are $I^V(\mathbb{Q})\times I^{V_0}(\mathbb{Q})$-equivariant.

Lemma 6.1. *a) The hermitian spaces $\tilde V$ and $\tilde V'$ are isomorphic at all finite places $\ell\neq p$. At the archimedean place $\tilde V$ has signature $(n-r,r)$ and Hasse invariant $(-1)^r$, whereas $\tilde V'$ has signature $(n,0)$ and Hasse invariant 1.*
b) $I^V(\mathbb{Q})\simeq U(\tilde V')(\mathbb{Q})$, and $I^{V_0}(\mathbb{Q})\simeq \boldsymbol{k}^1=\ker(\mathrm{Nm}_{\boldsymbol{k}/\mathbb{Q}})$.

Proof. Recall that $(A^o,\iota^o,\lambda^o)\in\mathcal{M}(n-r,r)^{\tilde V}(\mathbb{F})$ is supersingular; hence we can choose an $O_{\boldsymbol{k}}$-equivariant isogeny $A^o\simeq (E^o)^n$. This yields, in turn, compatible isomorphisms

$$\tilde V'(\mathbb{Q})=\mathrm{Hom}^0_{O_{\boldsymbol{k}}}(E^o,A^o)\simeq \mathrm{End}^0_{O_{\boldsymbol{k}}}(E^o)^n=\boldsymbol{k}^n,$$

and

$$\mathrm{End}^0_{O_{\boldsymbol{k}}}(A^o)\simeq M_n(\mathrm{End}^0_{O_{\boldsymbol{k}}}(E^o))=M_n(\boldsymbol{k}).$$

Hence also the natural map

$$\tilde V'\otimes\mathbb{A}_f^p\to \mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}\big(T^p(E^o)^0, T^p(A^o)^0\big) \tag{6.7}$$

is an isomorphism. This proves a).
For b) note that there are obvious homomorphisms $I^V\to U(\tilde V')$ and $I^{V_0}\to\ker(\mathrm{Nm}_{\boldsymbol{k}/\mathbb{Q}})$ of algebraic groups over $\mathbb{Q}$. Since these induce isomorphisms $I^V(\mathbb{A}_f^p)\simeq U(\tilde V')(\mathbb{A}_f^p)$, and $I^{V_0}(\mathbb{A}_f^p)\simeq(\boldsymbol{k}\otimes\mathbb{A}_f^p)^1$, these homomorphisms are isomorphisms, and induce on the $\mathbb{Q}$-points the isomorphisms in b). ∎

In order to state our uniformization result for special cycles, we need the following definition of special cycles in $\mathcal{N}\times\mathcal{N}^0$, which is a slight variant of Definition 3.2 of [39].

Definition 6.2. For a collection $\underline{\underline{\mathbf{x}}}\in\mathrm{Hom}_{O_{\boldsymbol{k}}\otimes\mathbb{Z}_p}(\mathbb{X}_0,\mathbb{X})^m$, let $\mathcal{Z}(\underline{\underline{\mathbf{x}}})$ be the subfunctor of $\mathcal{N}\times\mathcal{N}_0$, where $\mathcal{Z}(\underline{\underline{\mathbf{x}}})(S)$ is the set of isomorphism classes of collections $(X,\iota,\lambda_X,\rho_X;Y,\iota,\lambda_Y,\rho_Y)$ in $(\mathcal{N}\times\mathcal{N}_0)(S)$ such that the quasi-homomorphism

$$\rho_X^{-1}\circ\underline{\underline{\mathbf{x}}}\circ\rho_Y: Y^m\times_S\bar S\longrightarrow X\times_S\bar S$$

extends to a homomorphism from Y^m to X. Here $S\in\mathrm{Nilp}_W$, and $\bar S=S\times_W\mathbb{F}$ is the special fiber of S.

Proposition 6.3. *Fix a base point $(A^o,\iota^o,\lambda^o;E^o,\iota_0^o,\lambda_0^o)$ in $\mathcal{M}^{(V^\sharp,V_0),\mathrm{ss}}(\mathbb{F})$, and define $\tilde V'$ by (6.3). For $S\in\mathrm{Nilp}_W$, define $\mathrm{Inc}_p(T;V^\sharp,V_0)(S)$, the* incidence set*, inside*

$$\big((\mathcal{N}\times\mathcal{N}_0)(S)\times(G^V(\mathbb{A}_f^p)^0/K^{V^\sharp,p}\times G^{V_0}(\mathbb{A}_f^p)^0/K_0^p)\big)\times\tilde V'(\mathbb{Q})^m$$

to be the subset of collections $(\xi,\xi_0,gK^{V^\sharp,p},g_0K_0^p;\mathbf{x}^o)$ determined by the following incidence relations:

(a) $h'(\mathbf{x}^o,\mathbf{x}^o)=T$.
(b) $g^{-1}\circ\underline{\mathbf{x}}^o\circ g_0\in\mathrm{Hom}_{O_{\boldsymbol{k}}\otimes\widehat{\mathbb{Z}}^p}(T^p(E^o),T^p(A^o))^m$.
(c) $(\xi,\xi_0)\in\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)(S)\subset(\mathcal{N}\times\mathcal{N}_0)(S)$.

Then $\mathrm{Inc}_p(T; V^\sharp, V_0)(S)$ *is the set of* S*-points of the formal scheme*

$$\mathrm{Inc}_p(T; V^\sharp, V_0) = \coprod_{(gK^{V^\sharp,p},\, g_0K_0^p)} \coprod_{\mathbf{x}^o} \mathcal{Z}(\underline{\underline{\mathbf{x}}}^o),$$

where $(gK^{V^\sharp,p}, g_0K_0^p)$ *runs over* $G^V(\mathbb{A}_f^p)^0/K^{V^\sharp,p} \times G^{V_0}(\mathbb{A}_f^p)^0/K_0^p$*, and* $\mathbf{x}^o \in \tilde{V}'(\mathbb{Q})^m$ *runs over the set of* m*-tuples satisfying conditions (a) and (b). Moreover, there is an isomorphism of formal stacks over* W*, compatible with the uniformization isomorphism for* $\widehat{\mathcal{M}}^{(V,V_0),\mathrm{ss}}$ *in Corollary 5.7,*

$$\big(I^V(\mathbb{Q}) \times I^{V_0}(\mathbb{Q})\big)\backslash \mathrm{Inc}_p(T; V^\sharp, V_0) \xrightarrow{\ \sim\ } \widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T).$$

Proof. Let us recall our fixed trivialization (5.1) of the prime-to-p units. Then in the notation of Proposition 5.3, the pair $(\xi, gK^{V^\sharp,p})$, resp. $(\xi_0, g_0K_0^p)$ determines (A,ι,λ), resp. (E,ι_0,λ_0), and a quasi-isogeny $\phi : A \to \widetilde{A}^o_S$, resp. $\phi_0 : E \to \widetilde{E}^o_S$. Let $\mathbf{x} \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(E,A)^m$ be such that $(A,\iota,\lambda; E,\iota_0,\lambda_0;\mathbf{x})$ is in $\mathcal{Z}(T)(S)$.

Let

$$\mathbf{x}^o = \phi \circ \mathbf{x} \circ \phi_0^{-1} \in \mathrm{Hom}^0_{O_{\boldsymbol{k}}}(\widetilde{E}^o_S, \widetilde{A}^o_S)^m = \mathrm{Hom}^0_{O_{\boldsymbol{k}}}(E^o, A^o)^m = \tilde{V}'(\mathbb{Q})^m, \tag{6.8}$$

be the corresponding collection of quasi-isogenies, and let

$$\underline{\mathbf{x}}^o = \eta^o \circ \mathbf{x}^o \circ (\eta^o_0)^{-1} \in \mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{A}_f^p}(V_0(\mathbb{A}_f^p), V(\mathbb{A}_f^p))^m,$$

and let

$$\underline{\underline{\mathbf{x}}}^o \in \mathrm{Hom}^0_{O_{\boldsymbol{k}}\otimes\mathbb{Z}_p}(\mathbb{X}_0, \mathbb{X})^m$$

be the collection of quasi-isogenies of p-divisible groups induced by $\mathbf{x}^o$. By (i) of Proposition 5.3, we have

$$h'(\mathbf{x}^o, \mathbf{x}^o) = h(\mathbf{x}, \mathbf{x}) = T,$$

i.e., condition (a) holds. Condition (b) follows immediately from (ii) of Proposition 5.3. Finally, by (iii) of Proposition 5.3,

$$(X(A), \iota, \lambda, (\phi_*)_{\bar{S}}; X(E), \iota_0, \lambda_0, ((\phi_0)_*)_{\bar{S}}, \underline{\underline{\mathbf{x}}}) \in \mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)(S),$$

where $\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)$ is the cycle in $\mathcal{N} \times \mathcal{N}_0$ defined above.

Conversely, if a collection $\mathbf{x}^o$ satisfying (a), (b) and (c) is given, the collection

$$\mathbf{x} = \phi^{-1} \circ \mathbf{x}^o \circ \phi_0 \in \mathrm{Hom}^0_{O_{\boldsymbol{k}}}(E, A)^m$$

actually lies in $\mathrm{Hom}_{O_{\boldsymbol{k}}}(E,A)^m$ and satisfies $h(\mathbf{x},\mathbf{x}) = T$.

Finally, it is easy to check that dividing out by the action of $I^V(\mathbb{Q}) \times I^{V_0}(\mathbb{Q})$ yields an identification with $\widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T)$. □

Remark 6.4. When $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$, so that $\mathcal{Z}(T)$ has support in the supersingular locus, there is a decomposition

$$\mathcal{Z}(T) = \coprod_{(V^\sharp, V_0)} \mathcal{Z}^{(V^\sharp,V_0),\mathrm{ss}}(T). \tag{6.9}$$

Suppose that $\mathcal{Z}(T)^{(V^\sharp,V_0),\mathrm{ss}} \neq \emptyset$. As before, let $\tilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0, V)$ and let $\tilde{V}'$ be the unique positive definite hermitian space such that $\mathrm{inv}_\ell(\tilde{V}') = \mathrm{inv}_\ell(\tilde{V})$ for all $\ell \neq p$. Then $\tilde{V}' \simeq V_T$, where $V_T = \boldsymbol{k}^n$ with hermitian form given by T. Indeed, for any point $(A,\iota,\lambda; E,\iota_0,\lambda_0;\mathbf{x}) \in \mathcal{Z}(T)^{(V^\sharp,V_0),\mathrm{ss}}$, the last entry defines a $\boldsymbol{k}$-linear map $\boldsymbol{k}^n \to \tilde{V}' = \mathrm{End}^0_{\boldsymbol{k}}(E,A)$ which is an isometry. Thus, the index set in (6.9) runs over the pairs $(V^\sharp, V_0)$ such that $\tilde{V}' \simeq V_T$.

Part III: Eisenstein series

In the next four sections, we review some material concerning theta integrals, Eisenstein series, the Siegel-Weil formula, etc. that will be needed in formulating our main results. We refer to [16], [22], and [30] for more details.

7. The theta integral

For the moment, we shift notation and allow V to be any nondegenerate hermitian space over $\boldsymbol{k}$ of dimension m. Let $G_1 = \mathrm{U}(V)$ and let $H = \mathrm{U}(n,n) = \mathrm{U}(W_0)$, where W_0 is a split skew hermitian space of dimension $2n$. Let $W = V \otimes_{\boldsymbol{k}} W_0$, with symplectic form,

$$\langle\langle\, v_1 \otimes w_1, v_2 \otimes w_2 \,\rangle\rangle = \mathrm{tr}_{\boldsymbol{k}/\mathbb{Q}}((v_1, v_2)\langle\, w_1, w_2 \,\rangle),$$

as in [16]. There is a homomorphism $G_1 \times H \to \mathrm{Sp}(W)$ and (G_1, H) is a reductive dual pair.

We fix a character η of $\boldsymbol{k}_{\mathbb{A}}^\times$ whose restriction to $\mathbb{Q}_{\mathbb{A}}^\times$ is χ^m, where $m = \dim_{\boldsymbol{k}} V$ and χ is the global quadratic character attached to $\boldsymbol{k}$. As explained in [16], the choice of η determines a homomorphism

$$G_1(\mathbb{A}) \times H(\mathbb{A}) \longrightarrow \mathrm{Mp}(W)(\mathbb{A})$$

where $\mathrm{Mp}(W)(\mathbb{A})$ is the metaplectic cover of $\mathrm{Sp}(W)(\mathbb{A})$ and hence a Weil representation[5] ω of the group $G_1(\mathbb{A}) \times H(\mathbb{A})$ on the Schwartz space $S(V(\mathbb{A})^n)$. We normalize this so that the action of $G_1(\mathbb{A})$ is given by $(\omega(g,1)\varphi)(x) = \varphi(g^{-1}x)$. The theta function attached to $\varphi \in S(V(\mathbb{A})^n)$ is then

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})^n} \omega(h)\varphi(g^{-1}x),$$

where $g \in G_1(\mathbb{A})$ and $h \in H(\mathbb{A})$.

We now suppose that $m = n$ and that $\mathrm{sig}(V) = (n, 0)$. The theta integral is then

$$I(h;\varphi) = \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \theta(g, h; \varphi)\, dg,$$

where the Haar measure dg is taken so that $\mathrm{vol}(G_1(\mathbb{Q})\backslash G_1(\mathbb{A})) = 1$. We take $\varphi = \varphi_\infty \otimes \varphi_f$ where

$$\varphi_\infty(x) = e^{-2\pi \mathrm{tr}(x,x)} \tag{7.1}$$

is the Gaussian, and φ_f is the characteristic function of $(M \otimes \widehat{\mathbb{Z}})^n$ for an $O_{\boldsymbol{k}}$-lattice M in V. We take $K_1 \subset G_1(\mathbb{A}_f)$ to be the stabilizer of $M \otimes \widehat{\mathbb{Z}}$ and we note that

$$\varphi_\infty(g^{-1}x) = \varphi_\infty(x)$$

for $g \in G_1(\mathbb{R})$. Write

$$G_1(\mathbb{A}_f) = \coprod_j G_1(\mathbb{Q}) g_j K_1.$$

Then

$$I(h;\varphi) = \mathrm{vol}(G_1(\mathbb{R})K_1) \cdot \sum_j |\Gamma_j|^{-1} \sum_{x \in V(\mathbb{Q})^n} \omega(h)\varphi(g_j^{-1}x),$$

[5] We also fix the standard additive character ψ of $\mathbb{A}$, trivial on $\mathbb{Q}$ and on $\widehat{\mathbb{Z}}$ and such that, for $x \in \mathbb{R}$, $\psi(x) = e(x) = e^{2\pi i x}$.

where $\Gamma_j = G_1(\mathbb{Q}) \cap g_j K_1 g_j^{-1}$ is the group of isometries of the hermitian lattice $M_j = (g_j\, L \otimes \widehat{\mathbb{Z}}) \cap V(\mathbb{Q})$, and the lattice M_j runs over representatives for the classes in the G_1-genus of M. Note that

$$1 = \mathrm{vol}(G_1(\mathbb{Q})\backslash G_1(\mathbb{A})) = \mathrm{vol}(G_1(\mathbb{R})K_1) \cdot \sum_j |\Gamma_j|^{-1},$$

so that

$$\mathrm{mass}(M) := \sum_j |\Gamma_j|^{-1} = \mathrm{vol}(G_1(\mathbb{R})K_1)^{-1} \tag{7.2}$$

is the classical mass of the genus of M.

Taking $h_f = 1$, we have

$$I(h;\varphi) = \mathrm{mass}(M)^{-1} \sum_j |\Gamma_j|^{-1} \sum_{x \in L_j^n} \omega_\infty(h)\varphi_\infty(x).$$

Let $D(W_0)$ be the space of negative n-planes in $W_0(\mathbb{R})$, so that

$$D(W_0) \simeq \{\, z \in M_n(\mathbb{C}) \mid v(z) := (2i)^{-1}(z - {}^t\bar{z}) > 0 \,\}.$$

Write $z = u(z) + iv(z)$, with $u(z) = 2^{-1}(z + {}^t\bar{z})$, and let

$$h_z = \begin{pmatrix} 1_n & u(z) \\ & 1_n \end{pmatrix} \begin{pmatrix} a & \\ & {}^t\bar{a}^{-1} \end{pmatrix} \quad \in H(\mathbb{R}), \tag{7.3}$$

where $a \in \mathrm{GL}_n(\mathbb{C})$ with $v(z) = a\,{}^t\bar{a}$. Note that $h_z(i1_n) = z$. Now taking $h_\infty = h_z$, we have

$$\omega_\infty(h_z)\varphi_\infty(x) = \eta_\infty(\det(a))\, \det(v(z))^{\frac{n}{2}}\, q^{(x,x)}, \tag{7.4}$$

where, for $T \in \mathrm{Herm}_n(\mathbb{C})$, we write $q^T = e(\mathrm{tr}(Tz))$. Thus, we obtain the classical expression[6]

$$\begin{aligned} I(z;M) :&= \eta_\infty(\det(a))^{-1}\, \det(v(z))^{-\frac{n}{2}}\, I(h_z;\varphi) \qquad (7.5)\\ &= \mathrm{mass}(M)^{-1} \cdot \sum_{T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})} r_{\mathrm{gen}}(T,M)\, q^T. \end{aligned}$$

Here the T-th Fourier coefficient is the representation number

$$r_{\mathrm{gen}}(T,M) = \sum_j |\Gamma_j|^{-1}\, |\Omega(T,M_j)|, \tag{7.6}$$

where M_j runs over the classes of lattices in the G_1-genus of M, and

$$\Omega(T,M_j) = \{\, x \in M_j^n \mid (x,x) = T \,\}.$$

8. The Siegel formula

Now we assume that $\det(T) \neq 0$ and we express $I_T(z;M)$, the T-th Fourier coeffiicient of the theta integral defined in (7.5) in terms of local densities. This is done in detail in section 6 of Ichino, [22], as part of the proof of the regularized Siegel-Weil formula for unitary groups. Here we specialize his formulas to the case where $\dim(V) = n$.

