SPECIAL CYCLES ON UNITARY SHIMURA VARIETIES I. UNRAMIFIED LOCAL THEORY

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Abstract

The supersingular locus in the fiber at p of a Shimura variety attached to a unitary similitude group $\mathrm{GU}(1, n-1)$ over $\mathbb{Q}$ is uniformized by a formal scheme $\mathcal{N}$. In the case when p is an inert prime, we define special cycles $\mathcal{Z}(\mathbf{x})$ in $\mathcal{N}$, associated to collections $\mathbf{x}$ of m 'special homomorphisms' with fundamental matrix $T \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$. When $m = n$ and T is nonsingular, we show that the cycle $\mathcal{Z}(\mathbf{x})$ is either empty or is a union of components of the Ekedahl-Oort stratification, and we give a necessary and sufficient condition, in terms of T, for $\mathcal{Z}(\mathbf{x})$ to be irreducible. When $\mathcal{Z}(\mathbf{x})$ is zero dimensional – in which case it reduces to a single point – we determine the length of the corresponding local ring by using a variant of the theory of quasi-canonical liftings. We show that this length coincides with the derivative of a representation density for hermitian forms.

1. Introduction

A relation between a generating series constructed from arithmetic cycles on an integral model of a Shimura curve and the derivative of a Siegel Eisenstein series of genus 2 was established by one of us in [10]. There, the hope is expressed that such a relation should hold in greater generality for integral models of Shimura varieties attached to orthogonal groups of signature $(2, n-2)$ for any n. The case of Shimura curves corresponds to $n = 3$; the relation for $n = 2$ is established in [16], and the cases $n = 4$ and $n = 5$ are considered in [13, 15]. However, the case of arbitrary n seems out of reach at the present time, since these Shimura varieties do not represent a moduli problem of abelian varieties with additional structure of PEL-type, so that it is difficult to define and study integral models of them. For the low values of n mentioned above, the analysis depends on exceptional isomorphisms between orthogonal and symplectic groups.

In the present series of papers, we take up an idea already mentioned in a brief remark at the very end of section 16 of [10] and consider integral models of Shimura varieties attached to unitary similitude groups of signature $(1, n-1)$ over $\mathbb{Q}$. These varieties are moduli spaces for abelian varieties for arbitrary n, and there are good integral models for them, at least away from primes of (very) bad reduction, [8], [19], [20], [21], [22]. In a sequel to the present paper, we define special arithmetic cycles in a modular way and study the classes determined by such cycles, together with suitable Green currents, in the arithmetic Chow groups. The ultimate goal is to relate the generating series defined by the height pairings, or arithmetic intersection numbers, of such classes to special values of derivatives of Eisenstein series for the group $\mathrm{U}(n,n)$. It should be noted that the complex points of our cycles coincide with the cycles defined in [9], [11], and [12], where the modular properties of the generating functions for the associated cohomology classes are established.

In the present paper, as a first step in this study, we consider the local analogue, in which the Shimura variety is replaced by a formal moduli space of p-divisible groups, the special arithmetic cycles are replaced by formal subvarieties, and the special values of the derivative of the Eisenstein series are replaced by the derivatives of representation densities of hermitian forms. The link between this local situation and the global one is provided by the uniformization of the supersingular locus by the formal schemes introduced in [23]. A similar local analogue occurs in our earlier work [14], where the intersection numbers of formal arithmetic cycles on the Drinfeld upper half-space, which uniformizes the fibers of bad reduction of Shimura curves, are related to the derivative of representation densities of quadratic forms. The results of [14] are an essential ingredient in the global theory of arithmetic cycles on Shimura curves established in [17]. The results of the present paper will play a similar role in the unitary case.

It turns out that when two global cycles are disjoint on the generic fiber, their intersections are supported in the fibers at non-split primes. Thus, for primes not dividing the level, there are two cases, depending on whether the prime is inert or ramified in the imaginary quadratic field. As indicated in the title, in the present paper we handle the unramified primes.

We now describe our results in more detail.

Let $\boldsymbol{k} = \mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$ and let $O_{\boldsymbol{k}} = \mathbb{Z}_{p^2}$ be its ring of integers. Let $\mathbb{F} = \bar{\mathbb{F}}_p$ and write $W = W(\mathbb{F})$ for its ring of Witt vectors. There are two embeddings φ_0 and $\varphi_1 = \varphi_0 \circ \sigma$ of $\boldsymbol{k}$ into $W_{\mathbb{Q}} = W \otimes_{\mathbb{Z}} \mathbb{Q}$. Let Nilp_W be the category of W-schemes S on which p is locally nilpotent. For any scheme S over W, let $\bar{S} = S \times_{\operatorname{Spec} W} \operatorname{Spec} \mathbb{F}$ be its special fiber.

The formal scheme on which we work is defined as follows. We consider p-divisible groups X of dimension n and height $2n$ over W-schemes S, with an action $\iota : O_{\boldsymbol{k}} \to \operatorname{End}(X)$ satisfying the *signature condition* $(1, n-1)$,

$$\operatorname{char}(\iota(\alpha), \operatorname{Lie} X)(T) = (T - \varphi_0(\alpha))(T - \varphi_1(\alpha))^{n-1} \in \mathcal{O}_S[T], \tag{1.1}$$

and equipped with a p-principal polarization λ_X, for which the Rosati involution $*$ satisfies $\iota(\alpha)^* = \iota(\alpha^\sigma)$.

Up to isogeny, there is a unique such a triple $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ over $\mathbb{F}$ such that $\mathbb{X}$ is *supersingular*[1]. Fixing $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$, we denote by $\mathcal{N}$ the formal scheme over W which parametrizes the quadruples $(X, \iota, \lambda_X, \rho_X)$ over schemes S in Nilp_W, where (X, ι, λ_X) is as above, and where

$$\rho_X : X \times_S \bar{S} \to \mathbb{X} \times_{\mathbb{F}} \bar{S}$$

is a quasi-isogeny which respects the auxilliary structures imposed. (See section 2.1 and [23], section 1, for the precise definition of $\mathcal{N}$.) Then $\mathcal{N}$ is formally smooth of relative dimension $n-1$ over W, and the underlying reduced scheme $\mathcal{N}_{\text{red}}$ is a singular scheme of dimension $[(n-1)/2]$ over $\mathbb{F}$.

In order to explain our results, we need to recall some of the results on the structure of $\mathcal{N}_{\text{red}}$ due to Vollaard [29], as completed by Vollaard and Wedhorn [31]. To the polarized isocrystal N of $\mathbb{X}$ there is associated a hermitian vector space $(C, \{\ ,\ \})$ of dimension n over $\boldsymbol{k}$ satisfying the parity condition

$$\operatorname{ord} \det(C) \equiv n + 1 \bmod 2.$$

Here $\det(C) \in \mathbb{Q}_p^\times / N(\boldsymbol{k}^\times)$ is the coset determined by $\det((\{c_i, c_j\}))$ for any $\boldsymbol{k}$ basis $\{c_i\}$ for C. Note that this condition determines $(C, \{\ ,\ \})$ up to isomorphism. A *vertex of level* i is an $O_{\boldsymbol{k}}$-lattice Λ in C with

$$p^{i+1}\Lambda^\vee \subset \Lambda \subset p^i \Lambda^\vee,$$

where

$$\Lambda^\vee = \{\ x \in C \mid \{x, \Lambda\} \subset O_{\boldsymbol{k}}\ \}$$

is the dual lattice. Such lattices correspond to the vertices of the building $\mathcal{B}(\mathrm{U}(C))$ of the unitary group $U(C)$, hence the terminology. The *type* of a vertex Λ is the index $t(\Lambda)$ of $p^{i+1}\Lambda^\vee$ in Λ. In fact, $t(\Lambda)$ is always an odd integer between 1 and n. To every vertex Λ of level i, Vollaard and Wedhorn associate a locally closed irreducible subset $\mathcal{V}(\Lambda)^o$ of $\mathcal{N}_{\text{red}}$ of dimension

[1] Recall that this means that $\mathbb{X}$ is isogenous to the nth power of the p-divisible group of a supersingular elliptic curve.

$\frac{1}{2}(t(\Lambda)-1)$ with the following properties:

a) The closure $\mathcal{V}(\Lambda)$ of $\mathcal{V}(\Lambda)^o$ is the finite disjoint union

$$\mathcal{V}(\Lambda)=\bigcup_{\Lambda'\subset\Lambda}\mathcal{V}(\Lambda')^o,$$

where Λ' runs over all vertex lattices of level i contained in Λ. Note that for such vertices $t(\Lambda')\leq t(\Lambda)$.

b) The union of $\mathcal{V}(\Lambda)^o$, as Λ ranges over all vertices of level i, is a connected component $\mathcal{N}_i$ of $\mathcal{N}_{\mathrm{red}}$, and as i varies, all connected components of $\mathcal{N}_{\mathrm{red}}$ arise in this way.

Thus the combinatorics of the stratification of $\mathcal{N}_{\mathrm{red}}$ are controlled by the building $\mathcal{B}(\mathrm{U}(C))$, just as the (much simpler) stratification of the special fiber of the formal model of the Drinfeld half-space is controlled by the tree $\mathcal{B}(\mathrm{GL}_2(\mathbb{Q}_p))$.

We next define special cycles on $\mathcal{N}$. Let $(\mathbb{Y},\iota,\lambda_{\mathbb{Y}})$ be the basic object over $\mathbb{F}$ used in the definition of the signature $(1,n-1)$ moduli space $\mathcal{N}$ in the case $n=1$. Thus $\mathbb{Y}$ is a supersingular p-divisible group over $\mathbb{F}$ of dimension 1 with $O_{\boldsymbol{k}}$-action ι which satisfies the signature condition $(1,0)$ with its natural p-principal polarization $\lambda_{\mathbb{Y}}$. Next, let $(\overline{\mathbb{Y}},\iota,\lambda_{\overline{\mathbb{Y}}})$ be the triple obtained from $(\mathbb{Y},\iota,\lambda_{\mathbb{Y}})$ by changing ι to $\iota\circ\sigma$. The $O_{\boldsymbol{k}}$-action on $\overline{\mathbb{Y}}$ satisfies the signature condition $(0,1)$, and, again, the triple $(\overline{\mathbb{Y}},\iota,\lambda_{\overline{\mathbb{Y}}})$ is unique up to isogeny. Since $\overline{\mathbb{Y}}$ has height 2, the pair $(\overline{\mathbb{Y}},\iota)$ has a canonical lift $(\bar{Y},\iota)$ over W, [3].

The *space of special homomorphisms* is the $\boldsymbol{k}$-vector space

$$\mathbb{V}:=\mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}},\mathbb{X})\otimes_{\mathbb{Z}}\mathbb{Q}.$$

with $\boldsymbol{k}$-valued hermitian form given by

$$h(x,y)=\lambda_{\overline{\mathbb{Y}}}^{-1}\circ\hat{y}\circ\lambda_{\mathbb{X}}\circ x\in\mathrm{End}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}})\otimes\mathbb{Q}\xrightarrow[\sim]{\iota^{-1}}\boldsymbol{k}.$$

Here $\hat{y}$ is the dual of y. The parity of $\mathrm{ord}\det(\mathbb{V})$ is always odd.

For a pair of integers (i,j) and a special homomorphism $x\in\mathbb{V}$, there is an associated *special cycle* $\mathcal{Z}_{i,j}(x)$ where, for any $S\in\mathrm{Nilp}_W$, $\mathcal{Z}_{i,j}(x)(S)$ is the subset of points $(X,\iota,\lambda_X,\rho_X)$ in $\mathcal{N}_j(S)$ where the composition, a quasi-homomorphism,

$$\overline{\mathbb{Y}}\times_{\mathbb{F}}\bar{S}\xrightarrow{p^i\cdot x}\mathbb{X}\times_{\mathbb{F}}\bar{S}\xrightarrow{\rho_X^{-1}}X\times_S\bar{S}$$

extends to an $O_{\boldsymbol{k}}$-linear homomorphism

$$\bar{Y}\times_W S\longrightarrow X$$

from the canonical lift of $\overline{\mathbb{Y}}$ to X. Then $\mathcal{Z}_{i,j}(x)$ is a relative (formal) divisor in $\mathcal{N}_j$. More generally, for an m-tuple $\mathbf{x} = [x_1, \dots, x_m]$ of special homomorphisms $x_r \in \mathbb{V}$, the *associated special cycle* $\mathcal{Z}_{i,j}(\mathbf{x})$ is the intersection of the special cycles associated to the components of $\mathbf{x}$.

For a collection $\mathbf{x}$ of special homomorphisms, we define the *fundamental matrix*

$$T(\mathbf{x}) = h(\mathbf{x}, \mathbf{x}) = (h(x_r, x_s)) \in \mathrm{Herm}_m(\boldsymbol{k}).$$

If a pair (i, j) is fixed, we let $\widetilde{T} = p^{2i-j}\, T(\mathbf{x})$.

We now assume that $m = n$. This will be the situation that arises from the global setting when one considers the arithmetic intersections of cycles in complementary dimensions. First, we show that, if $\widetilde{T} \notin \mathrm{Herm}_n(O_{\boldsymbol{k}})$, then $\mathcal{Z}_{i,j}(\mathbf{x})$ is empty. Next, assume that $\det T(\mathbf{x}) \neq 0$, as will be the case in the global setting when the cycles do not meet on the generic fiber. When $\widetilde{T}$ is integral, our main result describes the structure of the cycle $\mathcal{Z}_{i,j}(\mathbf{x})$ in terms of the Jordan block structure of $\widetilde{T}$.

Theorem 1.1. *Suppose that* $\det T(\mathbf{x}) \neq 0$ *and that* $\widetilde{T} \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$. *Write* $\mathrm{red}(\widetilde{T})$ *for the image of* $\widetilde{T}$ *in* $\mathrm{Herm}_n(\mathbb{F}_{p^2})$.

(i) (compatibility with the stratification) $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ *is a union of strata* $\mathcal{V}(\Lambda)^o$ *where* Λ *ranges over a finite set of vertices of level* j *which can be explicitly described in terms of* $\mathbf{x}$.

(ii) (dimension) *Let* $r^0(\widetilde{T}) = n - \mathrm{rank}(\mathrm{red}(\widetilde{T}))$ *be the dimension of the radical of the hermitian form* $\mathrm{red}(\widetilde{T})$. *Then* $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ *is purely of dimension* $\left[(r^0(\widetilde{T}) - 1)/2\right]$.

(iii) (irreducibility) *Let*[2]

$$\tilde{T} \simeq \mathrm{diag}(1_{n_0}, p1_{n_1}, \dots, p^k 1_{n_k})$$

be a Jordan decomposition of $\tilde{T}$ *and let*

$$n^+_{\mathrm{even}} = \sum_{\substack{i \geq 2 \\ \mathrm{even}}} n_i \qquad \textit{and} \qquad n^+_{\mathrm{odd}} = \sum_{\substack{i \geq 3 \\ \mathrm{odd}}} n_i.$$

Then $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ *is irreducible if and only if*

$$\max(n^+_{\mathrm{even}}, n^+_{\mathrm{odd}}) \leq 1.$$

Moreover its dimension is then $\frac{1}{2}(n_1 + n^+_{\mathrm{odd}} - 1)$.

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[2]See Theorem 4.5 for the notation.

(iv) (zero-dimensional case) $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ *is of dimension zero if and only if* $\tilde{T}$ *is* $O_{\boldsymbol{k}}$*-equivalent to* $\mathrm{diag}(1_{n-2}, p^a, p^b)$*, where* $0 \le a < b$ *and where* $a+b$ *is odd. In this case,* $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ *consists of a single point* ξ*, and the length of the local ring* $\mathcal{O}_{\mathcal{Z}_{i,j}(\mathbf{x}),\xi}$ *is finite and given by*

$$\mathrm{length}_W(\mathcal{O}_{\mathcal{Z}_{i,j}(\mathbf{x}),\xi}) = \frac{1}{2}\sum_{l=0}^{a} p^l(a+b+1-2l)\ .$$

The right hand side of the last identity can be expressed in terms of representation densities of hermitian forms. Recall that, for nonsingular hermitian matrices $S \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$ and $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$, with $m \ge n$, the representation density $\alpha_p(S,T)$ is defined as

$$\alpha_p(S,T) = \lim_{k\to\infty} (p^{-k})^{n(2m-n))} |A_{p^k}(S,T)|,$$

where

$$A_{p^k}(S,T) = \{\ x \in M_{m,n}(O_{\boldsymbol{k}}/p^k O_{\boldsymbol{k}}) \mid S[x] \equiv T \mod p^k\ \},$$

with $S[x] = {}^t x S \sigma(x)$. The density depends only on the $\mathrm{GL}_m(O_{\boldsymbol{k}})$-equivalence class of S (resp. the $\mathrm{GL}_n(O_{\boldsymbol{k}})$-equivalence class of T). An explicit formula for $\alpha_p(S,T)$ has been given by Hironaka, [5]. For $r \ge 0$, let $S_r = \mathrm{diag}(S, 1_r)$. Then

$$\alpha_p(S_r, T) = F_p(S,T;(-p)^{-r})$$

for a polynomial $F_p(S,T;X) \in \mathbb{Q}[X]$, as can be seen immediately from Hironaka's formula.

If $m = n$ and $\mathrm{ord}(\det(S)) + \mathrm{ord}(\det(T))$ is odd, then $\alpha_p(S,T) = 0$. In this case, we define the derivative of the representation density

$$\alpha_p'(S,T) = -\frac{\partial}{\partial X} F_p(S,T;X)|_{X=1}.$$

The right hand side of the identity in (iv) of Theorem 1.1 is now expressed in terms of hermitian representation densities, as follows.

Proposition 1.2. *Let* $S = 1_n$ *and* $T = \mathrm{diag}(1_{n-2}, p^a, p^b)$ *for* $0 \le a < b$ *with* $a+b$ *odd. Then* $\alpha_p(S,T) = 0$ *and*

$$\frac{\alpha_p'(S,T)}{\alpha_p(S,S)} = \frac{1}{2}\sum_{\ell=0}^{a} p^\ell(a+b-2\ell+1),$$

where

$$\alpha_p(S,S) = \prod_{\ell=1}^{n}(1-(-1)^\ell p^{-\ell}).$$

In this form, the formula in (iv) should hold for any nonsingular fundamental matrix, that is, the relation between derivatives of representation densities and intersection multiplicities should continue to hold even in the case of improper intersections. More precisely:

Conjecture 1.3. *Let $\mathbf{x} = [x_1, \ldots, x_n] \in \mathbb{V}^n$ be such that $\mathcal{Z}_{i,j}(\mathbf{x}) \neq \emptyset$ and such that the fundamental matrix $T = T(\mathbf{x})$ is nonsingular. Let $\tilde{T} = p^{2i-j}T$. Then $\mathcal{Z}_{i,j}(\mathbf{x})$ is connected and*

$$\chi(\mathcal{O}_{\mathcal{Z}_{i,j}(x_1)} \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}_{i,j}(x_n)}) = \frac{\alpha_p'(S, \tilde{T})}{\alpha_p(S, S)} \ .$$

The Euler-Poincaré characteristic appearing here is indeed finite, since it can be shown that $\mathcal{O}_{\mathcal{Z}_{i,j}(\mathbf{x})}$ is annihilated by a power of p. In the case that $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ is of dimension zero, it can be shown that there are no higher Tor-terms on the LHS of the above identity [27], so that indeed the statement (iv) of the above theorem confirms the conjecture in this case. Note that the analogue of Conjecture 1.3 in the case of improper intersections of cycles on the Drinfeld space was proved in [14]. The case of improper intersections of arithmetic Hirzebruch-Zagier cycles is considered by U. Terstiege in [26] and [27].[3]

The layout of the paper is as follows. In section 2, we define the moduli space $\mathcal{N}$ and recall the results of Vollaard and Wedhorn concerning its structure. In section 3, we introduce special cycles $\mathcal{Z}_{i,j}(\mathbf{x})$ on $\mathcal{N}$. In section 4, we prove the statements on $\mathcal{Z}_{i,j}(\mathbf{x})_{\mathrm{red}}$ in our main theorem. The rest of the paper is concerned with the determination of the length of $\mathcal{O}_{\mathcal{Z}_{i,j}(\mathbf{x}),\xi}$ in the situation of (iv) of the main theorem. In section 5, this problem is reduced to a deformation problem on 2-dimensional formal p-divisible groups. In section 6, we introduce the analogue in our context of Gross's quasi-canonical divisors. In section 7, we solve a lifting problem of homomorphisms analogous to the one solved by Gross in the classical case. In section 8, we use the results of the previous two sections to solve the deformation problem of section 5. Finally, in section 9, we relate the RHS in (iv) of the main theorem to representation densities.

There are two important ingredients of algebraic-geometric nature that are used in our proofs. The first is the results of I. Vollaard and T. Wedhorn on the structure of $\mathcal{N}_{\mathrm{red}}$, cf. Theorem 2.7. The second is the determination, due to Th. Zink, of the length of a certain specific deformation space in

[3]Since this paper was submitted, Terstiege has proved the above conjecture in the case $n = 3$, cf. [28].

equal characteristic, cf. Proposition 8.2. Both are based on Zink's theory of displays and their windows, [35, 36].

