book?

Cycles on Siegel 3-folds and derivatives of Eisenstein series

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Introduction.

This is the first of two companion papers in which we try to generalize the results of one of us, [**19**], to higher dimensions. In that paper, it was established that the nonsingular Fourier coefficients of the derivative at 0 of certain incoherent Eisenstein series on the metaplectic group in 4 variables are closely related to the value of the height pairing of a pair of arithmetic cycles on a Shimura curve. Both sides of the identity to be proved turned out to be a sum of terms enumerated by the places of $\mathbb{Q}$. In the present papers we are only concerned with the situation at a nonarchimedean prime.

It is a hope, already expressed in [**19**], that a similar relation holds in general between the value of the derivative at 0 of certain incoherent Eisenstein series on the metaplectic group in $2n$ variables and the height pairing of suitable arithmetic cycles on Shimura varieties associated to orthogonal groups of signature $(n-1,2)$. This would constitute an arithmetic analogue of the result of the first author [**18**] which relates the *value* at 1/2 of certain *coherent* Eisenstein series with the intersection pairing on suitable classical cycles on these Shimura varieties.

A first difficulty with this general program is that models over the integers of the Shimura varieties associated to orthogonal groups are not well understood. For low values of n there are, however, exceptional isomorphisms which relate the groups in question to symplectic groups, and the Shimura varieties associated to them have integral models which one can investigate. In the present paper we are concerned with the exceptional isomorphism which relates the orthogonal group of signature $(3,2)$ with the symplectic group in 4 variables. In the companion paper [**21**] we are concerned with the Shimura variety associated to an orthogonal group of signature $(2,2)$ which is related to certain Hilbert-Blumenthal surfaces.

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Let us now be more specific about the contents of this paper. Let B be an indefinite quaternion algebra over $\mathbb{Q}$. Let $C = M_2(B)$ and put

$$V = \{x \in C \mid \; x' = x, \; \mathrm{tr}^0(x) = 0\} \quad , \tag{0.1}$$

where $x \mapsto x' = {}^t x^\iota$ is the involution on C induced by the main involution on B. Then (V, q), with q defined by $x^2 = q(x) \cdot 1$, is a quadratic space of signature $(3, 2)$ and the group $G = GSpin(V)$ of V can be identified with a twisted form of the group of symplectic similitudes in 4 variables. Let $\mathcal{D}$ be the space of oriented negative 2-planes in $V(\mathbb{R})$ and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then, the Shimura variety $\mathrm{Sh}(G, \mathcal{D})_K$, whose complex points are given by

$$\mathrm{Sh}(G, \mathcal{D})_K(\mathbb{C}) = G(\mathbb{Q}) \setminus [\mathcal{D} \times G(\mathbb{A}_f)/K], \tag{0.2}$$

is a (twisted) version of the Siegel 3-fold over $\mathbb{Q}$. For example, the case of the split quaternion algebra $B = M_2(\mathbb{Q})$ yields the usual Siegel modular variety of genus 2.
The exceptional isomorphism of $G = GSpin(V)$ with a form of GSp_4 plays a fundamental role throughout the paper. In particular, we use it to construct a good integral model of $\mathrm{Sh}(G, \mathcal{D})_K$. More precisely, we fix a prime $p > 2$ such that B is unramified at p and take K of the form $K = K^p.K_p$, where $K^p \subset G(\mathbb{A}_f^p)$ is sufficiently small and where K_p is the natural maximal compact open subgroup of $G(\mathbb{Q}_p)$. Then we use the modular interpretation of $\mathrm{Sh}(G, \mathcal{D})_K$ to construct a smooth model $\mathcal{M}$ over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$, as a parameter space of certain abelian varieties with additional structure.

Algebraic cycles on $\mathrm{Sh}(G, \mathcal{D})_K$ were defined analytically in [**18**] as follows. For $x \in V^n$ let $q(x) = \frac{1}{2}\big((x_i, x_j)\big) \in \mathrm{Sym}_n(\mathbb{Q})$, be the matrix of inner products of the compnents of x for the symmetric bilinear form $(\;,\;)$ associated to q. Assume that $d = q(x)$ is positive-definite (hence $n \leq 3$), and let $\mathcal{D}_x$ be the subspace of oriented negative 2-planes orthogonal to all entries of x. Let G_x be the pointwise stabilizer of x. Then $\mathrm{Sh}(G_x, \mathcal{D}_x)$ is a sub-Shimura variety of $\mathrm{Sh}(G, \mathcal{D})$, and thus defines a cycle of codimension n in $\mathrm{Sh}(G, \mathcal{D})_K$. These cycles are a special case of the totally geodesic cycles in locally symmetric spaces studied in [**20**] and elsewhere. A slight generalization of the previous construction yields a cycle $Z(d, \omega; K)$ of $\mathrm{Sh}(G, \mathcal{D})_K$ which is associated to any positive definite $d \in \mathrm{Sym}_n(\mathbb{Q})$ and any K-invariant compact open subset ω of $V(\mathbb{A}_f)^n$.
The next step is to give a modular definition of these cycles. First, for one of the abelian varieties parametrized by $\mathcal{M}$, we define the notion of a *special endomorphism* (Definition 2.1). The space of such endomorphisms is a finitely generated free $\mathbb{Z}$-module equipped with a quadratic form q. The cycle $Z(d, \omega; K)$ $(= Z(d, \omega)$ if K is fixed) is then obtained by imposing an n-tuple $\mathbf{j}$ of special endomorphisms such that $q(\mathbf{j}) = d$, and satisfying an additional compatibility with respect to ω.

If ω satisfies an integrality condition at p, this definition can be used to extend the cycle $Z(d,\omega)$ to a cycle $\mathcal{Z}(d,\omega)$ for the integral model $\mathcal{M}$ of the Shimura variety. Here, by a cycle on $\mathcal{M}$, we mean a scheme which maps by a finite unramified morphism to $\mathcal{M}$. At this point we meet a very important problem: in contrast to $\mathcal{M}$, the cycles $\mathcal{Z}(d,\omega)$ will no longer be smooth, in general. In fact, they often are not flat over $\mathbb{Z}_{(p)}$ and may even have embedded components. Our justification for our choice of this integral extension of the classical cycles is that their definition is very simple, has a nice inductive structure with respect to intersection, and that we are able to prove something about them. Before stating these results, we note that, while the arithmetic cycles $\mathcal{Z}(d,\omega)$ can be defined for any $d \in \mathrm{Sym}_n(\mathbb{Q})$, any n, they are nonempty only when d is positive semidefinite and with coefficients in $\mathbb{Z}_{(p)}$.

We fix positive integers $n_1, \dots, n_r$ with $n_1 + \ldots + n_r = 4$ and, for each i, we choose a positive definite $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Z}_{(p)})$ and a K-invariant open compact subset $\omega_i \in V(\mathbb{A})^{n_i}$. The resulting cycles $\mathcal{Z}(d_i,\omega_i)$ on $\mathcal{M}$ have generic fibres of codimension n_i. We form the fibre product

$$\mathcal{Z} = \mathcal{Z}(d_1,\omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r,\omega_r) \quad . \tag{0.3}$$

To each point ξ of $\mathcal{Z}$, we then associate its *fundamental matrix* $T_\xi \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{\geq 0}$, defined by $T_\xi = q(\mathbf{j})$ where $\mathbf{j} = (\mathbf{j}_1, \dots, \mathbf{j}_r)$ is the 4-tuple of special endomorphisms imposed at a point of the fiber product. Note that the diagonal blocks of T are $(d_1, \dots, d_r)$. The function $\xi \mapsto T_\xi$ is locally constant for the Zariski topology on $\mathcal{Z}$ and induces a disjoint sum decomposition in which the summands are again special cycles of a definite kind,

$$\mathcal{Z} = \coprod_{\substack{T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{\geq 0} \\ \mathrm{diag}(T) = (d_1, \dots, d_r)}} \mathcal{Z}(T,\omega) \quad . \tag{0.4}$$

Here $\omega = \omega_1 \times \ldots \times \omega_r$. The summand on the right corresponding to T is the set of points ξ where $T_\xi = T$. This decomposition illustrates the inductive nature of the special cycles mentioned above.

The decomposition (0.4) bears some formal similarity to the partitioning into isogeny classes that occurs in the approach of Langlands-Kottwitz to the calculation of the zeta function of a Shimura variety. In that approach the stable conjugacy class of the Frobenius endomorphism is the most basic invariant of an isogeny class. In our context this role is played by the fundamental matrix. One of our discoveries is that the fundamental matrix and more specifically its divisibility by p governs the intersection behaviour of the special cycles. Thus if $\det(T_\xi) \neq 0$, i.e. $T = T_\xi$ is positive definite, then the point ξ lies in characteristic p and is not the specialization of a point of $\mathcal{Z}$ in characteristic 0. In this case, the connected component $\mathcal{Z}(T,\omega)$ of $\mathcal{Z}$ containing ξ consists entirely of supersingular points of $\mathcal{M}$. Contrary to what one might expect, however, the condition $\det(T_\xi) \neq 0$ is not sufficient to ensure that ξ is an isolated point of intersection. One of our main results is the characterization of when this is the case.

Theorem 0.1. *Let $\xi \in \mathcal{Z}$ with $\det(T_\xi) \neq 0$. Then ξ is an isolated intersection point if and only if T_ξ represents 1 over $\mathbb{Z}_p$. In this case the underlying abelian variety is isomorphic to a power of a supersingular elliptic curve.*

When $T = T_\xi$ does not represent 1 over $\mathbb{Z}_p$ (but still is positive definite), then the connected component $\mathcal{Z}(T,\omega)$ of $\mathcal{Z}$ containing ξ is a union of projective lines and, in fact, one can enumerate these lines. It turns out that the more divisible T_ξ is by p, the more components there will be. A more thorough analysis of the set of irreducible components can be found in [**21**]. We point out that this phenomenon of excess intersection does not occur in the case of Shimura curves at a place of good reduction [**19**], but it does occur at a place of bad reduction [**22**].
With the previous notation let us put

$$< \mathcal{Z}(d_1,\omega_1), \ldots , \mathcal{Z}(d_r,\omega_r) >_p^{\text{proper}} = \sum_{\substack{\xi \in \mathcal{Z}, \\ \xi \text{ isolated}}} e(\xi) \quad , \tag{0.5}$$

where each isolated intersection point ξ appears with multiplicity $e(\xi) = \lg \mathcal{O}_{\mathcal{Z},\xi}$, the length of the local ring of $\mathcal{Z}$ at ξ.

We next come to the relation with Eisenstein series, for which we refer the reader to section 8 or the first part of [**19**] for more details. Let W be a symplectic space over $\mathbb{Q}$ of dimension 8 and let

$$W = W_1 + \ldots + W_r \tag{0.6}$$

be a decomposition of W into symplectic spaces W_i of dimension $2n_i$. Let

$$i : Mp_{1,\mathbb{A}} \times \ldots \times Mp_{r,\mathbb{A}} \to Mp_{\mathbb{A}} \tag{0.7}$$

be the corresponding embedding of metaplectic groups. Let $\Phi(s)$ be the standard section of the induced representation $I(s,\chi_V)$ of $Mp_{\mathbb{A}}$ which is of the form $\Phi(s) = \Phi_\infty(s) \otimes \Phi_f(s)$. Here the finite part is associated to the Schwartz function $\operatorname{char} \omega \in S(V(\mathbb{A}_f)^4)$ under the natural map $S(V(\mathbb{A}_f)^4) \to I_f(0,\chi)$ defined via the Weil representation. Similarly, the component $\Phi_\infty(s)$ at ∞ is associated to the Gaussian for the 5-dimensional quadratic space $V'(\mathbb{R})$ over $\mathbb{R}$ of signature $(5,0)$ under the map $S(V'(\mathbb{R})^4) \to I_\infty(0,\chi)$. Thus the section $\Phi(s)$ is determined by

$$\omega = \omega_1 \times \cdots \times \omega_r, \tag{0.8}$$

and is incoherent in the sense of [**19**]. In particular, for $h \in Mp_{\mathbb{A}}(W)$, the corresponding Eisenstein series $E(h,s,\Phi)$ vanishes at the center of symmetry $s=0$. For any $(h_1,\dots,h_r) \in Mp_{1,\mathbb{R}} \times \dots \times Mp_{r,\mathbb{R}}$ we put

(0.9)

$$F_{d_1,\dots,d_r}(h_1,\dots,h_r,\Phi)_p^{\text{proper}} = \sum_{\substack{T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0} \\ T \text{ represented by } V(\mathbb{A}_f^p), \\ \text{but not by } V(\mathbb{Q}_p) \\ T \text{ represents } 1 \text{ over } \mathbb{Z}_p}} E'_T(i(h_1,\dots,h_r),0,\Phi)$$

On the right, we sum over certain Fourier coefficients of the derivative at 0 of the Eisenstein series for $Mp_{\mathbb{A}}$. Our second main result is the following identity (Corollary 9.4).