We slightly shift notation and take dg to be Tamagawa measure on $G_1(\mathbb{A})$, defined with respect to a gauge form ν_{G_1} on G_1. Since the Tamagawa number of G_1 is 2, we include a factor of $\frac{1}{2}$ in the definition of the theta integral. For general factorizable $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A})^n)$, Ichino obtains

$$I_T(h;\varphi) = \frac{1}{2}\, L(1,\chi)^{-1} \prod_v \lambda_v^{-1} \int_{\Omega_T(\mathbb{Q}_v)} \omega(h_v)\varphi_v(x)\, d\mu_{T,v}(x),$$

[6] This is called the analytic genus invariant in Siegel [58] and Braun [5].

where the convergence factors are

$$\lambda_v^{-1} = L_v(1, \chi_v), \tag{8.1}$$

and where $\Omega_T(\mathbb{Q}_v)$ is the set of $\mathbb{Q}_v$-points of the variety

$$\Omega_T = \{x \in V^n \mid (x, x) = T\}.$$

The measures $d\mu_{T,v}$ are defined as follows. We choose $x \in V(\mathbb{Q}_v)^n$ with $(x, x) = T$, and define an isomorphism $i_x : G_1(\mathbb{Q}_v) \xrightarrow{\sim} \Omega_T(\mathbb{Q}_v)$ via $g_v \mapsto g_v \cdot x$. The measure $d\mu_{T,v}$ on $\Omega_T(\mathbb{Q}_v)$ is obtained, via this isomorphism, from the Tamagawa measure dg_v on $G_1(\mathbb{Q}_v)$ determined by ν_{G_1}. Note that, for a place $v \notin S$, for a sufficiently large finite set of places S including the archimedean place,

$$\int_{\Omega_T(\mathbb{Q}_v)} \omega(h_v)\varphi_v(x)\, d\mu_{T,v}(x) = \prod_{i=1}^{n} L_v(n - i + 1, \chi_v^{n-i+1})^{-1},$$

so that [7]

$$\begin{aligned} I_T(h; \varphi) &= \frac{1}{2}\, L(1, \chi)^{-1} \prod_{i=1}^{n-1} L^S(n - i + 1, \chi^{n-i+1})^{-1} \\ &\qquad\qquad \times \prod_{v \in S} \lambda_v^{-1} \int_{\Omega_T(\mathbb{Q}_v)} \omega(h_v)\varphi_v(x)\, d\mu_{T,v}(x) \\ &= \frac{1}{2} \prod_{i=1}^{n} L^S(n - i + 1, \chi^{n-i+1})^{-1} \prod_{v \in S} \int_{\Omega_T(\mathbb{Q}_v)} \omega(h_v)\varphi_v(x)\, d\mu_{T,v}(x). \end{aligned}$$

On the other hand, if $\Phi(s)$ is the Siegel-Weil section associated to a factorizable φ, we have a factorization, [22] [59],

$$E_T(h, s, \Phi) = L^S(2s, n, \chi)^{-1} \cdot \prod_{v \in S} W_{T,v}(h_v, s, \Phi_v),$$

of the T-th Fourier coefficient of the Eisenstein series again, for a sufficiently large finite set of places S, where, for convenience, we set

$$L^S(2s, n, \chi) = \prod_{i=1}^{n} L^S(2s + n - i + 1, \chi^{n-i+1}). \tag{8.2}$$

Here, for $\mathrm{Re}(s) > n$, the Whittaker function (discussed in more detail in section 10 below) is given by

$$W_{T,v}(h_v, s, \Phi_v) = \int_{\mathrm{Herm}_n(\boldsymbol{k}_v)} \Phi_v(w^{-1} n(b_v) h_v, s)\, \psi_v(-\mathrm{tr}(T b_v))\, db_v, \tag{8.3}$$

where db_v is the self-dual measure on $\mathrm{Herm}_n(\boldsymbol{k}_v)$ with respect to the pairing $\langle\, b_1, b_2 \,\rangle = \psi_v(\mathrm{tr}(b_1 b_2))$. By a standard argument due to Weil, [67], [50], see also [42], p. 127, at $s = 0$ we have

$$W_{T,v}(h_v, 0, \Phi_v) = \gamma_v \int_{\Omega_T(\mathbb{Q}_v)} \omega(h_v)\varphi_v(x)\, d\mu_{T,v}(x), \tag{8.4}$$

where γ_v is a Weil index. Note that, in this identity, the gauge form ν_{G_1} defining the measure $d\mu_{T,v}$ is the one determined by the moment map (10.4). This point is discussed in more detail in section 10, below. Since, for S sufficiently large,

$$\prod_{v \in S} \gamma_v = 1,$$

this proves that

$$2\, I_T(h; \varphi) = E_T(h, 0, \Phi). \tag{8.5}$$

[7] Here, as usual, the superscript S indicates that the Euler factors for the primes in S are omitted.

In particular, taking $h = h_z$ as in (7.3), we have

$$2\, I_T(z;\varphi) = \eta_\infty(\det(a))^{-1}\, \det(v(z))^{-\frac{n}{2}}\, E_T(h_z, 0, \Phi). \tag{8.6}$$

Let $\Phi^n_\infty(s)$ be the Siegel-Weil section associated to the Gaussian (7.1), and, for each prime ℓ, let $\Phi_\ell(s)$ be the Siegel-Weil section attached to φ_ℓ, the characteristic function of the set $(M \otimes \mathbb{Z}_\ell)^n$. We obtain the fundamental identity

$$2\,\mathrm{mass}(M)^{-1}\, r_{\mathrm{gen}}(T, M)\, q^T = L^S(0, n, \chi)^{-1} \cdot \prod_{\ell \in S_f} W_{T,\ell}(1, 0, \Phi_\ell) \cdot W_{T,\infty}(h_z, 0, \Phi^n_\infty), \tag{8.7}$$

for a sufficiently large set of primes $S = S_f \cup \{\infty\}$.

9. The incoherent case

We now turn to the incoherent case. Fix a hermitian space V with signature $(n-1, 1)$ and assume that V contains a self-dual lattice L, i.e., that V is relevant in the sense of section 2. Take $\varphi_f \in S(V(\mathbb{A}_f)^n)$ to be the characteristic function of $(L \otimes \widehat{\mathbb{Z}})^n$. The incoherent Eisenstein series is defined to be $E(h, s, \Phi)$ for $\Phi(s) = \Phi(s, L) = \Phi_\infty(s) \otimes \Phi_f(s, L)$ where $\Phi_f(s, L)$ is the Siegel-Weil section associated to φ_f and $\Phi_\infty(s) = \Phi^n_\infty(s)$ is the Siegel-Weil section attached to the Gaussian of a hermitian space of signature $(n, 0)$. As in the previous section, e.g., (8.6), we let

$$E(z, s, \Phi) = \eta_\infty(\det(a))^{-1}\, \det(v(z))^{-\frac{n}{2}}\, E(h_z, s, \Phi) \tag{9.1}$$

be the corresponding 'classical' Eisenstein series. As explained in [30] (in the orthogonal case), $E(z, 0, \Phi) = 0$ and we are interested in the central derivative $E'(z, 0, \Phi)$.

For $T \in \mathrm{Herm}_n(\boldsymbol{k})$ with $\det(T) \neq 0$, let

$$\mathrm{Diff}(T, V) = \{\, p < \infty \mid \chi_p(\det(T)) = -\chi_p(\det(V))\,\}. \tag{9.2}$$

Note that only ramified or inert primes can occur in this set. Moreover, if $T > 0$, then $1 = \chi_\infty(\det(T)) = -\chi_\infty(\det(V))$, and hence $\mathrm{Diff}(T, V)$ has odd cardinality, due to the product formula (1.2). Finally, since V contains a self dual lattice, an inert place p lies in $\mathrm{Diff}(T, V)$ if and only if $\mathrm{ord}_p(\det(T))$ is odd. By the analysis reviewed in the previous section,

$$E_T(h, s, \Phi) = L^S(2s, n, \chi)^{-1} \cdot \prod_{v \in S} W_{T,v}(h_v, s, \Phi_v),$$

for a sufficiently large finite set S of places including the archimedean place. We assume that S contains $\mathrm{Diff}(T, V)$. By (8.4),

$$W_{T,p}(h_p, 0, \Phi_p) = 0,$$

for all $p \in \mathrm{Diff}(T, V)$, since V_p does not represent T. This implies the following.

Lemma 9.1. *(i) If $T \in \mathrm{Herm}_n(\boldsymbol{k})_{>0}$ and $|\mathrm{Diff}(T, V)| > 1$, then*

$$E'_T(h, 0, \Phi) = 0.$$

(ii) If $T \in \mathrm{Herm}_n(\boldsymbol{k})_{>0}$ and $\mathrm{Diff}(T, V) = \{\, p\,\}$, then

$$E'_T(h, 0, \Phi) = W'_{T,p}(h_p, 0, \Phi_p) \cdot L^S(0, n, \chi)^{-1} \cdot \prod_{\substack{v \in S \\ v \neq p}} W_{T,v}(h_v, 0, \Phi_v).$$

(iii) If $\mathrm{Diff}(T, V)$ is empty, so that $\mathrm{sig}(T) = (n - r, r)$ for r odd, then

$$E'_T(h, 0, \Phi) = W'_{T,\infty}(h_\infty, 0, \Phi^n_\infty) \cdot L^S(0, n, \chi)^{-1} \cdot \prod_{\substack{v \in S \\ v \neq \infty}} W_{T,v}(h_v, 0, \Phi_v).$$

These formulas can be made more explicit; we will only do this in a special case. We take $h = h_z$, as above and note that, by analogy with Lemma 5.2.1 of [42], $E_T(h_z, s, \Phi)$ vanishes unless $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$.

Remark 9.2. In this situation, we will write $W_{T,p}(s, \Phi_p)$ for $W_{T,p}(1, s, \Phi_p)$ for a finite prime p.

We suppose that $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with $\mathrm{Diff}(T, V) = \{p\}$ for an odd inert prime p. Let $V' = V_T$. Note that, up to isometry, V' depends only on V and p, since it is the unique positive definite hermitian space with $\mathrm{inv}_p(V') = -\mathrm{inv}_p(V)$ and $\mathrm{inv}_\ell(V') = \mathrm{inv}_\ell(V)$ for all other finite primes. Fix an isomorphism $V'(\mathbb{A}_f^p) = V(\mathbb{A}_f^p)$, and let L' be the lattice in V' determined by $L' \otimes \widehat{\mathbb{Z}}^p = L \otimes \widehat{\mathbb{Z}}^p$ and $L'_p = \Lambda^*$ where $\Lambda \subset V'_p$ is a vertex of type 1 and level 0 as in (12.5). Let $\varphi'_p \in S((V'_p)^n)$ be the characteristic function of $(L'_p)^n$, and let $\varphi' = \varphi'_p \otimes \varphi^p$ where $\varphi^p \in S(V(\mathbb{A}_f^p)^n)$ is the characteristic function of $(L \otimes \widehat{\mathbb{Z}}^p)^n$. The following facts will be proved in the next section.

Proposition 9.3. *For $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with $\mathrm{Diff}(T, V) = \{p\}$ for an odd inert prime p, suppose that, under the action of $\mathrm{GL}_n(O_{\boldsymbol{k},p})$,*

$$T \sim \mathrm{diag}(1_{n-2}, p^a, p^b), \qquad 0 \le a < b.$$

Let $S = 1_n$ and $S' = \mathrm{diag}(1_{n-1}, p)$.
(i) Let $\Phi_p(s)$ be the Siegel-Weil section associated to the characteristic function φ_p of the set $(L \otimes \mathbb{Z}_p)^n$ in $V(\mathbb{Q}_p)^n$. Then

$$W'_{T,p}(0, \Phi_p) = \gamma_p(V)^n\, \alpha_p(S, S)\, \mu_p(T)\, \log p,$$

where

$$\mu_p(T) = \frac{1}{2} \sum_{\ell=0}^{a} p^\ell (a + b - 2\ell + 1).$$

and $\alpha_p(S, S)$ is the local density, cf. (10.1). Note that $\alpha_p(S, S) = L_p(0, n, \chi)^{-1}$, where $L_p(2s, n, \chi)$ is the local factor at p of the L-function (8.2).
(ii) Let $\Phi'_p(s)$ be the Siegel-Weil section associated to the characteristic function φ'_p of the set $(L'_p)^n$ in $(V'_p)^n$, where L'_p is as explained above. Then

$$W_{T,p}(0, \Phi'_p) = (-1)^n \gamma_p(V)^n\, p^{-n}\, \alpha_p(S', S').$$

In particular, this quantity is nonzero and independent of T.

Using the nonvanishing in (ii), we can write

$$\begin{aligned} E'_T(h, 0, \Phi) &= \frac{W'_{T,p}(0, \Phi_p)}{W_{T,p}(0, \Phi'_p)} \cdot L^S(0, n, \chi)^{-1} \cdot \prod_{v \in S} W_{T,v}(h_v, 0, \Phi'_v) \\ &= \frac{W'_{T,p}(0, \Phi_p)}{W_{T,p}(0, \Phi'_p)} \cdot 2\, I_T(h, \varphi'). \end{aligned}$$

Then, using (8.7) and Proposition 9.3, we obtain the expression

$$E'_T(z, 0, \Phi) = (-1)^n\, \mu_p(T)\, \log(p) \cdot C_p \cdot 2\, \mathrm{mass}(L')^{-1}\, r_{\mathrm{gen}}(T, L') \cdot q^T,$$

where

$$C_p = p^n \frac{\alpha_p(S, S)}{\alpha_p(S', S')}.$$

Finally, we write $G'_1 = U(V')$ and note that $G_1 = U(V)$ and G'_1 are inner forms of each other. Our fixed choice of a gauge form $\nu_1 = \nu_{G_1}$ on G_1, together with an isomorphism of inner forms

$\Psi : G' \xrightarrow{\sim} G$ defined over $\bar{\mathbb{Q}}$, determines a gauge form $\nu'_1 = \nu_{G'_1} = \Psi^*(\nu_1)$ on G'_1. This form is again defined over $\mathbb{Q}$ and hence there are associated Haar measures on $G_1(\mathbb{Q}_p)$ and $G'_1(\mathbb{Q}_p)$, cf. [26], p. 631. Now, by (7.2) for the positive definite space V', we have

$$2\,\mathrm{mass}(L')^{-1} = \mathrm{vol}(G'_1(\mathbb{R})K'_1, d\nu'_1)$$

where the volume is taken with respect to Tamagawa measure and our fixed convergence factors (8.1). By (iii) of Lemma 10.4, we have

$$C_p \cdot \mathrm{vol}(K'_{1,p}, d\nu'_1) = \mathrm{vol}(K_{1,p}, d\nu_1).$$

Thus

$$C_p \cdot 2\,\mathrm{mass}(L')^{-1} = \mathrm{vol}(G'_1(\mathbb{R}), d\nu'_1)\,\mathrm{vol}(K_1, d\nu_1).$$

As we will see in a moment, the quantity $\mathrm{vol}(G'_1(\mathbb{R}), d\nu'_1)$ is independent of p. For later convenience, we will now write

$$E(z, s, L) = E(z, s, \Phi) \tag{9.3}$$

to emphasize the dependence on the choice of the self-dual lattice L.

Corollary 9.4. *For $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with $\mathrm{Diff}(T, V) = \{\, p \,\}$ for an odd inert prime p, suppose that T satisfies the condition of Proposition 9.3. Then*

$$E'_T(z, 0, L) = (-1)^n\, C \cdot \mu_p(T)\,\log(p) \cdot r_{\mathrm{gen}}(T, L') \cdot q^T,$$

where L' is obtained from L as explained just before Proposition 9.3 and

$$C = \mathrm{vol}(G'_1(\mathbb{R}), d\nu'_1)\,\mathrm{vol}(K_1, d\nu_1).$$

Note that the factor $(-1)^n$ arises due to the switch from an incoherent to a coherent Eisenstein series, more precisely, from the sign change in the Weil invariant $\gamma_p(V'_p) = -\gamma_p(V_p)$ and the presence of the factor $\gamma_p(V)^n$ in the formula for the local Whittaker function in Proposition 10.1.

Finally, the constant C has the following nice interpretation.

Lemma 9.5.

$$2\,C^{-1} = (-1)^{n-1}\,\chi(\mathbb{P}^{n-1}(\mathbb{C}))^{-1} \cdot \chi_\bullet\big(\, G_1(\mathbb{Q})\backslash(D \times G_1(\mathbb{A}_f)/K_1)\,\big),$$

where, for an arithmetic group Γ of isometries of the $n-1$ ball D,

$$\chi_\bullet(\Gamma\backslash D) := \int_{\Gamma\backslash D} \Omega,$$

is the integral of the Gauss-Bonnet form Ω.

Remark 9.6. If Γ is torsion free, then $\chi_\bullet(\Gamma\backslash D)$ is the Euler characteristic of the manifold $\Gamma\backslash D$. Also note that $\mathbb{P}^{n-1}(\mathbb{C})$ is the compact dual of D and that $(-1)^{n-1}\,\chi_\bullet(\Gamma\backslash D)$ is positive.

Proof. First note that, for Tamagawa measure,

$$2 = \mathrm{vol}(G_1(\mathbb{Q})\backslash G_1(\mathbb{A}), d\nu_1) = \big(\sum_j \mathrm{vol}(\Gamma_j\backslash G_1(\mathbb{R}))\big) \cdot \mathrm{vol}(K_1),$$

where

$$G_1(\mathbb{A}) = \coprod_j G_1(\mathbb{Q})G_1(\mathbb{R})g_jK_1$$

and $\Gamma_j = G(\mathbb{Q}) \cap g_jK_1g_j^{-1}$. Thus

$$2\,\mathrm{vol}(K_1)^{-1} = \sum_j \mathrm{vol}(\Gamma_j\backslash G_1(\mathbb{R})),$$

and so

$$2\,C^{-1} = \mathrm{vol}(G_1'(\mathbb{R}))^{-1}.\sum_j \mathrm{vol}(\Gamma_j\backslash G_1(\mathbb{R})).$$

Here we are using the measure obtained from matching gauge forms on $G_1(\mathbb{R}) = U(n-1,1)$ and its compact dual $G_1'(\mathbb{R}) = U(n)$, as described above. But then, the ratio is independent of the choice of this gauge form. Now write the real Lie algebra of G_1 as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\dim\mathfrak{p} = 2n-2$ and $\mathfrak{g}' = \mathfrak{k} + i\mathfrak{p}$. We can use the standard recipe, described in Serre's article [55] on Euler-Poincare measures, pp.135-138, to give a top degree form Ω on $\mathfrak{p}$ (degree $2n-2$) such that, writing Ω for the corresponding $G_1(\mathbb{R})$-invariant form on D,

$$\int_{\Gamma\backslash D} \Omega = \chi(\Gamma\backslash D),$$

for $\chi(\Gamma\backslash D)$ the Euler characteristic of the manifold $\Gamma\backslash D$, when Γ is torsion free, i.e., Ω gives the Gauss-Bonnet form on D, [15]. We can extend this to $\mathfrak{g}$ by wedging with a top degree form η on $\mathfrak{k}$ which gives the corresponding maximal compact subgroup volume 1. According to [55] p.136, the resulting measure μ on $G_1(\mathbb{R})$ is an Euler-Poincare measure. Then, since the form on $i\mathfrak{p}$ corresponds to $(-1)^{n-1}$ times the Gauss-Bonnet form on $\mathbb{P}^{n-1}(\mathbb{C})$, the measure of $G_1'(\mathbb{R})$ is

$$\mathrm{vol}(G_1'(\mathbb{R}), \mu') = (-1)^{n-1}\chi(\mathbb{P}^{n-1}(\mathbb{C})).$$

$\square$

10. Representation densities

In this section, we derive the information about local Whittaker functions and their central derivatives summarized in Proposition 9.3 from various explicit formulas for representation densities of hermitian forms.