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2. Structure of the moduli space $\mathcal{N}$

In this section we recall some facts about the moduli space $\mathcal{N}$ from [29]. We write $\boldsymbol{k} = \mathbb{Q}_{p^2}$ for the unramified quadratic extension of $\mathbb{Q}_p$ and $O_{\boldsymbol{k}} = \mathbb{Z}_{p^2}$ for its ring of integers. We also write $\mathbb{F} = \bar{\mathbb{F}}_p$ and $W = W(\mathbb{F})$ for its ring of Witt vectors. There are two embeddings φ_0 and $\varphi_1 = \varphi_0 \circ \sigma$ of $\boldsymbol{k}$ into $W_{\mathbb{Q}} = W \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.1. **Definition of the moduli space.** Let $(\mathbb{X}, \iota)$ be a fixed supersingular p-divisible group of dimension n and height $2n$ over $\mathbb{F}$ with an action $\iota : O_{\boldsymbol{k}} \to \operatorname{End}(\mathbb{X})$ satisfying the *signature condition* $(r, n-r)$,

$$\operatorname{char}(\iota(\alpha), \operatorname{Lie}\mathbb{X})(T) = (T - \varphi_0(\alpha))^r (T - \varphi_1(\alpha))^{n-r} \in \mathbb{F}[T]. \tag{2.1}$$

Let $\lambda_{\mathbb{X}}$ be a p-principal polarization of $\mathbb{X}$ for which the Rosati involution $*$ satisfies $\iota(\alpha)^* = \iota(\alpha^\sigma)$. The data $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ is unique up to isogeny.

Let Nilp_W be the category of W-schemes on which p is locally nilpotent. The functor $\mathcal{N} = \mathcal{N}(r, n-r)$ associates to a scheme $S \in \mathrm{Nilp}_W$ the set of isomorphism classes of data $(X, \iota, \lambda_X, \rho_X)$. Here X is a p-divisible group over S with $O_{\boldsymbol{k}}$-action ι satisfying the signature condition $(r, n-r)$, and λ_X is a p-principal polarization of X/S, such that the Rosati involution determined by λ_X induces the Galois involution σ on $O_{\boldsymbol{k}}$. Finally,

$$\rho_X : X \times_S \bar{S} \to \mathbb{X} \times_{\mathbb{F}} \bar{S}$$

is an $O_{\boldsymbol{k}}$-linear quasi-isogeny such that $\rho_X^\vee \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X in $\operatorname{Hom}_{O_{\boldsymbol{k}}}(X, X^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$. Here, two such data $(X, \iota, \lambda_X, \rho_X)$ and $(X', \iota', \lambda_{X'}, \rho_{X'})$ are said to be isomorphic if there is a $O_{\boldsymbol{k}}$-linear isomorphism $\alpha : X \to X'$ with $(\alpha \times_W \mathbb{F}) \circ \rho_X = \rho_{X'}$, such that $\alpha^\vee \circ \lambda_{X'} \circ \alpha$ is a $\mathbb{Z}_p^\times$-multiple of λ_X. This functor is represented by a separated formal scheme $\mathcal{N}$, locally formally of finite type over W, [23]. Furthermore, $\mathcal{N}$ is formally smooth of dimension

$r(n-r)$ over W, i.e., all completed local rings of points in $\mathcal{N}(\mathbb{F})$ are isomorphic to $W[[t_1, \ldots, t_{r(n-r)}]]$. Moreover, there is a disjoint sum decomposition

$$\mathcal{N} = \coprod_{i \in \mathbb{Z}} \mathcal{N}_i \ ,$$

where $\mathcal{N}_i$ is the formal subscheme where the quasi-isogeny ρ_X has height ni.

We next review the structure of the underlying reduced subscheme $\mathcal{N}_{\mathrm{red}}$ of $\mathcal{N}$, following [29]. Note that $\mathcal{N}_{\mathrm{red}}(\mathbb{F}) = \mathcal{N}(\mathbb{F})$. Let $\mathbb{M}$ be the (covariant) Dieudonné module of $\mathbb{X}$ and let $N = \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated isocrystal. Then N has an action of $\boldsymbol{k}$ and a skew-symmetric $W_{\mathbb{Q}}$-bilinear form $\langle\ ,\ \rangle$ satisfying

$$\langle\, Fx, y\,\rangle = \langle\, x, Vy\,\rangle^{\sigma}, \tag{2.2}$$

and with $\langle\, \alpha x, y\,\rangle = \langle\, x, \alpha^{\sigma} y\,\rangle$, for $\alpha \in \boldsymbol{k}$.

Proposition 2.1. ([29], Lemmas 1.4, 1.6) *There is a bijection between $\mathcal{N}_i(\mathbb{F})$ and the set of W-lattices M in N such that M is stable under F, V, and $O_{\boldsymbol{k}}$, and with the following properties:*

$$\mathrm{char}(\alpha, M/VM)(T) = (T - \varphi_0(\alpha))^r (T - \varphi_1(\alpha))^{n-r} \in \mathbb{F}[T], \tag{2.3}$$

and

$$M = p^i M^{\perp} \ , \tag{2.4}$$

where

$$M^{\perp} = \{\, x \in N \mid \langle\, x, M\,\rangle \subset W\,\}. \tag{2.5}$$

The isomorphism $O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} W \oplus W$ yields a decomposition $M = M_0 \oplus M_1$ into rank n submodules, and F and V have degree 1 with respect to this decomposition. Also M_0 and M_1 are isotropic with respect to $\langle\ ,\ \rangle$. Moreover, the determinant condition (2.3) is equivalent to the chain condition

$$pM_0 \overset{n-r}{\subset} FM_1 \overset{r}{\subset} M_0, \tag{2.6}$$

or, equivalently,

$$pM_1 \overset{r}{\subset} FM_0 \overset{n-r}{\subset} M_1.$$

(Here the numbers above the inclusion signs indicate the lengths of the respective cokernels.) Note that $M_0 = M \cap N_0$ and $M_1 = M \cap N_1$ for the analogous decomposition $N = N_0 \oplus N_1$.

Since the isocrystal N is supersingular, the operator $\tau = V^{-1}F = pV^{-2}$ is a σ^2-linear automorphism of degree 0 and has all slopes 0. Let

$$C = N_0^{\tau=1}$$

be the space of τ-invariants, so that C is a n-dimensional $\mathbb{Q}_{p^2}$-vector space and

$$N_0 = C \otimes_{\mathbb{Q}_{p^2},\varphi_0} W_{\mathbb{Q}}.$$

For x and $y \in N$, let

$$\{x, y\} = \delta^{-1}\langle\, x, Fy\,\rangle,$$

where $\delta \in \mathbb{Z}_{p^2}^{\times}$ with $\delta^{\sigma} = -\delta$ is fixed once and for all. Note that the corresponding form in [29] is taken without the scaling by δ. By (2.2),

$$\{x, y\} = \{y, \tau^{-1}(x)\}^{\sigma},$$

and hence

$$\{\tau(x), \tau(y)\} = \{x, y\}^{\sigma^2}.$$

Thus, $\{\,,\}$ defines a $\mathbb{Q}_{p^2}$-valued hermitian form on $N^{\tau=1}$ and, in particular, on C. Note that, since the polarization form $\langle\,,\rangle$ is non-degenerate on N with N_0 and N_1 isotropic subspaces, and since $FN_0 = N_1$, the hermitian form $\{\,,\}$ is non-degenerate on C.

For a W-lattice $L \subset N_0$, let

$$L^{\vee} = \{\, x \in N_0 \mid \{x, L\} \subset W\,\} = (FL)^{\perp}. \tag{2.7}$$

Then $(L^{\vee})^{\vee} = \tau(L)$.

Lemma 2.2. *For an $O_{\boldsymbol{k}}$-stable Dieudonné module $M = M_0 \oplus M_1$,*

$$M = p^i M^{\perp} \qquad \Longleftrightarrow \qquad FM_1 = p^{i+1} M_0^{\vee}.$$

Proof. The condition $M = p^i M^{\perp}$ is equivalent to the condition

$$\langle\, M_0, M_1\,\rangle = p^i W. \tag{2.8}$$

Since $VF = FV = p$, this is, in turn, equivalent to $\langle\, FM_1, FM_0\,\rangle = p^{i+1}W$, by (2.2). This last identity can be rewritten as

$$\{FM_1, M_0\} = p^{i+1}W, \tag{2.9}$$

as claimed. □

Note that the lattice M_0 in N_0 determines the lattice M_1 in N_1, either by (2.8) or by the condition $FM_1 = p^{i+1}M_0^{\vee}$.

Proposition 2.3. ([29], Prop. 1.11, Prop. 2.6 a)) *There is a bijection between $\mathcal{N}_i(\mathbb{F})$ and the set of W-lattices*

$$\mathcal{D}_i = \mathcal{D}_i(C) = \big\{\ A \subset N_0 \ \big|\ p^{i+1}A^{\vee} \overset{r}{\subset} A \overset{n-r}{\subset} p^i A^{\vee}\ \big\},$$

given by mapping the Dieudonné lattice $M = M_0 \oplus M_1$ to the W-lattice $A = M_0$ in N_0. Under this correspondence

$$FM_1 = p^{i+1}A^{\vee}.$$

Remark 2.4. As observed in [29], Lemma 1.16, for an $O_{\boldsymbol{k}}$- lattice A in C with
$$pA^\vee \overset{r}{\subset} A \overset{n-r}{\subset} A^\vee,$$
there is a $O_{\boldsymbol{k}}$-basis for A for which the hermitian form $\{,\}$ has matrix $\mathrm{diag}(1_r, p1_{n-r})$.

Remark 2.5. We record the two simplest examples for later use. We first consider the case of signature $(1,0)$. Let us call $\mathbb{Y}$ the base point used to define $\mathcal{N}$ above. In this case we have $C = \mathbb{Q}_{p^2}\cdot 1_0$ with hermitian form given by $\{1_0, 1_0\} = 1$, and $N_0 = W_{\mathbb{Q}}\cdot 1_0$. Moreover, $\mathcal{N}_i(\mathbb{F}) \simeq \mathcal{D}_i$ is empty for i odd, and $\mathcal{N}_{2i}(\mathbb{F}) \simeq \mathcal{D}_{2i}$ consists of a single point corresponding to the W-lattice $A = p^i W\cdot 1_0$ in N_0, where $A^\vee = p^{-i}W\cdot 1_0$. Let $\mathbb{M}^0 = \mathbb{M}^0_0 + \mathbb{M}^0_1$ be the Dieudonné module of $\mathbb{Y}$. Then, $\mathbb{M}^0_0 = W\cdot 1_0$, $\mathbb{M}^0_1 = W\cdot 1_1$, $F1_1 = p1_0$, $F1_0 = 1_1$, and the polarization $\langle\ ,\ \rangle^0$ is given by
$$\langle\, 1_0, 1_1\,\rangle^0 = \delta. \tag{2.10}$$
Note that, on $N_0 = \mathbb{M}^0\otimes_{\mathbb{Z}}\mathbb{Q}$, $\{1_0, 1_0\} = \delta^{-1}\langle\, 1_0, F1_0\,\rangle^0 = 1$, as required. Also note that $\mathrm{End}(\mathbb{Y})$ can be identified with the ring of integers O_D in the quaternion division algebra over $\mathbb{Q}_p$, and that the endomorphism
$$\Pi:\ 1_0 \mapsto p1_1,\ 1_1 \mapsto 1_0 \tag{2.11}$$
is a uniformizer of O_D.
Now $\mathbb{Y}$ is a formal $O_{\boldsymbol{k}}$-module over $\mathbb{F}$. Hence there is a unique lift Y of $\mathbb{Y}$ to W as a formal $O_{\boldsymbol{k}}$-module. In particular, $\mathcal{N}_{2i} = \mathrm{Spf}\, W$ for every i.

In the case of signature $(0,1)$, again $C = \mathbb{Q}_{p^2}\cdot \bar{1}_0$ but now with hermitian form given by $\{\bar{1}_0, \bar{1}_0\} = p$. Moreover, $\mathcal{N}_i(\mathbb{F}) \simeq \mathcal{D}_i$ is empty for i odd, and $\mathcal{N}_{2i}(\mathbb{F}) \simeq \mathcal{D}_{2i}$ consists of a single point corresponding to the W-lattice $A = p^i W\bar{1}_0$ in N_0, where $A^\vee = p^{-i-1}W\bar{1}_0$. We write $\overline{\mathbb{Y}}$ for the p-divisible group over $\mathbb{F}$ corresponding to the unique point of $\mathcal{N}_0(\mathbb{F})$, and let $\overline{\mathbb{M}}^0 = \overline{\mathbb{M}}^0_0 + \overline{\mathbb{M}}^0_1$ be its Dieudonné module. Then, $\overline{\mathbb{M}}^0_0 = W\cdot\bar{1}_0$, $\overline{\mathbb{M}}^0_1 = W\cdot\bar{1}_1$, $F\bar{1}_1 = \bar{1}_0$, $F\bar{1}_0 = p\bar{1}_1$, and the polarization is given by
$$\langle\, \bar{1}_0, \bar{1}_1\,\rangle^0 = \delta. \tag{2.12}$$
On $\bar{N}_0 = \overline{\mathbb{M}}^0\otimes_{\mathbb{Z}}\mathbb{Q}$, we have $\{\bar{1}_0, \bar{1}_0\} = \delta^{-1}\langle\, \bar{1}_0, F\bar{1}_0\,\rangle^0 = p$.

Note that the formal $O_{\boldsymbol{k}}$-module $\overline{\mathbb{Y}}$ is obtained from $\mathbb{Y}$ by identifying the underlying p-divisible groups, but changing the $O_{\boldsymbol{k}}$-action by precomposing it with the Galois automorphism σ of $O_{\boldsymbol{k}}$. On the level of Dieudonné modules, this identification is given by switching the roles of $\mathbb{M}^0_0$ and $\mathbb{M}^0_1$. In particular, the canonical lift $\bar{Y}$ of $\overline{\mathbb{Y}}$ to W is isomorphic to the canonical lift Y, with the conjugate $O_{\boldsymbol{k}}$-action.

2.2. **The case of signature** $(1, n-1)$**.** From now on, we assume that $r = 1$ so that our signature is $(1, n-1)$.

The structure of $\mathcal{D}_i$ is best described in terms of strata associated to vertices of the building for the unitary group $\mathrm{U}(C)$ of the hermitian space $(C, \{,\})$. Recall [29] that a vertex of level i is a τ-invariant W-lattice Λ in N_0 with

$$p^{i+1}\Lambda^\vee \overset{t}{\subset} \Lambda \subset p^i\Lambda^\vee.$$

Here we are identifying lattices in C with τ-invariant lattices in N_0. The type of a vertex Λ is the index $t = 2d+1$ of $p^{i+1}\Lambda^\vee$ in Λ (which is always an odd integer between 1 and n).

Lemma 2.6. (quantitive version of Zink's Lemma, cf. [29], Lemma 2.2.) *For $A \in \mathcal{D}_i$, there exists a minimal d, $0 \le d \le \frac{n-1}{2}$, such that the lattice*

$$\Lambda = \Lambda(A) := A + \tau(A) + \cdots + \tau^d(A)$$

is τ-invariant. Then Λ is a vertex of level i, i.e.,

$$p^{i+1}\Lambda^\vee \subseteq \Lambda \subseteq p^i\Lambda^\vee \ .$$

Moreover

$$p^{i+1}\Lambda^\vee \subseteq p^{i+1}A^\vee \overset{1}{\subset} A \subseteq \Lambda \subseteq p^i\Lambda^\vee,$$

and the index of $p^{i+1}\Lambda^\vee$ in Λ, the type of Λ, is $2d+1$.

A lattice $A \in \mathcal{D}_i$ is *superspecial* if $\tau(A) = A$, so that $A = \Lambda$ is also a vertex of type 1. In general, $\Lambda(A)$ is the smallest τ-invariant lattice containing A and, equivalently, $p^{i+1}\Lambda^\vee$ is the largest τ-invariant lattice contained in $p^{i+1}A^\vee$.

For a vertex Λ of level i, let

$$\mathcal{V}(\Lambda) = \{\ A \in \mathcal{D}_i \ \mid \ A \subset \Lambda\ \}.$$

Equivalently, $p^{i+1}\Lambda^\vee \subset p^{i+1}A^\vee$.

As Λ ranges over the vertices of level i, the subsets $\mathcal{V}(\Lambda)$ of $\mathcal{D}_i$ are organized in a coherent way, as follows.

For two vertices Λ and Λ' of level i,

$$\mathcal{V}(\Lambda) \subset \mathcal{V}(\Lambda') \iff \Lambda \subset \Lambda', \tag{2.1}$$

cf. [29], Prop. 2.7. In this case we have $t(\Lambda) \le t(\Lambda')$ for the types.

Let

$$\mathcal{V}(\Lambda)^o = \mathcal{V}(\Lambda) \setminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathcal{V}(\Lambda').$$

Then by [29], Prop. 3.5

$$\mathcal{V}(\Lambda) = \bigcup_{\Lambda' \subset \Lambda} \mathcal{V}(\Lambda')^o, \tag{2.2}$$

where the (finite) union on the right hand side is disjoint and runs over all vertex lattices of level i contained in Λ (these are then of type $t(\Lambda') \leq t(\Lambda)$).

The subsets $\mathcal{V}(\Lambda)$ and $\mathcal{V}(\Lambda)^o$ have an algebraic-geometric meaning.

Theorem 2.7. (Vollaard, Wedhorn) *Let $\mathcal{N}_i$ be non-empty.*
(i) $\mathcal{N}_{i,\mathrm{red}}$ is connected.
(ii) For any vertex Λ of level i, $\mathcal{V}(\Lambda)$ is the set of $\mathbb{F}$-points of a closed irreducible subvariety of dimension $\frac{1}{2}(t(\Lambda)-1)$ of $\mathcal{N}_{i,\mathrm{red}}$, and the inclusions $\mathcal{V}(\Lambda') \subset \mathcal{V}(\Lambda)$ for $\Lambda' \subset \Lambda$ are induced by closed embeddings of algebraic varieties over $\mathbb{F}$.

Vollaard [29] has proved this for signature $(1, s)$ with $s = 1$ or 2. The general case is in [31]. In [31] it is also proved that the variety corresponding to $\mathcal{V}(\Lambda)$ is smooth.
We note that $\mathcal{N}_i$ is always non-empty if n is even; if n is odd, then $\mathcal{N}_i$ is non-empty if and only if i is even, cf. [29], Prop. 1.22.

We conclude this section with the following observation about scaling, which will be useful later.

Lemma 2.8. *(i) If $A \in \mathcal{D}_i$, then, for $k \in \mathbb{Z}$, $p^k A \in \mathcal{D}_{i+2k}$.*
(ii) If Λ is a vertex of level i, then $p^k\Lambda$ is a vertex of level $i + 2k$.
(iii) If Λ is a vertex of level i for the hermitian space $(C, \{,\})$, then Λ is a vertex of level $i + k$ for the hermitian space $(C, p^k\{,\})$.

Proof. Here, in cases (ii) and (iii), note that

$$p^{i+1}\Lambda^\vee \subset \Lambda \subset p^i\Lambda^\vee \quad \Longleftrightarrow \quad p^{i+1}W \subset \{\Lambda, \Lambda\} \subset p^i W.$$

□

3. Cycles in the moduli space $\mathcal{N}$.

Let $(\overline{\mathbb{Y}}, \iota)$ be the p-divisible group of dimension 1 and height 2 over $\mathbb{F}$, with an action $\iota : O_{\boldsymbol{k}} \to \mathrm{End}(\overline{\mathbb{Y}})$ of $O_{\boldsymbol{k}}$ and with principal polarization $\lambda_{\overline{\mathbb{Y}}}$ satisfying the signature condition $(0, 1)$, cf. Example 2.5. Let $\mathcal{N}^0 = \mathcal{N}(0, 1)$ be the corresponding moduli space as in section 1. Recall that, by Remark 2.5,

$$\mathcal{N}^0 = \coprod_{i \in \mathbb{Z}} \mathcal{N}^0_{2i},$$

and that $\mathcal{N}^0_{2i} = \mathrm{Spf}\, W$. For example, the unique point of $\mathcal{N}^0_0(\mathbb{F})$ corresponds to $(\overline{\mathbb{Y}}, \iota, \lambda_{\overline{\mathbb{Y}}}, \rho_{\overline{\mathbb{Y}}})$ where $\rho_{\overline{\mathbb{Y}}}$ is the identity map.

Definition 3.1. The *space of special homomorphisms* is the $\boldsymbol{k}$-vector space

$$\mathbb{V} := \mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For $x, y \in \mathbb{V}$, we let

$$h(x,y) = \lambda_{\overline{\mathbb{Y}}}^{-1} \circ \hat{y} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}) \otimes \mathbb{Q} \xrightarrow[\sim]{\iota^{-1}} \boldsymbol{k}. \tag{3.1}$$

This hermitian form is $O_{\boldsymbol{k}}$-valued on the lattice $\mathbb{L} := \mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}, \mathbb{X})$.