Theorem 0.2. *We have*

$$\begin{aligned} F_{d_1,\dots,d_r}(h_1,\dots,h_r,\Phi)_p^{\text{proper}} &= c\,W_{d_1}^{5/2}(h_1)\dots W_{d_r}^{5/2}(h_r)\cdot \log p \cdot \\ &\cdot \mathrm{vol}(\mathrm{pr}(K)) \cdot < \mathcal{Z}(d_1,\omega_1),\dots,\mathcal{Z}(d_r,\omega_r) >_p^{\text{proper}}, \end{aligned} \tag{0.10}$$

where $c = \frac{1}{2}\mathrm{vol}(SO(V'(\mathbb{R})))$.

Unexplained notation may be found in the body of the text. The identity is proved by unravelling both sides of (0.10), where, for the right side, we use the decomposition (0.4) and Theorem 0.1. The identity then reduces to the statement that, for $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ such that T is not represented by $V(\mathbb{Q}_p)$ and where T represents 1 over $\mathbb{Z}_p$, we have

$$\left[(\log p)^{-1} \cdot \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}\right] \left[\mathrm{vol}(K)^{-1} \cdot I_{T,f}(\varphi_f^{(p)})\right] = < \mathcal{Z}(T,\omega) >_p^{\text{proper}} \tag{0.11}$$

Here, in the first factor on the left, there appears a quotient of the *derivative* at 0 of a certain Whittaker function for the quadratic space $V(\mathbb{Q}_p)$ by the *value* at 0 of a Whittaker function for a twist $V'(\mathbb{Q}_p)$, and, in the second factor, a Fourier coefficient of a theta integral. It turns out that the first factor equals the multiplicity $e(\xi)$ of any point $\xi \in \mathcal{Z}(T,\omega)$ (which is constant), while the second factor is equal to the number of points in $\mathcal{Z}(T,\omega)$. For the multiplicity $e(\xi)$, the calculation can be reduced to a problem on one-dimensional formal groups of height 2 which has been solved by Gross and Keating [**7**]. For the calculation of the Whittaker functions we use the results of Kitaoka [**14**] on local representation densities. It should be pointed out that we are using here the length of the local ring $\mathcal{O}_{\mathcal{Z},\xi}$ as the multiplicity of a point ξ, whereas the sophisticated definition would also involve Tor-terms. It is a fundamental question whether these correction terms vanish. This question we have to leave open.

In summary, we may say that Theorem 0.2 is proved by explicitly computing both sides of (0.10) and comparing them. It would of course be highly desirable to find a more direct connection between the analytic side and the algebro-geometric side of this identity.

We now give an overview of the structure of this paper. In section 1, we introduce the Shimura variety and formulate the moduli problem solved by $\mathcal{M}$. Our special cycles are introduced in section 2. We define the fundamental matrix in section 3 and isolate there the part of $\mathcal{Z}$ lying purely in characteristic p. It is clear from the above description that to proceed further we need a thorough understanding of the supersingular locus of $\mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \operatorname{Spec} \mathbb{F}_p$. This is essentially due to Moret-Bailly [**23**] and Oort [**24**]. In section 4, we give a presentation of their results in terms of Dieudonné theory, better suited for our needs. A similar presentation was independently given by Kaiser [**11**] for a different purpose. The heart of the paper is section 5. In it we determine the space of special endomorphisms of certain Dieudonné modules and deduce the characterization of isolated intersection points (Theorems 5.11, 5.12 and 5.14). Here again the exceptional isomorphism plays a vital role. In section 6, we explain the reduction of the calculation of $e(\xi)$ to the result of Gross and Keating, and, in section 7, we explain how to count the number of isolated points. Section 8 is a review of the Fourier coefficients of Siegel Eisenstein Series. In section 9, we bring everything together and prove the identity (0.10) above. In section 10, we review some results of Kitaoka and show how they can be used to prove the formulas on Whittaker functions needed in section 9. Finally there is an appendix containing some facts on Clifford algebras in our special situation.

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§1. The Shimura variety.

In this section, we review the construction of the Siegel 3-folds associated to indefinite quaternion algebras over $\mathbb{Q}$, and the corresponding moduli problem. The use of the Clifford algebra is modeled on [**28**]. We refer to the appendix for some facts on those Clifford algebras that will be relevant for our purposes.

Let B be an indefinite quaternion algebra over $\mathbb{Q}$, let $C = M_2(B)$, with involution $x' = {}^t x^\iota$, and let

$$V = \{\ x \in C \ \mid\ x' = x \text{ and } \mathrm{tr}(x) = 0\ \}. \tag{1.1}$$

We define a quadratic form q on V by setting $x^2 = q(x) \cdot 1_2 \in M_2(B)$, cf. Appendix, A.3. Since B is indefinite, the signature of (V, q) is $(3, 2)$, cf. Appendix, A.6, and the Witt index of V over $\mathbb{Q}$ is 2 if $B = M_2(\mathbb{Q})$ and 1 if B is a division algebra, cf. Appendix, A.3. Let $C(V)$ be the Clifford algebra of the quadratic space (V, q). Since, for $x \in V \subset C$, $x^2 = q(x)$, there is a natural algebra homomorphism $C(V) \longrightarrow C$ extending the inclusion of V into C. The restriction of this map to the even Clifford algebra $C^+(V)$ induces an isomorphism

$$C^+(V) \simeq C. \tag{1.2}$$

Let

$$G = GSpin(V) = \{g \in C^\times \mid gg' = \nu(g)\ \}, \tag{1.3}$$

cf. Appendix, A.3, so that G is a twisted form over $\mathbb{Q}$ of GSp_4, cf. Appendix, A.2. The group G acts on $V \subset C$ by conjugation and this action yields an exact sequence

$$1 \longrightarrow Z \longrightarrow G \longrightarrow SO(V) \longrightarrow 1, \tag{1.4}$$

where Z is the center of G.

Let $\mathcal{D}$ be the space of oriented negative 2-planes in $V(\mathbb{R})$. This space has two connected components and the group $G(\mathbb{R})$ acts transitively on it, via its action on $V(\mathbb{R})$. For an oriented 2-plane $z \in \mathcal{D}$, let z_1, $z_2 \in z$ be a properly oriented basis such that the restriction of the quadratic form q from $V(\mathbb{R})$ to z has matrix -1_2 for the basis z_1, z_2. Let $j_z = z_1 z_2 \in C(\mathbb{R})$. Viewing j_z as the image of the element $z_1 z_2 \in C(V(\mathbb{R}))$, the Clifford algebra of $V(\mathbb{R})$, and recalling the commutative diagram of section A.3 of the Appendix, we see that $j_z' = -j_z$ and that $j_z^2 = -z_1^2 z_2^2 = -1$. Hence, $j_z j_z' = 1$ and so, $j_z \in G(\mathbb{R})$. There is an isomorphism of algebras over $\mathbb{R}$,

$$\mathbb{C} \xrightarrow{\sim} C^+(z) \qquad i \mapsto z_1 z_2, \tag{1.5}$$

where $C^+(z)$ is the even Clifford algebra of the real 2-plane z. The composition of this map with the map

$$C^+(z) \subset C^+(V(\mathbb{R})) \xrightarrow{\sim} C(\mathbb{R}) = M_2(B(\mathbb{R})) \tag{1.6}$$

induces a morphism, defined over $\mathbb{R}$, $h_z : \mathbb{S} \longrightarrow G$, where $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, as usual. Note that $h_z(i) = j_z$. The space $\mathcal{D}$ can thus be viewed as the space of conjugacy classes of such maps under the action of the group $G(\mathbb{R})$. The data $(G, \mathcal{D})$ or (G, h_z) defines a Shimura variety $Sh(G, \mathcal{D})$, [**2**], [**3**], whose canonical model is defined over $\mathbb{Q}$. Note that $\mathcal{D}$ is isomorphic to two copies of the Siegel space of genus 2, and, if $B = M_2(\mathbb{Q})$, $Sh(G, \mathcal{D})$ is just the Siegel modular variety of genus 2.

Since G satisfies the Hasse principle, the Shimura variety represents a certain moduli problem over $(Sch/\mathbb{Q})$, [**17**]. To define this we must introduce more notation.

Fix a maximal order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\iota = \mathcal{O}_B$, and let $\mathcal{O}_C = M_2(\mathcal{O}_B)$. Let $D(B)$ be the product of the primes p at which B_p is division, and, as in [**1**], choose $\tau \in B^\times$ such that $\tau^\iota = -\tau$, $\tau^2 = -D(B)$, and $\tau\mathcal{O}_B\tau^{-1} = \mathcal{O}_B$. By section A.7 of the Appendix, the map $x \mapsto x^* = \tau x^\iota \tau^{-1}$ is a positive involution of B preserving $\mathcal{O}_B$. Also, for

$$\alpha = \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \in M_2(B), \tag{1.7}$$

$\alpha' = -\alpha$ and $x^* = \alpha x' \alpha^{-1} = \alpha^{-1} x' \alpha$ is a positive involution of C, preserving $\mathcal{O}_C$.

Let $U = \mathcal{O}_C$, viewed as a module for $\mathcal{O}_C$ under both left and right multiplication. Define an alternating form:

$$< \, , \, > : U \times U \longrightarrow \mathbb{Z} \tag{1.8}$$

by

$$< x, y > = \mathrm{tr}(y' \alpha^{-1} x). \tag{1.9}$$

Then

$$< cx, y > = \mathrm{tr}(y'\alpha^{-1}cx) = \mathrm{tr}(y'\alpha^{-1}c\alpha\alpha^{-1}x) = < x, c^* y >, \tag{1.10}$$

and

$$< xc, y > = \mathrm{tr}(y'\alpha^{-1}xc) = \mathrm{tr}(cy'\alpha^{-1}x) = < x, yc' > . \tag{1.11}$$

Thus, if $g \in G$,

$$< xg, yg >= \nu(g) < x, y >, \tag{1.12}$$

and, in particular, for $z \in \mathcal{D}$,

$$< xj_z, yj_z > = < x, y > . \tag{1.13}$$

We fix a compact open subgroup $K \subset G(\mathbb{A}_f)$. The functor M_K associates to $S \in (\mathrm{Sch}/\mathbb{Q})$ the set of quadruples, $(A, \iota, \lambda, \bar{\eta})$, up to isomorphism, where

(i) A is an abelian scheme over S, up to isogeny,

(ii) $\iota : C \longrightarrow End^0(A)$ is a homomorphism such that

$$\det(\iota(c); \mathrm{Lie}(A)) = N^o(c)^2,$$

where $N^o(c)$ is the reduced norm on C.

(iii) λ is a $\mathbb{Q}$-class of polarizations on A which induce the involution $*$ on C:

$$\lambda \circ \widehat{\iota(c)} \circ \lambda^{-1} = \iota(c^*).$$

(iv) $\bar{\eta}$ is a K-class of isomorphisms

$$\eta : \hat{V}(A) \xrightarrow{\sim} U \otimes \mathbb{A}_f$$

which are C-linear (for the left module structure on U) and respect the symplectic forms on both sides up to a constant in $\mathbb{A}_f^\times$. Here

$$\hat{V}(A) = \prod_\ell T_\ell(A) \otimes \mathbb{Q}.$$

For the precise meaning of the datum (iv) we refer to [**17**], p. 390. In particular, if $S = \operatorname{Spec} k$ is the spectrum of a field, the K-class $\overline{\eta}$ is supposed to be stable under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ where $\overline{k}$ is the algebraic closure used to form the Tate module of A.