Recall that the classical representation densities are define as follows. For nonsingular matrices $S \in \mathrm{Herm}_m(O_{\boldsymbol{k},p})$ and $T \in \mathrm{Herm}_n(O_{\boldsymbol{k},p})$, let

$$A_{p^k}(S,T) = \{\ x \in M_{m,n}(O_{\boldsymbol{k}}/p^k O_{\boldsymbol{k}}) \mid S[x] \equiv T \mod p^k\,\mathrm{Herm}_n(O_{\boldsymbol{k}})^\vee\ \},$$

where

$$\mathrm{Herm}_n(O_{\boldsymbol{k}})^\vee = \{b \in \mathrm{Herm}_n(\boldsymbol{k}) \mid \mathrm{tr}(bc) \in O_{\boldsymbol{k}} \text{ for all } c \in \mathrm{Herm}_n(O_{\boldsymbol{k}})\ \}.$$

The representation density is then defined as the limit

$$\alpha_p(S,T) = \lim_{k\to\infty} (p^{-k})^{n(2m-n)}\,|A_{p^k}(S,T)|. \tag{10.1}$$

An explicit formula for $\alpha_p(S,T)$ has been given by Hironaka, [17], in the case of an inert prime p.

These quantities are related to the Whittaker functions $W_{T,p}(s,\Phi_p)$ where $\Phi_p(s)$ is the Siegel-Weil section determined by the characteristic function $\varphi_p \in S(V_p^n)$ of $(L_p)^n$ for a lattice L_p on which the hermitian form has matrix S. We sketch the argument, which is analogous to that given for quadratic forms in [30], in order to be precise about the various constants involved.

Recall that

$$W_{T,p}(s,\Phi_p) = \int_{\mathrm{Herm}_n(\boldsymbol{k}_p)} \Phi_p(w^{-1}n(b),s)\,\psi_p(-\mathrm{tr}(Tb))\,db. \tag{10.2}$$

where[8]

$$w = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}, \qquad n(b) = \begin{pmatrix} 1_n & b \\ & 1_n \end{pmatrix},$$

and db is the self-dual measure with respect to the pairing

$$\langle\, b, c\,\rangle = \psi_p(\mathrm{tr}(bc))$$

on $\mathrm{Herm}_n(\boldsymbol{k}_p)$. In particular, note that

$$\mathrm{vol}(\mathrm{Herm}_n(O_{\boldsymbol{k},p}), db) = |\Delta|_p^{n(n-1)/4}.$$

Proposition 10.1. *Let*

$$S_r = \begin{pmatrix} & & 1_r \\ & S & \\ 1_r & & \end{pmatrix}.$$

Then

$$W_{T,p}(r, \Phi_p) = \gamma_p(V)^n \, |N(\det S)|_p^{\frac{1}{2}n} \, |\Delta|_p^e \, \alpha_p(S_r, T),$$

where $\gamma_p(V)$ is an eight root of unity given by (10.3) and $e = \frac{1}{2}n(m+2r) + \frac{1}{4}n(n-1)$.

Proof. We first use Rallis's interpolation trick. Let $V_p^{[r]} = V_p \oplus V_{r,r}$ where $V_{r,r}$ is the split space of signature (r, r), and let

$$\varphi_p^{[r]} = \varphi_p \otimes \varphi_{r,r} \in S((V_p^{[r]})^n)$$

where $\varphi_{r,r}$ is the characteristic function of $(L_{r,r})^n$ for a self-dual lattice $L_{r,r}$ in $V_{r,r}$. Then, as in [30], p.642,

$$\Phi_p(h, r) = \omega(h)\varphi_p^{[r]}(0),$$

where ω is the Weil representation of $H(\mathbb{Q}_p)$ on $S((V_p^{[r]})^n)$. Note that we use the character η fixed above to define the Weil representation for all r. Recall, [29], that for $w = w_n \in \mathrm{U}(n,n)$ as above,

$$\omega(w^{-1})\varphi_p(x) = \gamma_p(V)^n \int_{V_p^n} \psi_p(-\mathrm{tr}_{\boldsymbol{k}/\mathbb{Q}_p}\mathrm{tr}(x,y))\, \varphi_p(y)\, dy,$$

where

$$\gamma_p(V) = (\Delta, \det(V_p))_p \, \gamma_p(-\Delta, \psi_{p,\frac{1}{2}})^m \, \gamma_p(-1, \psi_{p,\frac{1}{2}})^{-m}, \tag{10.3}$$

where $\gamma_p(a, \psi_{p,\frac{1}{2}})$ is the Weil index, [29] with the additive character $\psi_{p,\frac{1}{2}}$ given by $\psi_{p,\frac{1}{2}}(x) = \psi_p(\frac{1}{2}x)$, and dy is the self-dual measure with respect to the pairing

$$\langle\, x, y\,\rangle = \psi_p(\mathrm{tr}_{\boldsymbol{k}/\mathbb{Q}_p}\mathrm{tr}(x,y))$$

on V_p^n. Inserting this in (10.2) and writing N for $\mathrm{Herm}_n(\boldsymbol{k}_p)$, we have

$$\begin{aligned} W_{T,p}(r, \Phi_p) &= \int_N \psi_p(-tr(Tb))\, \Phi_p(w^{-1}n(b), r)\, db \\ &= \int_N \psi_p(-tr(Tb))\, \gamma_p(V)^n \int_{(V_p^{[r]})^n} \psi_p(\mathrm{tr}(b(x,x)))\, \varphi_p^{[r]}(x)\, dx\, db \\ &= \gamma_p(V)^n \, |\Delta|_p^{n(n-1)/4} \lim_{k\to\infty} p^{kn^2} \int_{\substack{x \in (V_p^{[r]})^n \\ (x,x)-T \in p^k \mathrm{Herm}_n(O_{\boldsymbol{k},p})^\vee}} \varphi_p^{[r]}(x)\, dx. \end{aligned}$$

Here, in the last step, we have computed the integral over N as the limit of the integrals over the sets $p^{-k}\,\mathrm{Herm}_n(O_{\boldsymbol{k},p}) \subset N$.

[8] Note that the conventions here differ slightly from those of [16], (6.5).

Take $\varphi_p = \varphi_{L_p}$ and choose an integral basis for $L_p \oplus L_{r,r}$ for which the hermitian form has matrix S_r, as above. Then $(L_p \oplus L_{r,r})^n = M_{m+2r,n}(O_{\boldsymbol{k},p})$, and we have

$$\mathrm{vol}(M_{m+2r,n}(O_{\boldsymbol{k},p}), dx) = |N(\det S_r)|_p^{\frac{n}{2}}\, |N(\partial_p)|_p^{\frac{n(m+2r)}{2}} = |\det S_0|_p^n\, |\Delta|_p^{\frac{n(m+2r)}{2}},$$

where $|\ |_p$ is the norm on $\mathbb{Q}_p$ and ∂_p is the different of $\boldsymbol{k}_p/\mathbb{Q}_p$. We break the integral up into cosets of $p^k M_{m+2r,n}(O_{\boldsymbol{k}})$ in $M_{m+2r,n}(O_{\boldsymbol{k}})$.

This gives

$$\begin{aligned} W_{T,p}(r, \Phi_p) &= \gamma_p(V)^n\, |\det S|_p^n\, |\Delta|_p^e \lim_{k\to\infty} (p^k)^{n^2-2(m+2r)n} |A_{p^k}(S_r, T)| \\ &= \gamma_p(V)^n\, |\det S|_p^n\, |\Delta|_p^e\, \alpha_p(S_r, T). \end{aligned}$$

where $e = \frac{n(m+2r)}{2} + n(n-1)/4$. $\square$

Proof of Proposition 9.3. In the case of an odd inert prime, the formula above reduces to

$$W_{T,p}(r, \Phi_p) = \gamma_p(V)^n\, |\det S|_p^n\, \alpha_p(S_r, T).$$

Now, by Proposition 9.1 of [39] (which uses Nagaoka's formula [45]), we have

$$\begin{aligned} W'_{T,p}(0, \Phi_p) &= \gamma_p(V)^n\, |\det S|_p^n\, \alpha'_p(S, T)\, \log(p) \\ &= \gamma_p(V)^n\, |\det S|_p^n\, \alpha_p(S, S)\, \mu_p(T)\, \log(p). \end{aligned}$$

This proves (i), where we note that (by Proposition 9.1 of [39])

$$\alpha_p(S, S) = \prod_{i=1}^n (1 - (-1)^i p^{-i}) = L_p(0, n, \chi)^{-1}.$$

To prove (ii), let $S' = \mathrm{diag}(1_{n-1}, p)$ and consider

$$W_{T,p}(0, \Phi'_p) = \gamma_p(V'_p)^n\, p^{-n}\, \alpha_p(S', T),$$

where, by (10.3), $\gamma_p(V'_p) = -\gamma_p(V_p)$. Note that S' is the matrix for the hermitian form of V'_p on the lattice Λ^*, where Λ is a vertex of type 1 and level 0 in V'_p. The analogue of the reduction formula of Proposition 9.3 of [39], which follows immediately from Corollary 9.12 of [39], implies that

$$\alpha_p(S', T) = \alpha_p(S', 1_{n-2})\, \alpha_p(S'', T''),$$

where $S'' = \mathrm{diag}(1, p)$ and $T'' = \mathrm{diag}(p^a, p^b)$.

We will give the proof of the following result after our discussion of gauge forms.

Proposition 10.2. *The quantity $\alpha_p(S'', T'')$ is independent of a and b.*

Corollary 10.3. *For $n \geq 2$,*

$$\alpha_p(S', T) = \alpha_p(S', S').$$

This finishes the proof of Proposition 9.3. $\square$

Next, we need to obtain some information about the dependence on p of the quantity $\mathrm{mass}(L')$. To do this, we first record a few facts about the moment map and gauge forms

$$h : V^n \longrightarrow \mathrm{Herm}_n, \qquad x \mapsto (x, x), \tag{10.4}$$

for a hermitian space V. Let

$$V^n_{\mathrm{reg}} = \{\, x \in V^n \mid \det(h(x)) \neq 0 \,\}.$$

Let $G_1 = \mathrm{U}(V)$ and note that for any $x \in V^n_{\mathrm{reg}}$ with $h(x) = T$, the map

$$i_x : G_1 \xrightarrow{\sim} \Omega_T = h^{-1}(T), \qquad g \mapsto g \cdot x$$

is an isomorphism.

Fix basis elements $\alpha \in \wedge^{\mathrm{top}}(V^n)^*$ and $\beta \in \wedge^{\mathrm{top}}(\mathrm{Herm}_n)^*$ and write α and β for the translation invariant forms they define. Here note that the exterior products are taken for the rational vector spaces V and Herm_n; in particular the forms α and β have degrees $2n^2$ and n^2 respectively. Then there is a form ν of degree n^2 on V^n_{reg} such that (i) $\alpha = h^*\beta \wedge \nu$, (ii) ν is invariant under the action of $G_1 \times \mathrm{GL}(n)$ on V^n and (iii) for all points $x \in V^n_{\mathrm{reg}}$, the restriction of ν to $\ker(dh_x)$ is nonzero. In particular, the pullback $\nu_{G_1} = i_x^*\nu$ is a gauge form on G_1 which is independent of the choice of x. It determines a Tamagawa measure on $G_1(\mathbb{A})$ via a choice of convergence factors λ_v, as in (8.1) above. The analogous facts in the orthogonal case are described in detail in Lemma 5.3.1 of [42].

In the local situation, we have the following useful observations. The analogue of Lemma 4.2 of [50] implies that the distribution

$$\varphi \mapsto W_{T,p}(0, \Phi_\varphi)$$

on $S(V_p^n)$ is proportional to the orbital integral

$$O_{T,p}(\varphi) = \int_{\Omega_T(\mathbb{Q}_p)} \varphi(x)\, d_{T,p}(x),$$

where $d_{T,p}(x)$ is the measure on $\Omega_T(\mathbb{Q}_p)$ determined by the restriction of ν to Ω_T. More precisely, by the same argument as in pp. 121-127 of [42], we have

$$W_{T,p}(0, \Phi_\varphi) = C_p(V, \alpha, \beta, \psi) \cdot O_{T,p}(\varphi),$$

where

$$C_p(V, \alpha, \beta, \psi) = \frac{\gamma_p(V)^n\, c_p(\beta, \psi)}{c_p(\alpha, \psi)}.$$

Here

$$d_{\alpha,p}x = c_p(\alpha, \psi)\, dx,$$

where $d_{\alpha,p}x$ is the measure on V_p^n determined by the gauge form α and dx is the self-dual measure with respect to the pairing $\langle\, x, y\,\rangle = \psi_p(\mathrm{tr}_{\boldsymbol{k}_p/\mathbb{Q}_p}\mathrm{tr}(x, y))$. Similarly,

$$d_{\beta,p}b = c_p(\beta, \psi)\, db$$

where $d_{\beta,p}x$ is the measure on $\mathrm{Herm}_n(\boldsymbol{k}_p)$ determined by the gauge form β and db is the self-dual measure with respect to the pairing $\langle\, b, c\,\rangle = \psi_p(\mathrm{tr}(bc))$. Taking $\omega_p(h)\varphi$ in place of φ, we get

$$W_{T,p}(h, 0, \Phi_\varphi) = C_p(V, \alpha, \beta, \psi) \cdot O_{T,p}(\omega(h)\varphi).$$

Note that, globally,

$$\prod_{p \le \infty} C_p(V, \alpha, \beta, \psi) = 1.$$

Lemma 10.4. *(i) Let $L_p \subset V_p$ be an $O_{\boldsymbol{k},p}$-lattice with basis $\{\, v_1, \dots, v_n\,\}$, and let $S = ((v_i, v_j))$ be the matrix for the hermitian form on L_p with respect to this basis. Let $K_p \subset U(V_p) = G_1(\mathbb{Q}_p)$ be the stabilizer of L_p. Then*

$$h^{-1}(S) \cap L_p^n = K_p \cdot \mathbf{v},$$

where $\mathbf{v} = [v_1, \dots, v_n] \in V_p^n$.
(ii) Let dg be the measure on $G_1(\mathbb{Q}_p)$ determined by the gauge form $\nu_{G_1} = i_{\mathbf{v}}^*\nu$. Let $\mathbf{v} \otimes \mathbf{e}$ be the $O_{\boldsymbol{k},p}$-basis $v_i \otimes e_j$ for $L_p^n = L_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^n$, where e_j is the standard basis of $\mathbb{Z}_p^n$. Then, for a

$\mathbb{Z}_p$-basis $1, \xi_p$ for $O_{\boldsymbol{k},p}$, $\mathbf{v}\otimes\mathbf{e}, \xi_p\mathbf{v}\otimes\mathbf{e}$ is a $\mathbb{Z}_p$-basis for L_p^n. Let $\mathbf{w} = [w_1, \dots, w_{n^2}]$ for a $\mathbb{Z}_p$-basis w_j for $\mathrm{Herm}_n(O_{\boldsymbol{k},p})$. Then

$$\mathrm{vol}(K_p, dg) = \frac{|\alpha(\mathbf{v}\otimes\mathbf{e}, \xi_p\mathbf{v}\otimes\mathbf{e})|_p}{|\beta(\mathbf{w})|_p} \cdot |\Delta|_p^{\frac{1}{2}n(n-1)} \cdot \alpha_p(S, S),$$

(iii)

$$\frac{\mathrm{vol}(K_p', dg)}{\mathrm{vol}(K_p, dg)} = \frac{|\det(S')|_p^n\, \alpha_p(S', S')}{|\det(S)|_p^n\, \alpha_p(S, S)}$$

Proof. To prove (i), take $\mathbf{x} \in h^{-1}(S) \cap L_p^n$, and write $\mathbf{x} = g \cdot \mathbf{v} = \mathbf{v} \cdot a$, with $g \in G_1(\mathbb{Q}_p)$ and $a \in \mathrm{GL}_n(\boldsymbol{k}_p) \cap M_n(O_{\boldsymbol{k},p})$. Let $L_p' \subset L_p$ be the $O_{\boldsymbol{k},p}$-lattice spanned by the components of $\mathbf{x}$. Then $L_p' = g \cdot L_p$ and $|L_p : L_p'| = |N(\det a)|_p^{-1}$. But

$$S = (\mathbf{x}, \mathbf{x}) = (\mathbf{v}a, \mathbf{v}a) = {}^t a(\mathbf{v}, \mathbf{v})\bar{a} = {}^t a S \bar{a},$$

so that $N(\det a) = 1$ and $gL_p = L_p$, i.e., $g \in K_p$. Conversely, if $k \in K_p$, then $\mathbf{x} = k \cdot \mathbf{v} \in h^{-1}(S) \cap L_p^n$.