Definition 3.2. (i) For a given special homomorphism $x \in \mathbb{V}$, define the *special cycle $\mathcal{Z}(x)$ associated to x in $\mathcal{N}^0 \times \mathcal{N}$* as the subfunctor of collections $\xi = (\bar{Y}, \iota, \lambda_{\bar{Y}}, \rho_{\bar{Y}}, X, \iota, \lambda_X, \rho_X)$ in $(\mathcal{N}^0 \times \mathcal{N})(S)$ such that the quasi-homomorphism

$$\rho_X^{-1} \circ x \circ \rho_{\bar{Y}} : \ \bar{Y} \times_S \bar{S} \longrightarrow X \times_S \bar{S}$$

extends to a homomorphism from $\bar{Y}$ to X. Here $\bar{S} = S \times_W \mathbb{F}$ is the special fiber of S.
(ii) More generally, for a fixed m-tuple $\mathbf{x} = [x_1, \dots . x_m]$ of special homomorphisms $x_i \in \mathbb{V}$, the *associated special cycle* $\mathcal{Z}(\mathbf{x})$ is the subfunctor of $\mathcal{N}^0 \times \mathcal{N}$ of collections $\xi = (\bar{Y}, \iota, \lambda_{\bar{Y}}, \rho_{\bar{Y}}, X, \iota, \lambda_X, \rho_X)$ in $(\mathcal{N}^0 \times \mathcal{N})(S)$ such that the quasi-homomorphism

$$\rho_X^{-1} \circ \mathbf{x} \circ \rho_{\bar{Y}} : \ \bar{Y}^m \times_S \bar{S} \longrightarrow X \times_S \bar{S}$$

extends to a homomorphism from $\bar{Y}^m$ to X.
(iii) For i and $j \in \mathbb{Z}$, let $\mathcal{Z}_{ij}(\mathbf{x})$ be the formal subscheme of $\mathcal{Z}(\mathbf{x})$ whose projection to $\mathcal{N}^0$ (resp. $\mathcal{N}$) lies in $\mathcal{N}^0_{2i}$ (resp. $\mathcal{N}_j$), i.e.,

$$\begin{array}{ccc} \mathcal{Z}_{ij}(\mathbf{x}) & \longrightarrow & \mathcal{Z}(\mathbf{x}) \\ \downarrow & & \downarrow \\ \mathcal{N}^0_{2i} \times \mathcal{N}_j & \longrightarrow & \mathcal{N}^0 \times \mathcal{N}. \end{array}$$

We note that $\mathcal{N}^0_{2i}$ has been identified with $\mathrm{Spf}\, W$, via the canonical lift $\bar{Y}$ of $\overline{\mathbb{Y}}$, with its $O_{\boldsymbol{k}}$-action. Hence $\mathcal{Z}_{ij}(\mathbf{x})$ can be identified with a closed formal subscheme of $\mathcal{N}_j$.

Remark 3.3. (i) It is clear from the definition that $\mathcal{Z}(\mathbf{x})$ depends only on the orbit of the vector $\mathbf{x}$ under the right action of $\mathrm{GL}_m(O_{\boldsymbol{k}})$, which acts as automorphisms of $\overline{\mathbb{Y}}^m$.

(ii) The definition of the special cycles is compatible with intersections. Specifically, the intersection of $\mathcal{Z}(\mathbf{x})$ and $\mathcal{Z}(\mathbf{y})$ is the locus where the whole collection $[\mathbf{x}, \mathbf{y}] = [x_1, \dots, x_m, y_1, \dots, y_{m'}]$ deforms, i.e.,

$$\mathcal{Z}(\mathbf{x}) \cap \mathcal{Z}(\mathbf{y}) = \mathcal{Z}([\mathbf{x}, \mathbf{y}]).$$

Remark 3.4. We note that $\mathcal{N}^0_{2i}$ has been identified with $\mathcal{N}^0_0$. Explicitly, $(\bar{Y}, \iota, \lambda_{\bar{Y}}, \rho_{\bar{Y}})$ in $\mathcal{N}^0_{2i}$ is sent to $(\bar{Y}, \iota, \lambda_{\bar{Y}}, p^{-i}\rho_{\bar{Y}})$. Under this identification the subfunctor $\mathcal{Z}_{ij}(\mathbf{x})$ of $\mathcal{N}^0_{2i} \times \mathcal{N}_j$ is identified with the subfunctor $\mathcal{Z}_{0,j-2i}(\mathbf{x})$ of $\mathcal{N}^0_0 \times \mathcal{N}_{j-2i}$. Here the point is that the compositions $\rho_X^{-1} \circ \mathbf{x} \circ \rho_{\bar{Y}}$ and $(p^{-i}\rho_X)^{-1} \circ \mathbf{x} \circ (p^{-i}\rho_{\bar{Y}})$ coincide.
For the same reason, $\mathcal{Z}_{i,j}(\mathbf{x})$ can be identified with $\mathcal{Z}_{0,j}(p^i\mathbf{x})$.

Proposition 3.5. *The functor $\mathcal{Z}(\mathbf{x})$ is represented by a closed formal subscheme of $\mathcal{N}^0 \times \mathcal{N}$. In fact, $\mathcal{Z}(x)$ is a relative divisor in $\mathcal{N}^0 \times \mathcal{N}$ (or empty) for any $x \in \mathbb{V} \setminus \{0\}$.*

Recall that a relative divisor is a closed formal subscheme, locally defined by one equation, which is neither a unit nor divisible by p.

Proof. The first statement follows from [23], Proposition 2.9. To prove the second statement, it suffices to prove that $\mathcal{Z}_{i,j}$ is a relative divisor in $\mathcal{N}_j$ (via the second projection in $\mathcal{N}^0 \times \mathcal{N}$). By following the proof of the corresponding statement in [26], Prop. 4.5, we see that, in order to prove that $\mathcal{Z}_{i,j}(x)$ is locally defined by the vanishing of one equation, it suffices to prove the following statement. Let A be a W-algebra and let $A_0 = A/I$, where the ideal I satisfies $I^2 = 0$. We equip I with trivial divided powers. We assume given a morphism $\phi : \operatorname{Spec} A \to \mathcal{N}_j$ whose restriction to $\operatorname{Spec} A_0$ factors through $\mathcal{Z}_{i,j}$. Then the obstruction to factoring the given morphism through $\mathcal{Z}_{i,j}$ is given by the vanishing of one element in I.

The value $\mathcal{D}_{\bar{Y}}$ of the Dieudonné crystal of $\bar{Y} \times_{\operatorname{Spec} W} \operatorname{Spec} A$ on (A, A_0) is given by $\overline{\mathbb{M}} \otimes_W A$, and is equipped with its Hodge filtration

$$0 \to \mathcal{F}_{\bar{Y}} \to \mathcal{D}_{\bar{Y}} \to \operatorname{Lie} \bar{Y} \otimes_W A \to 0, \tag{3.2}$$

where $\mathcal{F}_{\bar{Y}}$ is generated by the element $\bar{1}_0$. Similarly, the Dieudonné crystal $\mathcal{D}_X$ of the pullback to A of the universal object (X, ι, λ) over $\mathcal{N}$ comes with its Hodge filtration

$$0 \to \mathcal{F}_X \to \mathcal{D}_X \to \operatorname{Lie} X \to 0. \tag{3.3}$$

The fact that the restriction of ϕ to $\operatorname{Spec} A_0$ factors through $\mathcal{Z}_{i,j}(x)$ implies that there is an $O_{\boldsymbol{k}}$-linear homomorphism $\alpha : \mathcal{D}_{\bar{Y}} \to \mathcal{D}_X$ of A-modules, which respects the filtrations (3.2) and (3.3) after tensoring with A_0. We need to show that the condition that α respect the filtrations (3.2) and (3.3) is locally defined by one equation. However, this condition is obviously that $\alpha(\bar{1}_0) \in \mathcal{F}_X$, i.e., that the image of $\alpha(\bar{1}_0)$ in the degree zero component

(Lie $X)_0$ vanishes. However, thanks to the signature condition, (Lie $X)_0$ is a locally free A-module of rank 1. After choosing a local generator of (Lie $X)_0$, we may identify $\alpha(\bar{1}_0)$ with an element in A with zero image in A_0. Hence the condition is described by the vanishing of one element in the ideal I.

We still have to show that this element is non-trivial, and that it is not divisible by p. We first note the following simple fact.

Lemma 3.6. *Let $\mathfrak{X}$ be a formal scheme such that $\mathcal{O}_{\mathfrak{X},x}$ is factorial for each $x \in \mathfrak{X}_{\mathrm{red}}$. Let $g \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ with $g|\mathfrak{X}_{\mathrm{red}} \equiv 0$ and such that $g \in \mathcal{O}_{\mathfrak{X},x}$ is an irreducible element for each $x \in \mathfrak{X}_{\mathrm{red}}$. Let $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and consider the subset*

$$V = \{x \in \mathfrak{X}_{\mathrm{red}} \mid g \text{ divides } f \text{ in } \ \mathcal{O}_{\mathfrak{X},x}\} \ .$$

Then V is open and closed in $\mathfrak{X}_{\mathrm{red}}$.

Proof. Consider the ideal sheaf

$$\mathfrak{a} = \{h \in \mathcal{O}_{\mathfrak{X}} \mid hf \in g\mathcal{O}_{\mathfrak{X}}\} \ .$$

Then $x \in V \iff \mathfrak{a}_x = \mathcal{O}_{\mathfrak{X},x} \iff x \notin \mathrm{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathfrak{a})$. Hence V is open. To show that V is closed, let $x \in V$ and let x' be a specialization of x. In $\mathcal{O}_{\mathfrak{X},x}$ we have an identity

$$\frac{f}{g} = \frac{h}{s} \quad , \quad s \in \mathcal{O}_{\mathfrak{X},x'} \setminus \mathfrak{p}_x \ .$$

Hence we obtain an identity in $\mathcal{O}_{\mathfrak{X},x'}$,

$$fs = gh \ .$$

If $g \mid f$ in $\mathcal{O}_{\mathfrak{X},x'}$, then $x' \in V$. Otherwise, since g is irreducible in $\mathcal{O}_{\mathfrak{X},x'}$, we have $g \mid s$. But then $s \in g\mathcal{O}_{\mathfrak{X},x'} \subset \mathfrak{p}_x$, a contradiction. $\square$

Now we prove that a local equation for $\mathcal{Z}_{i,j}(x)$ is not divisible by p. Otherwise, by the previous lemma, and since, by Theorem 2.7 (i), $\mathcal{N}_j$ is connected, it would follow that $\mathcal{N}_j(\mathbb{F}) = \mathcal{Z}_{i,j}(x)(\mathbb{F})$. We now distinguish the cases $n \geq 3$ and $n \leq 2$.

In the case $n \geq 3$ we appeal to Proposition 3.10 below (of course, the proof of this proposition does not use the statement we are in the process of proving). According to this proposition, the inclusion $\mathcal{N}_j(\mathbb{F}) \subset \mathcal{Z}_{i,j}(x)(\mathbb{F})$ implies

$$x \in \bigcap_{\Lambda} p^{j+1}\Lambda^{\vee} \ , \tag{3.4}$$

where Λ runs through all vertices of level j. By the next lemma, the intersection (3.4) is trivial.

Lemma 3.7. *Let $n \geq 3$. Then*

$$\bigcap_{\Lambda} \Lambda \;= (0)\ .$$

Here the intersection runs over all vertices of level j.

Proof. By the reduction argument right after Corollary 3.11 below, we may assume $j = 0$. First let n be odd. By the parity condition of [29] recalled in (4.1) below, we may choose a basis of C as follows. Let $n - 1 = 2k$ with $k \geq 1$. We choose a basis $e, f_{\pm 1}, \ldots, f_{\pm k}$ with

$$\{e, e\} = 1,\ \{e, f_{\pm i}\} = 0\ , \quad \{f_i, f_j\} = p\delta_{i,-j}\ .$$

Let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$. Set

$$\Lambda = \Lambda_{\mathbf{a}} = [e, p^{a_1} f_1, \ldots, p^{a_k} f_k, p^{-a_k} f_{-k}, \ldots, p^{-a_1} f_{-1}]\ .$$

Then

$$\Lambda^{\vee} = [e, p^{a_1 - 1} f_1, \ldots, p^{a_k - 1} f_k, p^{-a_k - 1} f_{-k}, \ldots, p^{-a_1 - 1} f_{-1}]\ .$$

Hence Λ is a vertex of level 0. By varying $\mathbf{a} \in \mathbb{Z}^k$, the intersection of these vertices is zero, which implies the assertion in this case.

Now let n be even. Let $n - 1 = 2k + 1$, with $k \geq 1$. We choose a basis $e, f_{\pm 1}, \ldots, f_{\pm k}, g$ with

$$\{e, e\} = 1, \{g, g\} = p, \{e, g\} = \{e, f_{\pm i}\} = \{g, f_{\pm i}\} = 0, \{f_{\pm i}, f_{\pm j}\} = p\delta_{i,-j}\ .$$

Then for $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, the lattice

$$\Lambda_{\mathbf{a}} = [e, p^{a_1} f_1, \ldots, p^{a_k} f_k, g, p^{-a_k} f_{-k}, \ldots, p^{-a_1} f_{-1}]$$

is a vertex of level 0, and the assertion follows as before. □

This proves the assertion for $n \geq 3$. For $n = 1$, the assertion is trivial. Now let $n = 2$. In this case $\mathcal{N}_{\mathrm{red}}$ is a discrete set of points, but the local rings are two-dimensional, hence the assertion is non-trivial in this case. Let $y \in \mathcal{Z}_{i,j}(x)(\mathbb{F})$. By the uniformization theorem [23], Thm. 6.30, the complete local ring $\widehat{\mathcal{O}}_{\mathcal{N}_j, y}$ is isomorphic to the completion of the local ring of a closed point of the integral model of the Shimura variety attached to $\mathrm{GU}(1,1)$. Hence by Wedhorn's theorem [32], the pullback of the universal p-divisible group X to the generic point of Spec $(\widehat{\mathcal{O}}_{\mathcal{N}_j, y} \otimes \mathbb{F})$ is ordinary, i.e., isogenous to $\widehat{\mathbb{G}}_m^2 \times (\mathbb{Q}_p/\mathbb{Z}_p)^2$. Hence there is no non-trivial homomorphism from $\bar{\mathbb{Y}}$ into this p-divisible group, hence the closed subscheme of Spec $\widehat{\mathcal{O}}_{\mathcal{N}_j, y}$ cut out by an equation of $\mathcal{Z}_{i,j}(x)$ does not contain the special fiber, as was to be shown. □

To study the set $\mathcal{Z}(\mathbf{x})(\mathbb{F})$ of $\mathbb{F}$-points of a special cycle $\mathcal{Z}(\mathbf{x})$, we apply the construction of the previous section and reformulate things in terms of the hermitian space C.

We begin by describing the space $\mathbb{V}$ of special homomorphisms. Recall from Remark 2.5 the Dieudonné module $\overline{\mathbb{M}}^0 = \overline{\mathbb{M}}^0_0 + \overline{\mathbb{M}}^0_1 = W\bar{1}_0 + W\bar{1}_1$ of $\overline{\mathbb{Y}}$. A special homomorphism $x \in \mathbb{V}$ corresponds to a homomorphism, which we also denote by x, from $\bar{N}^0$ to N. This homomorphism has degree zero with respect to the grading given by the $O_{\boldsymbol{k}}$-action, and so we may write $x = x_0 + x_1$ where $x_0 : \bar{N}^0_0 \to N_0$ and $x_1 : \bar{N}^0_1 \to N_1$. Moreover, since x is F-linear, $x_1 F = F x_0$, so that x_0 determines x_1. In particular, x is determined by $x_0(\bar{1}_0)$. Note that $x_0(\bar{1}_0) \in C = N_0^{\tau=1}$, since x commutes with F and V, and $\bar{1}_0 \in C^0 = (\bar{N}^0_0)^{\tau=1}$.

The hermitian form on $\mathbb{V}$ defined by (3.1) can be written as

$$h(x,y) = \iota^{-1}(y^* \circ x),$$

where y^* is the adjoint of y with respect to the polarizations, i.e., for $u^0 \in \bar{N}^0$ and $u \in N$,

$$\langle\, y(u^0), u \,\rangle = \langle\, u^0, y^*(u) \,\rangle^0.$$

Lemma 3.8. *For $x = x_0 + x_1$ and $y = y_0 + y_1$ in $\mathbb{V}$,*

$$y^* \circ x = y_1^* \circ x_0 + y_0^* \circ x_1 \in O_{\boldsymbol{k}} \otimes W \simeq W \oplus W,$$

with

$$y_1^* \circ x_0 = p^{-1}\{x_0(\bar{1}_0), y_0(\bar{1}_0)\},$$

and

$$y_0^* \circ x_1 = p^{-1}\{y_0(\bar{1}_0), x_0(\bar{1}_0)\}.$$

Proof. Writing $y_1^* \circ x_0(\bar{1}_0) = \alpha\, \bar{1}_0$, we have

$$\begin{aligned} -\alpha\delta &= \langle\, \bar{1}_1, y_1^* \circ x_0(\bar{1}_0) \,\rangle^0 \\ &= \langle\, y_1(\bar{1}_1), x_0(\bar{1}_0) \,\rangle \\ &= \langle\, F y_0 F^{-1}(\bar{1}_1), x_0(\bar{1}_0) \,\rangle \\ &= -p^{-1}\delta\{x_0(\bar{1}_0), y_0(\bar{1}_0)\}. \end{aligned}$$

The component $y_0^* \circ x_1$ is found in the same way. □

In summary, we have proved the following.

Lemma 3.9. *There is an isomorphism*

$$\mathbb{V} = \mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes \mathbb{Q} \xrightarrow{\ \sim\ } C, \qquad x \mapsto x_0(\bar{1}_0).$$

The hermitian forms on the two spaces are related by

$$h(x,y) = p^{-1}\{x_0(\bar{1}_0), y_0(\bar{1}_0)\}.$$

□

Proposition 3.10. *For $\mathbf{x} = [x_1, \dots, x_m] \in \mathbb{V}^m$, let L be the W-submodule in N_0 spanned by the components of*

$$\mathbf{x}_0(\bar{1}_0) = [(x_1)_0(\bar{1}_0), \dots, (x_m)_0(\bar{1}_0)] \in C^m.$$

Then the image of the projection of $\mathcal{Z}_{i,j}(\mathbf{x})$ to $\mathcal{N}_j(\mathbb{F}) \simeq \mathcal{D}_j$ is

$$\mathcal{W}_{i,j}(\mathbf{x}) := \{\ A \in \mathcal{D}_j \mid p^i L \subset p^{j+1} A^\vee\ \}.$$

Proof. Recall that $\mathcal{N}_0^0(\mathbb{F})$ consists of a single point corresponding to $\overline{\mathbb{Y}}$ with Dieudonné lattice given above, cf. Remark 2.5. Similarly, $\mathcal{N}_{2i}^0(\mathbb{F})$ consists of a single point corresponding to $\bar{Y}$ with Dieudonné lattice $\bar{M}^0 = p^i\overline{\mathbb{M}}^0$ in $\bar{N}^0$. A special homomorphism $x \in \mathbb{V}$, extends to $\bar{Y} \to X$ if and only if $x(p^i\overline{\mathbb{M}}^0) \subset M$, where M is the Dieudonné lattice of X. The latter condition is equivalent to $p^i x_0(\bar{1}_0) \in M_0$ and $p^i x_1(\bar{1}_1) \in M_1$. Recall that $FM_1 = p^{j+1}A^\vee \subset A$. Then, $p^i x_1(\bar{1}_1) \in M_1$ if and only if $p^i F x_1(\bar{1}_1) \in FM_1 = p^{j+1}A^\vee$. But, since $Fx_1 = x_0 F$ and $F(\bar{1}_1) = \bar{1}_0$, this last condition is equivalent to $p^i x_0(\bar{1}_0) \in p^{j+1}A^\vee$, which, in turn, implies the condition $p^i x_0(\bar{1}_0) \in M_0 = A$. Thus, a collection $\mathbf{x}$ extends if, for each component x_r, $p^i(x_r)_0(\bar{1}_0)$ lies in $p^{j+1}A^\vee$, or, equivalently, $p^i L \subset p^{j+1}A^\vee$. □

We call the hermitian matrix

$$T = h(\mathbf{x},\mathbf{x}) = \big(h(x_i, x_j)\big) \in \mathrm{Herm}_m(\boldsymbol{k})$$

the *fundamental matrix* determined by $\mathbf{x}$. There is a variant of it which will also be useful in the sequel. Namely, when considering $\mathcal{Z}_{i,j}(\mathbf{x})$, where the fundamental matrix of $\mathbf{x}$ is T, we will call the matrix $\tilde{T} = p^{2i-j}T$ the *scaled fundamental matrix* attached to $\mathcal{Z}_{i,j}(\mathbf{x})$.

Corollary 3.11. *If $\mathcal{Z}_{i,j}(\mathbf{x})(\mathbb{F})$ is non-empty, then the corresponding scaled fundamental matrix $\tilde{T}$ is integral, i.e.,*

$$\tilde{T} \in \mathrm{Herm}_m(O_{\boldsymbol{k}}).$$

Proof. The components of the matrix $h(\mathbf{x},\mathbf{x})$ have the form $p^{-1}\{x_0(\bar{1}_0), y_0(\bar{1}_0)\}$ with $x_0(\bar{1}_0)$ and $y_0(\bar{1}_0)$ contained in $p^{j-i+1}A^\vee \subset p^{-i}A$. Note that the matrix determined by the components of $\mathbf{x}_0(\bar{1}_0)$ is then

$$\{\mathbf{x}_0(\bar{1}_0), \mathbf{x}_0(\bar{1}_0)\} = \big(\{(x_i)_0(\bar{1}_0), (x_j)_0(\bar{1}_0)\}\big) = p\,T \in p^{j-2i+1}\,\mathrm{Herm}_m(O_{\boldsymbol{k}}).$$

□

We note various 'scaling relations' among the cycles $\mathcal{Z}_{i,j}(\mathbf{x})$. First, there is an isomorphism

$$\mathcal{Z}_{i,j}(\mathbf{x})(\mathbb{F}) \xrightarrow{\sim} \mathcal{Z}_{0,j-2i}(\mathbf{x})(\mathbb{F}), \qquad (\bar{M}^0, M) \mapsto (\overline{\mathbb{M}}^0, p^{-i}M). \tag{3.5}$$

Next, note that

$$\mathcal{W}_{i,j}(\mathbf{x}) = \mathcal{W}_{i,j}(L) = \mathcal{W}_{0,j}(p^i L). \tag{3.6}$$

These two relations are simply the translation into the language of lattices of the scaling relations on the level of formal schemes in Remark 3.4. Finally, note that the set of lattices $\mathcal{D}_j$ for the hermitian space $(C, \{,\})$ coincides with the set of lattices $\mathcal{D}_0$ for the space $(C, p^j\{,\})$. Both sets are empty if n and j are both odd.