Note that the abelian scheme A will have relative dimension 8 over S.

Proposition 1.1. *For K neat this moduli problem is representable by a smooth quasi-projective scheme M_K over $\mathbb{Q}$ and*

$$M_K(\mathbb{C}) \simeq Sh(G, \mathcal{D})(\mathbb{C}).$$

Proof. For the representability, see [**17**]. We prove the last assertion in detail, since the conventions involved will be used later.

For $\tau \in B^{\times}$, as above, let

$$\tau_0 = D(B)^{-\frac{1}{2}}\tau \in B^{\times}(\mathbb{R}), \tag{1.14}$$

so that $\tau_0^2 = -1$. Choose $\beta \in B^{\times}$ such that

$$\beta\tau = -\tau\beta, \qquad \text{and} \qquad \beta^{\iota} = -\beta. \tag{1.15}$$

Since B is indefinite, $\beta^2 > 0$, and we can set

$$\beta_0 = (\beta^2)^{-\frac{1}{2}}\beta \in B^{\times}(\mathbb{R}), \tag{1.16}$$

so that $\beta_0^2 = 1$. The vectors

$$\begin{pmatrix} & \beta_0 \\ -\beta_0 & \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} & \tau_0\beta_0 \\ -\tau_0\beta_0 & \end{pmatrix} \in V(\mathbb{R}) \tag{1.17}$$

form a standard basis of an oriented negative 2-plane $z_0 \in \mathcal{D}$, and

$$j_{z_0} = \begin{pmatrix} & \beta_0 \\ -\beta_0 & \end{pmatrix}\begin{pmatrix} & \tau_0\beta_0 \\ -\tau_0\beta_0 & \end{pmatrix} = \begin{pmatrix} \tau_0 & \\ & \tau_0 \end{pmatrix} = D(B)^{-\frac{1}{2}}\alpha =: \alpha_0. \tag{1.18}$$

Lemma 1.2. *For any $z \in \mathcal{D}$,*

$$< xj_z, y > = < yj_z, x >,$$

and, for $x \in U(\mathbb{R})$, $x \neq 0$,

$$< xj_z, x > > 0,$$

if z lies in the same connected component of $\mathcal{D}$ as z_0, and

$$< xj_z, x > < 0,$$

if z and z_0 lie in opposite components.

Proof. For the first assertion:

$$< xj_z, y > = \; - < x, yj_z > = < yj_z, x > . \tag{1.19}$$

For the second, write $z = gz_0$ for $g \in G(\mathbb{R})$, so that

$$j_z = gj_{z_0}g^{-1} = g\alpha_0 g^{-1}. \tag{1.20}$$

Then, we have

$$\begin{aligned} < xj_z, x > &= < xg\alpha_0 g^{-1}, x > \\ &= \nu(g)^{-1} < xg\alpha_0, xg > \\ &= \nu(g)^{-1}\mathrm{tr}((xg)'\alpha^{-1}xg\alpha_0) \\ &= \nu(g)^{-1}D(B)^{-\frac{1}{2}}\,\mathrm{tr}(\alpha(xg)'\alpha^{-1}(xg)) \\ &= \nu(g)^{-1}D(B)^{-\frac{1}{2}}\,\mathrm{tr}((xg)^*(xg)). \end{aligned} \tag{1.21}$$

Since $x \mapsto x^*$ is a positive involution, this gives the claim. $\square$

Let $\mathcal{D}^+$ be the connected component of $\mathcal{D}$ containing z_0 and $\mathcal{D}^-$ the connected component of $\mathcal{D}$ not containing z_0. Then, for any $z \in \mathcal{D}^\pm$, we obtain a (principally) polarized abelian variety over $\mathbb{C}$,

$$A_z = (U(\mathbb{R}), j_z, U(\mathbb{Z}), \pm < \, , \, >) \tag{1.22}$$

with $\dim A_z = 8$ and with an action

$$\iota : \mathcal{O}_C \hookrightarrow \mathrm{End}(A_z). \tag{1.23}$$

Note that

$$\hat{V}(A_z) = U(\hat{\mathbb{Z}}) \otimes \mathbb{Q} = U(\mathbb{A}_f). \tag{1.24}$$

If

$$\gamma \in \Gamma = \{ \, g \in G(\mathbb{Q})^+ \mid U(\mathbb{Z})g = U(\mathbb{Z}) \, \}, \tag{1.25}$$

then right multiplication by γ^{-1} induces an isomorphism

$$A_z \xrightarrow{\sim} A_{\gamma z}. \tag{1.26}$$

Thus $\Gamma \backslash \mathcal{D}^+$ parametrizes such principally polarized abelian varieties, up to isomorphism.

More generally, to $(z, g) \in \mathcal{D} \times G(\mathbb{A}_f)$, we associate the collection $(A, \iota, \lambda, \bar{\eta})$ defined by:

- $(A, \iota) = (A_z, \iota)$, where A_z is taken up to isogeny.
- λ is the $\mathbb{Q}$-class of polarizations determined by $< \, , \, >$.
- $\bar{\eta}$ is the K-class containing the isomorphism:

$$\hat{V}(A_z) = U(\mathbb{A}_f) \ \overset{r(g)}{\xrightarrow{\sim}} \ U(\mathbb{A}_f).$$

Note that, if $\gamma \in G(\mathbb{Q})$ and $k \in K$, then $(\gamma z, \gamma g k)$ defines a collection isomorphic to that defined by (z, g), via the element of $\mathrm{Hom}^0(A_z, A_{\gamma z})$ given on $U(\mathbb{R})$ by right multiplication by γ^{-1}. The map

$$G(\mathbb{Q})(z, g)K \mapsto (A, \iota, \lambda, \bar{\eta}) / \sim \tag{1.27}$$

yields the isomorphism

$$G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f) / K \xrightarrow{\sim} M_K(\mathbb{C}). \qquad \square \tag{1.28}$$

We now turn to the construction of a p-integral model. Fix a prime p such that $p \nmid D(B)$, so that $C \otimes \mathbb{Q}_p \simeq M_4(\mathbb{Q}_p)$. Let $\mathcal{O}_C$ be the maximal order chosen above, and note that the maximal order $\mathcal{O}_C \otimes \mathbb{Z}_p$ in $C \otimes \mathbb{Q}_p$ is the stabilizer of the lattice $U_{\mathbb{Z}_p} = U \otimes \mathbb{Z}_p$ in $U \otimes \mathbb{Q}_p$ under both right and left multiplication. The choice of τ made before (1.7) ensures that $< \, , \, >$ defines a perfect pairing

$$< \, , \, >: U_{\mathbb{Z}_p} \times U_{\mathbb{Z}_p} \longrightarrow \mathbb{Z}_p. \tag{1.29}$$

Let K_p be the stabilizer of $U_{\mathbb{Z}_p}$ in $G(\mathbb{Q}_p)$, acting on $U_{\mathbb{Q}_p}$ via right multiplication. Let $K^p \subset G(\mathbb{A}_f^p)$ be compact open, and take $K = K_p \cdot K^p$.

We now want to formulate a moduli problem over $(\mathrm{Sch}/\mathbb{Z}_{(p)})$ which extends the previous one. The functor $\mathcal{M}_{K^p}$ associates to $S \in (\mathrm{Sch}/\mathbb{Z}_{(p)})$ the set of isomorphism classes of quadruples $(A, \iota, \lambda, \bar{\eta}^p)$ where

(i) A is an abelian scheme over S, up to prime to p isogeny

(ii) $\iota : \mathcal{O}_C \otimes \mathbb{Z}_{(p)} \longrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a homomorphism such that, for $c \in \mathcal{O}_C$,

$$\det(\iota(c); \mathrm{Lie}(A)) = N^o(c)^2,$$

where N^o is the reduced norm on C.

(iii) λ is a $\mathbb{Z}_{(p)}^\times$-class of isomorphisms $A \longrightarrow \hat{A}$ such that $n\lambda$, for a suitable natural number n, is induced by an ample line bundle on A.

(iv) $\bar{\eta}^p$ is a K^p-class of $\mathcal{O}_C$-linear isomorphisms (in the sense of Kottwitz)

$$\eta^p : \hat{V}^p(A) \xrightarrow{\sim} U \otimes \mathbb{A}_f^p,$$

which respects the symplectic form on both sides up to a constant in $(\mathbb{A}_f^p)^\times$. Here

$$\hat{V}^p(A) = \prod_{\ell \neq p} T_\ell(A) \otimes \mathbb{Q}.$$

In the determinant condition above, the equality is meant as an identity of polynomial functions. In the case at hand, it simply says $\dim A = 8$.

Proposition 1.3. *For K^p neat the above moduli problem is representable by a smooth quasiprojective scheme $\mathcal{M}_{K^p}$ over* Spec $\mathbb{Z}_{(p)}$. *Its generic fibre can be canonically identified with M_K,*

$$\mathcal{M}_{K^p} \times_{\mathrm{Spec}\ \mathbb{Z}_{(p)}} \mathrm{Spec}\ \mathbb{Q} = M_K \quad .$$

Let us briefly explain the last identification on geometric points. Let S be the spectrum of an algebraically closed field of characteristic 0. Let us consider

$(A, \iota, \lambda, \bar{\eta}^p) \in \mathcal{M}_{K^p}(S)$. Then the p-adic Tate module $T_p(A)$ is equipped with a perfect symplectic form, unique up to scaling by $\mathbb{Z}_p^\times$ and hence there is an $\mathcal{O}_C \otimes \mathbb{Z}_p$-linear isomorphism

$$\eta_p : T_p(A) \xrightarrow{\sim} U_{\mathbb{Z}_p},$$

which respects the symplectic forms up to $\mathbb{Z}_p^\times$. The set of such η_p's form a single orbit for K_p, which acts via right multiplication in $U_{\mathbb{Z}_p}$. Hence, from $(A, \iota, \lambda, \bar{\eta}^p)$, we obtain an object $(A \otimes \mathbb{Q}, \iota \otimes \mathbb{Q}, \lambda \otimes \mathbb{Q}, \bar{\eta}^p \cdot \bar{\eta}_p)$ of $M_K(S)$. Passage in the other direction is similar. For example, in the isogeny class A and for $\eta \in \bar{\eta}$, there is an abelian variety B, unique up to prime to p isogeny, such that $\eta_p(T_p(B)) = U_{\mathbb{Z}_p}$.

The above proposition tells us that, when $K = K^p \cdot K_p$, as above, then $\mathcal{M}_{K^p}$ provides us with a smooth model of $Sh(G, D)_K$ over $\mathbb{Z}_{(p)}$. From now on, we will use the same notation for both moduli problems, if this does not cause confusion.

§2. Special cycles.

In this section we give a modular definition of the special cycles in $Sh(G,\mathcal{D})$, which were defined analytically in [**18**]. We then explain the relation between the two definitions.

Recall that the quadratic form on the space $V \subset C = M_2(B)$ was defined by $x^2 = q(x)\cdot 1_2$. Let

$$(x,y) = q(x+y) - q(x) - q(y) \tag{2.1}$$

be the corresponding bilinear form, so that $q(x) = \frac{1}{2}(x,x)$. If $x = (x_1, x_2, \dots, x_n) \in V^n(\mathbb{Q})$, we let

$$q(x) = \frac{1}{2}((x_i,x_j))_{i,j} \in \mathrm{Sym}_n(\mathbb{Q}). \tag{2.2}$$

This defines a quadratic map $q : V^n \longrightarrow \mathrm{Sym}_n$.

Fix a positive integer n. For $d \in \mathrm{Sym}_n(\mathbb{Q})$ a symmetric rational matrix, let

$$\Omega_d = \{\ x \in V^n \ \mid \ q(x) = d\ \} \tag{2.3}$$

be the corresponding hyperboloid. The group G acts diagonally on V^n and preserves Ω_d.