To prove (ii), note that by (i),

$$\mathrm{vol}(K_p, dg) = O_{S,p}(\varphi_p),$$

where φ_p is the characteristic function of L_p^n. Thus,

$$\begin{aligned}\mathrm{vol}(K_p, dg) &= C_p(V, \alpha, \beta, \psi)^{-1}\, W_{S,p}(0, \Phi_{\varphi_p}) \\ &= C_p(V, \alpha, \beta, \psi)^{-1}\, \gamma_p(V)^n\, |\det S|_p^n\, |\Delta|_p^e\, \alpha_p(S, S) \qquad (10.5) \\ &= \frac{c_p(\alpha, \psi)}{c_p(\beta, \psi)}\, |\det S|_p^n\, |\Delta|_p^e\, \alpha_p(S, S).\end{aligned}$$

For an $O_{\boldsymbol{k},p}$-basis of L_p, $\mathbf{v}$ and a $\mathbb{Z}_p$-basis $1, \xi_p$ for $O_{\boldsymbol{k},p}$, write $\mathbf{v}\otimes\mathbf{e}$ for the $O_{\boldsymbol{k},p}$-basis $v_i \otimes e_j$ for $L_p^n = L_p \otimes \mathbb{Z}_p^n$, and note that $\mathbf{v}\otimes\mathbf{e}, \xi_p\mathbf{v}\otimes\mathbf{e}$ is a $\mathbb{Z}_p$-basis for L_p^n. Then

$$\mathrm{vol}(L_p^n, d_{\alpha,p}x) = |\alpha(\mathbf{v}\otimes\mathbf{e}, \xi_p\,\mathbf{v}\otimes\mathbf{e})|_p = c_p(\alpha, \psi)\, |\det S|_p^n |\Delta|_p^{\frac{1}{2}n^2},$$

so that

$$c_p(\alpha, \psi) = |\alpha(\mathbf{v}\otimes\mathbf{e}, \xi_p\,\mathbf{v}\otimes\mathbf{e})|_p\, |\det S|_p^{-n} |\Delta|_p^{-\frac{1}{2}n^2}.$$

Similarly, if $\mathbf{w} = [w_1, \dots, w_{n^2}]$ is a $\mathbb{Z}_p$-basis for $\mathrm{Herm}_n(O_{\boldsymbol{k},p})$, then

$$\mathrm{vol}(\mathrm{Herm}_n(O_{\boldsymbol{k},p}), d_{\beta,p}b) = |\beta(\mathbf{w})|_p = c_p(\beta, \psi)\, |\Delta|_p^{\frac{1}{4}n(n-1)}.$$

Using these expressions in (10.5), we obtain (ii).

Finally, to prove (iii), note that there exists a $\boldsymbol{k}$-linear isomorphism $\gamma : V'(\bar{\mathbb{Q}}) \xrightarrow{\sim} V(\bar{\mathbb{Q}})$ of $\bar{\mathbb{Q}}$-vector spaces such that $(\gamma(x), \gamma(y))_V = (x, y)_{V'}$, for all $x, y \in V'(\bar{\mathbb{Q}})$. This isomorphism induces an isomorphism $\mathrm{Ad}(\gamma) : G_1' \xrightarrow{\sim} G_1$, $g' \mapsto \gamma g' \gamma^{-1}$ of algebraic groups over $\bar{\mathbb{Q}}$. Note that the function $\sigma \mapsto \psi_\sigma = \sigma(\gamma) \circ \gamma^{-1} \in G_1(\bar{\mathbb{Q}})$ on $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ defines the 1-cocycle relating the inner forms G_1 and G_1'.

Let $\alpha' = \gamma^*(\alpha)$ be the top degree form on $(V')^n$ obtained by pulling back α. Since $G_1(\bar{Q})$ acts trivially on $\wedge^{\mathrm{top}}(V)$, this form is rational over $\mathbb{Q}$. Moreover, the gauge form $\nu_{G_1'}$ on G_1' determined by α' and β is the pullback via $\mathrm{Ad}(\gamma)$ of the gauge form ν_{G_1} on G_1 determined by α and β.

With respect to the $\boldsymbol{k}$-bases $\mathbf{v}$ and $\mathbf{v}'$ there exists a matrix $g = a + \delta b$ with $a, b \in M_n(\bar{\mathbb{Q}})$ such that

$$\gamma(\mathbf{v}') = \mathbf{v} \cdot g.$$

Then

$$S' = (\mathbf{v}', \mathbf{v}')_{V'} = (\gamma(\mathbf{v}'), \gamma(\mathbf{v}'))_V = {}^t g(\mathbf{v}, \mathbf{v})_V \bar{g} = {}^t g S \bar{g},$$

where $\bar{g} = a - \delta b$. In particular,

$$N(\det(g)) = \det(S') \det(S)^{-1}.$$

On the other hand,

$$\alpha'(\mathbf{v}' \otimes \mathbf{e}, \xi_p \mathbf{v}' \otimes \mathbf{e}) = \alpha(\gamma(\mathbf{v}') \otimes \mathbf{e}, \xi_p \gamma(\mathbf{v}') \otimes \mathbf{e}) = N(\det(g))^n \, \alpha(\mathbf{v} \otimes \mathbf{e}, \xi_p \mathbf{v} \otimes \mathbf{e}).$$

Using these in the ratio of the expressions in (ii), we obtain (iii). □

Proof of Proposition 10.2. Let V_p'' be the 2-dimensional hermitian space over $\boldsymbol{k}_p$ with $O_{\boldsymbol{k},p}$-lattice L_p'' spanned by the components of $\mathbf{v}'' = [v_1'', v_2'']$ with $(\mathbf{v}'', \mathbf{v}'') = S''$. Let α'' (resp. β'') be a top degree translation invariant form on $(V_p'')^2$ (resp. Herm_2), and let $\nu'' = \alpha''/h^*(\beta'')$ be the corresponding 4-form on $(V_p'')^2$, as above. Let

$$X = \{\mathbf{x} \in (L_p)^2 \mid {}^t\mathbf{x}'' S'' \bar{\mathbf{x}}'' = T \ \}.$$

Then

$$\alpha_p(S'', T'') = \mathrm{vol}(X, d\nu'') \cdot \frac{|\beta''(\mathbf{w}'')|_p}{|\alpha''(\mathbf{v}'' \otimes \mathbf{e}, \xi_p \mathbf{v}'' \otimes \mathbf{e})|_p},$$

where $\mathbf{w}'' = [w_1, \ldots, w_4]$ is a $\mathbb{Z}_p$-basis for $\mathrm{Herm}_2(O_{\boldsymbol{k},p})$. Note that we are now assuming that $\boldsymbol{k}_p/\mathbb{Q}_p$ is unramified. In particular, the space V_p'' is anisotropic. It is then easy to check that, for any $T = \mathrm{diag}(p^a, p^b)$ with $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b$ odd, the set X is non-empty and that

$$X = G_p'' \cdot \mathbf{x}_0'' \xleftarrow{\sim} G_p''$$

for any $\mathbf{x}_0'' \in X$, where $G_p'' = U(V_p'')$. Thus, cf. [11], p.500, for example,

$$\alpha_p(S'', T'') = \mathrm{vol}(G_p'', d\nu_{G_p''}) \cdot \frac{|\beta''(\mathbf{w}'')|_p}{|\alpha''(\mathbf{v}'' \otimes \mathbf{e}, \xi_p \mathbf{v}'' \otimes \mathbf{e})|_p},$$

where, as explained above, the gauge for $\nu_{G_p''}$ is independent of $\mathbf{x}_0'' \in (V_p'')^2$. In particular, $\alpha_p(S'', T'') = \alpha_p(S'', S'')$, as claimed. □

Remark 10.5. Although we do not need it, the value

$$\alpha_p(S', S') = p \cdot (1 + p^{-1}) \prod_{r=1}^{n-1} (1 - (-1)^r p^{-r})$$

is obtained from the general formula in [11], Theorem 7.3.

Part IV. Arithmetic intersection numbers

From now on, we restrict to the case of signature $(n-1, 1)$. Accordingly, we abbreviate $\mathcal{M}(n-1, 1)$ to $\mathcal{M}(n)$, and we write $\mathcal{M}$ for the base change $\mathcal{M}(n) \times_{Spec(O_{\boldsymbol{k}})} \mathcal{M}_0$ of $\mathcal{M}(n)$ to $\mathcal{M}_0$.

11. The main theorem and a conjecture

We first recall from the introduction the following definition.

Definition 11.1. A hermitian matrix $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})_{>0}$ (and the corresponding special cycle $\mathcal{Z}(T)$) is called *non-degenerate* if $\mathcal{Z}(T)$ is of pure dimension $n-m$.
In particular, in the special case $m=1$ any $T = t \in \mathbb{Z}_{>0}$ is non-degenerate.

The following proposition gives a partial characterization of non-degenerate $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$; it follows from [39] and Proposition 6.3 above. Recall that for a nonsingular T in $\mathrm{Herm}_n(O_{\boldsymbol{k}})$, $\mathrm{Diff}_0(T)$ is the set of inert primes p such that $\mathrm{ord}_p \det(T)$ is odd.

Proposition 11.2. *For $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$, suppose that $\mathcal{Z}(T) \neq \emptyset$.*

(i) Suppose that $\mathrm{Diff}_0(T) = \{p\}$ for $p > 2$. Let $r^0(T) = n - \mathrm{rank}(\mathrm{red}(T))$ be the dimension of the radical of the hermitian form $\mathrm{red}(T)$ over $O_{\boldsymbol{k}}/pO_{\boldsymbol{k}}$. Then $\mathcal{Z}(T)$ is equidimensional of dimension

$$\dim \mathcal{Z}(T) = \left[\frac{r^0(T)-1}{2}\right].$$

In particular, T is non-degenerate if and only if T is $\mathrm{GL}_n(O_{\boldsymbol{k},p})$-equivalent to $\mathrm{diag}(1_{n-2}, p^a, p^b)$ with $0 \le a < b$.

(ii) Conversely, suppose that $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ and that, for an odd unramified prime p, $\mathcal{Z}(T) \cap \mathcal{M}_p$ is a nonempty 0-cycle. Then T is non-degenerate with $\mathrm{Diff}_0(T) = \{p\}$.

Proof. By Proposition 2.22, the cycle $\mathcal{Z}(T)$ is empty unless $|\mathrm{Diff}_0(T)| \le 1$. When $|\mathrm{Diff}_0(T)| = 0$, then the cycle $\mathcal{Z}(T)$ is supported in the fibers at ramified primes. Thus, under either of the hypotheses in (i) or (ii), it follows that p is an odd inert prime and that $\mathrm{Diff}_0(T) = \{p\}$. Moreover, by Remark 6.4, $\mathrm{supp}(\mathcal{Z}(T))$ is contained in the component $\mathcal{M}_p^{\tilde{V},\mathrm{ss}}$, where $\tilde{V} \in \mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ is locally isomorphic to V_T at all finite places away from p. Now the completion of $\mathcal{Z}(T)$ along its supersingular locus breaks up into a disjoint sum (6.9) of $\widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T)$, where the pairs $(V^\sharp, V_0)$ run over $\mathcal{R}_{(n-1,1)}(\boldsymbol{k})^\sharp \times \mathcal{R}_{(1,0)}(\boldsymbol{k})$ such that $\mathrm{Hom}_{\boldsymbol{k}}(V_0, V) \simeq \tilde{V}$. By Proposition 6.3, $\widehat{\mathcal{Z}}^{(V^\sharp,V_0),\mathrm{ss}}(T)$ is the stack quotient of the formal scheme

$$\mathrm{Inc}_p(T; V^\sharp, V_0) = \coprod_{(gK^{V^\sharp,p}, g_0K_0^p)} \coprod_{\mathbf{x}^o} \mathcal{Z}(\underline{\underline{\mathbf{x}}}^o), \tag{11.1}$$

by the group $I^V(\mathbb{Q}) \times I^{V_0}(\mathbb{Q})$. It is shown in [39] that $\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)$ is equidimensional of dimension $[(r^0(T)-1)/2]$, and hence is a 0-cycle if and only if T is $\mathrm{GL}_n(O_{\boldsymbol{k},p})$-equivalent to $\mathrm{diag}(1_{n-2}, p^a, p^b)$ with $0 \le a < b$. □

Remark 11.3. To obtain a complete characterization of non-degenerate T's of maximal size it would be necessary to determine what happens for ramified primes and for $p = 2$, e.g., to extend the results of [63], [65] and [39] to such primes.

For positive integers $m_1, \ldots, m_r$ with $\sum_{i=1}^r m_i = n$, we fix $T_i \in \mathrm{Herm}_{m_i}(O_{\boldsymbol{k}})$, with corresponding special cycles $\mathcal{Z}(T_i)$. The fiber product of these cycles over $\mathcal{M}$ decomposes as

$$\mathcal{Z}(T_1) \times_{\mathcal{M}} \times \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r) = \coprod_T \mathcal{Z}(T), \tag{11.2}$$

where T runs over elements of $\mathrm{Herm}_n(O_{\boldsymbol{k}})_{\ge 0}$ with diagonal blocks $T_1, \ldots, T_r$. The decomposition is according to the *fundamental matrix* which, to an S-valued point $(A, \iota, \lambda; E, \iota_0, \lambda_0; \mathbf{x}_1, \ldots, \mathbf{x}_r)$

for a connected scheme S, attaches the matrix $T = h'(\mathbf{x}, \mathbf{x})$, where $\mathbf{x} = [\mathbf{x}_1, \dots, \mathbf{x}_r]$. Here h' is the hermitian form (2.4) on $\mathrm{Hom}_{O_{\boldsymbol{k}}}(E, A)$.

Definition 11.4. For $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with non-degenerate diagonal blocks $T_1, \dots, T_r$, let

$$\langle \mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r) \rangle_T = \sum_p \chi\big(\mathcal{Z}(T)_p, \mathcal{O}_{\mathcal{Z}(T_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(T_r)}\big) \log p. \tag{11.3}$$

Here $\mathcal{Z}(T)_p$ denotes the part of $\mathcal{Z}(T)$ with support in the fiber at p. Note that by Proposition 2.22, $\mathcal{Z}(T)$ has proper support in the special fiber of at most finitely many primes p, so that the sum and the Euler-Poincaré characteristics appearing here are finite. Note that, since $\mathcal{Z}(T)$ is a stack and not a scheme, we have to use here the 'stacky' definition of the Euler-Poincaré characteristic, cf. [10], VI 4.1.

Remark 11.5. Note that, if $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ is non-degenerate, then the diagonal blocks $T_1, \dots, T_r$ are automatically non-degenerate.

By Proposition 2.22, if $|\mathrm{Diff}_0(T)| > 1$, the cycle $\mathcal{Z}(T)$ is empty and the Euler-Poincaré characteristic is zero. On the other hand, if $\mathrm{Diff}_0(T) = \{p\}$, then $\mathcal{Z}(T)$ is supported in the fiber over p of $\mathcal{M}$, and

$$\langle \mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r) \rangle_T = \chi\big(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(T_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(T_r)}\big) \log p. \tag{11.4}$$

Finally, if $\mathrm{Diff}_0(T)$ is empty, then $\mathcal{Z}(T)$ is either empty or supported in the fibers for p ramified in $\boldsymbol{k}$, and the sum in (11.3) runs over such primes.

We expect that $\langle \mathcal{Z}(T_1), \dots, \mathcal{Z}(T_r) \rangle_T$ is the contribution of T to the arithmetic intersection number of the non-degenerate special cycles $\mathcal{Z}(T_1), \dots \mathcal{Z}(T_r)$.

Proposition 11.6. *Let $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with diagonal blocks $T_1, \dots, T_r$ and assume that T is non-degenerate. Then*

$$\mathcal{O}_{\mathcal{Z}(T_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(T_r)} = \mathcal{O}_{\mathcal{Z}(T_1)} \otimes \dots \otimes \mathcal{O}_{\mathcal{Z}(T_r)};$$

more precisely, $\mathcal{O}_{\mathcal{Z}(T_1)} \otimes \dots \otimes \mathcal{O}_{\mathcal{Z}(T_r)}$ represents the left hand side in the derived category.

Proof. We use the p-adic uniformization, Proposition 6.3 of special cycles. We are then reduced to the corresponding statement concerning closed formal subschemes of $\mathcal{N} \simeq \mathcal{N} \times \mathcal{N}_0$: we are given

$$\mathcal{Z}(\underline{\underline{\mathbf{x}}}_i) \subset \mathcal{N}, \quad \underline{\underline{\mathbf{x}}}_i \in \mathrm{Hom}_{O_{\boldsymbol{k}} \otimes \mathbb{Z}_p}(\mathbb{X}_0, \mathbb{X})^{n_i},$$

for $i = 1, \dots, r$. By assumption $\cap_{i=1,\dots,r} \mathcal{Z}(\underline{\underline{\mathbf{x}}}_i)$ is of dimension 0 (and in fact, by [39], Corollary 4.7 reduced to a single point ξ). But by Proposition 3.5 of [39], for any $x \in \mathrm{Hom}_{O_{\boldsymbol{k}} \otimes \mathbb{Z}_p}(\mathbb{X}_0, \mathbb{X})$, the formal subscheme $\mathcal{Z}(x)$ of $\mathcal{N}$ is defined by a single equation. It follows that, denoting by x_i^j $(j = 1, \dots, n_i)$ the components of $\underline{\underline{\mathbf{x}}}_i$, each $\mathcal{Z}(x_i^j)$ is a formal divisor in $\mathcal{N}$, and $\mathcal{Z}(\underline{\underline{\mathbf{x}}}_i)$ is the proper intersection of these divisors. Hence the local equations f_i^j $(j = 1, \dots, n_i)$ of the divisors $\mathcal{Z}(x_i^j)$ form a regular sequence in the regular local ring $\mathcal{O}_{\mathcal{N},\xi}$. This implies that $\mathcal{O}_{\mathcal{Z}(\underline{\underline{\mathbf{x}}}_i)}$ represents the derived tensor product

$$\mathcal{O}_{\mathcal{Z}(x_i^1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_i^{n_i})}.$$

Indeed, each $\mathcal{O}_{\mathcal{Z}(x_i^j)}$ is represented by the complex

$$[\mathcal{O} \xrightarrow{f_i^j} \mathcal{O}].$$

Hence the derived tensor product of all $\mathcal{O}_{\mathcal{Z}(x_i^j)}$ is represented by the Koszul complex [3] of the elements f_i^j $(j=1,\dots n_i)$, which, since the f_i^j form a regular sequence, has as its only homology group the module $\mathcal{O}_{\mathcal{Z}(\underline{\mathbf{x}}_i)}$.
Similarly, since all formal divisors $\mathcal{Z}(x_i^j)$ $(i=1,\dots,r; j=1,\dots,n_i)$ intersect properly, an analogous reasoning proves that all f_i^j form a regular sequence in $\mathcal{O}_{\mathcal{N},\xi}$ and hence the derived tensor product of all $\mathcal{O}_{\mathcal{Z}(x_i^j)}$ is represented by the Koszul complex of the elements f_i^j, which has as its only homology group the module $\mathcal{O}_{\cap\mathcal{Z}(x_i^j)}$. □

Corollary 11.7. *For a non-degenerate $T\in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with diagonal blocks $T_1,\dots,T_r$,*

$$\langle \mathcal{Z}(T_1),\dots,\mathcal{Z}(T_r)\rangle_T = \mathrm{length}(\mathcal{Z}(T))\cdot \log p =: \widehat{\deg}(\mathcal{Z}(T))$$

is the arithmetic degree of the 0-cycle $\mathcal{Z}(T)$. In particular, this quantity is independent of the choice of the sizes of the blocks T_i on the diagonal of T.