4. Hermitian lattices

In this section, we consider the cycle $\mathcal{Z}_{i,j}(\mathbf{x})$ determined by an n-tuple $\mathbf{x}$ of special homomorphisms with *nonsingular* fundamental matrix $T = h(\mathbf{x}, \mathbf{x}) \in \mathrm{Herm}_n(\boldsymbol{k})$. For global reasons (cf. the Introduction), the cycle $\mathcal{Z}_{i,j}(\mathbf{x})$ has empty generic fiber in this case, i.e., p is locally nilpotent on $\mathcal{Z}_{i,j}(\mathbf{x})$. We give a description of $\mathcal{Z}_{i,j}(\mathbf{x})(\mathbb{F})$ as a union of the strata in $\mathcal{N}^0_{2i} \times \mathcal{N}_j$ defined in [29].

Recall [29] that the hermitian space $(C, \{,\})$ is determined up to isomorphism by its dimension, since

$$\mathrm{ord}\det(C) \equiv \dim(C) + 1 \mod 2. \tag{4.1}$$

Let $\mathbf{x}_0(\bar{1}_0) = \big[(x_1)_0(\bar{1}_0), \ldots, (x_n)_0(\bar{1}_0)\big]$ be an n–tuple of vectors in C spanning a lattice L, and let $T' = \{\mathbf{x}_0(\bar{1}_0), \mathbf{x}_0(\bar{1}_0)\} = p\,T$ be the corresponding matrix of inner products. By Lemma 3.9, $\mathrm{ord}\det(T) = \mathrm{ord}\det(C) - n$, hence

$$\mathrm{ord}\det(T) \text{ is odd.} \tag{4.2}$$

The cycle $\mathcal{Z}_{i,j}(\mathbf{x})$ determined by $\mathbf{x}$ depends only on the $O_{\boldsymbol{k}}$-lattice L, and, by Proposition 3.10, the projection of $\mathcal{Z}_{i,j}(\mathbf{x})(\mathbb{F})$ to $\mathcal{N}_j(\mathbb{F}) \xrightarrow{\sim} \mathcal{D}_j$ is the set

$$\mathcal{W}_{i,j}(L) = \{\ A \in \mathcal{D}_j \mid p^i L \subset p^{j+1} A^\vee\ \}.$$

We first note that $\mathcal{W}_{i,j}(L)$ is a union of strata $\mathcal{V}(\Lambda)$.

Proposition 4.1.

$$\mathcal{W}_{i,j}(L) = \bigcup_{\substack{\Lambda \\ p^i L \subset p^{j+1}\Lambda^\vee}} \mathcal{V}(\Lambda),$$

where the Λ's are vertices of level j.

Proof. The lattice $\Lambda(A)$ associated to $A \in \mathcal{D}_j$ by Lemma 2.6 is the smallest τ-stable W-lattice containing A. By duality, $p^{j+1}\Lambda(A)^\vee$ is the largest τ-stable W-lattice contained in $p^{j+1}A^\vee$. Thus, $p^iL \subset p^{j+1}A^\vee$ if and only if $p^iL \subset p^{j+1}\Lambda(A)^\vee$. Thus, $A \in \mathcal{W}_{i,j}(L)$ if and only if $\mathcal{V}(\Lambda(A)) \subset \mathcal{W}_{i,j}(L)$. □

Our main results about the structure of $\mathcal{W}_{i,j}(L)$ are the following. In the rest of the section, $\tilde{T} = p^{2i-j}T$ will denote the corresponding scaled fundamental matrix.

Theorem 4.2. *(i) If $\tilde{T} \notin \mathrm{Herm}_n(O_{\boldsymbol{k}})$, then $\mathcal{W}_{i,j}(L)$ is empty.*
(ii) If $\tilde{T} \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$, let $\mathrm{red}(\tilde{T})$ be the image of $\tilde{T}$ in $\mathrm{Herm}_n(\mathbb{F}_{p^2})$, and let $t_0 = t_0(\tilde{T})$ be the largest odd integer less than or equal to $n - \mathrm{rank}(\,\mathrm{red}(\tilde{T})\,)$. Then

$$\mathcal{W}_{i,j}(L) = \bigcup_{\substack{\Lambda \\ p^iL \subset p^{j+1}\Lambda^\vee \\ t(\Lambda)=t_0}} \mathcal{V}(\Lambda).$$

By Theorem 2.7, (ii) we deduce from this theorem:

Corollary 4.3. *If it is non-empty, $\mathcal{W}_{i,j}(L)$ is the set of $\mathbb{F}$-points of a variety of pure dimension $\frac{1}{2}(t_0(\tilde{T}) - 1)$.* □

Remark 4.4. Note that t_0 only depends on the span L of the components of $\mathbf{x}_0(\bar{1}_0)$.

Theorem 4.5. *Let*[4]

$$\tilde{T} \simeq \mathrm{diag}(1_{n_0}, p1_{n_1}, \dots, p^k 1_{n_k})$$

be a Jordan decomposition of $\tilde{T}$ and let

$$n^+_{\text{even}} = \sum_{\substack{i \geq 2 \\ i \text{ even}}} n_i \qquad \text{and} \qquad n^+_{\text{odd}} = \sum_{\substack{i \geq 3 \\ i \text{ odd}}} n_i.$$

Then $\mathcal{W}_{i,j}(L) = \mathcal{V}(\Lambda)$ for a unique vertex Λ of level j and type $t_0(\tilde{T})$ if and only if

$$(\star) \qquad \max(n^+_{\text{even}}, n^+_{\text{odd}}) \leq 1.$$

By Theorem 2.7, (ii) we deduce from this theorem:

Corollary 4.6. *$\mathcal{W}_{i,j}(L)$ is the set of $\mathbb{F}$-points of an irreducible variety if and only if the condition $(\star)$ in the previous theorem is satisfied.* □

---

[4] Since p is odd, every $\mathrm{GL}_n(O_{\boldsymbol{k}})$-orbit in $\mathrm{Herm}_n(O_{\boldsymbol{k}})$ has a unique representative of this form.

Corollary 4.7. *$\mathcal{W}_{i,j}(L)$ consists of a single point if and only if*

$$n - \mathrm{rank}(\mathrm{red}(\tilde{T})) \leq 2.$$

$\square$

Remark 4.8. Is it true that $\mathcal{W}_{i,j}(L)$ is always connected?

Before proving these results, we make a few simple observations. First of all, since $\mathcal{W}_{i,j}(L) = \mathcal{W}_{0,j}(p^i L)$, it suffices to consider the case where $i = 0$. Second, for a lattice $A \subset C$, note that $p^j A^\vee$ is the dual lattice with respect to the scaled hermitian form $p^{-j}\{,\}$. Thus, if we denote by $C^{(j)}$ the scaled hermitian space $(C, p^{-j}\{,\})$, we have $\mathcal{D}_j = \mathcal{D}_0^{(j)}$ in the obvious notation. Thus,

$$\mathcal{W}_{i,j}(L) = \mathcal{W}_{0,0}^{(j)}(p^i L).$$

Moreover, Λ is a vertex of level j in C if and only if Λ is a vertex of level 0 in $C^{(j)}$. Finally, note that if $T = h(\mathbf{x}, \mathbf{x})$, then $p^{2i-j}T = p^{-j}h(p^i\mathbf{x}, p^i\mathbf{x})$. Thus it suffices to consider the case where $i = j = 0$. It is important to note that for n odd, $\mathcal{N}_j(\mathbb{F})$ is empty unless j is even. Thus, in all cases, the space $C^{(j)}$ again satisfies the parity relation (4.1).

From now on, in this section, we assume that $i = j = 0$ and write $\mathcal{W}(L)$ for $\mathcal{W}_{0,0}(L)$. Also, all vertices will have level 0, and we assume that $\tilde{T} = T = h(\mathbf{x}, \mathbf{x}) \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$.

First note that if $L \subset p\Lambda^\vee$ for some vertex Λ, then the inclusions

$$L \subset p\Lambda^\vee \overset{t}{\subset} \Lambda \subset pL^\vee,$$

show that $\Lambda/p\Lambda^\vee$ is a subquotient of $pL^\vee/L$ and, in particular, the type $t = t(\Lambda)$ is constrained by the structure of $pL^\vee/L$. More precisely, let $D = D(L) = pL^\vee/L$ with $\boldsymbol{k}/O_{\boldsymbol{k}}$-valued hermitian form determined by $h(\ ,\ ) = p^{-1}\{,\}$. Note that $pL^\vee$ is the dual lattice of L with respect to h, so that the resulting hermitian form on D is nondegenerate. Below we identify $O_{\boldsymbol{k}}/pO_{\boldsymbol{k}}$ with $\mathbb{F}_{p^2}$.

Lemma 4.9. *(i) Let*

$$m = \dim_{\mathbb{F}_{p^2}} D[p] = \dim_{\mathbb{F}_{p^2}} D/pD.$$

Then $m = n - \mathrm{rank}(\mathrm{red}(T))$.
(ii) There is a bijection between the sets

$$\mathrm{Vert}(L) := \{\Lambda \mid \Lambda \text{ a vertex with } L \subset p\Lambda^\vee\}$$

and

$$GrD := \left\{B \mid B \text{ an } O_{\boldsymbol{k}}\text{-submodule of } D \text{ with } pB \subset B^\perp \subset B\right\},$$

given by $\Lambda \mapsto \Lambda/L$. The type $t(\Lambda)$ of Λ is the dimension of the $\mathbb{F}_{p^2}$-vector space $B/B^\perp$. □

Example 4.10. Suppose that $T \simeq \mathrm{diag}(1_{n_0}, p\,1_{n_1})$, so that $D = D(L)$ is an $\mathbb{F}_{p^2}$-vector space of dimension n_1. Then GrD can be identified with the set of all isotropic subspaces U in D via the map $B \mapsto U = B^\perp$. Thus, there is a unique maximal vertex Λ in $\mathrm{Vert}(L)$ of type n_1 corresponding to $U = 0$, and $\mathcal{W}(L) = \mathcal{V}(\Lambda)$, as asserted in Theorem 4.5.

Proof of Theorem 4.2.

Lemma 4.11. *Suppose that $\Lambda \in \mathrm{Vert}(L)$ with $\dim_{\mathbb{F}_{p^2}}(\Lambda^\vee \cap pL^\vee)/\Lambda \geq 2$. Then there exists a lattice $\Lambda_1 \in \mathrm{Vert}(L)$ with $\Lambda \subset \Lambda_1$ and $t(\Lambda_1) > t(\Lambda)$.*

Proof. Note that the $\mathbb{F}_{p^2}$-vector space $\Lambda^\vee/\Lambda$ has a non- degenerate $\boldsymbol{k}/O_{\boldsymbol{k}}$-valued hermitian form determined by $\{,\}$. If $\dim_{\mathbb{F}_{p^2}}(\Lambda^\vee \cap pL^\vee)/\Lambda \geq 2$, then this subspace contains an isotropic line ℓ. Let $\Lambda_1 = \mathrm{pr}^{-1}(\ell)$ where $\mathrm{pr} : \Lambda^\vee \to \Lambda^\vee/\Lambda$. By construction, $\Lambda \subset \Lambda_1 \subset \Lambda^\vee \cap pL^\vee$, so that, in particular, $L \subset p\Lambda_1^\vee$ and $p\Lambda_1^\vee \subset p\Lambda^\vee \subset \Lambda \subset \Lambda_1$. Also, since $\ell \subset \ell^\perp$ in $\Lambda^\vee/\Lambda$, we have $\Lambda_1 \subset \Lambda_1^\vee = \mathrm{pr}^{-1}(\ell^\perp)$. Thus $\Lambda_1 \in \mathrm{Vert}(L)$ and the index t_1 of $p\Lambda_1^\vee$ in Λ_1 is strictly larger than t. □

Next, we record a few more general facts. For any $\Lambda \in \mathrm{Vert}(L)$, let $B \in GrD(L)$ be the $O_{\boldsymbol{k}}$-submodule of $D = D(L)$ associated to Λ. Since the image of $p\Lambda^\vee$ in D is $B^\perp$, the image of $\Lambda^\vee \cap pL^\vee$ in D is

$$\{\, x \in D \mid px \in B^\perp \} = (pB)^\perp.$$

In particular, note that $D[p] \subset (pB)^\perp$ and that the quotient $(pB)^\perp/B^\perp$ is killed by p. Since the pairing on D induced by h is perfect, we have

$$pB \overset{r}{\subset} B^\perp \overset{t}{\subset} B \overset{r}{\subset} (pB)^\perp,$$

where $r = \dim(\Lambda^\vee \cap pL^\vee)/\Lambda$. But the inclusion on the right implies that the subspace $B[p] \subset D[p]$ has codimension at most r, so that we obtain

$$m \geq t + r = \dim B/pB = \dim B[p] \geq m - r. \tag{4.3}$$

This gives

$$2r \geq m - t = m - t_0 + (t_0 - t).$$

Lemma 4.12. *If $\Lambda \in \mathrm{Vert}(L)$ with $t(\Lambda) < t_0$, then either*

$$\dim_{\mathbb{F}_{p^2}}(\Lambda^\vee \cap pL^\vee)/\Lambda \geq 2,$$

or the special case

$(\star\star)$ $\quad m = t_0$, $t = m - 2$, $r = 1$, *and* $\dim B[p] = m - 1$

holds.

Proof. By assumption, $t_0 - t \geq 2$ is even so that $r \geq 2$, as claimed, except in the special case. $\square$

Lemma 4.13. *In the special case* $(\star\star)$, *the lattice* $\Lambda_1 = \Lambda^\vee \cap pL^\vee$ *is in* $\mathrm{Vert}(L)$ *with* $t(\Lambda_1) = t_0$.

Proof. In this case, we have the picture.

$$pB \overset{1}{\subset} B^\perp \overset{m-2}{\subset} B \overset{1}{\subset} (pB)^\perp.$$

For any $x_0 \in pL^\vee$ whose image $\bar{x}_0$ in D lies in $D[p] \setminus B[p]$, we have $(pB)^\perp = B + O_{\boldsymbol{k}} \cdot \bar{x}_0$. Thus, $\Lambda_1 = \Lambda^\vee \cap pL^\vee = \Lambda + O_{\boldsymbol{k}} x_0$. As in the proof of Lemma 4.11, it follows that $L \subset p\Lambda_1^\vee$ and $p\Lambda_1^\vee \subset p\Lambda^\vee \subset \Lambda \subset \Lambda_1$. But now

$$\{\Lambda_1, \Lambda_1\} = \{\Lambda^\vee \cap pL^\vee, \Lambda + O_{\boldsymbol{k}} x_0\} \subset O_{\boldsymbol{k}}$$

because $\{\Lambda^\vee, \Lambda\} = O_{\boldsymbol{k}}$ and

$$\{pL^\vee, x_0\} = p\, h(pL^\vee, x_0) \subset h(pL^\vee, L) \subset O_{\boldsymbol{k}},$$

since $px_0 \in L$. Thus $\Lambda_1 \subset \Lambda_1^\vee$, so that Λ_1 is in $\mathrm{Vert}(L)$, as claimed. $\square$

The previous lemmas show that every $\Lambda \in \mathrm{Vert}(L)$ is contained in some $\Lambda_0 \in \mathrm{Vert}(L)$ with $t(\Lambda_0) = t_0$. On the other hand, by Lemma 4.9, the type of any lattice $\Lambda \in \mathrm{Vert}(L)$ is at most t_0. This completes the proof of Theorem 4.2. $\square$

Corollary 4.14. *Suppose that* $\Lambda \in \mathrm{Vert}(L)$ *with* $t_0 = t(\Lambda)$ *maximal and let* $B \in GrD(L)$ *be the associated* $O_{\boldsymbol{k}}$*-submodule. Then*

$$r = \dim_{\mathbb{F}_{p^2}} (\Lambda^\vee \cap pL^\vee)/\Lambda = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Moreover, $p^{-1}L \cap pL^\vee \subset \Lambda$.

Proof. If m is odd, then $t = t_0 = m$ in (4.3), so that $r = 0$, while, if m is even, then $t = t_0 = m - 1$ and (4.3) forces $r = 1$. The last assertion follows from the fact that $B[p] = D[p]$, since both have dimension m. $\square$

Proof of Theorem 4.5. Suppose that T has the given Jordan decomposition with respect to some basis $e_1, \dots, e_n$ of L, and note that $pL^\vee$ has basis $p^{-a_i} e_i$, where $h(e_i, e_i) = p^{a_i}$. If $\max(n^+_{\text{even}}, n^+_{\text{odd}}) \geq 2$, i.e., if

$$n^+_{\text{even}} = \sum_{\substack{i \geq 2 \\ i \text{ even}}} n_i \geq 2 \qquad (\text{resp.} \quad n^+_{\text{odd}} = \sum_{\substack{i \geq 3 \\ i \text{ odd}}} n_i \geq 2\ ), \tag{4.4}$$

we can scale the e_i's to a basis $f_1, \dots, f_n$ of C for which the hermitian form h has matrix

$$T_0 = \mathrm{diag}(1_{n'}, p\, 1_{n''}, p^a\, 1_2)$$

with $a = 2$ (resp. $a = 3$). Let L_0 be the lattice spanned by $f_1, \dots, f_n$, and note that

$$pL_0^\vee = [f_1, \dots, f_{n'}, p^{-1}f_{n'+1}, \dots, p^{-1}f_{n'+n''}, p^{-a}f_{n-1}, p^{-a}f_n].$$

Also note that $L \subset L_0$ so that $\mathrm{Vert}(L_0) \subset \mathrm{Vert}(L)$. Take $u \in \mathbb{Z}_{p^2}^\times$ with $uu^\sigma = -1$, and let

$$g_1 = f_{n-1} + uf_n, \qquad g_2 = f_{n-1} - uf_n.$$

These are isotropic vectors in L_0 with $h(g_1, g_2) = 2p^a$. Let

$$\Lambda_1 = [f_1, \dots, f_{n'}, p^{-1}f_{n'+1}, \dots, p^{-1}f_{n'+n''}, p^{-1}g_1, p^{-a}g_2],$$

and

$$\Lambda_2 = [f_1, \dots, f_{n'}, p^{-1}f_{n'+1}, \dots, p^{-1}f_{n'+n''}, p^{-a}g_1, p^{-1}g_2].$$

Then

$$p\Lambda_1^\vee = [f_1, \dots, f_{n'}, f_{n'+1}, \dots, f_{n'+n''}, g_1, p^{1-a}g_2],$$

and

$$p\Lambda_2^\vee = [f_1, \dots, f_{n'}, f_{n'+1}, \dots, f_{n'+n''}, p^{1-a}g_1, g_2],$$

so that Λ_1 and Λ_2 are vertices in $\mathrm{Vert}(L_0)$ of type $n'' + 2$. Suppose that Λ_1 and Λ_2 were contained in a common vertex $\Lambda \in \mathrm{Vert}(L)$. Then we would have

$$\Lambda_1 \subset \Lambda \subset \Lambda^\vee \subset \Lambda_1^\vee, \qquad \text{and} \qquad \Lambda_2 \subset \Lambda \subset \Lambda^\vee \subset \Lambda_2^\vee,$$

and hence $h(\Lambda_1, \Lambda_2) \subset p^{-1}O_{\boldsymbol{k}}$. But $h(p^{-a}g_2, p^{-a}g_1) = 2\,p^{-a}$, so this is not the case. Thus there is more than one vertex $\Lambda \in \mathrm{Vert}(L)$ with $t(\Lambda) = t_0$ when (4.4) holds.

Now we prove the converse. Suppose that $\max(n^+_{\text{even}}, n^+_{\text{odd}}) \le 1$. Then there are several cases for the Jordan decomposition of T. First suppose that $n^+_{\text{even}} = n^+_{\text{odd}} = 1$, so that T has Jordan decomposition $\mathrm{diag}(1_{n_0}, p1_{n_1}, p^a, p^b)$ with $2 \le a < b$ and $a + b$ odd. Thus, $L = [e_1, \dots, e_n]$ and

$$pL^\vee = [e_1, \dots, e_{n_0}, p^{-1}e_{n_0+1}, \dots, p^{-1}e_{n-2}, p^{-a}e_{n-1}, p^{-b}e_n].$$

Recall that, by (4.2), $\mathrm{ord}\det(T)$ is odd. Thus n_1 must be even, $t_0 = n_1 + 1$, and any $\Lambda \in \mathrm{Vert}(L)$ with $t(\Lambda) = t_0$ contains the lattice

$$p^{-1}L \cap pL^\vee = [e_1, \dots, e_{n_0}, p^{-1}e_{n_0+1}, \dots, p^{-1}e_{n-2}, p^{-1}e_{n-1}, p^{-1}e_n].$$

Let $L' = [e_{n-1}, e_n]$, a lattice in the two-dimensional hermitian vector space V' spanned by e_{n-1}, e_n. Then the map

$$\Lambda \mapsto \Lambda \cap V'$$

gives a bijection between the lattices $\Lambda \in \mathrm{Vert}(L)$ with $t(\Lambda) = t_0$, and the lattices $\Lambda' \in \mathrm{Vert}(L')$ with $t(\Lambda') = 1$.