Cycles in $Sh(G,\mathcal{D})$ were defined analytically in [**18**] as follows. For $x \in \Omega_d(\mathbb{Q})$, let $< x > \subset V$ be the $\mathbb{Q}$-subspace spanned by the components of x, and let $V_x = < x >^{\perp}$ be its orthogonal complement. Let $\mathcal{D}_x$ denote the space of oriented negative 2-planes in $V_x(\mathbb{R})$, and let G_x be the pointwise stabilizer of $< x >$ in G. Note that $G_x \simeq GSpin(V_x)$, and that $\mathcal{D}_x \subset \mathcal{D}$. Moreover, for $z \in \mathcal{D}_x$, the homomorphism h_z factors through $G_x(\mathbb{R})$. Thus there is a natural morphism of Shimura varieties, rational over $\mathbb{Q}$,

$$Sh(G_x,\mathcal{D}_x) \longrightarrow Sh(G,\mathcal{D}). \tag{2.4}$$

If the space $< x >$ is not positive-definite, then $\mathcal{D}_x = \emptyset$. If $< x >$ is positive-definite of dimension r then d is positive semi-definite of rank r, $\mathrm{sig}(V_x) = (3-r, 2)$ and $\mathcal{D}_x$ has codimension r in $\mathcal{D}$. Hence the previous construction is only interesting when d is positive semi-definite and even only when d is positive definite with $n \le 3$.

For a fixed compact open subgroup $K \subset G(\mathbb{A}_f)$ and for $h \in G(\mathbb{A}_f)$, there is a cycle, namely the image of the map

(2.5)

$$Z(x,h;K) : G_x(\mathbb{Q})\backslash\mathcal{D}_x \times G_x(\mathbb{A}_f)/(G_x(\mathbb{A}_f)\cap hKh^{-1}) \longrightarrow G(\mathbb{Q})\backslash\mathcal{D}\times G(\mathbb{A}_f)/K$$

given by $(z,g)\mapsto(z,gh)$. This map is finite and generically injective, hence the cycle image is taken with multiplicity 1. This cycle of codimension $r=\mathrm{rk}(d)$ on $Sh(G,\mathcal{D})_K$ is rational over $\mathbb{Q}$.

Assume that $\Omega_d(\mathbb{Q})\neq\phi$ and fix $x_0\in\Omega_d(\mathbb{Q})$. Let $\varphi\in S(V(\mathbb{A}_f)^n)^K$ be a Schwartz function which is K-invariant, and write

$$\mathrm{supp}(\varphi)\cap\Omega_d(\mathbb{A}_f)=\coprod_r Kh_r^{-1}x_0 \tag{2.6}$$

for elements $h_r\in G(\mathbb{A}_f)$. Then define the **weighted cycle**:

$$Z(d,\varphi;K)=\sum_r\varphi(h_r^{-1}x_0)\cdot Z(x_0,h_r;K). \tag{2.7}$$

This cycle is independent of the choice of x_0 and of the orbit representatives h_r. It is a (weighted linear combination of) cycle(s) of codimension $r=\mathrm{rk}(d)$ on $Sh(G,\mathcal{D})_K$ and is rational over $\mathbb{Q}$.

If φ is the characteristic function of a K-invariant compact open subset ω of $V(\mathbb{A}_f)^n$, then $Z(d,\omega;K)=Z(d,\varphi;K)$ can be considered as a disjoint union of maps (2.5), or as the union of the images of these maps.

We introduce the following definition, which will play a key role throughout the paper.

Definition 2.1. *Let* $(A,\iota,\lambda,\overline{\eta})\in M_K(S)$ *. A* **special endomorphism of** $(A,\iota,\lambda,\overline{\eta})$ *is an element* $j\in\mathrm{End}^0_S(A,\iota)$ *which satisfies*

$$j^*=j \quad \textit{and} \quad \mathrm{tr}^0(j)=0 \quad . \tag{2.8}$$

Here $*$ denotes the Rosati involution of λ. Also note that $\mathrm{End}^0(A,\iota)$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra, so that the reduced trace appearing here makes sense. Indeed, this is well-known when S is the spectrum of a field. The case when S is irreducible follows by reduction to its generic point, and the general case follows by considering the irreducible components of S.

Lemma 2.2. *Let* j *be a special endomorphism of* $(A,\iota,\lambda,\overline{\eta})\in M_K(S)$ *, where* S *is connected. Then*

$$j^2=q(j)\cdot\mathrm{id} \quad , \tag{2.9}$$

with $q(j)\in\mathbb{Q}$ *.*

Proof. Again we may reduce first to the case where S is irreducible and then to the case when S is the spectrum of a field. However for $\eta\in\overline{\eta}$ let $x=\eta^*(j)\in$

$\mathrm{End}_C(U(\mathbb{A}_f)) = C(\mathbb{A}_f)$. Under the last identification the adjoint involution $*$ w.r.t. $<\ ,\ >$ corresponds to the involution $'$ on $C(\mathbb{A})$, cf. (1.11). Hence x lies in $V(\mathbb{A}_f)$ and the assertion follows, cf. appendix A.3. $\square$

The previous Lemma justifies the following definitions. Let S be a connected scheme and $\xi = (A, \iota, \lambda, \overline{\eta}) \in M_K(S)$ be an S-valued point of M_K. Let

$$C_\xi^0 = \mathrm{End}_S^0(A, \iota)^{\mathrm{op}} \tag{2.10}$$

and

$$V_\xi^0 = \{x \in C_\xi^0 \mid x^* = x \text{ and } \mathrm{tr}^0(x) = 0\} \quad . \tag{2.11}$$

Then V_ξ^0 is the finite-dimensional $\mathbb{Q}$-vector space of special endomorphisms with quadratic form

$$q_\xi : V_\xi^0 \longrightarrow \mathbb{Q} \tag{2.12}$$

given by $x^2 = q_\xi(x) \cdot \mathrm{id}_A$. By the universal property of the Clifford algebra of (V_ξ^0, q_ξ) there is a natural homomorphism

$$C(V_\xi^0, q_\xi) \longrightarrow C_\xi^0 \quad . \tag{2.13}$$

This structure is compatible with specialization. If $S' \subset S$ is a connected closed subscheme, let $\xi' \in M_K(S')$ be the restriction of ξ. Then we have a homomorphism of $\mathbb{Q}$-algebras

$$C_\xi^0 = \mathrm{End}_S^0(A, \iota)^{\mathrm{op}} \hookrightarrow \mathrm{End}_{S'}(A, \iota)^{\mathrm{op}} = C_{\xi'}^0 \tag{2.14}$$

inducing a map

$$V_\xi^0 \hookrightarrow V_{\xi'}^0$$

of quadratic spaces.

Let us spell out these concepts in the classical case.

Lemma 2.3. *Let $\xi \in M_K(\mathbb{C})$ with parameter (z, g) in $Sh(G, \mathcal{D})_K$. Let $A_z^{\mathrm{top}} = U(\mathbb{R})/U(\mathbb{Z})$ be the real torus underlying A_z.*

(i)

$$C(\mathbb{Q}) \xrightarrow{\sim} \mathrm{End}^0(A_z^{\mathrm{top}}, \iota)^{\mathrm{op}}, \qquad y \mapsto r(y),$$

where $r(y)$ denotes the action of $y \in C(\mathbb{Q})$ on $U(\mathbb{R}) \supset U(\mathbb{Q})$ by right multiplication. Moreover, $r(y)^* = r(y')$.

(ii)

$$C_\xi^0 \simeq Cent_{C(\mathbb{Q})}(j_z), \qquad \text{and} \qquad V_\xi^0 \simeq \{\ x \in V(\mathbb{Q}) \mid x j_z = j_z x\ \}.$$

(iii)

$$Cent_{C(\mathbb{R})}(j_z) \cap V(\mathbb{R}) = z^{\perp}.$$

In particular,

$$V_\xi^0 = V(\mathbb{Q}) \cap z^{\perp},$$

and so $0 \leq \dim_{\mathbb{Q}} V_\xi^0 \leq 3$.

Proof. The first two assertions are obvious by (1.11). To prove the last assertion let $z_1, z_2 \in z$ be a properly oriented basis such that the restriction of the quadratic form q to z has matrix -1_2 in terms of this basis. Let $v \in V(\mathbb{R})$ with $(v, z_i) = a_i$, $i = 1, 2$. Then

$$v \cdot j_z = v \cdot (z_1 \cdot z_2) = z_1 z_2 v - a_2 z_1 + a_1 z_2 = j_z v - a_2 z_1 + a_1 z_2.$$

Hence $v \in \mathrm{Cent}_{C(\mathbb{R})}(j_z)$ iff $a_1 = a_2 = 0$, i.e. iff $v \in z^{\perp}$. □

Let us return to the abstract situation.

Lemma 2.4. *Let* $\xi = (A, \iota, \lambda, \overline{\eta}) \in M_K(S)$ *be a point with values in a connected scheme* S *. The quadratic space* V_ξ^0 *is positive-definite.*

Proof. We may assume that S is the spectrum of a field. The assertion follows from the positivity of the Rosati involution, since

$$q(x) \cdot \mathrm{id}_A = x^2 = x \cdot x^* \quad , \quad x \in V_\xi^0 \quad . \quad \square$$

We next give a modular definition of the cycles introduced above. We take here the point of view that a cycle is given by a finite unramified morphism into the ambient scheme. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup, and let $\omega \subset V(\mathbb{A}_f)^n$ be a K-invariant compact open subset. Consider the functor on (Sch/$\mathbb{Q}$) which associates to a scheme S the set of isomorphism classes of 5-tuples $(A, \iota, \lambda, \overline{\eta}; \mathbf{j})$ where $(A, \iota, \lambda, \overline{\eta}) \in M_K(S)$. Here the additional element $\mathbf{j} = (j_1, \ldots, j_n) \in \mathrm{End}^0(A, \iota)^n$ is an n-tuple of special endomorphisms of A , satisfying the following conditions.

(2.15) For some (and hence for all) $\eta \in \bar{\eta}$, the element $\eta^*(\mathbf{j}) \in \mathrm{End}_C(U(\mathbb{A}_f))^n$ lies in ω .

(2.16) $q(\mathbf{j}) = d$.

Let us explain the condition (2.15). As in the proof of Lemma 2.2 above, for any $\eta \in \overline{\eta}$

$$x = \eta^*(\mathbf{j}) \in V(\mathbb{A}_f)^n \subset C(\mathbb{A}_f)^n = \mathrm{End}_C(U(\mathbb{A}_f))^n \quad .$$

The condition imposes that $x \in \omega$. If η is changed to $r(k) \circ \eta$, with $k \in K$ and $r(k) \in \mathrm{End}_C(U(\mathbb{A}_f))$ the endomorphism defined by right multiplication by k, then

$$(r(k) \circ \eta)^*(\mathbf{j}) = r(k) \circ \eta^*(\mathbf{j}) \circ r(k)^{-1}. \tag{2.17}$$

The condition (2.15) asserts that $\eta^*(\mathbf{j}) = r(x)$ for some $x \in \omega$. If this is the case, then

$$(r(k) \circ \eta)^*(\mathbf{j}) = r(k) \circ \eta^*(\mathbf{j}) \circ r(k)^{-1} = r(k^{-1}xk), \tag{2.18}$$

and $k^{-1}xk \in \omega$. Thus the condition (2.15) depends only on $\bar{\eta}$.
To interpret condition (2.16) we may assume S to be connected. Let $(\ ,\ )$ be the bilinear form on the space of special endomorphisms of $(A, \iota, \lambda, \overline{\eta})$ associated to the quadratic form q of lemma 2.2. Then $q(\mathbf{j}) = \frac{1}{2}((j_i, j_j))_{i,j} \in \mathrm{Sym}_n(\mathbb{Q})$ is defined as in (2.2). The condition (2.16) requires that $q(\mathbf{j}) = d$.