Note that the blocks T_i here are automatically nondegenerate, cf. Remark 11.5.

The proof of the following explicit formula for the length will be given in the next section. Before stating the formula, we recall that, for $T\in\mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with $\mathrm{Diff}_0(T)=\{p\}$, an $O_{\boldsymbol{k}}$-lattice M in the positive definite hermitian space V_T determined by T is called *nearly self-dual* if $M\subset M^*$ with $M^*/M\simeq O_{\boldsymbol{k}}/pO_{\boldsymbol{k}}$. If M is such a lattice, then $r_{\mathrm{gen}}(T,M)$, the representation number of T by the $\mathrm{U}(V_T)$-genus of M, is defined by (7.6). Finally, let

$$r_{\mathrm{gen}}(T,V_T)=\sum_{[[M]]} r_{\mathrm{gen}}(T,M),$$

In the sum, $[[M]]$ runs over the genera of nearly self-dual lattices in V_T.

Theorem 11.8. *Let $T\in\mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ be non-degenerate with $\mathcal{Z}(T)\neq\emptyset$ and with $\mathrm{Diff}_0(T)=\{p\}$ for some $p>2$. Then*

$$\mathrm{length}(\mathcal{Z}(T))=\mu_p(T)\cdot\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot r_{\mathrm{gen}}(T,V_T),$$

with

$$\mu_p(T)=\frac{1}{2}\sum_{l=0}^{a}p^l(a+b+1-2l),$$

where T is $\mathrm{GL}_n(O_{\boldsymbol{k},p})$-equivalent to $\mathrm{diag}(1_{n-2},p^a,p^b)$ with $0\le a<b$.

Note that the factor $\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}$ is the degree of the stack $\mathcal{M}_0=\left[\,O_{\boldsymbol{k}}^\times\backslash \mathrm{Spec}\, O_H\,\right]$ over $\mathrm{Spec}\, O_{\boldsymbol{k}}$.

We may now formulate our main result. For each self-dual lattice L in a relevant hermitian space $V\in\mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ there is an incoherent Eisenstein series $E(z,s,L)$ defined by (9.1); it depends only on the G_1^V-genus $[[L]]$ of L. Let

$$E(z,s,V)=\sum_{[[L]]}E(z,s,L), \tag{11.5}$$

be the sum of these series over the G_1^V-genera of self-dual lattices in V, cf. Corollary 2.16. We also consider the Eisenstein series

$$E(z,s)=\sum_{V\in\mathcal{R}_{(n-r,r)}(\boldsymbol{k})}E(z,s,V)$$

obtained by summing over all self-dual genera.

Theorem 11.9. *Let $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ be non-degenerate with $\mathcal{Z}(T) \neq \emptyset$ and with $\mathrm{Diff}_0(T) = \{p\}$ for some $p > 2$. Let V_T be the positive definite hermitian space of dimension n determined by the matrix T. Then*

$$E'_T(z, 0) = E'_T(z, 0, V) = C_1 \cdot \widehat{\deg}(\mathcal{Z}(T)) \cdot q^T,$$

where $V \in \mathcal{R}_{(n-1,1)}(\boldsymbol{k})$ is the unique relevant hermitian space that is locally isomorphic to V_T at all primes other than ∞ and p, and $E(z, s, V)$ is the corresponding incoherent Eisenstein series defined by (11.5). Here

$$C_1 = (-1)^n \frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \operatorname{vol}(G_1^{V_T}(\mathbb{R})) \operatorname{vol}(K_1),$$

for $K_1 \subset G_1^V(\mathbb{A}_f)$ the stabilizer of a self-dual $O_{\boldsymbol{k}}$-lattice L in V and for the measure normalized as in sections 7 and 8.

Note that the factor $\operatorname{vol}(G_1^{V_T}(\mathbb{R})) \operatorname{vol}(K_1)$ is a (stacky) Euler characteristic, cf. Lemma 9.5.

Proof. This result is an immediate consequence of the formulas for the two sides given in Corollary 9.4 and Theorem 11.8 respectively. ∎

Note that this identity can be rephrased as saying that, for T as in the theorem,

$$E'_T(z, 0, V) = C_1 \cdot \langle \mathcal{Z}(T_1), \ldots, \mathcal{Z}(T_r) \rangle_T \, q^T,$$

where $T_1, \ldots, T_r$ are diagonal blocks of T, as above.

We have the following conjecture.

Conjecture 11.10. *Let $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0}$ with non-degenerate diagonal blocks $T_1, \ldots, T_r$ and assume that $\mathrm{Diff}_0(T) = \{p\}$ for $p > 2$. Suppose that $\mathcal{Z}(T) \neq \emptyset$. Let V_T, V, and $E(z, s, V)$ be as in Theorem 11.9. Then*

$$E'_T(z, 0) = E'_T(z, 0, V) = C_1 \cdot \langle \mathcal{Z}(T_1), \ldots, \mathcal{Z}(T_r) \rangle_T \cdot q^T.$$

The point here is that, when the cycle $\mathcal{Z}(T)$ has positive dimension, there is no intrinsic definition of an arithmetic degree $\widehat{\deg}\, \mathcal{Z}(T)$. Instead, as in Conjecture 11.10, we assign to T the following quantity. Let $t_1, \ldots, t_n \in \mathbb{Z}_{>0}$ be the diagonal entries of T. Then consider

$$\widehat{\deg}\, \mathcal{Z}(T) := \langle \mathcal{Z}(t_1), \ldots, \mathcal{Z}(t_r) \rangle_T. \tag{11.6}$$

The conjecture includes the assertion that $\langle \mathcal{Z}(T_1), \ldots, \mathcal{Z}(T_r) \rangle_T$ should always be equal to this quantity, no matter what ordered partition of n is used to divide T up into blocks, as long as the blocks $T_1, \ldots, T_r$ on the diagonal are non-degenerate. Also note that due to the invariance property of the Fourier coefficients $E'_T(z, 0, V)$ under the action of $\mathrm{GL}_n(O_{\boldsymbol{k}})$ coming from the transformation law, the conjecture also implies that the following invariance property of the arithmetic intersection numbers (11.6) should hold. If $T' = gT{}^t\bar{g}$ for some $g \in \mathrm{GL}_n(O_{\boldsymbol{k}})$, then $\widehat{\deg}\, \mathcal{Z}(T) = \widehat{\deg}\, \mathcal{Z}(T')$. The picture is consistent with what is proved in [37] in the case of degenerate intersections of special divisors on Shimura curves and in [60] and [61] in the case of degenerate intersections of special divisors on Hilbert modular surfaces.

12. The arithmetic degree in the non-degenerate case

In this section we prove Theorem 11.8. Recall that there is a decomposition

$$\widehat{\mathcal{Z}}(T) = \coprod_{(V^\sharp, V_0)} \widehat{\mathcal{Z}}^{(V^\sharp, V_0),\mathrm{ss}}(T). \tag{12.1}$$

We use the p-adic uniformization of the special cycle $\widehat{\mathcal{Z}}^{(V^\sharp, V_0),\mathrm{ss}}(T)$ given in Proposition 6.3. As in [39], we identify $\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)$ with a closed formal subscheme of $\mathcal{N}$. By Theorem 5.1 of [39], the length of $\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)$ depends only on T and is equal to

$$\mathrm{length}(\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)) = \mu_p(T) = \frac{1}{2}\sum_{l=0}^{a} p^l\,(a+b+1-2l). \tag{12.2}$$

Therefore, we have

$$\mathrm{length}(\widehat{\mathcal{Z}}^{(V^\sharp, V_0),\mathrm{ss}}(T)) = \mu_p(T)\cdot |\big[(I^V(\mathbb{Q})\times I^{V_0}(\mathbb{Q}))\backslash \mathrm{Inc}_p(T; V^\sharp, V_0)(\mathbb{F})\big]|, \tag{12.3}$$

where the second factor on the right side is the (stack) cardinality of the quotient.

Recall from Lemma 6.1, a) and Remark 6.4 that the hermitian space $\tilde{V}' = \mathrm{Hom}^0_{O_{\boldsymbol{k}}}(E^o, A^o)$ is positive definite and has invariants $\mathrm{inv}_p(\tilde{V}') = -\mathrm{inv}_p(\tilde{V}) = -1$, and $\mathrm{inv}_\ell(\tilde{V}') = \mathrm{inv}_\ell(\tilde{V})$ for $\ell \neq p$, where we recall that $\tilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0, V)$. Let $G' = \mathrm{GU}(\tilde{V}')$ and $G'_1 = \mathrm{U}(\tilde{V}')$, and recall from Lemma 6.1, b) that $I^V(\mathbb{Q}) \simeq G'_1(\mathbb{Q})$ and $I^{V_0}(\mathbb{Q}) = \boldsymbol{k}^1$.

We also recall that, by Theorem 4.5 of [39],

$$\mathcal{Z}(\underline{\underline{\mathbf{x}}}^o)(\mathbb{F}) = \mathcal{V}(\Lambda)(\mathbb{F}), \tag{12.4}$$

is a single point, where Λ is the unique vertex of level 0 and type 1 in[9] $\tilde{V}'_p$ such that Λ^* contains the components of $\underline{\underline{\mathbf{x}}}^o$ and where $\mathcal{V}(\Lambda)$ is the stratum associated to Λ in $\mathcal{N}$. Recall here from [39] that a *vertex of level j and type t* is a lattice Λ in $\tilde{V}'_p$ such that

$$p^j\Lambda^* \overset{t}{\subset} \Lambda \subset p^{j-1}\Lambda^*, \tag{12.5}$$

where[10]

$$\Lambda^* = \{\, x \in \tilde{V}'_p \mid (\Lambda, x) \subset O_{\boldsymbol{k},p} \,\},$$

and where the notation $p^j\Lambda^* \overset{t}{\subset} \Lambda$ means that $\Lambda/p^j\Lambda^* \simeq (O_{\boldsymbol{k}}/pO_{\boldsymbol{k}})^t$. Also note that, if $g \in G'(\mathbb{Q}_p)$ with $\mathrm{ord}_p\nu(g) = r$, then $\nu(g)(g\Lambda)^* = g(\Lambda^*)$ and hence $g\Lambda$ has level $j+r$ and type t. In particular the group $G'(\mathbb{Q}_p)^0$ (the subgroup of elements with scale factor a p-adic unit) acts on the set of lattices of level 0 and type t. The following fact is easily checked, cf. [24], section 7.

Lemma 12.1. *Let $\mathcal{L}_t$ be the set of lattices in $\tilde{V}'_p$ of level 0 and type t. Then $G'(\mathbb{Q}_p)^0$ acts transitively on $\mathcal{L}_t$.* □

We fix a lattice $\Lambda \in \mathcal{L}_1$ and let K'_p be its stabilizer in $G'(\mathbb{Q}_p)$. Note that $K'_p \subset G'(\mathbb{Q}_p)^0$.

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[9] Here we are rephrasing the result of [39] in terms of $\tilde{V}'_p$. Note that, by Lemma 3.9 in [39], the space C, $\{\ ,\ \}$ coincides with the space $\mathbb{V} = \tilde{V}'_p$ but with the hermitian form scaled by p, i.e., $\{\ ,\ \} = p\,(\ ,\ )$. Since $\mathrm{ord}_p \det C$ is 0 if n is odd and 1 if n is even, we have $\mathrm{inv}_p(\tilde{V}') = -1$, as required.

[10] In [39], this relation is expressed in terms of

$$p^{-1}\Lambda^* = \Lambda^\vee = \{\, x \in C \mid \{\Lambda, x\} \subset O_{\boldsymbol{k},p} \,\}.$$

Recall that K^p is the stabilizer of the $\widehat{\mathbb{Z}}^p$-lattice $\mathrm{Hom}_{O_{\boldsymbol{k}}\otimes\widehat{\mathbb{Z}}^p}(T^p(E^o), T^p(A^o))$, where we write $K^p = K^{V^\sharp,p}$ to simplify the notation. Thus, we must compute the stack cardinality of the quotient of

$$\mathrm{Inc}_p(T; V^\sharp, V_0)(\mathbb{F}) = \coprod_{(gK^p,\, g_0K_0^p)} \ \coprod_{g'_pK'_p\in G'(\mathbb{Q}_p)^0/K'_p} \ \coprod_{\mathbf{x}^o} \{\mathrm{pt}\} \tag{12.6}$$

by the action of $G'_1(\mathbb{Q}) \times I^{V_0}(\mathbb{Q})$. Here, the last union runs over the set of $\mathbf{x}^o \in V'(\mathbb{Q})^n$ such that

(i) $h'(\mathbf{x}^o, \mathbf{x}^o) = T$,
(ii) $g^{-1} \circ \underline{\mathbf{x}}^o \circ g_0 \in \big(\mathrm{Hom}_{O_{\boldsymbol{k}}\otimes\widehat{\mathbb{Z}}^p}(T^p(E^o), T^p(A^o))\big)^n$,
(iii) $\underline{\underline{\mathbf{x}}}^o \in g'_p\Lambda^*$.

The combination of (6.2) and (6.7) determines an isometry $\tilde{V}(\mathbb{A}_f^p) \simeq \tilde{V}'(\mathbb{A}_f^p)$ compatible with the action of $I^V(\mathbb{Q})$ on both sides. Define a lattice $\tilde{L}'$ in $\tilde{V}'$ by taking $\tilde{L}'_p = \Lambda^*$ and

$$\tilde{L}' \otimes \widehat{\mathbb{Z}}^p = \mathrm{Hom}_{O_{\boldsymbol{k}}\otimes\widehat{\mathbb{Z}}^p}(T^p(E^o), T^p(A^o)).$$

The stabilizer of $\tilde{L}' \otimes \widehat{\mathbb{Z}}$ in $G'(\mathbb{A}_f)$ is $K' = K'_pK'^p$, where $K'^p = K^p$ under the identification of $V(\mathbb{A}_f^p)$ with $V'(\mathbb{A}_f^p)$. Note that this lattice is nearly self-dual, i.e.,

$$(\tilde{L}')^*/\tilde{L}' \simeq O_{\boldsymbol{k}}/pO_{\boldsymbol{k}}. \tag{12.7}$$

Next observe that

$$\nu(K') = \nu(G'(\mathbb{A}_f)^0),$$

so that

$$G'(\mathbb{A}_f)^0/K' \simeq G'_1(\mathbb{A}_f)/K'_1$$

for $K'_1 = K' \cap G'_1(\mathbb{A}_f)$. The cosets $g'K'_1$ in $G'_1(\mathbb{A}_f)/K'_1$ correspond to the lattices $\tilde{L}'' = g'\tilde{L}'$ in the G'_1-genus of $\tilde{L}'$, via

$$\tilde{L}'' = \tilde{V}' \cap \big(g' \cdot (\tilde{L}' \otimes \widehat{\mathbb{Z}})\big).$$

For any such lattice, $\tilde{L}'' \subset \tilde{V}'$, corresponding to $g'K'_1$, we let

$$\Omega(T, \tilde{L}'') = \{\, \mathbf{x} \in (\tilde{L}'')^n \ \mid\ h'(\mathbf{x}, \mathbf{x}) = T \,\}$$

and

$$\Gamma(\tilde{L}'') = G'_1(\mathbb{Q}) \cap g'K'_1(g')^{-1}.$$

Note that this is a finite group.

Observe that since the components of any $\mathbf{x} \in \Omega(T, \tilde{L}'')$ span $\tilde{V}'$, the stabilizer $\Gamma(\tilde{L}'')_{\mathbf{x}}$ is trivial. This implies that

$$\Big[\, G'_1(\mathbb{Q})\backslash \coprod_{gK^p} \ \coprod_{g'_pK'_p} \ \coprod_{\mathbf{x}^o} \{\mathrm{pt}\} \,\Big] = \coprod_{\tilde{L}''} \Gamma(\tilde{L}'')\backslash\Omega(T, \tilde{L}'')$$

as orbifold quotients. Here, on the right hand side, $\tilde{L}''$ runs over the classes of lattices in the G'_1-genus of $\tilde{L}'$. Thus

$$\Big|\, \Big[\, G'_1(\mathbb{Q})\backslash \coprod_{gK^p} \ \coprod_{g'_pK'_p} \ \coprod_{\mathbf{x}^o} \{\mathrm{pt}\} \,\Big] \,\Big| = \sum_{\tilde{L}''} |\Gamma(\tilde{L}'')|^{-1}\, |\Omega(T, \tilde{L}'')\,| = r_{\mathrm{gen}}(T, \tilde{L}').$$

This is not yet the cardinality of the stack quotient of (12.6); we have to take into account the role of the cosets $g_0K_0^p$ and the action of $I^{V_0}(\mathbb{Q})$. Here $g_0K_0^p$ runs over $G^{V_0}(\mathbb{A}_f^p)^0/K_0^p \simeq$

$G_1^{V_0}(\mathbb{A}_f^p)/K_{0,1}^p$. For any $g_0 \in G_1^{V_0}(\mathbb{A}_f^p) = \boldsymbol{k}^1_{\mathbb{A}_f^p}$, as the lattice $\tilde{L}''$ runs over a set of representatives for the G_1'-genus of $\tilde{L}'$, so does the lattice $\tilde{L}'' g_0^{-1}$. Note that

$$\big|\,[\,I^{V_0}(\mathbb{Q})\backslash G_1^{V_0}(\mathbb{A}_f^p)/K_{0,1}^p\,]\,\big| = \frac{1}{w_{\boldsymbol{k}}}\cdot|\boldsymbol{k}^1\backslash\boldsymbol{k}^1_{\mathbb{A}_f}/\widehat{O}_{\boldsymbol{k}}^\times| = \frac{h_{\boldsymbol{k}}}{2^{\delta-1}\,w_{\boldsymbol{k}}}.$$

Thus, we obtain the following explicit formula for the stack cardinality.

Lemma 12.2.

$$\big|\,[\,(I^V(\mathbb{Q})\times I^{V_0}(\mathbb{Q}))\backslash \mathrm{Inc}_p(T;V^\sharp,V_0)(\mathbb{F})\,]\,\big| = \frac{h_{\boldsymbol{k}}}{2^{\delta-1}\,w_{\boldsymbol{k}}}\cdot r_{\mathrm{gen}}(T,\tilde{L}').$$

Note that the right side here depends only on $\tilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0,V)$ and the type of the genus in $V^\sharp = (V,[[L]])$. Moreover, $r_{\mathrm{gen}}(T,\tilde{L}')$ vanishes unless $\tilde{V}$ represents T, i.e., $\tilde{V}\simeq V_T$. The number of pairs (V,V_0) occurring in the decomposition (12.1) such that $\tilde{V} = \mathrm{Hom}_{\boldsymbol{k}}(V_0,V)\simeq V_T$ is $2^{\delta-1}$. As we vary the choice of $[[L]]$ in V for a given such pair (V,V_0), $\tilde{L}'$ runs over the $G_1^{\tilde{V}}$-genera of nearly self-dual lattices in $\tilde{V}$. Taking these facts into account, Theorem 11.8 then follows from Lemma 12.2 together with the formulas (12.2) and (12.3).