Thus, we may assume that $n = 2$ and that $T = \mathrm{diag}(p^a, p^b)$ where $2 \leq a$, b and a is even. We proceed by explicit computation. Suppose that $L = [e_1, e_2]$ and write $p\Lambda^\vee = [e_1, e_2]S$ for $S \in \mathrm{GL}_2(\boldsymbol{k})$ unique up to right multiplication by an element of $\mathrm{GL}_2(O_{\boldsymbol{k}})$. Then we have $\Lambda = [e_1, e_2]T^{-1}{}^t\bar{S}^{-1}$ and $pL^\vee = [e_1, e_2]T^{-1}$, and the various inclusions amount to the following conditions:

$$\begin{array}{ccccll} L & \subset & p\Lambda^\vee & \iff & S^{-1} & \in M_2(O_{\boldsymbol{k}}) \\ p\Lambda^\vee & \subset & \Lambda & \iff & {}^t\bar{S}TS & \in M_2(O_{\boldsymbol{k}}) \\ \Lambda & \subset & \Lambda^\vee & \iff & pS^{-1}T^{-1}{}^t\bar{S}^{-1} & \in M_2(O_{\boldsymbol{k}}). \end{array}$$

Moreover, Λ has type $t(\Lambda) = t_0 = 1$ if and only if $\mathrm{ord}(\det {}^t\bar{S}TS) = 1$. Assuming that this is the case, we may modify S on the right by an element of $\mathrm{GL}_2(O_{\boldsymbol{k}})$ so that $T_1 := {}^t\bar{S}TS = \mathrm{diag}(1, p)$. Note that the last of the above conditions is then immediate. Write $S = \mathrm{diag}(p^{-a/2}, p^{-(b-1)/2})S_0$. Then ${}^t\bar{S}_0T_1S_0 = T_1$ so that $u = \det(S_0)$ has norm 1 and hence is a unit. After replacing S_0 by $S_1 = \mathrm{diag}(1, \bar{u})S_0$, so that $\det(S_1) = 1$, a short calculation shows that

$$S_1 = \begin{pmatrix} \alpha & \beta \\ -p^{-1}\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

for α and $\beta \in \boldsymbol{k}$ with $1 = \alpha\bar{\alpha} + p^{-1}\beta\bar{\beta}$. Since the two terms on the right side of this last identity have ord's of opposite parity, we must have $\mathrm{ord}(\alpha) = 0$ and $\mathrm{ord}(\beta) \geq 1$, so that S_1 and S_0 lie in $\mathrm{GL}_2(O_{\boldsymbol{k}})$. Thus

$$\Lambda = [p^{-a/2}e_1, p^{-(b+1)/2}e_2]$$

is the unique vertex in $\mathrm{Vert}(L)$ with $t(\Lambda) = 1$. Of course, this argument just amounts to the fact that the isometry group of T_1 is anisotropic so that the building of this group reduces to a single point.

The cases where $(n^+_{\mathrm{even}}, n^+_{\mathrm{odd}}) = (1, 0)$, $(0, 1)$, or $(0, 0)$, i.e., where $T = \mathrm{diag}(1_{n_0}, p1_{n_1}, p^a)$ or $T = \mathrm{diag}(1_{n_0}, p1_{n_1})$ are easier and will be omitted. The $(0, 0)$ case is discussed in Example 4.10.

This completes the proof of Theorem 4.5. □

5. Intersection multiplicities

In this section, we fix i, j and consider a non-empty special cycle $\mathcal{Z}_{i,j}(\mathbf{x})$, where we assume that $\mathbf{x}$ is an n-tuple of special homomorphisms whose scaled fundamental matrix $\tilde{T} = p^{2i-j}h(\mathbf{x}, \mathbf{x}) = p^{2i-j-1}\{\mathbf{x}, \mathbf{x}\}$, which is still assumed to be non-degenerate, satisfies

$$\mathrm{rank}(\mathrm{red}(\tilde{T})) \geq n-2.$$

By Corollary 4.7, this implies that the cycle $\mathcal{Z}_{i,j}(\mathbf{x})$ in $\mathcal{N}^0_{2i} \times \mathcal{N}_j$ has underlying reduced scheme of dimension 0 which, in fact, reduces to a single point. Write $\mathcal{Z}_{i,j}(\mathbf{x}) = \mathrm{Spec}\, R(\mathbf{x})$ for a local W-algebra $R(\mathbf{x})$ with residue field $\mathbb{F}$. The arithmetic degree of $\mathcal{Z}_{i,j}(\mathbf{x})$ is then, by definition,

$$\widehat{\deg}(\mathcal{Z}_{i,j}(\mathbf{x})) = \mathrm{length}_W\, R(\mathbf{x}) \cdot \log p,$$

where $\mathrm{length}_W\, R(\mathbf{x})$ is the length of $R(\mathbf{x})$ as a W- module.

Theorem 5.1. *Suppose that $\tilde{T}$ is $\mathrm{GL}_n(O_{\boldsymbol{k}})$- equivalent to $\mathrm{diag}(1_{n-2}, p^a, p^b)$, where $0 \leq a \leq b$. Then $R(\mathbf{x})$ is of finite length and*

$$\widehat{\deg}(\mathcal{Z}_{i,j}(\mathbf{x})) = \log p \cdot \frac{1}{2} \sum_{l=0}^{a} p^l(a+b+1-2l).$$

Note that $a+b \equiv \mathrm{ord}(\det(T)) \bmod 2$ is odd, cf. (4.2) and the remarks after Corollary 4.7, so that, in fact, $0 \leq a < b$. As in the previous section, we may reduce to the case $i = j = 0$, which we assume from now on. Accordingly we write $\mathcal{Z}(\mathbf{x})$ for $\mathcal{Z}_{0,0}(\mathbf{x})$.

The first step in the proof is to reduce to the case $n = 2$.

Lemma 5.2. *Suppose that $\mathbf{y} = \mathbf{x}g$ for $g \in \mathrm{GL}_n(O_{\boldsymbol{k}})$ has matrix of inner products $h(\mathbf{y}, \mathbf{y}) = \mathrm{diag}(1_{n-2}, p^a, p^b)$ where a is even and b is odd* (but a and b are not ordered by size). *Then $\mathcal{Z}(\mathbf{x})(\mathbb{F}) = \mathcal{Z}(\mathbf{y})(\mathbb{F})$ corresponds to the lattice $A = \tau(A)$ given by*[5]

$$A = Wy_1 + \cdots + Wy_{n-2} + Wp^{-a/2}y_{n-1} + Wp^{-(b+1)/2}y_n.$$

If $(\mathbf{X}, \iota, \lambda_{\mathbf{X}}, \rho_{\mathbf{X}})$ is the corresponding p-divisible group, then there is an isomorphism

$$\underbrace{\overline{\mathbb{Y}} \times \cdots \times \overline{\mathbb{Y}}}_{n-1} \times \mathbb{Y} \xrightarrow{\sim} \mathbf{X}$$

such that, as elements of $\mathrm{Hom}_{O_{\boldsymbol{k}}}(\overline{\mathbb{Y}}, \mathbf{X})$,

$$\rho_{\mathbf{X}}^{-1} \circ y_i = \begin{cases} \mathrm{inc}_i & \text{if } i \leq n-2, \\ \mathrm{inc}_i \circ \Pi^a & \text{if } i = n-1, \\ \mathrm{inc}_i \circ \Pi^b & \text{if } i = n, \end{cases}$$

where inc_i denotes the inclusion into the i-th factor of the product. Here Π denotes the fixed uniformizer in $O_D = \mathrm{End}(\overline{\mathbb{Y}})$, cf. Remark 2.5.

[5]Here we have written y_i for $(y_i)_0(\bar{1}_0)$, so that $y_i \in C$.

Proof. Let $a = 2r$ and $b = 2s+1$. For the given lattice A, we have

$$pA^\vee = Wy_1 + \cdots + Wy_{n-2} + Wp^{-r}y_{n-1} + Wp^{-s}y_n.$$

so that $pA^\vee \subset A$ with index 1. Moreover, since $\tau(y_i) = y_i$, we have $\tau(A) = A$ so that $A = \Lambda(A)$ is a vertex and $\mathcal{Z}(\mathbf{x}) = \mathcal{V}(\Lambda(A))$ as claimed. For convenience, we let $u_i = y_i$, for $i \le n-2$, $u_{n-1} = p^{-r}y_{n-1}$ and $u_n = p^{-s-1}y_n$. The Dieudonné module $M = M_0 + M_1$ associated to A in Proposition 2.3 has $M_0 = A$ and $M_1 = Wv_1 + \cdots + Wv_n$, where $v_i = F^{-1}u_i$, for $i \le n-1$ and $v_n = pF^{-1}u_n$. Here recall that $FM_1 = pA^\vee$. Then, since

$$\langle\, u_i, Vu_j \,\rangle = \langle\, u_i, Fu_j \,\rangle = \delta\,\{u_i, u_j\} = \delta\,\delta_{ij}\begin{cases} p & \text{if } i \le n-1, \\ 1 & \text{if } i = n, \end{cases}$$

we have $\langle\, u_i, v_j \,\rangle = \delta\,\delta_{ij}$ for all i and j.

Recalling that $\overline{\mathbb{Y}}$ has Dieudonné module $\overline{\mathbb{M}}^0 = W\bar{1}_0 + W\bar{1}_1$, with $F\bar{1}_0 = p\bar{1}_1$, $F\bar{1}_1 = \bar{1}_0$ and $\langle\, \bar{1}_0, \bar{1}_1 \,\rangle^0 = \delta$, we see that

$$M = \oplus_{i=1}^n (Wu_i + Wv_i) \ \simeq\ \underbrace{\overline{\mathbb{M}}^0 \oplus \cdots \oplus \overline{\mathbb{M}}^0}_{n-1} \oplus \mathbb{M}^0$$

as polarized Dieudonné modules. The inclusion maps are then given by

$$\mathrm{inc}_i : W\bar{1}_0 + W\bar{1}_1 \longrightarrow M, \qquad \bar{1}_0 \mapsto u_i, \quad \bar{1}_1 \mapsto v_i,$$

for $i \le n-1$, resp.

$$\mathrm{inc}_i : W1_0 + W1_1 \longrightarrow M \qquad 1_0 \mapsto v_n, \quad 1_1 \mapsto u_n,$$

for $i = n$.

Finally, recalling that we have already identified the isocrystal of $\mathbf{X}$ with that of $\mathbb{X}$ via $\rho_{\mathbf{X}}$, we see that the morphism $y_i : \overline{\mathbb{Y}} \to \mathbf{X}$ does indeed yield the given elements $y_i = y_{i0}(\bar{1}_0)$ of N_0. On the other hand, $y_{i1}F = Fy_{i0}$, so that $py_{i1}(\bar{1}_1) = y_{i1}(F\bar{1}_0) = Fy_{i0}(\bar{1}_0) = Fy_i$. Hence $y_i : \overline{\mathbb{M}}^0 \to M$ is given by

$$\bar{1}_0 \mapsto y_i = \begin{cases} u_i & \text{if } i \le n-2, \\ p^r u_i & \text{if } i = n-1, \\ p^{s+1}u_i & \text{if } i = n, \end{cases} \qquad \bar{1}_1 \mapsto \begin{cases} v_i & \text{if } i \le n-2, \\ p^r v_i & \text{if } i = n-1, \\ p^s v_i & \text{if } i = n. \end{cases}$$

Thus, for $i = n$, we have

$$y_n = p^s\,\mathrm{inc}_n \circ \Pi$$

where $\Pi : \bar{1}_0 \mapsto p1_1$, $\bar{1}_1 \mapsto 1_0$ and is W-linear. The cases $i \le n-1$ are clear. $\square$

Let $\mathrm{Def}\,(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{x})$ be the universal formal deformation ring of the collection $(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{x})$, where $\mathbf{x}$ is our given n-tuple of special homomorphisms

$x_i : \overline{\mathbb{Y}} \to \mathbf{X}$. Since, as in the previous lemma, it is equivalent to deform the linear combination $\mathbf{y} = \mathbf{x}g$ for $g \in \mathrm{GL}_n(O_{\boldsymbol{k}})$, we have

$$\mathrm{Def}\,(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{x}) = \mathrm{Def}\,(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{y}). \tag{5.1}$$

We now want to calculate the length of this deformation ring.

If $(X, \iota, \lambda_X; \mathbf{y})$ is any deformation, the element

$$e_0 = \sum_{i=1}^{n-2} y_i \circ y_i^*$$

is an idempotent in $\mathrm{End}_{O_{\boldsymbol{k}}}(X)$ so that $X = e_0X \times (1 - e_0)X$, where e_0X (resp. $(1 - e_0)X$) is a deformation of $e_0\mathbf{X}$ (resp. of $\mathbf{X}' := (1 - e_0)\mathbf{X}$). Furthermore, the polarization decomposes into the product of polarizations of e_0X and $(1-e_0)X$. But the collection $[y_1, \dots, y_{n-2}]$ defines an isomorphism $\bar{Y}^{n-2} \xrightarrow{\sim} e_0X$, compatibly with polarizations. Thus, the deformations of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{y})$ are in bijection with those of $(\mathbf{X}', \iota, \lambda_{\mathbf{X}'}; \mathbf{y}')$, where $\mathbf{X}' \simeq \overline{\mathbb{Y}} \times \mathbb{Y}$, and where $\mathbf{y}' = [y_{n-1}, y_n]$. Hence we have

$$\mathrm{Def}\,(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{y}) = \mathrm{Def}\,(\mathbf{X}', \iota, \lambda_{\mathbf{X}'}; \mathbf{y}').$$

Thus, it suffices to compute the length of the deformation ring (5.1) in the case $n = 2$, where $\mathbf{y} = [y_1, y_2]$. By the previous lemma, we have

$$\mathbf{X} = \overline{\mathbb{Y}} \times \mathbb{Y}, \quad y_1 = \mathrm{inc}_1 \circ \Pi^a, \quad y_2 = \mathrm{inc}_2 \circ \Pi^b, \tag{5.2}$$

where a is even and b is odd (but a and b are not ordered by size).

Let $\mathcal{M}$ denote the universal deformation space of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}})$. Then $\mathcal{M} \simeq \mathrm{Spf}\, W[[t]]$ and the locus $\mathcal{Z}(y_1)$ (resp. $\mathcal{Z}(y_2)$) where y_1 (resp. y_2) deforms is a (formal) divisor on this 2-dimensional regular (formal) scheme. The problem is now to determine the length

$$\mathrm{length}_W \mathrm{Def}(\mathbf{X}, \iota, \lambda_{\mathbf{X}}; \mathbf{y}) = \mathcal{Z}(y_1) \cdot \mathcal{Z}(y_2) \tag{5.3}$$

of the intersection of these two formal subschemes of $\mathcal{M}$.

This problem is solved in section 8 as an application of a variant of Gross's theory of quasicanonical liftings described in the next two sections.

6. Quasi-canonical liftings

In this section, we first explain a general construction of extending the endomorphism ring of a p-divisible group. We then apply this construction to quasi-canonical lifts, and show how this can be used to construct liftings of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}})$ to finite extensions of W.

For a p-divisible group X, we define the p-divisible group $O_{\boldsymbol{k}} \otimes X$ in the standard way as an exterior tensor product, e.g. [2], p. 131. Explicitly, after choosing a $\mathbb{Z}_p$-basis $\mathbf{e} = (e_1, e_2)$ of $O_{\boldsymbol{k}}$, we define $O_{\boldsymbol{k}} \otimes X$ to be $X \times X$. Any other choice $\mathbf{e}'$ of a $\mathbb{Z}_p$-basis differs from the first choice by a matrix $g \in \mathrm{GL}_2(\mathbb{Z}_p)$. Then g defines an automorphism $\alpha_{\mathbf{e},\mathbf{e}'} : X \times X \to X \times X$. Since $\alpha_{\mathbf{e},\mathbf{e}''} = \alpha_{\mathbf{e},\mathbf{e}'} \circ \alpha_{\mathbf{e}',\mathbf{e}''}$, we obtain a system of compatible isomorphisms. Hence we obtain in an unambiguous way a p-divisible group $O_{\boldsymbol{k}} \otimes X$, unique up to unique isomorphism, with isomorphisms $\beta_{\mathbf{e}} : O_{\boldsymbol{k}} \otimes X \to X \times X$, for any choice of a basis $\mathbf{e}$, such that $\beta_{\mathbf{e}'} = \alpha_{\mathbf{e},\mathbf{e}'} \circ \beta_{\mathbf{e}}$. The action by left translation of $O_{\boldsymbol{k}}$ on itself defines an action $O_{\boldsymbol{k}} \to \mathrm{End}(O_{\boldsymbol{k}} \otimes X)$. We obtain in this way a functor from the category of p-divisible groups to the category of p-divisible groups with $O_{\boldsymbol{k}}$-action. It is compatible with base change,

$$(O_{\boldsymbol{k}} \otimes X) \times_S S' = O_{\boldsymbol{k}} \otimes (X \times_S S').$$

We mention the following properties of this functor.

Lemma 6.1. *(i) The functor $X \mapsto O_{\boldsymbol{k}} \otimes X$ from the category of p-divisible groups to the category of p-divisible groups with $O_{\boldsymbol{k}}$-action is left adjoint to the functor forgetting the $O_{\boldsymbol{k}}$-action.*
(ii) For the Lie algebras

$$\mathrm{Lie}(O_{\boldsymbol{k}} \otimes X) = O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p} \mathrm{Lie}\ X.$$

Similarly, for the p-adic Tate modules,

$$T_p(O_{\boldsymbol{k}} \otimes X) = O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p} T_p(X).$$

There is an analogous statement for the Dieudonné module, if X is a p-divisible group over a perfect field.
(iii) For two p-divisible groups X and Y,

$$\mathrm{Hom}(O_{\boldsymbol{k}} \otimes X, O_{\boldsymbol{k}} \otimes Y) \simeq M_2(\mathrm{Hom}(X, Y)),$$

and

$$\mathrm{Hom}_{O_{\boldsymbol{k}}}(O_{\boldsymbol{k}} \otimes X, O_{\boldsymbol{k}} \otimes Y) = O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p} \mathrm{Hom}(X, Y).$$

Proof. The first two assertions follow immediately from the definitions. For the first isomorphism in (iii), choose a $\mathbb{Z}_p$-basis of $O_{\boldsymbol{k}}$ which identifies $O_{\boldsymbol{k}} \otimes X$ and $O_{\boldsymbol{k}} \otimes Y$ with X^2 and Y^2. For the second isomorphism, note that the left hand side is then identified with the matrices in $\mathrm{M}_2(\mathrm{Hom}(X, Y)) = \mathrm{Hom}(X^2, Y^2)$ that commute with the matrix for multiplication by δ. $\square$

Another useful fact is the following.

Lemma 6.2. *Let $p \neq 2$. If (X, ι) is a p-divisible group with $O_{\boldsymbol{k}}$-action, then there is an isomorphism*

$$X \times \bar{X} \xrightarrow{\sim} O_{\boldsymbol{k}} \otimes X$$

given by

$$(z_1, z_2) \mapsto \alpha_+(1 \otimes z_1) + \alpha_-(1 \otimes z_2),$$

where

$$\alpha_\pm = \delta \otimes 1 \pm 1 \otimes \iota(\delta) \in \mathrm{End}(O_{\boldsymbol{k}} \otimes X).$$

Here $\bar{X}$ denotes the group X with $O_{\boldsymbol{k}}$-action given by $\iota \circ \sigma$. This isomorphism is $O_{\boldsymbol{k}}$-linear, where $\alpha \in O_{\boldsymbol{k}}$ acts on $O_{\boldsymbol{k}} \otimes X$ via $\alpha \otimes 1$.

Proof. We simply observe the following facts. First,

$$(\delta \otimes 1) \circ \alpha_\pm = \pm \alpha_\pm \circ (1 \otimes \iota(\delta)) = \alpha_\pm \circ (\delta \otimes 1). \tag{6.1}$$

Also $\alpha_+ + \alpha_- = 2\delta \otimes 1$ is an automorphism of $O_{\boldsymbol{k}} \otimes X$ with $\alpha_+ \circ \alpha_- = 0$, and $(\alpha_\pm)^2 = (2\delta \otimes 1)\, \alpha_\pm$. □

Finally, we will need the following construction of polarizations on $O_{\boldsymbol{k}} \otimes X$. We consider the standard hermitian form on $O_{\boldsymbol{k}}$, with $\{1, 1\} = 1$, and the associated perfect alternating $\mathbb{Z}_p$-valued form

$$\langle\, \alpha, \beta\, \rangle = \mathrm{tr}_{O_{\boldsymbol{k}}/\mathbb{Z}_p}(\alpha \delta^{-1} \beta^\sigma).$$

Using this form on the first factor, we have a canonical perfect pairing

$$\langle\ ,\ \rangle :\ (O_{\boldsymbol{k}} \otimes X) \times (O_{\boldsymbol{k}} \otimes X^\vee)\ \to \hat{\mathbb{G}}_m, \tag{6.2}$$

satisfying the identity on points

$$\langle \alpha\, x, y \rangle = \langle x, \alpha^\sigma\, y \rangle.$$

Using this pairing, we may identify the Cartier dual $(O_{\boldsymbol{k}} \otimes X)^\vee$ of $O_{\boldsymbol{k}} \otimes X$ with $O_{\boldsymbol{k}} \otimes X^\vee$. Hence, starting with a polarization $\lambda : X \to X^\vee$, we obtain a $O_{\boldsymbol{k}}$-linear polarization

$$\mathrm{id}_{O_{\boldsymbol{k}}} \otimes \lambda : O_{\boldsymbol{k}} \otimes X \to O_{\boldsymbol{k}} \otimes X^\vee = (O_{\boldsymbol{k}} \otimes X)^\vee,$$

such that the associated Rosati involution induces the Galois automorphism σ on $O_{\boldsymbol{k}}$.