Proposition 2.5. *The above functor has a coarse moduli scheme $\mathcal{Z}(d, \omega)$. If K is neat, then $\mathcal{Z}(d, \omega)$ is a fine moduli scheme and the forgetful morphism*

$$\mathcal{Z}(d, \omega) \longrightarrow M_K \tag{2.19}$$

is finite and unramified. Furthermore $\mathcal{Z}(d, \omega)(\mathbb{C}) = Z(d, \omega, K)$.

Proof. The first statement follows easily from the second. Let us assume that K is neat. The relative representability of the forgetful morphism by a morphism of finite type follows in a standard way from Grothendieck's theory of Hilbert schemes since M_K may be considered as a moduli scheme of polarized abelian varieties with additional structure. By the Néron universal property the valuative criterion for properness is satisfied. Since the matrix d gives the squares j_i^2 of the special endomorphisms, the morphism is quasi-finite and hence finite. The unramifiedness follows from the rigidity theorem for abelian varieties.
The last statement is to be interpreted as an equality between the image of (2.5) and $\mathcal{Z}(d, \omega)(\mathbb{C})$, and follows easily from Lemma 2.3 above.

We now assume that $p \nmid 2D(B)$ and that $K = K^p \cdot K_p$ with K^p neat, as in Proposition 1.3, and we formulate a p-integral version of the previous moduli problem.

Before doing this let us point out that for a point $\xi = (A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}_{K^p}(S)$ of the p-integral version of our moduli problem with values in a connected scheme S we may transpose the concepts above. Hence we introduce the $\mathbb{Z}_{(p)}$-algebra

$$C_\xi = \mathrm{End}_S(A, \iota)^{\mathrm{op}} \otimes \mathbb{Z}_{(p)} \tag{2.20}$$

and

$$V_\xi = \{x \in C_\xi;\ x^* = x \text{ and } \mathrm{tr}^0(x) = 0\} \quad . \tag{2.21}$$

The latter is a $\mathbb{Z}_{(p)}$-module with a $\mathbb{Z}_{(p)}$-valued positive definite quadratic form. The elements of V_ξ will again be called the special endomorphisms of $(A, \iota, \lambda, \overline{\eta}^p)$.

Let now again $d \in \mathrm{Sym}_n(\mathbb{Q})$. Let $\omega^p \subset V(\mathbb{A}_f^p)^n$ be a K^p-invariant open compact subset. Then a point of the corresponding moduli problem $\mathcal{Z}(d, \omega^p)$ on a $\mathbb{Z}_{(p)}$-scheme S is an isomorphism class of 5-tuples $(A, \iota, \lambda, \overline{\eta}^p; \mathbf{j})$ where $(A, \iota, \lambda, \overline{\eta}^p)$ is an object of $\mathcal{M}_{K^p}(S)$ and where $\mathbf{j} \in \big(\mathrm{End}(A, \iota) \otimes \mathbb{Z}_{(p)}\big)^n$ is an n-tuple of special endomorphisms which satisfies (2.16) above and, in addition,

$$(\eta^p)^*(\mathbf{j}) \in \omega^p. \tag{2.22}$$

These conditions are to be interpreted in the same way as (2.15)-(2.16) above.

To clarify the relation between the p-integral version $\mathcal{Z}(d, \omega^p)$ and the previous $\mathcal{Z}(d, \omega)$, let

$$\omega_p = V(\mathbb{Z}_p)^n \tag{2.23}$$

where $V(\mathbb{Z}_p) = V(\mathbb{Q}_p) \cap (\mathcal{O}_C \otimes \mathbb{Z}_p)$, the intersection taking place inside of $C \otimes \mathbb{Q}_p$. Let

$$\omega = \omega_p \times \omega^p \quad , \tag{2.24}$$

a K-invariant open compact subset of $V(\mathbb{A}_f)^n$.

Proposition 2.6. *If K^p is neat, the functor $\mathcal{Z}(d, \omega^p)$ is representable by a scheme which maps by a finite unramified morphism to $\mathcal{M}_{K^p}$. Furthermore, there is an identification*

$$\mathcal{Z}(d, \omega^p) \times_{\mathrm{Spec}\ \mathbb{Z}_{(p)}} \mathrm{Spec}\, \mathbb{Q} = \mathcal{Z}(d, \omega) \quad .$$

Remark 2.7: By Lemma 2.4 the scheme $\mathcal{Z}(d, \omega^p)$ is empty unless d is positive semi-definite. Similarly $\mathcal{Z}(d, \omega^p) = \emptyset$, unless $d \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})$. Note that it may well happen that $\mathcal{Z}(d, \omega^p)$ is non-empty but where both sides of the equality in Proposition 2.6 are empty. In fact we will later consider cases in which $d \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})$ is positive definite so that $\mathcal{Z}(d, \omega) = \emptyset$ and when $\mathcal{Z}(d, \omega^p) \neq \emptyset$.

¿From now on, since we will be interested in the arithmetic situation, we will simplify our notation by denoting ω what is denoted by ω^p above, i.e.,

$$\omega \subset V(\mathbb{A}_f^p)^n \tag{2.25}$$

is a K^p-invariant open compact subset.

§3. The intersection problem.

We continue to fix $p \nmid 2D(B)$ and a neat open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$ as at the end of section 2. Then $\mathcal{M} = \mathcal{M}_{K^p}$ is a regular noetherian scheme of dimension 4. We wish to consider the intersection of the cycles introduced in a modular way in the previous section. Let us set up our problem in a more precise way.

We fix integers $n_1, \dots, n_r$ with $1 \leq n_i \leq 4$ and with $n_1 + \cdots + n_r = 4$. For each i, we choose $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})$ positive definite, and a K^p-invariant open compact subset $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$. Let

$$\mathcal{Z} = \mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r) \tag{3.1}$$

be the fiber product of the corresponding special cycles.

By what has been said in section 2, since the codimensions of the generic fibres of our special cycles add up to the arithmetic dimension of $\mathcal{M}_{K^p}$, one might expect that $\mathcal{Z}$ consists of finitely many points of characteristic p. We will see that this is in fact quite false, but we will be able to determine that part of $\mathcal{Z}$ which lies purely in characteristic p and also determine the isolated points of $\mathcal{Z}$.

Let ξ be a point of $\mathcal{Z}$, with corresponding point $(A_\xi, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}$. We denote by C_ξ and (V_ξ, q_ξ) the $\mathbb{Z}_{(p)}$-algebra and the quadratic $\mathbb{Z}_{(p)}$-module associated to $(A_\xi, \iota, \lambda, \overline{\eta}^p)$, cf. (2.20). The projections $\mathcal{Z} \to \mathcal{Z}(d_i, \omega_i)$ define n_i-tuples of special endomorphisms

$$\mathbf{j}_i \in V_\xi^{n_i} \quad , \quad i = 1, \dots, r \quad . \tag{3.2}$$

Let

$$T_\xi = \frac{1}{2} \begin{pmatrix} (\mathbf{j}_1, \mathbf{j}_1)_\xi & \dots & (\mathbf{j}_1, \mathbf{j}_r)_\xi \\ \vdots & & \vdots \\ (\mathbf{j}_r, \mathbf{j}_1)_\xi & \dots & (\mathbf{j}_r, \mathbf{j}_r)_\xi \end{pmatrix} \in Sym_4(\mathbb{Z}_{(p)}), \tag{3.3}$$

where $(\ ,\ )_\xi$ is the bilinear form associated to q_ξ. Here, as always, $p \neq 2$. The matrix T_ξ is called the **fundamental matrix associated to the intersection point** ξ of the special cycles $\mathcal{Z}(d_1, \omega_1), \dots, \mathcal{Z}(d_r, \omega_r)$. We note that the blocks on the diagonal of T_ξ are $d_1, \dots, d_r$. By the results of section 2, the function $\xi \mapsto T_\xi$ is constant on each connected component of $\mathcal{Z}$. Therefore, for $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})$ we may introduce

$$\begin{aligned} \mathcal{Z}_T = (\mathcal{Z}(d_1, \omega_1) \cap \cdots \cap \mathcal{Z}(d_r, \omega_r))_T = \\ \text{union of the connected components of } \mathcal{Z} \\ \text{consisting of the points } \xi \text{ with } T_\xi = T \end{aligned} \tag{3.4}$$

We note here the hereditary nature of our construction, given by

$$\mathcal{Z}(T, \omega_1 \times \cdots \times \omega_r) = (\mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r))_T \quad , \tag{3.5}$$

valid provided that the blocks on the diagonal of T are $d_1, \ldots, d_r$. We may therefore write

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r) \\ &= \coprod_T \mathcal{Z}_T \\ &= \coprod_{\substack{T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{\geq 0} \\ \mathrm{diag}(T) = (d_1, \ldots, d_r)}} \mathcal{Z}(T, \omega) \quad . \end{aligned} \tag{3.6}$$

Here $\omega = \omega_1 \times \ldots \times \omega_r$.

We shall see that the fundamental matrix governs the intersection behaviour of our special cycles. We first note the following result.

Proposition 3.1. *Let $\xi \in \mathcal{Z} = \mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r)$ where $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$ and $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})$ positive-definite with $n_1 + \ldots + n_r = 4$. Suppose that $\det(T_\xi) \neq 0$. Then ξ lies in the special fiber of $\mathcal{Z}$, and ξ does not lie in the closure of any point of $\mathcal{Z}$ in the generic fiber.*

Proof. By Lemma 2.4 the assumption on T_ξ means that $T_\xi \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})$ is positive definite. However, for a point of $\mathcal{M}$ in characteristic zero, the space of special endomorphisms is contained in a 3-dimensional positive definite quadratic space. $\square$

Next suppose that $\xi \in \mathcal{M}(\bar{\mathbb{F}}_p)$. In this case, the standard Honda-Tate results yield information about the possibilities for C_ξ^0. We have $\iota : M_2(B) = C \hookrightarrow \mathrm{End}^0(A_\xi)$, so that, up to isogeny $A_\xi \simeq A \times A$ where $\dim A = 4$ and there is an embedding $B \hookrightarrow \mathrm{End}^0(A)$.

Lemma 3.2. *Suppose that $p \nmid D(B)$. Then there are no simple abelian varieties A_0 over $\bar{\mathbb{F}}_p$ with $\dim A_0 = 2$ or 4 and with $B \hookrightarrow \mathrm{End}^0(A_0)$.*

Proof. If A_0 is simple over $\mathbb{F}_q$, then $E = \mathrm{End}^0(A_0)$ is a central simple algebra over $F = \mathbb{Q}(\pi_{A_0})$, and

$$2 \dim A_0 = [E : F]^{\frac{1}{2}} \cdot [F : \mathbb{Q}]. \tag{3.7}$$

Here π_{A_0} denotes as usual the Frobenius endomorphism. If $\dim A_0 \geq 2$ and A_0 remains simple over $\bar{\mathbb{F}}_p$, then F is a CM field. Suppose that $\dim A_0 = 2$, so that $[F:\mathbb{Q}] = 2$ or 4. The second case is excluded, since then $E = F$ is commutative. In the first case, E is a division quaternion algebra over F ramified only at places over p. Thus p splits in F and $inv_v(E) = inv_{\bar{v}}(E) = \frac{1}{2}$ for $v \mid p$. But the embedding $B \hookrightarrow E$ yields an isomorphism $B \otimes_{\mathbb{Q}} F \simeq E$. This is possible only if $p \mid D(B)$ and F splits B at all other primes.

If $\dim A = 4$, then $[F:\mathbb{Q}] = 2$, 4 or 8, and the last case is again excluded since $E = F$. In the case $[F:\mathbb{Q}] = 4$, E is a quaternion algebra over F, ramified only at primes lying over p, and $B \otimes_{\mathbb{Q}} F \simeq E$. This cannot occur if $p \nmid D(B)$. Finally, if $[F:\mathbb{Q}] = 2$, then p splits in F and E is a division algebra over F of dimension 16 with invariants $\frac{1}{4}$ and $\frac{3}{4}$ at the primes over p. There is no homomorphism from a quaternion algebra $B \otimes_{\mathbb{Q}} F$ into such an algebra. $\square$

Returning to A, and assuming that $p \nmid D(B)$, we see that A cannot be simple and that any simple factor of A of dimension 1 or 2 must occur with multiplicity at least 2. Thus we have various possibilities for A, up to isogeny:

(3.8.i) $A \simeq A_2 \times A_2$, with $\dim A_2 = 2$ simple and $\mathrm{End}^0(A_2) \simeq F$ for a CM field F with $[F:\mathbb{Q}] = 4$ which splits B, i.e., such that $B \otimes_{\mathbb{Q}} F \simeq M_2(F)$. Then, $\mathrm{End}^0(A) \simeq M_2(F)$, $C_\xi^0 = \mathrm{End}^0(A,\iota) \simeq F$, and $V_\xi^0 = \mathbb{Q}$.