Part V. Examples

13. Level structures

In this section we define variants of our moduli problem $\mathcal{M}(\boldsymbol{k};n-r,r)$, and discuss how to extend our main results to these cases. These variants actually show up in the examples discussed in the following sections.

13.1. **Special parahoric level structures at ramified primes.** For every $p\mid\Delta$, we fix an *even* integer $t(p)$ with $0\le t(p)\le n$ (the *type* of p). We denote by $\mathbf{t}$ the function $p\mapsto t(p)$ on the set of divisors of Δ. We then introduce the following variant $\mathcal{M}(\boldsymbol{k},\mathbf{t};n-r,r)^{*,\mathrm{naive}}$ of the stack $\mathcal{M}(\boldsymbol{k};n-r,r)^{\mathrm{naive}}$ over $(\mathrm{Sch}/\mathrm{Spec}\,O_{\boldsymbol{k}})$ of section 2. It parametrizes objects (A,ι,λ) as in the definition of $\mathcal{M}(\boldsymbol{k};n-r,r)^{\mathrm{naive}}$, except that the condition that the polarization λ be principal is replaced by the following condition. We require that $\ker\lambda\subset A[\Delta]$, so that $O_{\boldsymbol{k}}/(\Delta)$ acts on $\ker\lambda$. In addition, we require that this action factors through the factor ring $\prod_{p\mid\Delta}\mathbb{F}_p$ of $O_{\boldsymbol{k}}/(\Delta)$, and that the height of $\ker\lambda$ be equal to $\prod_{p\mid\Delta}p^{t(p)}$. Note that if $\mathbf{t}=0$, then $\mathcal{M}(\boldsymbol{k},\mathbf{t};n-r,r)^{*,\mathrm{naive}}$ coincides with $\mathcal{M}(\boldsymbol{k};n-r,r)^{\mathrm{naive}}$. When $\boldsymbol{k}$ and $\mathbf{t}$ are understood, we simply write $\mathcal{M}(n-r,r)^{*,\mathrm{naive}}$ for this stack.

Then $\mathcal{M}(n-r,r)^{*,\mathrm{naive}}$ is a Deligne-Mumford stack over $\mathrm{Spec}\,O_{\boldsymbol{k}}$ which is smooth of relative dimension $(n-r)r$ over $\mathrm{Spec}\,O_{\boldsymbol{k}}[\Delta^{-1}]$, provided it is not empty (this may happen, see below). At the primes $p\mid\Delta$, the stack is not flat in general. We define $\mathcal{M}(n-r,r)^*$ to be the flat closure of $\mathcal{M}(n-r,r)^{*,\mathrm{naive}}\times_{\mathrm{Spec}\,O_{\boldsymbol{k}}}\mathrm{Spec}\,O_{\boldsymbol{k}}[\Delta^{-1}]$ in $\mathcal{M}(n-r,r)^{*,\mathrm{naive}}$.

Proposition 13.1. *Let $r=1$ and assume $2\nmid\Delta$.*

(i) If $\mathbf{t}=0$, then $\mathcal{M}(\boldsymbol{k},\mathbf{t};n-1,1)^*=\mathcal{M}(\boldsymbol{k};n-1,1)$.

(ii) At all primes $p\mid\Delta$ with $t(p)=0$, $\mathcal{M}(\boldsymbol{k},\mathbf{t};n-1,1)^*$ is Cohen-Macaulay, normal, and regular outside finitely many points. If $n\ge3$, the special fibers at such primes p are irreducible, reduced,

normal and with isolated rational singularities. If $n = 2$, then $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-1, 1)^*$ has semi-stable, but non-smooth, reduction at such primes.

(iii) If n is even, then $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-1, 1)^*$ is smooth at all primes $p \mid \Delta$ with $t(p) = n$.

(iv) If n is odd, then $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-1, 1)^*$ is smooth at all primes $p \mid \Delta$ with $t(p) = n - 1$.

Proof. Statement (i) is Pappas' Proposition 2.5, which is [46], Theorem 4.5, and statement (ii) is also contained in loc. cit.; statement (iii) is [49], §5.c., and statement (iv) is [2], Prop. 4.16. □

We extend the definition of the special cycles in the obvious way: for a hermitian matrix $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$, we define the Deligne-Mumford stack $\mathcal{Z}(T)^*$ equipped with a natural morphism to $\mathcal{M}^* = \mathcal{M}(n-r, r)^* \times \mathcal{M}_0$ as the stack that parametrizes collections $(A, \iota, \lambda; E, \iota_0, \lambda_0; \mathbf{x})$, where $\mathbf{x} = [x_1, \dots, x_m] \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(E, A)^m$ is an m-tuple of homomorphisms such that $h'(\mathbf{x}, \mathbf{x}) = T$. If S is not connected, we require these conditions on each connected component.

We expect that with these definitions all results of the previous sections can be transposed to these cases. Let us illustrate this by discussing the complex uniformization of $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^*$.

Definition 13.2. An $O_{\boldsymbol{k}}$-module L equipped with a $\boldsymbol{k}$-valued hermitian form of signature $(n - r, r)$ is called[11] *of type* $\mathbf{t}$ if $L \subset L^\vee \subset \Delta^{-1}L$ and $L^\vee/L \simeq \prod_{p\mid\Delta} \mathbb{F}_p^{t(p)}$.

In particular, an $O_{\boldsymbol{k}}$-module of type $\mathbf{t} = 0$ is a self-dual hermitian $O_{\boldsymbol{k}}$-module. We note that for any $O_{\boldsymbol{k}}$-module of type $\mathbf{t}$, the hermitian space $V = L \otimes \mathbb{Q}$ is relevant. Indeed, to check the existence of a self-dual lattice in V, it suffices to check this locally at any inert prime p. But for such p a self-dual lattice in V_p is given by $L \otimes \mathbb{Z}_p = L^\vee \otimes \mathbb{Z}_p$. We also note that, if $2 \nmid \Delta$, lattices of type $\mathbf{t}$ always exist in a given relevant hermitian space V, except in the case when n is even and $t(p) = n$ for some $p \mid \Delta$ with $\mathrm{inv}_p(V) = -1$. Indeed, this is a local question at primes p dividing Δ. Fix a self-dual lattice Λ in V_p. Then, up to conjugacy under the unitary group $\mathrm{U}(V_p)$, the lattices L in V_p with $L \subset L^\vee \subset \pi^{-1}L$ such that $L^\vee/L \simeq \mathbb{F}_p^{t(p)}$ correspond to the totally isotropic subspaces of $\Lambda/\pi\Lambda$ of dimension $t(p)/2$ with respect to the non-degenerate symmetric form induced by the hermitian form. Such subspaces always exist except in the case when n is even and $t(p) = n$, in which case they exist if and only if $\mathrm{inv}_p(V) = 1$.

Let $\mathcal{L}_{(n-r,r)}(\boldsymbol{k}, \mathbf{t})$ be the set of isomorphism classes of hermitian $O_{\boldsymbol{k}}$-modules of signature $(n-r, r)$ and type $\mathbf{t}$. Now the complex uniformization of $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^*$ is given by the following analogue of Proposition 3.1.

Proposition 13.3. *There is an isomorphism of orbifolds*

$$\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^*(\mathbb{C}) \xrightarrow{\sim} \coprod_L \, [\Gamma_L \backslash D(L)],$$

where L runs over $\mathcal{L}_{(n-r,r)}(\boldsymbol{k}, \mathbf{t})$. □

Remark 13.4. By our remarks above, the set $\mathcal{L}_{(n-r,r)}(\boldsymbol{k}, \mathbf{t})$ is either empty or is in bijective correspondence with $\mathcal{L}_{(n-r,r)}(\boldsymbol{k})$, provided that $2 \nmid \Delta$.

[11] Hopefully, there will be no confusion between this notion of type and the type of a G_1^V-genus which occurs in earlier sections.

13.2. **Level structures at unramified primes.** Now we discuss how to introduce level structures at unramified primes.

Definition 13.5. Let N be an odd positive integer prime to Δ. A *level N-structure* on an object (A, ι, λ) of $\mathcal{M}(\boldsymbol{k}; n-r, r)(S)$ is an $O_{\boldsymbol{k}}$-linear isomorphism of finite group schemes

$$A[N] \xrightarrow{\sim} (O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n_S,$$

compatible with the hermitian form associated to the Riemann form corresponding to λ on the LHS and the standard hermitian form on the RHS, up to a scalar in $(\mathbb{Z}/N\mathbb{Z})^\times$.

Here the standard hermitian form h on $(O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n$ with values in $O_{\boldsymbol{k}}/NO_{\boldsymbol{k}}$ is given in terms of the canonical basis by $h(e_i, e_{n-j+1}) = \delta_{ij}, \forall i, j = 1, \ldots, n$. It induces a similar form on the constant group scheme over S. Note that a level N-structure can only exist if N is invertible on S. The compatibility condition is independent of the choice of trivialization of the N-th roots of unity needed for the comparison with the standard form. We also note that if L is a self-dual $O_{\boldsymbol{k}}$-module, then L/NL is isomorphic to $(O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n$ with the standard form.

We obtain a Deligne-Mumford stack $\mathcal{M}(\boldsymbol{k}; n-r, r)_{(N)}$ over $\mathcal{M}(\boldsymbol{k}; n-r, r)[N^{-1}]$ which parametrizes collections $(A, \iota, \lambda, \eta)$ where η is a level N-structure. It is smooth of relative dimension $(n-r)r$ over $\operatorname{Spec} O_{\boldsymbol{k}}[(N\Delta)^{-1}]$. Note that if $N \geq 3$, then, by Serre's Lemma, $\mathcal{M}(\boldsymbol{k}; n-r, r)_{(N)}$ is a scheme.

Remark 13.6. A variant of the preceding definition arises from a subgroup $\bar{K}$ of $\mathrm{GU}\big((O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n\big)$, defining a level $\bar{K}$-structure to be an isomorphism between $A[N]$ and $(O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n_S$, given modulo $\bar{K}$. In this way we obtain a Deligne-Mumford stack $\mathcal{M}(\boldsymbol{k}; n-r, r)_{(N),\bar{K}}$ over $\operatorname{Spec} O_{\boldsymbol{k}}[N^{-1}]$. For instance, if $\bar{K}$ is the subgroup preserving the standard flag spanned by $e_1, e_2, \ldots, e_n$, then a level $\bar{K}$-structure corresponds to a complete flag of primitive $O_{\boldsymbol{k}}$-submodules in $A[N]$, which is self-dual for the Riemann form.

There are adelic variants of these definitions. Let $\mathbb{Q}_N = \prod_{\ell | N} \mathbb{Q}_\ell$ and $\mathbb{Z}_N = \prod_{\ell | N} \mathbb{Z}_\ell$. Then a $\Gamma(N)$-*level structure* on an object (A, ι, λ) is a class of $\boldsymbol{k}$-linear isomorphisms which respect the hermitian forms up to a scalar in $\mathbb{Q}_N^\times$,

$$\eta : T_N(A)^o \xrightarrow{\sim} \boldsymbol{k}^n \otimes \mathbb{Q}_N,$$

given modulo $\Gamma(N)$. Here $T_N(A)^o = \prod_{\ell | N} T_\ell(A)^o$ is the product of the rational Tate modules for $\ell \mid N$, and $\Gamma(N)$ denotes the principal congruence subgroup of level N of $\Gamma(1) = \mathrm{GU}\big(O_{\boldsymbol{k}}^n \otimes \mathbb{Z}_N\big)$. A level N-structure is equivalent to a $\Gamma(N)$-level structure η such that $\eta\big(T_N(A)\big) = O_{\boldsymbol{k}}^n \otimes \mathbb{Z}_N$. More generally, if K is an open compact subgroup of $\mathrm{GU}(k^n \otimes \mathbb{Q}_N)$, one defines the notion of a K-*level structure* as the datum of η as above, given modulo K. For a subgroup $\bar{K} \subset \mathrm{GU}\big((O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})^n\big)$, a level $\bar{K}$-structure is equivalent to giving a K-level structure η with $\eta\big(T_N(A)\big) = O_{\boldsymbol{k}}^n \otimes \mathbb{Z}_N$, where K is the inverse image of $\bar{K}$ in $\Gamma(1)$.

The definition of our special cycles can now be extended as follows to the cases with level structures. Let $K_0 \subset \mathrm{GU}\big(\boldsymbol{k} \otimes \mathbb{Q}_N\big)$ and $K \subset \mathrm{GU}\big(\boldsymbol{k}^n \otimes \mathbb{Q}_N\big)$ be open compact subgroups, with associated moduli stacks $\mathcal{M}(\boldsymbol{k}; 1, 0)_{K_0}$ and $\mathcal{M}(\boldsymbol{k}; n-r, r)_K$. Let m be a positive integer, and fix a compact open subset

$$\omega \subset \mathrm{Hom}_{\boldsymbol{k}}\big(\boldsymbol{k} \otimes \mathbb{Q}_N, \boldsymbol{k}^n \otimes \mathbb{Q}_N\big)^m$$

stable under the action of $K_0 \times K$. Then for $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}}[N^{-1}])$, we define the stack $\mathcal{Z}(T, \omega)$ over $\mathcal{M}(\boldsymbol{k}; n-r, r)_K \times \mathcal{M}(\boldsymbol{k}; 1, 0)_{K_0}$ as parametrizing the collections $(A, \iota, \lambda, \eta, E, \iota_0, \lambda_0, \eta_0; \mathbf{x})$, where $\mathbf{x} \in \big(\mathrm{Hom}_{O_{\boldsymbol{k}}}(E, A) \otimes O_{\boldsymbol{k}}[N^{-1}]\big)^m$ satisfies the following two conditions

(i) $h'(\mathbf{x},\mathbf{x}) = T$

(ii) $\eta \circ \mathbf{x} \circ \eta_0^{-1} \in \omega$.

This definition is analogous to the definition of special cycles adopted in our previous papers [30, 36, 38]. Note that if K_0 and K are trivial, i.e., equal to $\Gamma(1)_0$ and $\Gamma(1)$ resp., so that $\mathcal{M}(\boldsymbol{k}; n-r, r)_K \times \mathcal{M}(\boldsymbol{k}; 1, 0)_{K_0} = \mathcal{M}(\boldsymbol{k}; n-r, r) \times \mathcal{M}(\boldsymbol{k}; 1, 0)$, and if $\omega = \mathrm{Hom}_{O_{\boldsymbol{k}}}\big(O_{\boldsymbol{k}} \otimes \mathbb{Z}_N, O_{\boldsymbol{k}}^n \otimes \mathbb{Z}_N\big)^m$ and $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$, then the stack $\mathcal{Z}(T, \omega)$ is equal to the previously defined stack $\mathcal{Z}(T)$.

Remark 13.7. Of course, one may also mix the two kinds of level structures and define in this way stacks $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^*_K$ over $\mathrm{Spec}\, O_{\boldsymbol{k}}[N^{-1}]$, and corresponding special cycles.

We expect that the results of the previous sections can be extended to the moduli stacks $\mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^*_K$. For the analogues of Theorems 11.8 and 11.9, one has to assume that $\mathrm{Diff}_0(T) = \{p\}$, where $p \nmid N$ is odd.

14. The case $n = 2$

In this section, we illustrate our general results in the case $n = 2$. In particular, we explain the relation of our special cycles to the Heegner points defined in the classic paper of Gross and Zagier, [14].

Let $\mathcal{E}$ be the moduli stack of elliptic curves over $\mathrm{Spec}\, \mathbb{Z}$. For a scheme S, an object $E \in \mathcal{E}(S)$, and an $O_{\boldsymbol{k}}$-module M, we obtain an abelian scheme $M \otimes_{\mathbb{Z}} E$ over S by the Serre construction, [53]. Then $M \otimes_{\mathbb{Z}} E$ has a natural $O_{\boldsymbol{k}}$-action, and the dual abelian scheme is given by

$$(M \otimes_{\mathbb{Z}} E)^\vee \simeq M^\vee \otimes_{\mathbb{Z}} E^\vee,$$

where $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ with its natural $O_{\boldsymbol{k}}$-action. If we take a fractional ideal $M = \mathfrak{a}$, then the abelian scheme $\mathfrak{a} \otimes_{\mathbb{Z}} E$ has relative dimension 2 over S and the $O_{\boldsymbol{k}}$-action satisfies the $(1,1)$-signature condition

$$\mathrm{char}(T, \iota(a) \mid \mathrm{Lie}\, A) = T^2 - \mathrm{tr}(a)\, T + N(a) \ \in \mathbb{Z}[T], \qquad a \in O_{\boldsymbol{k}}.$$

Note that there is an $O_{\boldsymbol{k}}$-antilinear isomorphism

$$\partial^{-1}\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^\vee, \qquad a \mapsto N(\mathfrak{a})^{-1}\, (\cdot, a)_{\boldsymbol{k}},$$

where $(x, y)_{\boldsymbol{k}} = \mathrm{tr}(x y^\sigma)$ is the trace form of $\boldsymbol{k}/\mathbb{Q}$, and ∂^{-1} is the inverse different. The canonical principal polarization $\lambda_E : E \xrightarrow{\sim} E^\vee$ gives a polarization

$$\lambda : \mathfrak{a} \otimes E \xrightarrow{\sim} \mathfrak{a} \otimes E^\vee \longrightarrow \partial^{-1}\mathfrak{a} \otimes E^\vee \simeq \mathfrak{a}^\vee \otimes E^\vee,$$

where middle arrow is induced by the inclusion $\mathfrak{a} \subset \partial^{-1}\mathfrak{a}$, and hence has degree $|\Delta|$. Thus, we obtain a functor

$$j_{\mathfrak{a}} : \mathcal{E} \longrightarrow \mathcal{M}^{\mathrm{spl}}(1,1)^*, \qquad E \mapsto (\mathfrak{a} \otimes_{\mathbb{Z}} E, \iota, \lambda). \tag{14.1}$$

where $\mathcal{M}^{\mathrm{spl}}(1,1)^*$ is a moduli stack that we now explain.