We now apply this construction to Gross's quasi-canonical lifts. Let us recall briefly a few facts from Gross's theory, [3], [1], that we will use in what follows.

Let G be a p-divisible formal group of height 2 and dimension 1 over $\mathbb{F}$. Then G is uniquely determined up to isomorphism, and $\mathrm{End}(G) = O_D$, the maximal order in the quaternion division algebra D over $\mathbb{Q}_p$. After fixing an embedding $i : O_{\boldsymbol{k}} \hookrightarrow O_D$, G becomes a formal $O_{\boldsymbol{k}}$-module of height 2. There is a unique lifting F_0 of this formal $O_{\boldsymbol{k}}$-module to W, the *canonical lift.*

Remark 6.3. Note that after fixing the embedding i of $O_{\boldsymbol{k}}$ into O_D we may identify G with the group $\mathbb{Y}$ of the previous sections. Let Π be a uniformizer of D that normalizes $O_{\boldsymbol{k}}$ and with $\Pi^2 = p$. Then the embedding $i \circ \sigma$ of $O_{\boldsymbol{k}}$ into O_D is the conjugate of i by Π, and the group G with the embedding $i \circ \sigma$ is the group $\bar{\mathbb{Y}}$. The formal $O_{\boldsymbol{k}}$-modules $\mathbb{Y}$ and $\bar{\mathbb{Y}}$ are not isomorphic[6] and Π gives an isogeny between them of degree p.

For an integer $s \geq 1$, let

$$O_{\boldsymbol{k},s} = \mathbb{Z}_p + p^s O_{\boldsymbol{k}}.$$

be the order of conductor s in $O_{\boldsymbol{k}}$.

A *quasi-canonical lift F_s of level s* of G is a lifting of G to some finite extension A of W with endomorphism ring equal to $O_{\boldsymbol{k},s}$, such that the induced action of $O_{\boldsymbol{k},s}$ on Lie F_s is through the embedding $O_{\boldsymbol{k},s} \subset W \subset A$ and such that the image of $O_{\boldsymbol{k},s}$ in $\mathrm{End}(G)$ is contained in the image of the fixed embedding of $O_{\boldsymbol{k}}$ in O_D, cf. [33], Definition 3.1[7]. Any quasi-canonical lift of level s is defined over W_s, the ring of integers in the ring class field M_s of $O_{\boldsymbol{k},s}^{\times}$, i.e., the finite abelian extension of $M = W_{\mathbb{Q}}$ corresponding to the subgroup $O_{\boldsymbol{k},s}^{\times}$ of $O_{\boldsymbol{k}}^{\times}$ under the reciprocity isomorphism. Note that M_s is a totally ramified extension of M of degree

$$e_s = p^{s-1}(p+1).$$

Quasi-canonical lifts of level s always exist; they are not unique, but any two are conjugate under the Galois group $\mathrm{Gal}(M_s/M)$, and, in fact, this Galois group acts in a simply transitive way on all quasi-canonical lifts of level s. Quasi-canonical lifts are isogenous to the canonical lift. More precisely, there exists an isogeny $\psi_s : F_0 \to F_s$ of degree p^s defined over W_s. In terms of the Tate modules of the generic fibers, the isogeny ψ_s can be described as follows. There exists a generator t of the free $O_{\boldsymbol{k}}$-module $T_p(F_0)$, such that

$$T_p(F_s) = (\mathbb{Z}_p \cdot p^{-s} + O_{\boldsymbol{k}}) \cdot t.$$

We note that when s is even, the isogeny ψ_s is compatible with the embedding of $O_{\boldsymbol{k}}$ in $O_D = \mathrm{End}(G)$, whereas when s is odd, it is compatible with $\iota \circ \sigma$. More precisely,

Lemma 6.4. *Let Π be a uniformizer of O_D which normalizes $O_{\boldsymbol{k}}$, as above. Given a quasi-canonical lift F_s, there exists a unique $O_{\boldsymbol{k}}$-linear isogeny*

[6] The analogous groups for a ramified extension $\boldsymbol{k}$ are isomorphic.

[7] More precisely, the induced embedding of $O_{\boldsymbol{k},s}$ coincides with the restriction to $O_{\boldsymbol{k},s} \subset O_{\boldsymbol{k}}$ of i, when s is even, or $i \circ \sigma$, when s is odd. This definition differs in fact from that given in loc.cit, where it is required that the induced embedding of $O_{\boldsymbol{k},s}$ in $\mathrm{End}(G)$ is equal to the restriction of the fixed embedding i of $O_{\boldsymbol{k}}$ into O_D. However, this condition is too strong since it would not allow quasi-canonical lifts for odd s, cf. for example Lemma 6.4.

$\psi_s : F_0 \to F_s$ which induces the endomorphism Π^s on the special fiber G. Moreover, the set $H_{0,s} \subset O_D$ of homomorphisms from $F_0 \otimes \mathbb{F} = G$ to $F_s \otimes \mathbb{F} = G$ that lift to homomorphisms from F_0 to F_s is precisely $\Pi^s O_{\boldsymbol{k}}$. $\square$

We will refer to the canonical lift F_0 as a quasi-canonical lift of level 0.

Let $X^{(s)} = O_{\boldsymbol{k}} \otimes F_s$, for some quasi-canonical lift F_s of level s. Then $X^{(s)}$ is a p-divisible group with $O_{\boldsymbol{k}}$-action ι over W_s, with Lie $X^{(s)} = O_{\boldsymbol{k}} \otimes_{\mathbb{Z}_p}$ Lie F_s, hence $X^{(s)}$ satisfies the signature condition $(1,1)$. Let λ be a p-principal polarization of F_s. Note that λ is unique up to a scalar in $\mathbb{Z}_p^\times$, since it induces a perfect alternating form on the 2-dimensional $\mathbb{Z}_p$-module $T_p(F_s)$. By the construction outlined above, we obtain a p-principal $O_{\boldsymbol{k}}$- linear polarization $\lambda^{(s)} = \mathrm{id}_{O_{\boldsymbol{k}}} \otimes \lambda$ on $X^{(s)}$ for which the Rosati involution induces the Galois conjugation on $O_{\boldsymbol{k}}$. The special fiber of $X^{(s)}$ is equal to $O_{\boldsymbol{k}} \otimes (F_s \otimes_{W_s} \mathbb{F})$. Now $F_s \otimes_{W_s} \mathbb{F}$ is equal to G; however, as a formal $O_{\boldsymbol{k}}$-module it is equal to $\mathbb{Y}$ when s is even, and equal to $\overline{\mathbb{Y}}$ when s is odd. Applying Lemma 6.2, we obtain identifications

$$X^{(s)} \otimes_{W_s} \mathbb{F} \;=\; \begin{cases} \mathbb{Y} \times \overline{\mathbb{Y}} & s \text{ even} \\ \overline{\mathbb{Y}} \times \mathbb{Y} & s \text{ odd}\,. \end{cases} \tag{6.3}$$

Recall from (5.2) that $\mathbf{X} = \overline{\mathbb{Y}} \times \mathbb{Y}$. We now identify $X^{(s)} \otimes_{W_s} \mathbb{F}$ with $\mathbf{X}$ by using the switch isomorphism $\mathbb{Y} \times \overline{\mathbb{Y}} \simeq \overline{\mathbb{Y}} \times \mathbb{Y}$ when s is even, resp. the identity map on $\overline{\mathbb{Y}} \times \mathbb{Y}$ when s is odd. In all cases we have therefore obtained canonical $O_{\boldsymbol{k}}$-linear isomorphisms

$$\rho^{(s)} : X^{(s)} \otimes_{W_s} \mathbb{F} \simeq \mathbf{X}. \tag{6.4}$$

By pulling back the polarization $\lambda_{\mathbf{X}}$ to $X^{(s)} \otimes_{W_s} \mathbb{F}$, we obtain a $O_{\boldsymbol{k}}$-linear p-principal polarization on $X^{(s)} \otimes_{W_s} \mathbb{F}$ which differs from $\lambda^{(s)}$ by a scalar in $\mathbb{Z}_p^\times$ (same argument as above, using the Dieudonné module instead of the Tate module). Hence we may change $\lambda^{(s)}$ by this scalar such that these polarizations coincide. Hence $(X^{(s)}, \iota, \lambda^{(s)})$ is a deformation of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}})$ to W_s. We therefore obtain a morphism

$$\varphi_s : \mathrm{Spf}\ W_s \to \mathcal{M}. \tag{6.5}$$

Lemma 6.5. *The morphism φ_s is a closed immersion.*

Proof. Denoting by R the affine ring of $\mathcal{M}$, we have to show that the morphism $\varphi_s^* : R \to W_s$ is surjective. Now $R \simeq W[[t]]$, and W_s is a finite totally ramified extension of W. To show that $\varphi^* : R \to W_s$ is surjective, it therefore suffices to show that $\varphi^*(t)$ is a uniformizing parameter, i.e., that $W_s \otimes_R \mathbb{F} = \mathbb{F}$. By the universal property of $\mathcal{M}$, this says that the locus in Spec W_s/pW_s where there exists an $O_{\boldsymbol{k}}$-linear isomorphism

$\alpha : O_{\boldsymbol{k}} \otimes F_s \to O_{\boldsymbol{k}} \otimes G$ is equal to $\operatorname{Spec} \mathbb{F}$. According to Lemma 6.1 we can write $\alpha = 1 \otimes \alpha_0 + \delta \otimes \alpha_1$, where α_0 and α_1 are homomorphisms from F_s to G. By identifying the special fibers of F_s and G, α becomes a unit in O_D. But then α_0 or α_1 is a unit in O_D, and hence one of them defines an isomorphism from $F_s \longrightarrow F_0$ over this locus. By [33], Cor. 4.7, this implies that the locus in question is reduced to the special point. □

Definition 6.6. Let $\mathcal{Z}_s$ be the divisor on $\mathcal{M}$ defined by the image of φ_s.

7. Deformations of homomorphisms

We consider the following lifting problem. As in [33], suppose that A is a finite extension of W with uniformizer λ, and let $A_m = A/\lambda^{m+1}$. We also let e be the ramification index of A over W and denote by ord_A the discrete valuation on A with $\operatorname{ord}_A(\lambda) = 1/e$. For integers r, $s \geq 0$ let F_r and F_s be quasi-canonical liftings defined over A. Suppose that a homomorphism

$$\mu : (O_{\boldsymbol{k}} \otimes F_0) \otimes_W \mathbb{F} \longrightarrow (O_{\boldsymbol{k}} \otimes F_s) \otimes_{W_s} \mathbb{F}$$

is given. Let $m_s(\mu)$ be the maximum m such that μ lifts to a homomorphism from $(O_{\boldsymbol{k}} \otimes F_0) \otimes_W A_m$ to $(O_{\boldsymbol{k}} \otimes F_s) \otimes_{W_s} A_m$. Also, as in the first part of (iii) of Lemma 6.1, write

$$\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}, \tag{7.1}$$

with $\mu_i \in \operatorname{Hom}(F_0 \otimes_W \mathbb{F}, F_s \otimes_{W_s} \mathbb{F}) = O_D$.

Our aim is to prove the following theorem.

Theorem 7.1. *Write μ as in (7.1), and suppose that*

$$\mu_i \in (\Pi^s O_{\boldsymbol{k}} + \Pi^{l_i} O_D) \setminus (\Pi^s O_{\boldsymbol{k}} + \Pi^{l_i+1} O_D)$$

for integers $l_i \geq 0$. Let $l = \min_i \{l_i\}$. Then

$$m_s(\mu) = e/e_s \cdot \begin{cases} \dfrac{p^{l+1} - 1}{p - 1} & \text{if } l < s, \\ \dfrac{p^s - 1}{p - 1} + \frac{1}{2}(l + 1 - s)\, e_s & \text{if } l \geq s. \end{cases}$$

Proof. Note that μ lifts to A_m if and only if the components μ_i all lift to homomorphisms $F_0 \otimes_W A_m \to F_s \otimes_{W_s} A_m$. Recall from [34], section 1.4, that for a homomorphism $\psi : F_r \otimes_{W_r} \mathbb{F} \longrightarrow F_s \otimes_{W_s} \mathbb{F}$, $n_{r,s}(\psi)$ is defined to be the maximum m such that ψ lifts to a homomorphism $F_r \otimes_{W_r} A_m \to F_s \otimes_{W_s} A_m$. Thus,

$$m_s(\mu) = \min_i \{n_{0,s}(\mu_i)\}.$$

So, we are reduced to determining the quantities $n_{0,s}(\mu_i)$. To this end, we prove the next Proposition, a slight extension of [34], Proposition 1.2.

As in [34], let $H_{r,s} \subset D$ be the subset of elements ϕ that lift to homomorphisms from F_r to F_s. For example, $H_{s,s} = O_{\mathbf{k},s}$. In general, if $s \geq r$, with the conventions introduced in the previous section, $H_{r,s} = \Pi^{s-r} O_{\mathbf{k},r}$ and there is an isomorphism

$$H_{r,s} \xrightarrow{\sim} H_{r,s+1}, \qquad \phi \mapsto \Pi\psi, \tag{7.2}$$

cf. [34], Proposition 1.1, 1). Also, passage to dual isogenies shows that $n_{r,s}(\psi) = n_{s,r}(\psi^*)$. Hence we may always assume that $s \geq r$, as we shall do from now on. For $\psi \in O_D \setminus H_{r,s}$, let

$$l_{r,s}(\psi) = \max\{\, v(\psi + \phi) \mid \phi \in H_{r,s} \,\} \tag{7.3}$$

where v is the valuation on D with $v(\Pi) = 1$. More explicitly, $l = l_{r,s}(\psi)$ is the positive integer such that

$$\psi \in (\Pi^{s-r} O_{\mathbf{k},r} + \Pi^l O_D) \setminus (\Pi^{s-r} O_{\mathbf{k},r} + \Pi^{l+1} O_D). \tag{7.4}$$

Note that, if $v(\psi) < s - r$, then $l_{r,s}(\psi) = v(\psi)$. If $v(\psi) \geq s + r$, then $l_{r,s}(\psi) + r - s \geq 2r$ is odd, cf. [30], Remark 2.2.

Proposition 7.2. *Let $l = l_{0,s}(\psi)$. Then*

$$n_{0,s}(\psi) = e/e_s \cdot \begin{cases} \dfrac{p^{l+1} - 1}{p - 1} & \text{if } l < s, \\[2ex] \dfrac{p^s - 1}{p - 1} + \frac{1}{2}\,(l + 1 - s)\, e_s & \text{if } l \geq s. \end{cases}$$

Proof. In fact, we will determine $n_{r,s}(\psi)$ for any $r \leq s$. First, by [30], Theorem 2.1, if $\psi \in O_D \setminus H_{r,r}$ with $l_{r,r}(\psi) = l$, then

$$n_{r,r}(\psi) = e/e_r \cdot \begin{cases} 2\,\dfrac{p^{l/2+1} - 1}{p - 1} - p^{l/2} & \text{if } l \leq 2r \text{ is even}, \\[2ex] 2\,\dfrac{p^{(l+1)/2} - 1}{p - 1} & \text{if } l \leq 2r \text{ is odd}, \\[2ex] 2\,\dfrac{p^r - 1}{p - 1} + \frac{1}{2}(l + 1 - 2r) e_r & \text{if } l \geq 2r - 1. \end{cases}$$

Next, we recall that Lemma 3.6 of [24] is the following (note that $e/e_{s+1} = \mathrm{ord}_A(\pi_{s+1})$ where π_{s+1} is a uniformizer of W_{s+1}):

Lemma 7.3. *Suppose that F_r, F_s and F_{s+1} are defined over A and that $\psi \in O_D \setminus H_{r,s}$ for $r \leq s$. Then*

$$n_{r,s+1}(\Pi\psi) = n_{r,s}(\psi) + e/e_{s+1}.$$

$\square$

Now suppose that $s > r$ and that $\psi \in O_D \setminus H_{r,s}$ with $l = l_{r,s}(\psi) \geq s - r$. Then, by (7.2) and (7.4), $\Pi^{r-s}\psi \in O_D \setminus H_{r,r}$, and, by Lemma 7.3, we have

$$n_{r,s}(\psi) = \frac{e}{e_s} + \frac{e}{e_{s-1}} + \cdots + \frac{e}{e_{r+1}} + n_{r,r}(\Pi^{r-s}\psi) \ . \tag{7.5}$$

Next suppose that $s > r$ and that $l = l_{r,s}(\psi) < s - r$. Then we may assume that $v(\psi) = l = l_{r,s}(\psi)$. In this case, we may pull out Π^l of ψ, and obtain

$$n_{r,s}(\psi) = \frac{e}{e_s} + \frac{e}{e_{s-1}} + \cdots + \frac{e}{e_{s-l+1}} + n_{r,s-l}(\Pi^{-l}\psi) \ . \tag{7.6}$$

Now $\Pi^{-l}\psi \in O_D^\times$, so that a lift of $\Pi^{-l}\psi$ over A_m defines an isomorphism $F_r \otimes A_m \xrightarrow{\sim} F_s \otimes A_m$.

Lemma 7.4. *Suppose that $l_{r,s}(\psi) = 0$. Then*

$$n_{r,s}(\psi) = (e/e_s) \cdot e_r.$$

Proof. This follows from [17], Prop. 7.7.7. □

Thus, if $\psi \in O_D \setminus H_{r,s}$ with $l = l_{r,s}(\psi) < s - r$, we obtain

$$n_{r,s}(\psi) = \frac{e}{e_s} + \frac{e}{e_{s-1}} + \cdots + \frac{e}{e_{s-l+1}} + (e/e_s) \cdot e_r \ .$$

Since, for $0 \leq k < s$, $e_s/e_{s-k} = p^k$ and $e_0 = 1$, we obtain the expressions claimed in Proposition 7.2. Theorem 7.1 follows immediately. □

□

8. Computation of intersection multiplicities

We now return to the situation at the end of section 5. To compute the intersection number (5.3), we first decompose the divisors $\mathcal{Z}(y_i)$.

Proposition 8.1. *As divisors on $\mathcal{M}$*

$$\mathcal{Z}(y_1) = \sum_{\substack{s=0\\ \text{even}}}^{a} \mathcal{Z}_s \qquad \textit{and} \qquad \mathcal{Z}(y_2) = \sum_{\substack{s=1\\ \text{odd}}}^{b} \mathcal{Z}_s.$$

Here $\mathcal{Z}_s$ is the divisor on $\mathcal{M}$ given in Definition 6.6.

Proof. We begin by showing that each $\mathcal{Z}_s$ for s in the given range lies in $\mathcal{Z}(y_i)$. Suppose that F_s is a quasi-canonical lift of level s. There is then a unique isogeny $\psi_s : F_0 \to F_s$ of degree p^s such that the reduction

$$\bar{\psi}_s : F_0 \otimes_W \mathbb{F} \longrightarrow F_s \otimes_{W_s} \mathbb{F}$$

is equal to Π^s. Here recall that, for any quasicanonical lift F_s, an identification $F_s \otimes_{W_s} \mathbb{F} = G$ is given so that $\bar{\psi}_s$ is identified with an element of O_D. We also write ψ_s for the corresponding isogeny

$$\psi_s : X^{(0)} \longrightarrow X^{(s)}.$$

There is an isomorphism

$$\gamma : \bar{Y} \times Y \xrightarrow{\sim} X^{(0)}$$

given by composing the isomorphism $Y \times \bar{Y} \xrightarrow{\sim} X^{(0)}$ of Lemma 6.2 with the switch of factors. We then obtain an $O_{\boldsymbol{k}}$-linear isogeny

$$\psi_s \circ \gamma : \bar{Y} \times Y \longrightarrow X^{(s)}.$$

Note that the diagram

$$\begin{array}{ccc} X^{(0)} \otimes \mathbb{F} & \xrightarrow{\bar{\psi}_s} & X^{(s)} \otimes \mathbb{F} \\ \bar{\gamma} \uparrow & & \uparrow \rho^{(s)} \\ \bar{\mathbb{Y}} \times \mathbb{Y} & \xrightarrow{g_o^s} & \bar{\mathbb{Y}} \times \mathbb{Y} \end{array}$$

is commutative where $g_o = \mathrm{sw} \circ (\Pi \times \Pi)$ where sw is the switch of factors. The map

$$\psi_s \circ \gamma \circ \mathrm{inc}_1 : \bar{Y} \longrightarrow X^{(s)}$$

is an $O_{\boldsymbol{k}}$-linear homomorphism whose reduction is

$$\overline{\psi_s \circ \gamma \circ \mathrm{inc}_1} = \begin{cases} \mathrm{inc}_1 \circ \Pi^s & \text{if } s \text{ is even,} \\ \mathrm{inc}_2 \circ \Pi^s & \text{if } s \text{ is odd.} \end{cases}$$

Now let

$$\begin{cases} \tilde{y}_1 = p^{(a-s)/2} \psi_s \circ \mathrm{inc}_1 & \text{if } s \text{ is even} \\ \tilde{y}_2 = p^{(b-s)/2} \psi_s \circ \mathrm{inc}_1 & \text{if } s \text{ is odd.} \end{cases} \tag{8.1}$$

Then, for s of the correct parity, $\tilde{y}_i : \bar{Y} \to X^{(s)}$ is a lift of y_i. This shows that the divisor $\mathcal{Z}_s$ is a component of $\mathcal{Z}(y_1)$ (resp. $\mathcal{Z}(y_2)$) for $0 \le s \le a$ even (resp. $1 \le s \le b$, odd). Hence we obtain inequalities of divisors on $\mathrm{Spec}\, W[[t]]$,

$$\sum_{\substack{s=0 \\ \text{even}}}^{a} \mathcal{Z}_s \le \mathcal{Z}(y_1) \ , \ \text{resp.} \ \sum_{\substack{s=1 \\ \text{odd}}}^{b} \mathcal{Z}_s \le \mathcal{Z}(y_2).$$

In order to show that these inequalities are equalities, it suffices to show that the intersection multiplicities of both sides with the special fiber $\mathcal{M}_p = \mathrm{Spec}\, k[[t]]$ are the same. For the LHS we obtain for these intersection multiplicities

$$\sum_{\substack{s=0 \\ \text{even}}}^{a} \mathcal{Z}_s \cdot \mathcal{M}_p = \sum_{\substack{s=0 \\ \text{even}}}^{a} e_s = 1 + (p+p^2) + \ldots + (p^{a-1} + p^a) = \frac{p^{a+1}-1}{p-1}$$

resp.

$$\sum_{\substack{s=1\\ \text{odd}}}^{b} \mathcal{Z}_s \cdot \mathcal{M}_p = \sum_{\substack{s=1\\ \text{odd}}}^{b} e_s = (1+p) + (p^2+p^3) + \ldots + (p^{b-1}+p^b) = \frac{p^{b+1}-1}{p-1}\ .$$

The assertion now follows from the following proposition. $\square$

Let $y : \overline{\mathbb{Y}} \to \mathbf{X}$ be an $O_{\boldsymbol{k}}$-linear homomorphism with $y^* \circ y \neq 0$. Consider the universal deformation of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}})$ in equal characteristic over $\mathcal{M}_p = \mathrm{Spf}\ \mathbb{F}[[t]]$, and the maximal closed formal subscheme $\mathcal{Z}(y)_p$ of $\mathcal{M}_p$, where y deforms into a homomorphism $y : \overline{\mathbb{Y}} \times_{\mathrm{Spec}\,\mathbb{F}} \mathcal{Z}(y)_p \to X \times_{\mathcal{M}_p} \mathcal{Z}(y)_p$. The proof of the following proposition is due to Th. Zink.