(3.8.ii) $A \simeq A_2 \times A_2$, with $\dim A_2 = 2$ simple and $\mathrm{End}^0(A_2) \simeq E$, where E is a quaternion algebra over a CM field F with $[F:\mathbb{Q}] = 2$. More precisely, p splits in F and $E \simeq H_p \otimes_{\mathbb{Q}} F$, where H_p is the quaternion algebra over $\mathbb{Q}$ ramified at ∞ and p. Let B' be the quaternion algebra over $\mathbb{Q}$ whose invariants agree with those of B except at ∞ and p. Then $\mathrm{End}^0(A) \simeq M_2(E)$, $\mathrm{End}^0(A,\iota) \simeq B' \otimes_{\mathbb{Q}} F$, and $V_\xi^0 = \{x \in B' \mid \mathrm{tr}(x) = 0\}$. Here note that $B \otimes_{\mathbb{Q}} B' \simeq M_2(H_p)$ and hence that $(B \otimes_{\mathbb{Q}} F) \otimes_F (B' \otimes_{\mathbb{Q}} F) \simeq M_2(E)$.

(3.8.iii) $A \simeq A_0^2 \times A_1^2$ where A_0 and A_1 are non-isogenous ordinary elliptic curves. Then $\mathrm{End}^0(A) \simeq M_2(F_0) \times M_2(F_1)$ for imaginary quadratic fields F_0 and F_1, which split B. Then, $\mathrm{End}^0(A,\iota) \simeq F_0 \times F_1$, and $V_\xi^0 = \mathbb{Q}$.

(3.8.iv) $A \simeq A_0^4$, for an ordinary elliptic curve A_0. Then $\mathrm{End}^0(A) \simeq M_4(F_0)$ where the imaginary quadratic field F_0 splits B, $\mathrm{End}^0(A,\iota) \simeq M_2(F_0)$ and $V_\xi^0 \simeq \{x \in M_2(F_0) \mid {}^t\bar{x} = x,\ \mathrm{tr}(x) = 0\}$.

(3.8.v) $A \simeq A_0^2 \times A_1^2$, where A_0 is a supersingular elliptic curve and A_1 is an ordinary elliptic curve. Then $\mathrm{End}^0(A) \simeq M_2(H_p) \times M_2(F_1)$, $\mathrm{End}^0(A,\iota) \simeq B' \times F_1$. Since the Rosati involution acts on $\mathrm{End}^0(A,\iota)$ by $(b,a) \mapsto (b^\iota, \bar{a})$, the conditions $x^* = x$ and $\mathrm{tr}(x) = 0$ force $V_\xi^0 \simeq \mathbb{Q}$.

(3.8.vi) $A \simeq A_0^4$, for a supersingular elliptic curve A_0. Then $\mathrm{End}^0(A) \simeq M_4(H_p)$, $\mathrm{End}^0(A,\iota) \simeq M_2(B')$, and

$$V_\xi^0 = \{x \in M_2(B') \mid x' = {}^t x^\iota = x,\ \mathrm{tr}(x) = 0\} = V'.$$

For the last identification we are using the proposition in section A.4 of the Appendix. Indeed, the Rosati involution on $\mathrm{End}^0(A_\xi) \simeq M_8(H_p)$ is of main type by A.4 and its restriction to $M_2(B)$ of neben type. Hence the involution on $M_2(B')$ is indeed of main type.

Note that $\dim V_\xi^0 \leq 3$, with the exception of the supersingular case (3.8.vi). As a consequence, we have the following:

Proposition 3.3. *Let* $T \in Sym_4(\mathbb{Z}_{(p)})$ *and* $\omega \subset V(\mathbb{A}_f^p)^4$ *with corresponding special cycle* $\mathcal{Z}(T,\omega)$ *. If* $\det(T) \neq 0$ *, then the point set underlying* $\mathcal{Z}(T,\omega)$ *maps to the supersingular locus of* $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$ *. In particular,* $\mathcal{Z}(T,\omega)$ *is proper over* $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ *with support in the special fibre.*

Proof. Indeed the previous results imply that this is true for closed points.

Corollary 3.4. *For* $i = 1, \ldots, r$ *, let* $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})$ *be positive definite with* $n_1 + \ldots + n_r = 4$ *, and let* $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$ *with corresponding special cycles* $\mathcal{Z}(d_i, \omega_i)$ *. For* $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})$ *with diagonal blocks* $d_1, \ldots, d_r$ *, let* $\mathcal{Z}_T$ *be the union of the connected components of* $\mathcal{Z}(d_1,\omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r,\omega_r)$ *where the fundamental matrix has value* T *. If* $\det(T) \neq 0$ *then the point set underlying* $\mathcal{Z}_T$ *lies over the supersingular locus of* $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$ *.* $\square$

Having answered these very crude questions on the intersection behaviour of our special cycles, we are led to ask more precise questions. Again for $i = 1, \ldots, r$ let $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})$ be positive-definite with $n_1 + \ldots + n_r = 4$ and let $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$ with corresponding cycles $\mathcal{Z}(d_1,\omega_1), \ldots, \mathcal{Z}(d_r,\omega_r)$. We then ask:

a) Under which conditions do the cycles $\mathcal{Z}(d_1,\omega_1), \ldots, \mathcal{Z}(d_r,\omega_r)$ intersect properly? More precisely, can one parametrize the isolated points of $\mathcal{Z} = \mathcal{Z}(d_1,\omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r,\omega_r)$ and calculate at such an isolated point y,

$$e(y) = \lg_{\mathcal{O}_{\mathcal{Z},y}}(\mathcal{O}_{\mathcal{Z},y}) \quad ? \tag{3.9}$$

b) Let Y be a connected component of $\mathcal{Z} = \mathcal{Z}(d_1,\omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r,\omega_r)$ lying over the supersingular locus of $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$. The intersection number along Y is

$$\chi(Y, \mathcal{O}_{\mathcal{Z}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \ldots \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}_r}) \quad , \tag{3.10}$$

cf. [**22**], [**27**]. An important question to answer is when the derived tensor product here can be replaced by an ordinary tensor product, i.e. by $\mathcal{O}_{\mathcal{Z}}$. In the case when

Y is an isolated point this would mean that the length in (3.9) is in fact the intersection number of $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ at y. In particular one may ask, when does the intersection number along Y depend only on the scheme Y? Related to this question is the problem of the singularities of the schemes $\mathcal{Z}(d, \omega)$: under which conditions are they Cohen-Macaulay, or even locally complete intersections? In general they are neither [**21**].

Our next task will be to investigate the structure of the supersingular locus $\mathcal{M}^{ss} \subset \mathcal{M} \times_{\mathrm{Spec}\ \mathbb{Z}_{(p)}} \mathrm{Spec}\ \mathbb{F}_p$.

§4. Structure of the supersingular locus.

As mentioned in the introduction, the results of this section are a presentation of results of Moret-Bailly [**23**] and Oort [**24**]. A similar presentation was independently given by Kaiser [**11**].

We put $\mathbb{F} = \bar{\mathbb{F}}_p$, and let $W = W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$ and $\mathcal{K} = W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ its quotient field. Also write $W[F,V]$ for the Cartier ring of $\mathbb{F}$.

Throughout this section, we assume that $p \nmid D(B)$, and we fix an isomorphism $O_C \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p)$.

Suppose that $\xi = (A, \iota, \lambda, \bar{\eta}^p) \in \mathcal{M}^{\mathrm{ss}}(\mathbb{F})$, and let $A(p)$ be the p-divisible (formal) group of A. The action of $O_C \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p)$ on $A(p)$ then induces a decomposition $A(p) \simeq A_0(p)^4$, where $A_0(p)$ is a p-divisible formal group of dimension 2 and height 4. Let L_0 be the (contravariant) Dieudonné module of $A_0(p)$ and let $\mathcal{L} = L_0 \otimes_W \mathcal{K}$ be the associated isocrystal. This does not depend on the choice of $\xi \in \mathcal{M}^{\mathrm{ss}}(\mathbb{F})$, up to isomorphism.
More precisely, we fix a base point $\xi_o = (A_o, \iota_o, \lambda_o, \eta_o^p) \in \mathcal{M}^{ss}(\mathbb{F})$ and let $\mathcal{L} = L_0 \otimes_W \mathcal{K}$ be the isocrystal associated to it. The isocrystal $\mathcal{L}$ has a polarization $< \, , \, >$, is isoclinic with slope $\frac{1}{2}$, and has $\dim_{\mathcal{K}} \mathcal{L} = 4$. Then F is σ-linear, $V = pF^{-1}$ is σ^{-1}-linear, and

$$(4.1) \qquad\qquad < Fx, y > = < x, Vy >^{\sigma} .$$

If $\xi = (A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}^{ss}(\mathbb{F})$ is another point, then the choice of an isogeny between ξ and ξ_o defines a W-lattice $L \subset \mathcal{L}$.

For a W-lattice $L \subset \mathcal{L}$ of rank 4, set

$$(4.2) \qquad\qquad L^{\perp} = \{ \ x \in \mathcal{L} \ \ | < x, L > \subset W \ \}.$$

Definition 4.1. *a) A W-lattice L in $\mathcal{L}$ is* **special** *if and only if $L = c \cdot L^{\perp}$, for some $c \in \mathcal{K}^{\times}$. Otherwise, L is* **non-special**.
b) A W-lattice L in $\mathcal{L}$ is **admissible** *if*

$$L \supset FL \supset pL \quad .$$

We note that, if L is an admissible lattice, then, since $\mathcal{L}$ is isoclinic of slope $1/2$, we have

$$\dim_{\mathbb{F}} \, L/FL = 2 \quad .$$

We define a set of lattices as follows:

$$(4.3) \qquad\qquad X = \{ L \subset \mathcal{L}; \ L \text{ admissible and special} \} \quad .$$

If $L \in X$ then $FL \in X$. This follows from $(FL)^\perp = V^{-1} \cdot L^\perp$, cf. (4.1).

The conditions in our moduli problem imply that the lattice $L \subset \mathcal{L}$ associated to $\xi \in \mathcal{M}^{ss}(\mathbb{F})$ and an isogeny between ξ and ξ_o actually lies in X. Note that each admissible lattice is the Dieudonné module of a p-divisible formal group of dimension 2 and height 4 over $\mathbb{F}$.

Let $L \subset \mathcal{L}$ be a lattice. Then L is special or non-special depending on whether the generalized index $\left[L^\perp : L\right] \in \mathbb{Z}$ is divisible by 4 or not. In any case this index is always even. If L is special and $L' \subset L$ is an inclusion of index 1, then L' is non-special and conversely.

If L is special, we can replace L by $\alpha \cdot L$, for $\alpha \in \mathcal{K}^\times$ to obtain a lattice with $L = L^\perp$ or $L = pL^\perp$. If L is non-special, we can scale to obtain a lattice with either

$$L \underset{\neq}{\subset} L^\perp \underset{\neq}{\subset} p^{-1}L, \text{ or } L^\perp \underset{\neq}{\subset} L \underset{\neq}{\subset} p^{-1}L^\perp, \tag{4.4}$$

with all indices equal to 2. We will call lattices scaled in this way **standard**.