Let $\mathcal{M}(1,1)^*$ be the moduli stack over $\mathrm{Spec}\, \mathbb{Z}$ for triples $(A, \iota, \lambda)/S$, where A is an abelian scheme over S, ι is an action of $O_{\boldsymbol{k}}$ on A satisfying the $(1,1)$-signature condition, and $\lambda : A \longrightarrow A^\vee$ is a polarization with corresponding Rosati involution satisfying $\iota(a)^* = \iota(a^\sigma)$. Finally, we require that $\ker(\lambda) = A[\partial]$. If $\Delta = N(\partial)$ is odd, this moduli problem was introduced in the previous section, and corresponds to the function $\mathbf{t}(p) = 2$, for all $p \mid \Delta$. Since the polarization λ is principal away from Δ, there is a decomposition

$$\mathcal{M}(1,1)^* = \coprod_V \mathcal{M}^V(1,1)^*,$$

where V runs over isomorphism classes $\mathcal{R}_{(1,1)}(\boldsymbol{k})$ of relevant hermitian spaces. These can be described as follows. For a quaternion algebra $B/\mathbb{Q}$ with an embedding $\boldsymbol{k} \to B$, write $B = \boldsymbol{k} \oplus B_-$, where B_- is the set of $b \in B$ such that $ba = a^\sigma b$ for all $a \in \boldsymbol{k}$. The space $V = V^B = B$, viewed as a left vector space over $\boldsymbol{k}$, has a hermitian form $(x, y) = (xy^\iota)_+$, where $b \mapsto b^\iota$ is the main involution of B, and where for $x \in B$, we denote by $x_+ \in \boldsymbol{k}$ its component in the above direct sum decomposition. Let $D(B)$ be the product of the primes p for which $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra.

Lemma 14.1.

$$\mathcal{R}_{(1,1)}(\boldsymbol{k}) = \{\ [V^B] \mid B \text{ indefinite},\ D(B) \mid \Delta\ \}.$$

□

Note that the quaternion algebra $M_2(\mathbb{Q})$ always occurs here and that $V^{\mathrm{spl}} := V^{M_2(\mathbb{Q})}$ is the split hermitian space. We write

$$\mathcal{M}^{\mathrm{spl}}(1,1)^* = \mathcal{M}^{V^{\mathrm{spl}}}(1,1)^*.$$

When $2 \mid \Delta$ and we work over $\operatorname{Spec} O_{\boldsymbol{k}}[\frac{1}{2}]$, the 'type' of the hermitian form on the 2-adic Tate module $T_2(A)$ gives a finer decomposition. Let π be a uniformizer of $O_{\boldsymbol{k},2}$, and note that $T_2(A)$ is then a π^i-modular lattice for $i = \mathrm{ord}_2(\Delta)$, in the sense of the lemma below. When 2 is ramified, the isometry types of such lattices are given as follows.

Lemma 14.2. *Let L be an $O_{\boldsymbol{k},2}$-lattice of rank 2 that is π^i-modular, i.e., $L^\vee = \pi^{-i}L$ as lattices in $V_2 = L \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$, where $i = \mathrm{ord}_2(\Delta)$.*
(a) Suppose that V_2 is isotropic. Then representatives for the isometry types of π^i-modular lattices are given by

$$2\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad 2\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

for $i = 2$, and

$$2\begin{pmatrix} 0 & \pi \\ \pi^\sigma & 0 \end{pmatrix}, \qquad 2\begin{pmatrix} 2 & \pi \\ \pi^\sigma & 0 \end{pmatrix},$$

for $i = 3$.
(b) Suppose that V_2 is anisotropic. Then representatives for the isometry types of π^i-modular lattices are given by

$$2\begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

for $i = 2$, and

$$2\begin{pmatrix} 2 & \pi \\ \pi^\sigma & 4 \end{pmatrix},$$

for $i = 3$.

Remark 14.3. We will refer to the first of the two lattices in each line of the lists in case (a) as 'type II' lattices.

It is easily checked that, for $p \mid \Delta$ with $p \neq 2$, and π_p a uniformizer of $O_{\boldsymbol{k},p}$, there is a unique isometry class of π_p-modular hermitian lattices of rank 2.

We denote by $\mathcal{R}_{(1,1)}(\boldsymbol{k})^*$ the isomorphism classes of pairs $V^* = (V, [[L]])$, where V is a relevant hermitian space and $[[L]]$ is a $G_1^V(\mathbb{A}_f)$-genus of ∂-modular lattices in V. By the previous lemma, the map from $\mathcal{R}_{(1,1)}(\boldsymbol{k})^*$ to $\mathcal{R}_{(1,1)}(\boldsymbol{k})$ has fibers of cardinality 1 or 2. The latter occurs precisely when $2 \mid \Delta$ and for those $V = V^B$ where B is split at 2.

Then there is a decomposition

$$\mathcal{M}(1,1)^*[\frac{1}{2}] = \coprod_{V^*} \mathcal{M}^{V^*}(1,1)^*[\frac{1}{2}],$$

where V^* runs over $\mathcal{R}_{(1,1)}(\boldsymbol{k})^*$. Here $\mathcal{M}^{V^*}(1,1)^*[\frac{1}{2}]$ is the open and closed substack where the 2-adic Tate module is of the type determined by V^*.

Proposition 14.4. *(i) The morphism $j_{\mathfrak{a}}$ of (14.1) has image contained in $\mathcal{M}^{\rm spl}(1,1)^*$.*
(ii) If $2 \nmid \Delta$, the morphisms $j_{\mathfrak{a}}$ induce an isomorphism

$$\coprod_{[\mathfrak{a}]\in C(\boldsymbol{k})} j_{\mathfrak{a}} : \coprod_{[\mathfrak{a}]\in C(\boldsymbol{k})} \mathcal{E} \xrightarrow{\sim} \mathcal{M}^{\rm spl}(1,1)^*.$$

(iii) If $2 \mid \Delta$, then, over $\operatorname{Spec} O_{\boldsymbol{k}}[\frac{1}{2}]$, the morphisms $j_{\mathfrak{a}}$ induce an isomorphism

$$\coprod_{[\mathfrak{a}]\in C(\boldsymbol{k})} j_{\mathfrak{a}} : \coprod_{[\mathfrak{a}]\in C(\boldsymbol{k})} \mathcal{E}[\frac{1}{2}] \xrightarrow{\sim} \mathcal{M}^{\rm spl,\ II}(1,1)^*[\frac{1}{2}].$$

where $\mathcal{M}^{\rm spl,\ II}(1,1)^*[\frac{1}{2}]$ denotes the locus in $\mathcal{M}^{\rm spl}(1,1)^*[\frac{1}{2}]$ where the 2-adic Tate module has type II.

Proof. Part (i) follows from the fact that for the rational Tate module we have $T^p(\mathfrak{a}\otimes E)^0 \simeq \boldsymbol{k}\otimes_{\mathbb{Q}} T^p(E)^o \simeq \mathrm{M}_2(\mathbb{A}_f^p)$.

(ii) Each morphism $j_{\mathfrak{a}}$ is proper, as one checks easily using the valuative criterion for properness. Also, $j_{\mathfrak{a}}$ is a bijective map onto its image, in the stack sense. Indeed, for a pair E, E' of elliptic curves over a base scheme S, any isomorphism $\mathfrak{a}\otimes E \xrightarrow{\sim} \mathfrak{a}\otimes E'$ induces an isomorphism $E \xrightarrow{\sim} E'$, after choosing a $\mathbb{Z}$-basis of $\mathfrak{a}$, which allows us to identify $\mathfrak{a}\otimes E$ with $E\times E$, and $\mathfrak{a}\otimes E'$ with $E'\times E'$. By Proposition 13.1, (iii), the stack $\mathcal{M}^{\rm spl}(1,1)^*$ is regular; hence we may apply Zariski's Main theorem, and the morphism $j_{\mathfrak{a}}$ induces an isomorphism of $\mathcal{E}$ with a union of connected components of $\mathcal{M}^{\rm spl}(1,1)^*$.

When $2 \mid \Delta$, over $\operatorname{Spec} O_{\boldsymbol{k}}[\frac{1}{2}]$, a simple calculation of the hermitian form on $T_2(j_{\mathfrak{a}}(E)) \simeq \mathfrak{a}\otimes_{\mathbb{Z}} T_2(E)$ shows that this hermitian lattice has type II and hence the image of $j_{\mathfrak{a}}(\mathcal{E}[\frac{1}{2}])$ lies in $\mathcal{M}^{\rm spl,II}(1,1)^*[\frac{1}{2}]$.

Since $\mathcal{E}$ is connected, to prove the isomorphism in (ii), it suffices to prove that $\pi_0(\mathcal{M}^{\rm spl}(1,1)^*) \simeq C(\boldsymbol{k})$, or also, since $\mathcal{M}^{\rm spl}(1,1)^*$ is regular, that $\pi_0(\mathcal{M}^{\rm spl}(1,1)^*_{\mathbb{C}}) \simeq C(\boldsymbol{k})$. This follows from Proposition 4.4 and (4.5), where we note that $\nu(K) = \widehat{\mathbb{Z}}$ and that the image of K in first factor of $T(\mathbb{A}_f) \simeq \boldsymbol{k}^1_{\mathbb{A}_f} \times \mathbb{Q}^\times_{\mathbb{A}_f}$ is the image U of $\widehat{O}^\times_{\boldsymbol{k}}$ under the map $u \mapsto u\bar{u}^{-1}$. Also note that this last map induces an isomorphism $C(\boldsymbol{k}) = \boldsymbol{k}^\times\backslash \boldsymbol{k}^\times_{\mathbb{A}_f}/\widehat{O}^\times_{\boldsymbol{k}} \simeq \boldsymbol{k}^1\backslash \boldsymbol{k}^1_{\mathbb{A}_f}/U$.

The proof of (iii) is similar and omitted. □

Now we turn to the special cycles. To lighten notation, we now denote by $\mathcal{E}$ (resp. $\mathcal{M}^{\rm spl}(1,1)^*$) the base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec} O_{\boldsymbol{k}}$, and write

$$\mathcal{M}^* = \mathcal{M}^{\rm spl}(1,1)^* \times \mathcal{M}_0,$$

where the fiber product is taken over $\operatorname{Spec} O_{\boldsymbol{k}}$. For a positive integer m, we define the special cycle

$$\mathcal{Z}^*(m) \longrightarrow \mathcal{M}^*$$

as the stack of collections $(A,\iota,\lambda,E_0,\iota_0,\lambda_0;x)$ where (A,ι,λ) is a object of $\mathcal{M}^{\rm spl}(1,1)^*(S)$, (E_0,ι_0,λ_0) is an object of $\mathcal{M}_0(S)$, and

$$x \in \operatorname{Hom}_S((E_0,\iota_0),(A,\iota))\otimes\mathbb{Q}$$

is an $O_{\boldsymbol{k}}$-linear quasi-homomorphism such that

$$x^* = \lambda_0^{-1}\circ x^\vee\circ\lambda \in \operatorname{Hom}_S((A,\iota),(E_0,\iota_0))$$

is a homomorphism[12] with $h(x,x) = x^*\circ x = \lambda_0^{-1}\circ x^\vee\circ\lambda\circ x = m$.

We want to determine the pullback of $\mathcal{Z}^*(m)$ under $j_{\mathfrak{a}}\times 1$, i.e., the fiber product

$$\begin{array}{ccc} (j_{\mathfrak{a}}\times 1)^*\mathcal{Z}^*(m) & \longrightarrow & \mathcal{Z}^*(m) \\ \downarrow & & \downarrow \\ \mathcal{E}\times\mathcal{M}_0 & \longrightarrow & \mathcal{M}^*. \end{array}$$

Let $\mathcal{T}(m)$ be the stack for which the objects of $\mathcal{T}(m)(S)$ are triples (E,E',ψ), where E and E' are elliptic schemes over S and $\psi: E\to E'$ is an m-isogeny. Let $s:\mathcal{T}(m)\to\mathcal{E}$ (resp. $t:\mathcal{T}(m)\to\mathcal{E}$) be the morphism defined by sending (E,E',ψ) to E (resp. E'). Consider the fiber product

$$\begin{array}{ccc} \mathcal{T}(m)_{\Delta,\mathfrak{a}} & \longrightarrow & \mathcal{T}(m) \\ \downarrow & & \downarrow {\scriptstyle (s,t)} \\ \mathcal{E}\times\mathcal{M}_0 & \xrightarrow{1\times i_{\mathfrak{a}}} & \mathcal{E}\times\mathcal{E}. \end{array}$$

where $i_0:\mathcal{M}_0\to\mathcal{E}$ is the morphism that sends (E_0,ι_0) to E_0 and $i_{\mathfrak{a}} = i_0\circ t_{\mathfrak{a}}$ where $t_{\mathfrak{a}}:\mathcal{M}_0\to\mathcal{M}_0$ sends E_0 to $\mathfrak{a}^{-1}\otimes_{O_{\boldsymbol{k}}} E_0$.

Proposition 14.5. *There is a natural isomorphism*

$$(j_{\mathfrak{a}}\times 1)^*\mathcal{Z}^*(m) \xrightarrow{\sim} \mathcal{T}(m)_{\Delta,\mathfrak{a}}$$

over $\mathcal{E}\times\mathcal{M}_0$.

Proof. An object of $(j_{\mathfrak{a}}\times 1)^*\mathcal{Z}^*(m)(S)$ is a collection $(E,E_0,\iota_0,\lambda_0;x)$ where $x: E_0\longrightarrow\mathfrak{a}\otimes_{\mathbb{Z}}E$ is an $O_{\boldsymbol{k}}$-linear quasi-homomorphism with x^* integral and with $h(x,x) = x^*\circ x = m$. Here $x^* = \lambda_0^{-1}\circ x^\vee\circ\lambda : \mathfrak{a}\otimes_{\mathbb{Z}}E\to E_0$. Let $x_0^*: E\to\mathfrak{a}^{-1}\otimes_{O_{\boldsymbol{k}}}E_0$ be the image of x^* under the natural isomorphism

$$\operatorname{Hom}_{O_{\boldsymbol{k}}}(\mathfrak{a}\otimes_{\mathbb{Z}}E,E_0) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(E,\mathfrak{a}^{-1}\otimes_{O_{\boldsymbol{k}}}E_0) \tag{14.2}$$

arising from the Serre construction. On the other hand, an object of $\mathcal{T}(m)_{\Delta,\mathfrak{a}}(S)$ is a collection $(E,E_0,\iota_0,\lambda_0;y_0)$ where $y_0: E\to\mathfrak{a}^{-1}\otimes_{O_{\boldsymbol{k}}}E_0$ is an isogeny of degree m. The proposition is then an immediate consequence of the following result.

Lemma 14.6. *If* $y\in\operatorname{Hom}_{O_{\boldsymbol{k}}}(\mathfrak{a}\otimes_{\mathbb{Z}}E,E_0)$ *and* $y_0\in\operatorname{Hom}_{\mathbb{Z}}(E,\mathfrak{a}^{-1}\otimes_{O_{\boldsymbol{k}}}E_0)$ *correspond under (14.2), then* $h(y,y) = y\circ y^* = \deg(y_0)$.

[12] In the case of a principal polarization, the integrality of x^* is equivalent to the integrality of x. In general, x need not be integral, so that we are slightly extending the earlier definition of section 13.

Proof. For an isogeny $y_0 \in \mathrm{Hom}_{\mathbb{Z}}(E, \mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)$, the corresponding $y \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(\mathfrak{a} \otimes_{\mathbb{Z}} E, E_0)$ is given by

$$y : \mathfrak{a} \otimes_{\mathbb{Z}} E \xrightarrow{1 \otimes y_0} \mathfrak{a} \otimes_{\mathbb{Z}} (\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0) \longrightarrow E_0, \tag{14.3}$$

where the second map arises by multiplication $a \otimes (b \otimes x) \mapsto ab \cdot x$. To compute the desired relation between $\deg y_0$ and $h(y, y) = y \circ y^*$, we can pass to the rational Tate modules at a prime ℓ different from the characteristic. On the rational Tate modules $V_\ell(E) = T_\ell(E)^0$, $V_\ell(\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)$ and $V_\ell(E_0)$, there are nondegenerate $\mathbb{Q}_\ell$-valued alternating forms ρ_E, $\rho_{\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0}$, and ρ_{E_0} induced by the canonical principal polarizations and a fixed trivialization $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$. For example, $\rho_E(z_1, z_2) = e_E(z_1, \lambda_E(z_2))$, where e_E is the Weil pairing on $V_\ell(E) \times V_\ell(E^\vee)$ and λ_E is the canonical polarization. The pullback of $\rho_{\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0}$ under the isomorphism

$$y_{0,\ell} : V_\ell(E) \xrightarrow{\sim} V_\ell(\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)$$

is $\deg(y_0) \cdot \rho_E$, while the pullback of ρ_{E_0} under the isomorphism

$$m_{\mathfrak{a},\ell} : V_\ell(\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0) \xrightarrow{\sim} V_\ell(E_0)$$

associated to the quasi-isogeny $\mathfrak{a}^{-1} \otimes E_0 \longrightarrow E_0$, given by multiplication, is $N(\mathfrak{a})^{-1} \cdot \rho_{\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0}$. Similarly, setting $A = \mathfrak{a} \otimes_{\mathbb{Z}} E$, the non-degenerate $\mathbb{Q}_\ell$-valued alternating pairing on $V_\ell(A) = V_\ell(\mathfrak{a} \otimes_{\mathbb{Z}} E) \simeq \boldsymbol{k} \otimes_{\mathbb{Q}} V_\ell(E)$ determined by the polarization $\lambda = \lambda_A$ is given by

$$\rho_A(a \otimes z, a' \otimes z') = N(\mathfrak{a})^{-1}\, (a, a')_{\boldsymbol{k}}\, \rho_E(z, z').$$

From (14.3), we have

$$\begin{aligned} y_\ell : \boldsymbol{k} \otimes_{\mathbb{Q}} V_\ell(E) &\overset{\xi}{\xrightarrow{\sim}} V_\ell(E_0) \oplus \overline{V_\ell(E_0)} \longrightarrow V_\ell(E_0), \\ a \otimes z &\mapsto \big(a \cdot m_{\mathfrak{a},\ell} \circ y_{0,\ell}(z), \bar{a} \cdot m_{\mathfrak{a},\ell} \circ y_{0,\ell}(z)\big) \mapsto a \cdot m_{\mathfrak{a},\ell} \circ y_{0,\ell}(z). \end{aligned} \tag{14.4}$$

Here, to arrive at the middle entry, we have used the isomorphism

$$V_\ell(\mathfrak{a} \otimes_{\mathbb{Z}} (\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)) \simeq \boldsymbol{k} \otimes_{\mathbb{Q}} (\boldsymbol{k} \otimes_{\boldsymbol{k}} V_\ell(E_0)) \simeq V_\ell(E_0) \oplus (\boldsymbol{k} \otimes_{\boldsymbol{k},\sigma} V_\ell(E_0)).$$

A short calculation shows that the pullback of the diagonal form $\rho_{E_0} \oplus \rho_{E_0}$ on the middle term under the isomorphism ξ is $\deg y_0 \cdot \rho_A$. If we identify $V_\ell(A)$ and $V_\ell(E_0) \oplus \overline{V_\ell(E_0)}$ via ξ, then $y_\ell^* = \deg y_0 \cdot \mathrm{inc}_1$, where inc_1 is the inclusion of $V_\ell(E_0)$ into the first factor and the adjoint y_ℓ^* is defined with respect to ρ_A. Hence $y \circ y^*$ is multiplication by $\deg y_0$ and the lemma is proved. □

This completes the proof of Proposition 14.5. □

For comparison with the classical theory, we add a little more level. We fix a positive integer N and work over $\mathrm{Spec}\, \mathbb{Z}[N^{-1}]$. Let $\mathcal{E}_0(N)$ be the moduli stack over $\mathrm{Spec}\, \mathbb{Z}[N^{-1}]$ for pairs of elliptic curves (E, E', ϕ) with a cyclic N-isogeny ϕ. Again applying the Serre construction, we obtain a morphism

$$j_{\mathfrak{a},N} : \mathcal{E}_0(N) \longrightarrow \mathcal{M}^{\mathrm{spl}}(1,1)_0^*(N),$$

where $\mathcal{M}^{\mathrm{spl}}(1,1)_0^*(N)$ is the following moduli stack. For a locally noetherian base S, an object in $\mathcal{M}^{\mathrm{spl}}(1,1)_0^*(N)(S)$ is a collection $(\xi, \xi'; \phi)$ where $\xi = (A, \iota_A, \lambda_A)$ and $\xi' = (A', \iota_{A'}, \lambda_{A'})$ are objects in $\mathcal{M}^{\mathrm{spl}}(1,1)^*(S)$, and $\phi : A \to A'$ is an $O_{\boldsymbol{k}}$-linear isogeny with $\phi^*(\lambda_{A'}) = N\lambda_A$ and such that locally in the fppf topology, $\ker(\phi) \simeq (O_{\boldsymbol{k}}/NO_{\boldsymbol{k}})_S$.