Proposition 8.2. *The length of the Artin scheme $\mathcal{Z}(y)_p$ is equal to $\frac{p^{v+1}-1}{p-1}$, where v is the D-valuation of the element $y^* \circ y \in O_D$ (maximal power of Π dividing $y^* \circ y$).*

Proof. (*Zink*): We are going to use the theory of displays [35] and their windows [36]. Let $R = \mathbb{F}[[t]]$ and $A = W[[t]]$. We extend the Frobenius automorphism σ on W to A by setting $\sigma(t) = t^p$. For any $a \geq 1$, we set $R_a = \mathbb{F}[[t]]/t^a$ and $A_a = A/t^a$. Then A is a frame for R, resp. A_a is a frame for R_a, with augmentation ideal generated by p.

We consider the category of p-divisible groups over R which have no étale part modulo t or, in other words, the category of formal groups over R which are p-divisible modulo t. For simplicity of expression we call them formal p-divisible groups over R.

Formal p-divisible groups over R are classified by A–R-windows (M, M_1, ϕ, ϕ_1), which satisfy a nilpotence condition, [36], Thm. 4. Recall that an $A-R$-window consists of a 4-tuple (M, M_1, ϕ, ϕ_1), where M is a free A-module of finite rank and M_1 is a submodule containing pM such that M/M_1 is a free R-module. Furthermore, $\phi : M \to M$ is a σ-linear endomorphism such that $\phi(M_1) \subset pM$ and such that $\phi(M_1)$ generates pM as an A-module (this last condition is easily seen to be equivalent to condition (ii) in [36], Def. 2). Finally $\phi_1 = \frac{1}{p}\ \phi : M_1 \to M$.

There is a base change functor from $A-R$-windows to $W-\mathbb{F}$-windows. This is compatible with base-changing formal p-divisible groups. (To see this, one first passes to the display associated to the window, then applies the base change property of displays, cf. [35], Definition 20, and then passes back to the associated window. We are using here that the frames for R and for $\mathbb{F}$ are chosen in a compatible way.) The category of $W-\mathbb{F}$-windows is isomorphic to the category of ordinary Dieudonné modules over k. The nilpotence condition says that V is topologically nilpotent on $M_{\mathbb{F}}$. Since we will only consider deformations of formal p-divisible groups, the nilpotence condition is always automatically satisfied and we will therefore ignore it.

We denote by

$$\phi_1^\sharp : M_1^{(\sigma)} \to M$$

the linearization of ϕ_1, where $M_1^{(\sigma)} = A \otimes_{A,\sigma} M_1$.

Lemma 8.3. *$\phi_1^\sharp$ is an isomorphism.*

Proof. Choosing a normal decomposition, we have

$$M = T \oplus L \quad , \quad M_1 = pT \oplus L \ .$$

The assertion follows since we see that $\phi_1^\sharp$ induces a surjection between free A-modules of the same rank. □

We obtain from (M, M_1, ϕ, ϕ_1) the free A-module M_1 and the A-linear homomorphism $\alpha : M_1 \to M_1^{(\sigma)}$ as the composition

$$\alpha : \ M_1 \hookrightarrow M \overset{(\phi_1^\sharp)^{-1}}{\longrightarrow} M_1^{(\sigma)} \ .$$

In this way, the category of formal p-divisible groups over R becomes equivalent to the category of pairs (M_1, α), consisting of a free A-module of finite rank and an A-linear injective homomorphism $\alpha : M_1 \to M_1^{(\sigma)}$, such that Coker α is an R-module which is free. An analogous description holds for the category of formal p-divisible groups over R_a. Under this equivalence the category of formal p-divisible groups with an $O_{\boldsymbol{k}}$-action becomes equivalent to the category of pairs (M_1, α), such that M_1 is $\mathbb{Z}/2\mathbb{Z}$-graded and α is a homogeneous morphism of degree 1.

Consider the p-divisible group $\overline{\mathbb{Y}}$ over $\mathbb{F}$ with its action ι of $O_{\boldsymbol{k}}$. It corresponds to the pair[8] (N, β), where N is the $\mathbb{Z}/2$-graded free W-module of rank 2 with

$$N^0 = W(k) \cdot n_0 \quad , \quad N^1 = W(k) \cdot n_1$$

and

$$\beta(n_0) = p \otimes n_1 \quad , \quad \beta(n_1) = 1 \otimes n_0 \ .$$

By base change $W \to A$ we obtain the pair $(\mathcal{N}, \beta)$ over A with the same defining relations which corresponds to the constant p-divisible group $\overline{\mathbb{Y}}$ over R.

[8]In fact, for convenience, we are writing here $M_1[1]$ (degree shift by 1) for the situation at hand. This has no effect on the outcome of the calculation.

The p-divisible group $\mathbf{X}$ over $\mathbb{F}$ corresponds to the A_1-module $M = N \oplus \overline{N}$, where

$$M = M^0 \oplus M^1 \quad \text{and}$$

$$M^0 = A \cdot f_0 \oplus A \cdot e_0 \quad , \quad M^1 = A \cdot f_1 \oplus Ae_1 \ ,$$

and to the graded map $\alpha : M \to M^{(\sigma)}$ given by

$$\begin{array}{lcl} \alpha(f_0) = p \otimes f_1 & , & \alpha(e_0) = 1 \otimes e_1 \\ \alpha(f_1) = 1 \otimes f_0 & , & \alpha(e_1) = p \otimes e_0 \ . \end{array}$$

We consider the deformation of $(\mathbf{X}, \iota)$ given by the free A-module $\mathcal{M}$ with the same generators as for M and the homomorphism α_t (in terms of the ordered basis f_0, e_0, e_1, f_1),

$$\alpha_t = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & p & t \\ \hline 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ p & -t & & \end{array} \right) = \begin{pmatrix} 0 & U \\ \breve{U} & 0 \end{pmatrix} \ .$$

Here

$$U = \begin{pmatrix} 0 & 1 \\ p & t \end{pmatrix} \ , \ \text{resp. } \breve{U} = \begin{pmatrix} 0 & 1 \\ p & -t \end{pmatrix} \ .$$

One checks that this deformation respects the polarization $\lambda_{\mathbf{X}}$ of $\mathbf{X}$ (rewrite the deformation in terms of the original display (M, M_1, ϕ, ϕ_1) and use [32], Cor. 3.29). In fact, $(\mathcal{M}, \alpha_t)$ defines the universal deformation of $(\mathbf{X}, \iota, \lambda_{\mathbf{X}})$, cf. [32].

Now let y correspond to the graded A_1-linear homomorphism,

$$\gamma : N \to M \ .$$

Then the length ℓ of the deformation space of γ is the maximal a such that there exists a lift

$$\tilde{\gamma} : \mathcal{N} \otimes R_a \to \mathcal{M} \otimes R_a \ ,$$

making the diagram

$$\begin{tikzcd} \mathcal{N} \otimes R_a \arrow[r, "\beta"] \arrow[d, "\tilde{\gamma}"'] & \mathcal{N}^{(\sigma)} \otimes R_a \arrow[d, "\tilde{\gamma}^{(\sigma)}"] \\ \mathcal{M} \otimes R_a \arrow[r, "\alpha_t"] & \mathcal{M}^{(\sigma)} \otimes R_a \ . \end{tikzcd}$$

commute.

To calculate ℓ, we distinguish cases. First let $v = 2r$ be even. Applying Lemma 5.2, in this case we may postcompose y with an automorphism of $\mathbf{X}$

such that $y = p^r \cdot \mathrm{inc}_1$, i.e. γ is given by $\gamma = (\gamma^0, \gamma^1) = (X(0), Y(0))$, with

$$X(0) = \begin{pmatrix} p^r & 0 \\ 0 & 0 \end{pmatrix} \ , \ Y(0) = \begin{pmatrix} 0 & 0 \\ 0 & p^r \end{pmatrix} \ .$$

In order to lift γ mod t^p, we search for matrices $X(1), Y(1) \in \mathrm{M}_2(A_p)$ such that

$$X(1) \equiv X(0) \text{ in } A_1$$
$$Y(1) \equiv Y(0) \text{ in } A_1$$

and satisfying the identities

$$\sigma(X(1)) \cdot S = U \cdot Y(1) \ , \ \sigma(Y(1)) \cdot S = \breve{U} \cdot Y(1) \ .$$

Here

$$S = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \ .$$

Note that σ can be viewed as a map $A_1 \to A_p$. Since $\sigma(X(1)) = \sigma(X(0))$ and $\sigma(Y(1)) = \sigma(Y(0))$, we obtain the identities

$$\begin{aligned} \sigma(X(0)) \cdot S &= U \cdot Y(1) \\ \sigma(Y(0)) \cdot S &= \breve{U} \cdot X(1) \ . \end{aligned} \tag{8.2}$$

Since A_p has no p-torsion, we obtain as unique solution

$$\begin{aligned} Y(1) &= U^{-1} \cdot \sigma(X(0)) \cdot S \\ X(1) &= \breve{U}^{-1} \cdot \sigma(Y(0)) \cdot S \ , \end{aligned}$$

provided that the matrices on the RHS have coefficients which are integral, i.e. which lie in $W[t]/t^p$. More precisely, we obtain $\ell \geq p$ if these coefficients are integral; otherwise ℓ is the maximum power t^a such that the coefficients mod t^a are integral.

Inserting the values for $X(0)$ and $Y(0)$, an easy calculation shows

$$X(1) = X(0) \quad \text{and} \quad Y(1) = \begin{pmatrix} 0 & -p^{r-1}t \\ 0 & p^r \end{pmatrix} \ . \tag{8.3}$$

If follows that $\ell = 1$ if $r = 0$ and $\ell \geq p$ if $r \geq 1$. This proves the assertion for $v = 0$.

In the next step we try to lift γ from A_p to A_{p^2}, in the next step from A_{p^2} to A_{p^3} and inductively from A_{p^n} to $A_{p^{n+1}}$ for any $n \geq 1$. At each step we use the map $\sigma : A_{p^n} \to A_{p^{n+1}}$. This gives at each step the recursive identities

$$\begin{aligned} Y(n+1) &= U^{-1} \cdot \sigma(X(n)) \cdot S \\ X(n+1) &= \breve{U}^{-1} \cdot \sigma(Y(n)) \cdot S \end{aligned} \tag{8.4}$$

which can be solved after inverting p for every n.

Claim: a) $X(2i) = X(2i+1)$ *and* $Y(2i+1) = Y(2i+2)$ for all $i \geq 0$.

b) *There exist polynomials* $P_0, P_1, \ldots, Q_0, Q_1, \ldots$ *in* $W[t]$ *such that*

$$X(2s) = \begin{pmatrix} \pm p^{r-s} \cdot t^{p^{2s-1}+\ldots+p+1} + p^{r-s+1} \cdot P_{2s} & 0 \\ p^{r-s+1} \cdot Q_{2s} & 0 \end{pmatrix}$$
$$Y(2s+1) = \begin{pmatrix} 0 & \pm p^{r-s-1} \cdot t^{p^{2s}+\ldots+p+1} + p^{r-s} \cdot P_{2r+1} \\ 0 & p^{r-s} \cdot Q_{2r+1} \end{pmatrix} .$$

Indeed, (8.3) shows this for the beginning terms with $P_0 = Q_0 = 0$ and $P_1 = 0, Q_1 = 1$. The higher terms follow by induction from the recursive relations (8.4).

The claim shows that γ deforms to $A_{p^{2r}}$, but not to $A_{p^{2r+1}}$. In fact the upper right coefficient of $Y(2r+1)$ shows that γ deforms precisely to $A_{p^{2r}+\ldots+p+1}$, which proves the proposition in this case.

Next we consider the case when $v = 2r+1$ is odd. In this case, Lemma 5.2 shows that we may postcompose y with an automorphism of $\mathbf{X}$ such that $y = p^r \cdot \ \mathrm{inc}_2 \circ \Pi$. Hence in this case

$$X(0) = \begin{pmatrix} 0 & 0 \\ p^{r+1} & 0 \end{pmatrix} \quad , \quad Y(0) = \begin{pmatrix} 0 & p^r \\ 0 & 0 \end{pmatrix} .$$

In this case an easy calculation using the identities (8.4) shows that

$$X(1) = \begin{pmatrix} p^r t & 0 \\ p^{r+1} & 0 \end{pmatrix} \quad , \quad Y(1) = Y(0) .$$

Inductively one shows as before

Claim: a) $X(2i+1) = X(2i+2)$ *and* $Y(2i) = Y(2i+1)$ for all $i \geq 0$.

b) *There exist polynomials* $P_0, P_1, \ldots, Q_0, Q_1, \ldots$ *in* $W[t]$ *such that*

$$X(2s+1) = \begin{pmatrix} \pm p^{r-s} \cdot t^{p^{2s}+\ldots+p+1} + p^{r-s+1} \cdot P_{2s+1} & 0 \\ p^{r-s+1} \cdot Q_{2s+1} & 0 \end{pmatrix}$$
$$Y(2s) = \begin{pmatrix} 0 & \pm p^{r-s} \cdot t^{p^{2s-1}+\ldots+p+1} + p^{r-s+1} P_{2s} \\ 0 & p^{r-s+1} \cdot Q_{2s} \end{pmatrix} .$$

By looking at the upper right coefficient of $Y(2(r+1))$, we see that the deformation locus of γ is given by $t^{p^{2r+1}+\ldots+p+1} = 0$, which proves the proposition in this case.

$\square$

By Proposition 8.1,

$$\mathcal{Z}(y_1)\cdot\mathcal{Z}(y_2) = \sum_{\substack{s=0\\ s\,\text{even}}}^{a} \mathcal{Z}_s\cdot\mathcal{Z}(y_2) = \sum_{\substack{s=1\\ s\ \text{odd}}}^{b} \mathcal{Z}(y_1)\cdot\mathcal{Z}_s. \tag{8.5}$$

Let $m_s(\mathbf{y})$ be the maximum m such that the $O_{\boldsymbol{k}}$-linear homomorphism

$$\bar{\mathbb{Y}}\times\bar{\mathbb{Y}} \xrightarrow{\mu(\mathbf{y})} \bar{\mathbb{Y}}\times\mathbb{Y} = \mathbf{X} = X^{(s)}\otimes\mathbb{F} \tag{8.6}$$

lifts to a homomorphism

$$\bar{Y}\times\bar{Y} \dashrightarrow X^{(s)}$$

over $W_s/\pi_s^m W_s$, where $\mu(\mathbf{y})$ has matrix $\mathrm{diag}(\Pi^a,\Pi^b)$. Then

$$m_s(\mathbf{y}) = \begin{cases} \mathcal{Z}_s\cdot\mathcal{Z}(y_2) & \text{for } s \text{ even},\\ \mathcal{Z}(y_1)\cdot\mathcal{Z}_s & \text{for } s \text{ odd}.\end{cases}$$

We can write (8.6) as

$$X^{(0)}\otimes\mathbb{F} = \bar{\mathbb{Y}}\times\mathbb{Y} \xrightarrow{\mu(\mathbf{y})} \bar{\mathbb{Y}}\times\mathbb{Y} = X^{(s)}\otimes\mathbb{F}\ , \tag{8.7}$$

where we have simply taken the conjugate linear $O_{\boldsymbol{k}}$-action on the second factor of the source $\bar{\mathbb{Y}}\times\bar{\mathbb{Y}}$. The matrix for $\mu(\mathbf{y})$ is unchanged, and $m_s(\mathbf{y})$ is the maximum m such that this map lifts to a map

$$X^{(0)} \dashrightarrow X^{(s)}$$

over $W_s/\pi_s^m W_s$.

To apply Theorem 7.1, we need to write $\mu(\mathbf{y})$ in the form (7.1). We take 1 and δ as $\mathbb{Z}_p$-basis for $O_{\boldsymbol{k}}$, and hence, for any p-divisible group X, we have an identification $O_{\boldsymbol{k}}\otimes X = X\times X$. If X is a p-divisible group with $O_{\boldsymbol{k}}$-action, then the isomorphism of Lemma 6.2

$$X\times\bar{X} \xrightarrow{\sim} O_{\boldsymbol{k}}\otimes X = X\times X$$

has matrix

$$C = \begin{pmatrix}\delta & -\delta\\ 1 & 1\end{pmatrix}.$$

Thus, the matrix for $\mu(\mathbf{y})$ is

$$\frac{1}{2}\begin{pmatrix}\Pi^a-\Pi^b & \delta(\Pi^b-\Pi^a)\\ (\Pi^b-\Pi^a)\delta^{-1} & \Pi^a+\Pi^b\end{pmatrix},$$

if s is even, and

$$\frac{1}{2}\begin{pmatrix}(\Pi^b-\Pi^a)\delta^{-1} & \Pi^a+\Pi^b\\ \Pi^a-\Pi^b & \delta(\Pi^b-\Pi^a)\end{pmatrix},$$

if s is odd. Now if $s \leq a$ is even, then $\Pi^a \in \Pi^s O_{\boldsymbol{k}}$ so that $l = b$ in Theorem 7.1. If $s \leq b$ is odd, then $\Pi^b \in \Pi^s O_{\boldsymbol{k}}$ and $l = a$. This yields the following result.

Proposition 8.4. *For $s \leq a$ even,*

$$\mathcal{Z}_s \cdot \mathcal{Z}(y_2) = \begin{cases} \dfrac{p^{b+1}-1}{p-1} & \text{if } b < s, \\ \dfrac{p^s-1}{p-1} + \frac{1}{2}\,(b+1-s)\,e_s & \text{if } b \geq s. \end{cases}$$

For $s \leq b$ odd,

$$\mathcal{Z}(y_1) \cdot \mathcal{Z}_s = \begin{cases} \dfrac{p^{a+1}-1}{p-1} & \text{if } a < s, \\ \dfrac{p^s-1}{p-1} + \frac{1}{2}\,(a+1-s)\,e_s & \text{if } a \geq s. \end{cases}$$

Here recall that $e_s = p^{s-1}(p+1)$ for $s \geq 1$ and $e_0 = 1$. Also, if

$$\mathcal{Z}_s \cap \mathcal{Z}(y_2) = \operatorname{Spec} W_s/\pi_s^{\ell}\ ,$$

then $\operatorname{ord}_A(\pi_s^{\ell}) = (e/e_s) \cdot \ell$.

Corollary 8.5. *Let $r \neq s$. Then*

$$\mathcal{Z}_s \cdot \mathcal{Z}_t = e_{\min\{s,t\}}.$$

By summing the $\mathcal{Z}_s \cdot \mathcal{Z}(y_2)$'s (resp. the $\mathcal{Z}(y_1) \cdot \mathcal{Z}_s$'s) of Proposition 8.4 as in (8.5), we obtain the expression in Theorem 5.1.

9. Representation densities of hermitian forms

In this section, we show that the expression given in Theorem 5.1 for the intersection multiplicity in the case where the scaled fundamental matrix $\tilde{T}$ is $\mathrm{GL}_n(O_{\boldsymbol{k}})$- equivalent to $\operatorname{diag}(1_{n-2}, p^a, p^b)$ coincides, up to an elementary factor, with the derivative of a certain representation density associated to $\tilde{T}$. As explained in the introduction, this relation is the component at p of an identity between a global arithmetic intersection number or height, and a Fourier coefficient of the derivative of an Eisenstein series on $\mathrm{U}(n,n)$. To avoid introducing additional notation, we continue to suppose that $\boldsymbol{k} = \mathbb{Q}_{p^2}$ is the unramified quadratic extension of $\mathbb{Q}_p$.