Recall from (3.8.vi) that $\mathrm{End}^0(A_{\xi_o}, \iota)^{\mathrm{op}} =: C' \simeq M_2(B')$, where B' is the definite quaternion algebra over $\mathbb{Q}$ with the same local invariants as B at all primes $\ell \neq p$. As before, let $V' = \{ x \in M_2(B') \mid x' = x \text{ and } \mathrm{tr}(x) = 0 \}$. Let

$$G' = \{ g \in C'^{,\times} \mid gV'g^{-1} = V' \text{ and } gg' = \nu(g) \}. \tag{4.5}$$

Note that the action of $G'(\mathbb{Q}_p)$ on $A_{\xi_o}(p)$ up to isogeny passes to $\mathcal{L}$. In fact,

$$G'(\mathbb{Q}_p) \simeq \{ g \in GL(\mathcal{L}) \mid < gx, gy >= \nu(g) < x, y >, \quad Fg = gF \}. \tag{4.6}$$

Here $\nu(g) \in \mathcal{K}^\times$.
The action of $G'(\mathbb{Q}_p)$ preserves the set of lattices X. Fix an isomorphism $B(\mathbb{A}_f^p) \simeq B'(\mathbb{A}_f^p)$ and, hence, an isomorphism $G(\mathbb{A}_f^p) \simeq G'(\mathbb{A}_f^p)$. Then, the usual analysis identifies $G'(\mathbb{Q})$ with the group of self-isogenies of ξ_o and yields an isomorphism

$$\mathcal{M}^{\mathrm{ss}}(\mathbb{F}) \simeq G'(\mathbb{Q}) \backslash \Big(X \times G'(\mathbb{A}_f^p)/K^p \Big). \tag{4.7}$$

We will now describe the lattices in X in more detail.

Definition 4.2. *For $L \in X$, let*

$$a(L) = \dim_{\mathbb{F}} L/(FL + VL) \quad .$$

Since $a(L) = \dim_{\mathbb{F}}\mathrm{Hom}_{W[F,V]}(L, \mathbb{F})$, we see that $a(L)$ is the a-number, [**24**], of the p-divisible group $A_0(p)$ associated to L , i.e.

$$a(L) = \mathrm{Hom}_{\mathbb{F}}(\alpha_p, A_0(p)) \quad .$$

Since

$$\begin{array}{ccccc} & & L & & \\ & & \uparrow & & \\ & & FL+VL & & \\ & \nearrow & & \nwarrow & \\ FL & & & & VL, \\ & \nwarrow & & \nearrow & \\ & & FL\cap VL & & \\ & & \uparrow & & \\ & & pL & & \end{array} \tag{4.8}$$

we have

$$a(L) = \begin{cases} 2 & \text{if } FL = VL, \\ 1 & \text{if } [L : FL+VL] = 1. \end{cases} \tag{4.9}$$

Let

$$X_0 = \{\ L \in X \mid a(L) = 2\ \}. \tag{4.10}$$

Such lattices will be called **superspecial**.

In addition to the superspecial lattices, the following type of lattice will play a key role in the description of the structure of X .

Definition 4.3. *A lattice $\tilde{L} \subset \mathcal{L}$ is* **distinguished** *if $\tilde{L}$ is admissible and $F\tilde{L} = c\tilde{L}^{\perp}$ for some $c \in \mathcal{K}^{\times}$.*

We denote by $\tilde{X}$ the set of distinguished lattices. Obviously, if $\tilde{L} \in \tilde{X}$ is distinguished, the index $[\tilde{L}^{\perp} : \tilde{L}]$ is congruent to 2 mod 4. Thus $\tilde{L}$ is non-special. Note that if $\tilde{L}$ is distinguished, then $F\tilde{L} = V\tilde{L}$. Indeed, by (4.1) for any lattice $\tilde{L}$ we have $(F\tilde{L})^{\perp} = V^{-1}\tilde{L}^{\perp}$. Hence if $F\tilde{L} = c \cdot \tilde{L}^{\perp}$ we get

$$\tilde{L} = c(F\tilde{L})^{\perp} = cV^{-1}\tilde{L}^{\perp} = cV^{-1}c^{-1}F\tilde{L} = V^{-1}F\tilde{L} \quad ,$$

i.e. $V\tilde{L} = F\tilde{L}$, as claimed. Similarly one sees that if $\tilde{L} \in \tilde{X}$, then $F\tilde{L} \in \tilde{X}$.

We note that if, in the identity defining a distinguished lattice $\tilde{L}$, the order of c is odd, then $\tilde{L}$ may be scaled to be standard in the sense of the first alternative of (4.4) above. If the order of c is even, then $\tilde{L}$ can be scaled to be standard in the sense of the second alternative of (4.4), and hence, $F\tilde{L}$ can be scaled to be standard in the sense of the first alternative of (4.4).

For any $\tilde{L} \in \tilde{X}$ and for any $\mathbb{F}$-line $\ell \subset \tilde{L}/F\tilde{L}$, let $L = L(\ell)$ be the inverse image of ℓ in $\tilde{L}$. Thus

$$\begin{array}{ccccc} \tilde{L} & \supset & L & \supset & F\tilde{L} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{L}/F\tilde{L} & \supset & \ell & \supset & 0 \end{array} \tag{4.11}$$

Lemma 4.4. *For* $\ell \subset \tilde{L}/F\tilde{L}$, $L = L(\ell) \in X$.

Proof. First, since $F\tilde{L} = V\tilde{L}$, we have

$$FL \subset F\tilde{L} \subset L, \tag{4.12}$$

and

$$FL \supset FV\tilde{L} = p\tilde{L} \supset pL. \tag{4.13}$$

Hence L is admissible.
Next, we have

$$\tilde{L} \supset L \supset F\tilde{L}, \tag{4.14}$$

where all inclusions have index 1. But $\tilde{L}$ is non-special and hence L is special. $\square$

The above proof in fact shows the following. Suppose that $\tilde{L} \in \tilde{X}$ with $F\tilde{L} = p \cdot \tilde{L}^{\perp}$. Then $L(\ell)^{\perp} = pL(\ell)$. If $F\tilde{L}^{\perp} = p\tilde{L}$, then $L(\ell)^{\perp} = L(\ell)$.

Thus to any distinguished $\tilde{L}$ we have associated a projective line $\mathbb{P}(\tilde{L}/F\tilde{L})$ and a family of admissible special lattices parametrized by the $\mathbb{F}$-points of this projective line. These projective lines have a natural $\mathbb{F}_{p^2}$-structure which we now describe.

For any W-lattice L in $\mathcal{L}$, we have

$$FL = VL \iff F^2L = FVL = pL \iff p^{-1}F^2L = L. \tag{4.15}$$

Lemma 4.5. *Suppose that* $p^{-1}F^2L = L$, *and let*

$$L_0 = \{\ x \in L \mid p^{-1}F^2x = x\ \}.$$

Then L_0 *is a* $\mathbb{Z}_{p^2}$*-module and*

$$L_0 \otimes_{\mathbb{Z}_{p^2}} W \simeq L. \qquad \square$$

If $\tilde{L} \in \tilde{X}$ is distinguished, then $\tilde{L}$ is preserved by the σ^2-linear endomorphism $p^{-1}F^2$, and we have $\tilde{L} \simeq \tilde{L}_0 \otimes_{\mathbb{Z}_{p^2}} W$. Moreover, $F\tilde{L}$ is also preserved by $p^{-1}F^2$, and $(F\tilde{L})_0 = F(\tilde{L}_0)$. Thus, the two dimensional $\mathbb{F}$-vector space $\tilde{L}/F\tilde{L}$ has a natural $\mathbb{F}_{p^2}$-structure:

$$\tilde{L}/F\tilde{L} \simeq \tilde{L}_0/F\tilde{L}_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F}. \tag{4.16}$$

We may then view any line ℓ as an element of $\mathbb{P}(\tilde{L}_0/F\tilde{L}_0)(\mathbb{F})$. We denote by $\mathbb{P}_{\tilde{L}}$ the projective line $\mathbb{P}(\tilde{L}_0/F\tilde{L}_0)$ over $\mathbb{F}_{p^2}$.

Lemma 4.6. *Under the isomorphism*

$$\tilde{L}/F\tilde{L} \simeq \tilde{L}_0/F\tilde{L}_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F},$$

the automorphism induced by $p^{-1}F^2$ *on* $\tilde{L}/F\tilde{L}$ *coincides with* $1 \otimes \sigma^2$ *on* $\tilde{L}_0/F\tilde{L}_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$. *Hence,*

$$p^{-1}F^2(L(\ell)) = L(\sigma^2(\ell)),$$

where ℓ *is identified with a point in* $\mathbb{P}_{\tilde{L}}(\mathbb{F})$. $\square$

Corollary 4.7. *A lattice* $L(\ell)$ *associated to a distinguished* $\tilde{L}$ *is superspecial, i.e., has* $a(L(\ell)) = 2$, *if and only if* $\ell \in \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^2})$.

Proposition 4.8. *Suppose that* $L \in X$ *with* $a(L) = 1$, *and let*

$$\tilde{L} = F^{-1}(FL + VL) \quad .$$

Then $\tilde{L}$ *is distinguished and* $L = L(\ell)$ *for a unique line* $\ell \in \mathbb{P}_{\tilde{L}}(\mathbb{F}) \setminus \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^2})$.

Proof. Let $L^{\perp} = c \cdot L$. Then

$$(F\tilde{L})^{\perp} = (FL)^{\perp} \cap (VL)^{\perp} = V^{-1}L^{\perp} \cap F^{-1}L^{\perp} = p^{-1}c \cdot (FL \cap VL).$$

On the other hand, $F^2\tilde{L} = F^2L + pL$. Let $S = L/pL$, and let f and v be the σ-linear resp. σ^{-1}-linear endomorphisms of S induced by F and V. Since $FV = VF = p$, we have $fv = vf = 0$ and so $\ker(f) = \mathrm{im}(v)$ and $\ker(v) = \mathrm{im}(f)$ are 2-dimensional subspaces of S. However, for any $L \in X$ there is some $j \geq 2$ with $F^jL \subset pL$ and hence f is nilpotent. If $f^2 = 0$, then $F^2L = pL$ since both lattices have index 4 in L and this would imply $a(L) = 2$, contrary to our assumption. Therefore, since $\mathrm{im}(f)$ is 2-dimensional we must have that $\mathrm{im}(f^2)$ is one-dimensional and $\mathrm{im}(f^2) = \mathrm{im}(f) \cap \mathrm{im}(v)$. Hence

$$F^2\tilde{L} = F^2L + pL = FL \cap VL \quad .$$

It follows that $F(F\tilde{L}) = p \cdot c^{-1}(F\tilde{L})^{\perp}$. On the other hand, $F\tilde{L}$ is admissible, since

$$pF\tilde{L} = p(FL + VL) \subset pL \subset F^2\tilde{L} = FL \cap VL \subset FL \subset F\tilde{L} = FL + VL$$

where all inclusions are of index 1. It follows that $F\tilde{L} \in \tilde{X}$ and hence also $\tilde{L} \in \tilde{X}$. Finally $L = L(\ell)$ for the line

$$\ell = L/F\tilde{L} \subset \tilde{L}/F\tilde{L} \quad . \qquad \square$$

We summarize the above construction in the following theorem.

Theorem 4.9. *There is a natural* $G'(\mathbb{Q}_p)$ *-equivariant map*

$$\coprod_{\tilde{L} \in \tilde{X}} \mathbb{P}_{\tilde{L}}(\mathbb{F}) \longrightarrow X$$

which induces a bijection

$$\coprod_{\tilde{L}} (\mathbb{P}_{\tilde{L}}(\mathbb{F}) \setminus \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^2})) \xrightarrow{\sim} X \setminus X_0 \quad .$$

The map associates to $(\tilde{L}, \ell)$ *, where* $\ell \subset \tilde{L}/F\tilde{L}$ *is a line, the element* $L = L(\ell) \in X$ *.*

The action of $g \in G'(\mathbb{Q}_p)$ *on the index set of the left hand side is lifted in the obvious way to the whole set appearing on the left hand side.*

Remark 4.10. It can be shown that the map above is in fact a morphism, i.e., is the map on $\mathbb{F}$-points induced by a morphism of schemes over $\mathrm{Spec}\ \mathbb{F}_p$,

$$\coprod_{\tilde{L} \in \tilde{X}} \mathbb{P}_{\tilde{L}} \longrightarrow \mathcal{M}^{ss} \quad .$$

This can be shown by the method of Oort, [**24**], or using Cartier theory, as in Stamm, [**30**]. Using either of these methods one can construct a morphism of schemes over $\mathrm{Spec}\, \mathbb{F}_p$,

$$G'(\mathbb{Q}) \setminus \left[(\coprod_{\tilde{L} \in \tilde{X}} \mathbb{P}_{\tilde{L}}) \times G'(\mathbb{A}_f^p)/K^p \right] \longrightarrow \mathcal{M}^{ss}$$

which turns out to be the normalization of the curve $\mathcal{M}^{ss}$.