As before we tacitly effect a base change from $\mathrm{Spec}\, \mathbb{Z}[N^{-1}]$ to $\mathrm{Spec}\, O_{\boldsymbol{k}}[N^{-1}]$, and write

$$\mathcal{M}^{**} = \mathcal{M}^{\mathrm{spl}}(1,1)_0^*(N) \times \mathcal{M}_0.$$

We define the special cycle $\mathcal{Z}^{**}(m) \to \mathcal{M}^{**}$ as the stack of collections $(\xi, \xi', \phi; E_0, \iota_0, \lambda_0; x)$, where $x : E_0 \to A$ is a quasi-homomorphism with x^* integral, etc. just as before, but with the additional requirement that, fppf locally, $x^*(\ker(\phi))$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})_S$.

To describe the pullback $(j_{\mathfrak{a},N} \times 1)^*(\mathcal{Z}^{**}(m))$ over $\mathcal{E}_0(N) \times \mathcal{M}_0$, let $\mathcal{T}(m)_0(N)$ be the stack whose objects over S are collections (E, E', ϕ, E'', ψ), where (E, E', ϕ) is an object of $\mathcal{E}_0(N)$ and $\psi : E \to E''$ is an isogeny of degree m such that the intersection $\ker(\psi) \cap \ker(\phi)$ is trivial. There are again source and target morphisms to $\mathcal{E}_0(N)$, where the target is the object $(E'', E''/\psi(\ker \phi), \phi')$ with ϕ' the quotient map.

Finally, as in [14], we make the assumption that all primes dividing N split in $\boldsymbol{k}$ and choose an integral ideal $\mathfrak{n}$ with $N(\mathfrak{n}) = N$. There is a resulting morphism

$$i_{0,\mathfrak{n}} : \mathcal{M}_0 \longrightarrow \mathcal{E}_0(N), \qquad E_0 \mapsto (E_0 \xrightarrow{\phi} E_0/E_0[\mathfrak{n}]),$$

and its twist $i_{\mathfrak{a},\mathfrak{n}} = i_{0,\mathfrak{n}} \circ t_{\mathfrak{a}}$.

Proposition 14.7. *There is a natural isomorphism*

$$(j_{\mathfrak{a},N} \times 1)^*(\mathcal{Z}^{**}(m)) \xrightarrow{\sim} \coprod_{\mathfrak{n}} \mathcal{T}(m)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}},$$

over $\mathcal{E}_0(N) \times \mathcal{M}_0$, where

$$\begin{array}{ccc} \mathcal{T}(m)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}} & \longrightarrow & \mathcal{T}(m)_0(N) \\ \downarrow & & \downarrow {\scriptstyle (s,t)} \\ \mathcal{E}_0(N) \times \mathcal{M}_0 & \xrightarrow{1\times i_{\mathfrak{a},\mathfrak{n}}} & \mathcal{E}_0(N) \times \mathcal{E}_0(N), \end{array}$$

is the fiber product. □

Remark 14.8. This proposition can be interpreted as follows. First suppose that $m = 1$. Then, over a base S on which N is invertible, an object of $\mathcal{T}(1)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}}(S)$ is a collection $(E \xrightarrow{\phi} E', E_0, \psi)$, where $(E \xrightarrow{\phi} E')$ is an object of $\mathcal{E}_0(N)(S)$ and ψ is an isomorphism

$$(E \xrightarrow{\phi} E') \xrightarrow{\sim} (\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0 \longrightarrow \mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0/(\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)[\mathfrak{n}]).$$

Hence, $\mathcal{T}(1)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}}$ can be viewed as the graph of the morphism of stacks

$$\begin{aligned} i_{\mathfrak{a},\mathfrak{n}} : \mathcal{M}_0 &\longrightarrow \mathcal{E}_0(N), \\ E_0 &\mapsto (\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0 \longrightarrow \mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0/(\mathfrak{a}^{-1} \otimes_{O_{\boldsymbol{k}}} E_0)[\mathfrak{n}]) \end{aligned}$$

Passing to the coarse moduli schemes, for each $\mathfrak{a}$ and $\mathfrak{n}$, we have a morphism

$$i_{\mathfrak{a},\mathfrak{n}} : \mathrm{Spec}\,(O_H[N^{-1}]) \longrightarrow \underline{\mathcal{E}_0(N)}$$

where O_H is the ring of integers in H, the Hilbert class field of $\boldsymbol{k}$. Finally, passing to the generic fiber, we obtain morphisms

$$i_{\mathfrak{a},\mathfrak{n},\mathbb{Q}} : \mathrm{Spec}\,(H) \longrightarrow \underline{\mathcal{E}_0(N)}_{\mathbb{Q}} = X_0(N). \tag{14.5}$$

Corollary 14.9. *(i) The point (14.5) in $X_0(N)(H)$ determined by $\mathcal{T}(1)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}}$ is the Heegner point associated to the ideal class $\mathfrak{a}$ and the primitive ideal $\mathfrak{n}$ of norm N as in* [14]*, p.227.*
(ii) For $m \geq 1$, the set of points on $X_0(N)$ similarly determined by $\mathcal{T}(m)_0(N)_{\Delta,\mathfrak{a},\mathfrak{n}}$ is the image of this Heegner point under the m-th Hecke operator.

Thus, in this special case, our cycles $\mathcal{Z}^{**}(m)$ provide an integral version of the images under the Hecke operators of the Heegner points considered in [14].

So far we have discussed the case $B = M_2(\mathbb{Q})$. This case corresponds to the condition that all prime divisors of N are split in $\boldsymbol{k}$, imposed by Gross and Zagier to ensure that their Heegner points lie on the modular curve. As observed in [14] p.313, if this condition is relaxed, the Heegner points lie on Shimura curves, as we now explain from our point of view. Suppose that B is an indefinite division quaternion algebra with a maximal order O_B and with $D(B) \mid |\Delta|$. Fix an embedding $O_{\boldsymbol{k}} \to O_B$. Let $\mathcal{M}^B$ be the Drinfeld stack over $\operatorname{Spec}\mathbb{Z}$ parametrizing two-dimensional abelian schemes with a special O_B-action, cf. [4]. Choose an element $\xi \in O_B$ with $\xi^2 = -D(B)$, [4], and define the involution $b \mapsto b' := \xi b^\iota \xi^{-1}$ where $b \mapsto b^\iota$ is the main involution of B. Given an object (A, ι_B) of $\mathcal{M}^B(S)$, there is a unique principal polarization λ_A of A for which the Rosati involution induces the involution $'$ on O_B, [4], Prop. (3.3), p.134. Since the element $\sqrt{\Delta}\,\xi^{-1} \in O_B$ is invariant under $'$, the composition $\lambda := \lambda_A \circ \iota_B(\sqrt{\Delta}\,\xi^{-1})$ is a polarization of A whose Rosati involution $*$ satisfies $\iota(a)^* = \iota(a^\sigma)$ for all $a \in O_{\boldsymbol{k}}$. Here we have written ι for the restriction of ι_B to $O_{\boldsymbol{k}}$. Then the collection (A, ι, λ) is an object of the moduli space $\mathcal{M}(1,1)^*_{D(B)}$ which parametrizes two-dimensional abelian schemes with $O_{\boldsymbol{k}}$-action of signature $(1,1)$ and polarization λ of degree $|\Delta|/D(B)$. If Δ is odd, $\mathcal{M}(1,1)^*_{D(B)}$ coincides with the moduli problem $\mathcal{M}(\mathbf{t};1,1)^*$ defined in the previous section where

$$t(p) = \begin{cases} 2 & \text{if } p \mid \Delta,\ p \nmid D(B), \\ 0 & \text{if } p \mid \Delta,\ p \mid D(B). \end{cases}$$

Thus we obtain a morphism $j^B : \mathcal{M}^B \to \mathcal{M}(1,1)^*_{D(B)}$ and twists

$$j^B_{\mathfrak{a}} : \mathcal{M}^B \to \mathcal{M}(1,1)^*_{D(B)}, \qquad j^B_{\mathfrak{a}} = \mathfrak{a} \otimes_{O_{\boldsymbol{k}}} j^B,$$

for any fractional ideal $\mathfrak{a}$. When Δ is odd, these give an isomorphism

$$\coprod_{[\mathfrak{a}] \in C(\boldsymbol{k})} \mathcal{M}^B \xrightarrow{\sim} \mathcal{M}^V(1,1)^*_{D(B)}, \tag{14.6}$$

where $V = V^B$, generalizing that in (ii) of Proposition 14.4 where $B = M_2(\mathbb{Q})$. As in the case of a split quaternion algebra, we can now pull back $\mathcal{Z}^*(m)$ to $\mathcal{M}^B \times \mathcal{M}_0$, via $j^B_{\mathfrak{a}} \times 1$. This parametrizes tuples $(A, \iota_B, E_0, \iota_0, x)$ where $x \in \operatorname{Hom}_{O_{\boldsymbol{k}}}(E_0, \mathfrak{a} \otimes_{O_{\boldsymbol{k}}} A) \otimes \mathbb{Q}$ is such that x^* is integral with $x^* \circ x = m$. Of course, here the adjoint x^* is formed using the canonical polarization λ of $\mathfrak{a} \otimes_{O_{\boldsymbol{k}}} A$ constructed above. And we may add level structures as in the case of the split quaternion algebra.

We end this section by explaining the relations among the $\mathcal{M}(\mathbf{t};1,1)^*$ for varying $\mathbf{t}$.

Proposition 14.10. *Suppose that $\mathbf{t}$ is given and that $t(p) = 0$ for some prime $p \neq 2$ with $p \mid \Delta$. Define $\mathbf{t}'$ by $t'(p') = t(p')$ for $p \neq p'$ and $t'(p) = 2$. Then there exists an étale Galois covering of degree 2*

$$\mathcal{M}(\mathbf{t};1,1)^{*,p} \longrightarrow \mathcal{M}(\mathbf{t};1,1)^*,$$

equipped with a proper morphism

$$\varphi : \mathcal{M}(\mathbf{t};1,1)^{*,p} \longrightarrow \mathcal{M}(\mathbf{t}';1,1)^*,$$

which is finite of degree $p+1$ outside the fibers over p and which contracts some lines in the special fiber of $\mathcal{M}(\mathbf{t};1,1)^{*,p}$ at p, (cf. Remark 14.11 below).

Proof. Let $\mathcal{M}(\mathbf{t};1,1)^{*,p}$ be the stack of objects $\xi = (A, \iota_A, \lambda_A)$ of $\mathcal{M}(\mathbf{t};1,1)^*$, together with an abelian scheme A' with $O_{\boldsymbol{k}}$-action of signature $(1,1)$ and a $O_{\boldsymbol{k}}$-linear isogeny $\mu : A' \longrightarrow A$ of degree p such that the pullback polarization $\lambda_{A'} = \mu^*(\lambda_A)$ has kernel contained in $A'[\sqrt{\Delta}]$. The morphism φ maps an object $(A, \iota_A, \lambda_A, \mu)$ to $(A', \iota_{A'}, \lambda_{A'})$ and is obviously proper. Let us determine the fiber of $\mathcal{M}(\mathbf{t};1,1)^{*,p}$ over a geometric point $\xi=(A,\iota_A,\lambda_A)\in\mathcal{M}(\mathbf{t};1,1)^*(k)$. If $\operatorname{char} k\neq p$, then $\mu: A'\longrightarrow A$ is given by the p-adic Tate module of A' which is an $O_{\boldsymbol{k}_p}$-stable submodule Λ of $T_p(A)$ such that $\Lambda\subset T_p(A)\subset\Lambda^\vee=\pi^{-1}\Lambda$, where π is a uniformizer in $\boldsymbol{k}_p$. However, the hermitian form h_A on $T_p(A)$ arising from the polarization λ_A induces a non-degenerate $\mathbb{F}_p$-valued *symmetric* form on $\pi^{-1}T_p(A)/T_p(A)$. The set of Λ as above corresponds in a one-to-one way to the set of isotropic lines in this two-dimensional $\mathbb{F}_p$-vector space, via $\Lambda\mapsto\Lambda^\vee/T_p(A)$. Since $p\neq 2$, there are precisely two such lines. If $\operatorname{char} k=p$, we use Dieudonné modules instead of Tate modules. The Dieudonné module of A is a module M over $O_{\boldsymbol{k}_p}\otimes W(k)$, and $\pi^{-1}M/M$ is equipped with a non-degenerate k-valued symmetric form. Then $\mu: A'\longrightarrow A$ is given by the Dieudonné module Λ of A' which is a $O_{\boldsymbol{k}_p}\otimes W(k)$-submodule with $\Lambda\subset M\subset\Lambda^\vee=\pi^{-1}\Lambda$. Just as before Λ corresponds to one of the two isotropic lines in $\pi^{-1}M/M$.

Now let us determine the fiber of φ over a geometric point $(A',\iota_{A'},\lambda_{A'})\in\mathcal{M}(\mathbf{t}';1,1)^*(k)$. If $\operatorname{char} k\neq p$, then the points in the fiber correspond to the $O_{\boldsymbol{k}_p}$-lattices L in the rational Tate module $V_p(A')$ with $T_p(A')\subsetneq L\subsetneq T_p(A')^\vee=\pi^{-1}T_p(A')$ (these are automatically self-dual). Hence there are $p+1$ of them. If $\operatorname{char} k=p$, then the points in the fiber correspond to $O_{\boldsymbol{k}_p}$-stable Dieudonné lattices M in the rational Dieudonné module of A', containing the Dieudonné module $M(A')$ of A', with $M(A')\subsetneq M\subsetneq M(A')^\vee=\pi^{-1}M(A')$. Now there are two cases. First suppose that $F\big(\pi^{-1}M(A')\big)\neq M(A')$, or equivalently, $V\big(\pi^{-1}M(A')\big)\neq M(A')$. Then either $F\big(\pi^{-1}M(A')\big)\subset M$, or $V\big(\pi^{-1}M(A')\big)\subset M$, and M is uniquely determined as $M=M(A')+F\big(\pi^{-1}M(A')\big)$ or $M=M(A')+V\big(\pi^{-1}M(A')\big)$, respectively. Hence in this case there is a unique point in the fiber. Next suppose that $F\big(\pi^{-1}M(A')\big)=M(A')=V\big(\pi^{-1}M(A')\big)$. Then there are no constraints on M and M corresponds to an arbitrary point in the projective line $\mathbb{P}\big(\pi^{-1}M(A')/M(A')\big)$.

Remark 14.11. Note that the points $\xi\in\mathcal{M}(\mathbf{t}';1,1)^*(k)$ with a fiber of positive dimension have a Dieudonné module $M(A')$ satisfying $F\big(\pi^{-1}M(A')\big)=M(A')=V\big(\pi^{-1}M(A')\big)$. Let ξ lie in the component $\mathcal{M}^V(\mathbf{t}';1,1)^*(k)$. Then this condition signifies that ξ is supersingular if $\operatorname{inv}_p(V_p)=1$ (resp. is superspecial in the sense of Drinfeld if $\operatorname{inv}_p(V_p)=-1$).

By Proposition 13.1, $\mathcal{M}(\mathbf{t};1,1)^*$ and $\mathcal{M}(\mathbf{t}';1,1)^*$ are regular; then $\mathcal{M}(\mathbf{t};1,1)^{*,p}\longrightarrow\mathcal{M}(\mathbf{t};1,1)^*$ is an étale Galois covering by EGA IV, 18.10.16; furthermore, φ is the composition of a blow-up morphism and a finite morphism. □

More generally, for given $\mathbf{t}$, let $S\subset\{p\mid p\neq 2, p\mid\Delta, t(p)=0\}$, and define $\mathbf{t}'_S$ by $t'_S(p)=t(p)$ for $p\notin S$ and $t'_S(p)=2$ for $p\in S$. Then there exists an étale Galois covering $\mathcal{M}(\mathbf{t};1,1)^{*,S}$ of $\mathcal{M}(\mathbf{t};1,1)^*$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^S$ and a morphism

$$\varphi_S:\mathcal{M}(\mathbf{t};1,1)^{*,S}\longrightarrow\mathcal{M}(\mathbf{t}'_S;1,1)^*,$$

which is finite over $\operatorname{Spec}\mathbb{Z}\setminus S$, and contracts lines in the fibers of $\mathcal{M}(\mathbf{t};1,1)^{*,S}$ over primes $p\in S$.

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