First recall that, for nonsingular matrices $S \in \mathrm{Herm}_m(O_{\boldsymbol{k}})$ and $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$, with $m \geq n$, the representation density $\alpha_p(S,T)$ is defined as

$$\alpha_p(S,T) = \lim_{k\to\infty} (p^{-k})^{n(2m-n))} |A_{p^k}(S,T)|, \tag{9.1}$$

where

$$A_{p^k}(S,T) = \{\ x \in M_{m,n}(O_{\boldsymbol{k}}/p^k O_{\boldsymbol{k}}) \mid S[x] \equiv T \mod p^k\ \},$$

where $S[x] = {}^t x S \sigma(x)$. The density depends only on the $\mathrm{GL}_m(O_{\boldsymbol{k}})$- (resp. $\mathrm{GL}_n(O_{\boldsymbol{k}})$-) equivalence class of S (resp. T). An explicit formula for $\alpha_p(S,T)$ has been given by Hironaka, [5].

Let $\ell(T)$ be the smallest ℓ such that $p^\ell T^{-1} \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$. In fact, for $k > \ell(T)$, the quantity $(p^{-k})^{n(2m-n))}|A_{p^k}(S,T)|$ is constant and is non-zero if and only if there exists an $x \in M_{m,n}(O_{\boldsymbol{k}})$ such that $S[x] = T$. For $r \geq 0$, let $S_r = \mathrm{diag}(S, 1_r)$. Then

$$\alpha_p(S_r, T) = F_p(S,T;(-p)^{-r})$$

for a polynomial $F_p(S,T;X) \in \mathbb{Q}[X]$, as can be seen immediately from Hironaka's formula.

Recall that the isometry class of a non-degenerate hermitian space V of dimension n over $\boldsymbol{k}$ is determined by its determinant $\det(V) \in \mathbb{Q}_p^\times / N(\boldsymbol{k}^\times)$. Thus, if S and $T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$ are non-singular with $\mathrm{ord}(\det(S)) + \mathrm{ord}(\det(T))$ odd, then $\alpha_p(S,T) = 0$. In this case, we define the derivative of the representation density

$$\alpha_p'(S,T) = -\frac{\partial}{\partial X} F_p(S,T;X)|_{X=1}.$$

The main result of this section is the following.

Proposition 9.1. *Let $S = 1_n$ and $T = \mathrm{diag}(1_{n-2}, p^a, p^b)$ for $0 \leq a < b$ with $a+b$ odd. Then $\alpha_p(S,T) = 0$ and*

$$\frac{\alpha_p'(S,T)}{\alpha_p(S,S)} = \frac{1}{2} \sum_{\ell=0}^{a} p^\ell (a+b-2\ell+1),$$

where

$$\alpha_p(S,S) = \prod_{\ell=1}^{n} (1 - (-1)^\ell p^{-\ell}).$$

Comparing this expression with that given in Theorem 5.1, we find the following relation between the derivative of the hermitian representation density and the arithmetic intersection multiplicity.

Corollary 9.2. *Let $\mathcal{Z}_{i,j}(\mathbf{x})$ be non-empty, with associated scaled fundamental matrix $\tilde{T} = p^{2i-j} h(\mathbf{x},\mathbf{x}) \in \mathrm{Herm}_n(O_{\boldsymbol{k}})$. Suppose that $\tilde{T}$ is $\mathrm{GL}_n(O_{\boldsymbol{k}})$-equivalent to $\mathrm{diag}(1_{n-2}, p^a, p^b)$ with $a+b$ odd. Then*

$$\widehat{\deg}(\mathcal{Z}_{i,j}(\mathbf{x})) = \log(p) \cdot \frac{\alpha_p'(S,\tilde{T})}{\alpha_p(S,S)}.$$

Proof of Proposition 9.1. The first step is the following reduction formula, which is the hermitian analogue of Corollary 5.6.1 in [7]. For the convenience of the reader, we will sketch the proof below.

Proposition 9.3. *Let* $S' = 1_2$ *and* $T' = \mathrm{diag}(p^a, p^b)$*. Then*

$$\alpha_p(S_r, T) = \alpha_p(S_r, 1_{n-2})\,\alpha_p(S'_r, T').$$

It follows that

$$\alpha'_p(S, T) = \alpha_p(S, 1_{n-2})\,\alpha'_p(S', T').$$

By Hironaka's formula or the classical literature, [25],

$$F_p(1_n, 1_n; X) = \prod_{\ell=1}^{n}(1 - (-1)^{\ell} p^{-\ell} X),$$

so that

$$\alpha_p(1_n, 1_{n-2}) = \prod_{\ell=1}^{n-2}(1 - (-1)^{\ell} p^{-\ell-2}).$$

On the other hand, Nagaoka, [18], proved the following in the binary case.

Proposition 9.4 (Nagaoka)**.** *Suppose that* $S' = 1_2$ *and that* $T' = \mathrm{diag}(p^a, p^b)$ *with* $0 \le a \le b$*, but with no condition on the parity of* $a + b$*. Then*

$$F_p(S', T'; X) = (1 + p^{-1}X)(1 - p^{-2}X)\sum_{\ell=0}^{a}(pX)^{\ell}\Bigg(\sum_{k=0}^{a+b-2\ell}(-X)^k\Bigg).$$

Corollary 9.5. *If* $a + b$ *is odd, then*

$$(1 - p^{-2})^{-1}(1 + p^{-1})^{-1}\alpha'_p(S', T') = \frac{1}{2}\sum_{\ell=0}^{a} p^{\ell}\ (a + b - 2\ell + 1).$$

This completes the proof of Proposition 9.1. □

Proof of Proposition 9.3. The proof is just the hermitian version of the argument given by Kitaoka, [7], pp. 104–107.

First we pass to the lattice formulation in the standard way. Viewing S as the matrix of inner products $((v_i, v_j))$ for a basis $\mathbf{v} = [v_1, \dots, v_m]$ of an $O_{\boldsymbol{k}}$-lattice M and T as the matrix of inner products $((u_i, u_j))$ for a basis $\mathbf{u} = [u_1, \dots, u_n]$ for an $O_{\boldsymbol{k}}$-lattice L, we have a bijection of $A_{p^r}(S, T)$ with the set

$$(9.2)\quad I_k(L, M) = \{\ \varphi \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(L, M/p^k M) \mid (\varphi(x), \varphi(y)) \equiv (x, y) \mod p^k,\ \forall x, y \in L\ \}.$$

Then

$$\alpha_p(S,T) = \alpha_p(L,M) = (p^{-k})^{n(2m-n)}|I_k(L,M)|,$$

for k sufficiently large.

We need the following preliminary results.

Lemma 9.6. *Suppose that $N \subset M$ is a regular*[9] *sublattice with $O_{\boldsymbol{k}}$-basis $\{v_i\}$, so that $N = [v_1, \dots, v_r]$. Suppose that $w_i \in M$ is sufficiently close to v_i. Then there is an isometry $\eta \in \mathrm{U}(M)$ with $\eta(N) = [w_1, \dots, w_r]$.*

Lemma 9.7. *If $\varphi : L \to M$ with $(\varphi(x), \varphi(y)) \equiv (x,y) \mod p^k$, for some sufficiently large k, then there is an isometry $\eta : L \to \varphi(L) \subset M$.*

Lemma 9.8. *Suppose that φ_1 and φ_2 are two homomorphisms satisfying the conditions of the previous lemma. Also suppose that $\varphi_1 \equiv \varphi_2 \mod p^k$, for some sufficiently large k. Then there is an isometry $\gamma \in \mathrm{U}(M)$ such that $\varphi_2(L) = \gamma(\varphi_1(L))$.*

For given L and M and a sublattice $N \subset M$ such that N is isometric to L, we let

$$\tilde{I}_k(L,M) = \{\ \varphi \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(L,M) \mid (\varphi(x), \varphi(y)) \equiv (x,y) \mod p^k,\ \forall x,\ y \in L\ \},$$

and define

$$\tilde{I}_k(L,M;N) = \{\ \varphi \in \tilde{I}_k(L,M) \mid \exists\, \eta \in \mathrm{U}(M) \text{ with } \varphi(L) = \eta(N)\ \}$$

and

$$I_k(L,M;N) = \{\ \varphi \in I_k(L,M) \mid \exists\, \eta \in \mathrm{U}(M) \text{ with } \tilde{\varphi}(L) = \eta(N)\ \}.$$

In this last set $\tilde{\varphi} \in \tilde{I}_k(L,M)$ is a preimage of φ. By the preliminary lemmas, these sets are well defined for k sufficiently large.

Proposition 9.9. *Suppose that $L = L_1 \perp L_2$ with L_j of rank n_j. Let $\{N_i\}$ be a set of representatives for the $\mathrm{U}(M)$-orbits in the set of all sublattices $N \subset M$ such that N is isometric to L_1. Then*

$$|I_k(L,M)| = \sum_i |I_k(L_1, M; N_i)| \times |\{\varphi_2 \in I_k(L_2, M) \mid (\varphi_2(L_2), N_i) \equiv 0 \mod p^k\ \}|. \tag{9.3}$$

Proof. First note that, for k sufficiently large, for any $\varphi_1 \in \tilde{I}_k(L_1, M)$, $\varphi_1(L_1)$ is isometric to L_1. For each $\varphi_1 \in \tilde{I}_k(L_1, M)$ choose an isometry

[9]Here, following the terminology in [7], regular means that the restriction of the hermitian form to N is non-degenerate.

$\gamma = \gamma(\varphi_1) \in \mathrm{U}(M)$ such that $N_i = \gamma(\varphi_1(L_1))$, for some i. Also, for each $\varphi_1 \in I_k(L_1, M)$, choose a preimage $\tilde{\varphi}_1 \in \tilde{I}_k(L_1, M)$. There is then a bijection

$$I_k(L, M) \xrightarrow{\sim} \coprod_i I_k(L_1, M; N_i) \times \{\varphi_2 \in I_k(L_2, M) \mid (\varphi_2(L_2), N_i) \equiv 0 \mod p^k\}$$

given by $\varphi \mapsto (\varphi_1, \varphi_2)$ with $\varphi_1 = \varphi|_{L_1}$ and $\varphi_2 = \gamma(\tilde{\varphi}_1) \circ \varphi|_{L_2}$. □

Lemma 9.10.

$$\begin{aligned}&|\{\varphi_2 \in I_k(L_2, M) \mid (\varphi_2(L_2), N_i) \equiv 0 \mod p^k\ \}| \\ &\qquad\qquad = |N_i^\vee : N_i|^{n_2} |M : N_i \perp N_i^\perp|^{-n_2} |I_k(L_2, N_i^\perp)|.\end{aligned}$$

Proof. As in Kitaoka,

$$\{x \in M \mid (x, N_i) \equiv 0 \mod p^k\} = p^k N_i^\vee \perp N_i^\perp.$$

where $N_i^\perp = (\boldsymbol{k} N_i)^\perp \cap N$, provided $p^k N_i^\vee \subset N_i$. Thus

$$\begin{aligned}(9.4)\quad &\{\varphi_2 \in I_k(L_2, M) \mid (\varphi_2(L_2), N_i) \equiv 0 \mod p^k\} \\ &= \{\varphi_2 : L_2 \to p^k N_i^\vee \perp N_i^\perp \mod p^k M \mid (\varphi_2(x), \varphi_2(y)) \equiv (x, y) \mod p^k\ \}.\end{aligned}$$

Next, we replace $p^k M$ by $p^k(p^a N_i^\vee \perp N_i^\perp)$, so that the cosets diagonalize, i.e., we consider the set

$$\begin{aligned}(9.5)\quad &\{\varphi_2 : L_2 \to p^k N_i^\vee \perp N_i^\perp \mod p^k(p^a N_i^\vee \perp N_i^\perp) \mid \\ &\qquad\qquad (\varphi_2(x), \varphi_2(y)) \equiv (x, y) \mod p^k\ \}.\end{aligned}$$

Write $\varphi_2 = \psi_1 + \psi_2$ with $\psi_1 : L_2 \to p^k N_i^\vee / p^{k+a} N_i^\vee$ and $\psi_2 : L_2 \to N_i^\perp / p^k N_i^\perp$. Since we are assuming that $p^k N_i^\vee \subset N_i$, we have $(\psi_1(x), \psi_1(y)) \in (N_i, p^k N_i^\vee) \subset p^k O_{\boldsymbol{k}}$. Thus the condition on φ_2 in (9.5) just amounts to the condition $(\psi_2(x), \psi_2(y)) \equiv (x, y) \mod p^k$, with no restriction on $\psi_1 \in \mathrm{Hom}_{O_{\boldsymbol{k}}}(L_2, p^k N_i^\vee / p^{k+a} N_i^\vee)$. This yields the claimed expression, once the various lattice indices are taken into account. □

Corollary 9.11. *With the notation of the previous proposition,*

$$\alpha_p(L, M) = \sum_i |M : N_i \perp N_i^\perp|^{-n_2} |N_i^\vee : N_i|^{n_2}\, \alpha_p(L_1, M; N_i)\, \alpha_p(L_2, N_i^\perp).$$

Now suppose that L_1 is unimodular, so that any $N \subset M$ isometric to L_1 is unimodular. Then, for any such N, $M = N \perp N^\perp$. Moreover, since $\boldsymbol{k}$ is unramified, if

$$M = N_1 \perp N_1^\perp = N_2 \perp N_2^\perp$$

are two such decompositions, then $N_1^\perp \simeq N_2^\perp$. Thus $\mathrm{U}(M)$ acts transitively on such N's.

Corollary 9.12. *With the notation of the previous proposition, suppose that L_1 is unimodular. Then*

$$\alpha_p(L, M) = \alpha_p(L_1, M)\, \alpha_p(L_2, N^{\perp}).$$

This proves Proposition 9.3. □

Correction to [17] and [24].

We take this occasion to close a gap in [17], where we inadvertently omitted the proof of Lemma 7.7.3. This lemma is also implicitly used in [24] (the equality of divisors right after Lemma 3.1). We formulate here the lemma and give the proof.

Lemma 9.13. *Let G be the formal p-divisible group of dimension 1 and height 2 over $\mathbb{F}$. Let $\mathcal{M} = \operatorname{Spec} W[[t]]$ be the universal deformation space of G. Let $\varphi \in \operatorname{End}(G)$ be an endomorphism which generates an order of conductor c in a quadratic extension $\boldsymbol{k}$ of $\mathbb{Q}_p$. Let $\mathcal{T} = \mathcal{T}(\varphi)$ be the deformation locus of φ (a relative divisor on $\mathcal{M}$, by [24], Prop. 1.4). Then there is an equality of divisors on $\mathcal{M}$,*

$$\mathcal{T} = \sum_{s=0}^{c} \mathcal{W}_s(\varphi)\ ,$$

where $\mathcal{W}_s(\varphi)$ denotes the quasi-canonical divisor of level s (relative to $\boldsymbol{k}$).

Proof. All quasi-canonical divisors $\mathcal{W}_s(\varphi)$ are prime divisors which are pairwise distinct and with $\mathcal{W}_s(\varphi) \subset \mathcal{T}$ for $0 \le s \le c$. Hence we have an inequality of relative divisors

$$\sum_{s=0}^{c} \mathcal{W}_s(\varphi) \le \mathcal{T}\ .$$

In order to show equality here, it suffices to compare the intersection multiplicities with the special fiber $\mathcal{M}_p = \operatorname{Spec} \mathbb{F}[[t]]$. For the LHS, this is equal to

$$\sum_{s=0}^{c} e_s = \begin{cases} 2 \cdot \sum_{i=0}^{c-1} p^i + p^c & \boldsymbol{k}/\mathbb{Q}_p \ \text{unramified} \\ 2 \cdot \sum_{i=0}^{c} p^i & \boldsymbol{k}/\mathbb{Q}_p \ \text{ramified}\ . \end{cases} \tag{9.6}$$

Here $e_s = [W_s : W]$ denotes the absolute ramification index.

To determine $\mathcal{M}_p \cdot \mathcal{T}$, we first note that $\varphi \in (\mathbb{Z}_p + \Pi^{\ell} O_D) \setminus (\mathbb{Z}_p + \Pi^{\ell+1} O_D)$, where $\ell = 2c$ in case $\boldsymbol{k}$ is unramified over $\mathbb{Q}_p$, and $\ell = 2c+1$ in case $\boldsymbol{k}$ is ramified over $\mathbb{Q}_p$, comp. [34], 1.2. Now we use the result of Keating [6], Theorem 1.1 (see also [30], Theorem 2.1), which gives as the length of the deformation locus of φ in $\mathcal{M}_p$ exactly the expression on the RHS of equation (9.6) above. $\square$

References

[1] ARGOS Seminar on intersections of modular correspondences, Astérisque **312** (2007).

[2] P. Deligne, J. Milne, A. Ogus, K.-y. Shih, Hodge cycles, motives, and Shimura varieties. Lecture Notes in Mathematics, **900**. Springer-Verlag, Berlin-New York, 1982.

[3] B. Gross, *On canonical and quasi-canonical liftings*, Invent. math., **84** (1986), 321–326.

[4] B. Gross and K. Keating, *On the intersection of modular correspondences*, Invent. math., **112** (1993), 225–245.

[5] Y. Hironaka, *Local zeta functions for hermitian forms and its application to local densities*, J. Number Theory, **71** (1998), 40–64.

[6] K. Keating, *Lifting endomorphisms of formal A-modules*, Compositio Math., **67** (1988), 211–239.

[7] Y. Kitaoka, Arithmetic of Quadratic Forms. Cambridge Tracts in Math. **106**, Cambridge University Press, Cambridge 1993.

[8] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Am. Math. Soc. **5** (1992), 373–444.

[9] S. Kudla, *Intersection numbers for quotients of the complex 2-ball and Hilbert modular forms*, Invent. math., **47** (1978), 189–208.

[10] ———, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math., **146** (1997), 545–646.

[11] S. Kudla and J. Millson, *The theta correspondence and harmonic forms, I*, Math. Annalen, **274** (1986), 353–378.

[12] ———, *Intersection numbers for cycles in locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several variables*, Pub. math. IHES, **71** (1990), 121–172.

[13] S. Kudla and M. Rapoport, *Arithmetic Hirzebruch-Zagier cycles*, J. Reine Angew. Math., **515** (1999), 155–244.

[14] ———, *Height pairings on Shimura curves and p-adic uniformization*, Invent. math., **142** (2000), 153–223.

[15] ———, *Cycles on Siegel threefolds and derivatives of Eisenstein series*, Ann. Sci. Ecole Norm. Sup. (4), **33** (2000), 695–756.

[16] S. Kudla, M. Rapoport, T. Yang, *On the derivative of an Eisenstein series of weight one*, Internat. Math. Res. Notices, **7** (1999), 347–385.

[17] ———, Modular Forms and Special Cycles on Shimura Curves, Annals of Mathematics Studies, **161**, Princeton University Press, Princeton 2006.

[18] S. Nagaoka, *An explicit formula for Siegel series*, Abh. Math. Sem. Univ. Hamburg, **59** (1989), 235–262.

[19] G. Pappas, *Local structure of arithmetic moduli of PEL Shimura varieties*, J. Alg. Geom., **9** (2000), 577–605.

[20] G. Pappas and M. Rapoport, *Local models in the ramified case. I: The EL-case*, J. Alg. Geom., **12** (2003), 107–145.

[21] ———, *Local models in the ramified case. II: Splitting models*, Duke Math. J., **127** (2005), 193–250.

[22] ———, *Local models in the ramified case. III: Unitary groups*, J. Inst. Math. Jussieu, **8** (2009), 507–564.

[23] M. Rapoport, T. Zink, Period spaces for p-divisible groups. Annals of Mathematics Studies, **141**, Princeton University Press, Princeton, 1996.

[24] M. Rapoport, *Deformations of formal groups*, in [1], p. 139–170.
[25] G. Shimura, *On Eisenstein series*, Duke Math. J., **50** (1983), 417–476.
[26] U. Terstiege, *Antispecial cycles on the Drinfeld upper half plane and degenerate Hirzebruch-Zagier cycles*, Manuscripta math., **125** (2008), 191–223.
[27] U. Terstiege, *Intersections of arithmetic Hirzebruch-Zagier cycles*, arXiv:0902.1921, to appear in Math. Ann.
[28] U. Terstiege, *Intersections of special cycles on the Shimura variety for GU(1,2)*, arXiv:1006.2106
[29] I.Vollaard, *The supersingular locus of the Shimura variety for GU(1,s)*, Canad. J. Math., **62** (2010), 668–720.
[30] I. Vollaard, *Endomorphisms of quasi-canonical lifts*, in [1], p. 105–112.
[31] I. Vollaard, T. Wedhorn, *The supersingular locus of the Shimura variety for GU(1,n-1), II.*, arXiv:0804.1522. To appear in Invent. math., DOI: 10.1007/s00222-010-0299-y.
[32] T. Wedhorn, *Ordinariness in good reductions of Shimura varieties of PEL-type*, Ann. Sci. Ecole Norm. Sup. (4), **32** (1999), 575–618.
[33] S. Wewers, *Canonical and quasi-canonical liftings*, in [1], p. 67–86.
[34] S. Wewers, *An alternative approach using ideal bases*, in [1], p. 171–178.
[35] T. Zink, *The display of a formal p-divisible group*, in: Cohomologies p-adiques et applications arithmétiques, I. Astérisque **278** (2002), 127–248.
[36] T. Zink, *Windows for displays of p-divisible groups*, in: Moduli of abelian varieties (Texel Island, 1999), 491–518, Progr. Math., **195**, Birkhäuser, Basel, 2001.

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