The ‘distinguished curves’ cross at the superspecial points. To describe this, it will be useful to have a normal form for superspecial lattices.

Lemma 4.11. *Fix* $\delta \in \mathbb{Z}_{p^2}^{\times}$ *with* $\delta^\sigma = -\delta$ *. Let* $L \in X_0$ *be superspecial and standard.*
(i) Suppose that $L = L^\perp$ *. Then there is a basis* e_1 *,* e_2 *,* e_3 *,* e_4 *for* L *over* W *such that* $e_3 = Fe_1$ *,* $e_4 = Fe_2$ *,* $Fe_3 = pe_1$ *,* $Fe_4 = pe_2$ *and such that the matrix for the polarization is*

$$(< e_i, e_j >)_{i,j} = \delta \cdot \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix},$$

(ii) If $L = pL^\perp$ *, then* $L = FL'$ *where* $L' \in X_0$ *with* $L' = (L')^\perp$ *.*

Proposition 4.12. *Suppose that* $L \in X_0$ *is superspecial and standard.*
(i) If $L^\perp = L$ *, consider lattices* $\tilde{L}$ *such that* $L \underset{\neq}{\supset} \tilde{L} \underset{\neq}{\supset} FL$ *and such that* $F\tilde{L} = p\tilde{L}^\perp$ *. Such* $\tilde{L}$ *'s are distinguished; there are* $p{+}1$ *of them, and they can be described explicitly as follows. Let* $e_1, \dots, e_4$ *be a standard basis as in Lemma 4.11. Then the distinguished* $\tilde{L}$ *'s have the form*

$$\tilde{L} = W(e_1 + \mu e_2) + FL$$

where $\mu \in \mathbb{Z}_{p^2}^{\times}$ *such that* $\mu\mu^\sigma \equiv -1 \mod p$ *.*
(ii) If $L = pL^\perp$ *, then the distinguished* $\tilde{L}$ *'s containing* L *with index* 1 *are those associated, as in (i), to* $L' = F^{-1}L$ *.*

Proof of Lemma 4.11. Since $p^{-1}F^2$ is a σ^2-linear automorphism of L, we can write $L = L_0 \otimes_{\mathbb{Z}_{p^2}} W$ for the rank 4 lattice L_0 of fixed points of $p^{-1}F^2$. Let $S_0 = L_0/pL_0$, a 4-dimensional symplectic vector space over $\mathbb{F}_{p^2}$, and note that FL_0/pL_0 is an isotropic 2-plane in S_0, which is paired with the quotient L_0/FL_0. We can then choose e_1 and $e_2 \in L_0$ whose images form a basis for L_0/FL_0 and such that $< e_1, e_2 >= 0$, after modification by elements of FL_0, if necesssary. The elements e_1, e_2, $e_3 := Fe_1$ and $e_4 := Fe_2$ then give a W-basis for L, and $Fe_3 = F^2 e_1 = pe_1$, and $Fe_4 = F^2 e_2 = pe_2$, as required, since e_1 and $e_2 \in L_0$. The matrix for the polarization is then

$$\begin{pmatrix} 0 & A \\ -{}^tA & 0 \end{pmatrix} \qquad \text{where } A =< \underline{e}, F\underline{e} >, \text{ with } \underline{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{4.17}$$

Note that $\det(A) \in \mathbb{Z}_{p^2}^{\times}$, and that

$$-{}^tA^\sigma =< F\underline{e}, \underline{e} >^\sigma =< \underline{e}, V\underline{e} >=< \underline{e}, F\underline{e} >= A, \tag{4.18}$$

since $V = pF^{-1}$ and so, on L_0, $V = F^2 \cdot F^{-1} = F$. If we change the vector $\underline{e}$ to $a \cdot \underline{e}$, for $a \in GL_2(\mathbb{Z}_{p^2})$, then A changes to $aA{}^ta^\sigma$. Since $\det(A) \in \mathbb{Z}_{p^2}^{\times}$ and since the norm map $N : \mathbb{Z}_{p^2}^{\times} \longrightarrow \mathbb{Z}_p^{\times}$ is surjective, it is easy to check that, for a suitable choice of a we can obtain $aA{}^ta^\sigma = \delta \cdot 1_2$. $\square$

Proof of Proposition 4.12. Let us prove (i). Using the standard basis of Lemma 4.11, we have $L = [e_1, e_2, e_3, e_4]$ (the square brackets indicate the W-span) and $FL = [pe_1, pe_2, e_3, e_4]$. Any lattice $\tilde{L}$ with $L \supset \tilde{L} \supset FL$ and with $[L : \tilde{L}] = 1$ has the form

$$\tilde{L} = W \cdot (ae_1 + be_2) + FL, \tag{4.19}$$

where at least one of a and $b \in W$ is a unit. If a is a unit, we can write $\tilde{L} = [e_1 + \mu e_2, pe_2, e_3, e_4]$. Then

$$F\tilde{L} = [e_3 + \mu^\sigma e_4, pe_4, pe_1, pe_2] \quad \text{and} \quad p\tilde{L}^\perp = [e_4 - \mu e_3, pe_4, pe_1, pe_2]. \tag{4.20}$$

Comparing, we see that μ must be a unit and that $\mu\mu^\sigma \equiv -1 \mod p$, as claimed. It is easy to check that the case in which a is not a unit yields no solutions. The assertion (ii) is trivial. $\square$

Corollary 4.13. *The map appearing in Theorem 4.9. is surjective. Any lattice in X_0 has $p+1$ preimages which all lie on distinct lines. In fact, the preimages of $L \in X_0$ correspond to the distinguished lattices $F^{-1}\tilde{L}$ where $\tilde{L}$ ranges over the lattices associated to L in (i) of Proposition 4.12 (resp. to distinguished lattices $\tilde{L}$ associated to L in (ii) of Proposition 4.12). Finally, the images of two distinct lines $\mathbb{P}_{\tilde{L}}$ and $\mathbb{P}_{\tilde{L}'}$ have at most one lattice in common which then lies in X_0.*

Proof. The last assertion follows since, if $L, L' \in X$, $L \neq L'$, both lie on $\mathbb{P}_{\tilde{L}}$, then $\tilde{L} = L + L'$. $\square$

The next result gives a standard basis for a distinguished lattice.

Lemma 4.14. *Let $\tilde{L}$ be a distinguished lattice which is standard.*
(i) If $F\tilde{L} = p\tilde{L}^\perp$, then there exists a W-basis $e_1, \dots, e_4$ of $\tilde{L}$ such that $e_3 = Fe_1$, $e_4 = Fe_2$, $Fe_3 = pe_1$, $Fe_4 = pe_2$, and such that the polarization has matrix

$$(< e_i, e_j >)_{i,j} = \delta \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -p \\ & & p & \end{pmatrix}.$$

(ii) If $F\tilde{L}^\perp = p\tilde{L}$, then $\tilde{L} = F\tilde{L}'$ where $\tilde{L}' \in \tilde{X}$ with $F\tilde{L}' = p \cdot \tilde{L}'^\perp$.

Proof of (i). Let $\tilde{L}_0$ be the fixed points of $p^{-1}F^2$ on $\tilde{L}$. Since $F\tilde{L} = p\tilde{L}^\perp$, $< , >$ induces a nondegenerate symplectic form on the two dimensional $\mathbb{F}_{p^2}$-vector space $\tilde{L}_0/F\tilde{L}_0$. Choose e_1 and $e_2 \in \tilde{L}_0$ whose images in $\tilde{L}_0/F\tilde{L}_0$ are a basis for this space and such that $< e_1, e_2 >= \delta$. Let $e_3 = Fe_1$ and $e_4 = Fe_2$, so that, as in

Lemma 4.11, $Fe_3 = F^2 e_1 = pe_1$ and $Fe_4 = F^2 e_2 = pe_2$. The polarization then has matrix

$$\begin{pmatrix} \delta J & A \\ -{}^t A & -p\delta J \end{pmatrix} \tag{4.21}$$

where $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $A = < \underline{e}, F\underline{e} > = -{}^t A^\sigma$, as in the proof of Lemma 4.11. In the present case, however, $A \equiv 0 \mod p$. A Hensel's Lemma argument shows that we can replace $\underline{e}$ by $a\underline{e} + bF\underline{e}$ with $a \in GL_2(\mathbb{Z}_{p^2})$ and $b \in M_2(\mathbb{Z}_{p^2})$ to achieve $A = 0$, while preserving the condition $< \underline{e}, \underline{e} > = \delta J$. $\square$

Recall that $G'(\mathbb{Q}_p)$, given by (4.6) above, acts on the set of admissible lattices. For any lattice L, $(gL)^\perp = \nu(g)^{-1} g(L^\perp)$. If $L \in X$ is a special lattice, with $L = c \cdot L^\perp$, then $gL = \nu(g)c \cdot (gL)^\perp$, so that gL is again special. Moreover, $a(gL) = a(L)$ so that the subset of superspecial lattice is preserved. Also, if $\tilde{L}$ is distinguished, and if $g \in G'(\mathbb{Q}_p)$, then $g\tilde{L}$ is again distinguished. Since the valuation of $\nu(g)$ is an arbitrary integer, for any $L \in X$ (resp. $\tilde{L} \in \tilde{X}$) there is $g \in G'(\mathbb{Q}_p)$ such that gL (resp. $g\tilde{L}$) is standard with $(gL)^\perp = gL$ (resp. $F(g\tilde{L}) = p \cdot (g\tilde{L})^\perp$). By Lemmas 4.11 and 4.14, we have:

Corollary 4.15. *$G'(\mathbb{Q}_p)$ acts transitively on the set of superspecial lattices and on the set of distinguished lattices.*

We would finally like to compute the stablizers in $G'(\mathbb{Q}_p)$ of the superspecial and distinguished lattices.

Let B' be as above, and, identifying $\mathbb{Q}_{p^2}$ with a subfield of B'_p, write $B'_p = \mathbb{Q}_{p^2} + \Pi\mathbb{Q}_{p^2}$ for an element $\Pi \in B'^{,\times}_p$ with $\Pi^2 = p$ and such that $\Pi a = a^\sigma \Pi$, for $a \in \mathbb{Q}_{p^2}$. Let $\mathcal{L}_0$ be the fixed set for the automorphism $p^{-1}F^2$ of $\mathcal{L}$, and let Π operate on $\mathcal{L}_0$ by F. By construction, $\Pi^2 = p$, and so $\mathcal{L}_0$ is naturally a left vector space over B'_p of dimension 2.

Lemma 4.16. *Let*

$$\mathrm{End}_{\mathcal{K}}(\mathcal{L}, F) := \{\alpha \in \mathrm{End}_{\mathcal{K}}(\mathcal{L}) \mid F\alpha = \alpha F \}.$$

Then,

$$\mathrm{End}_{\mathcal{K}}(\mathcal{L}, F) = \mathrm{End}_{\mathbb{Q}_{p^2}}(\mathcal{L}_0, F) = \mathrm{End}_{B'_p}(\mathcal{L}_0).$$

The polarization on $\mathcal{L}$ induces a $\mathbb{Q}_{p^2}$-bilinear symplectic form on $\mathcal{L}_0$, which still satisfies $< Fx, y > = < x, Vy >^\sigma$.

Lemma 4.17. *Let U be a left B'_p -vector space with a B'_p -Hermitian form $(\ ,\ ) : U \times U \to B'_p$. Thus $(bx, cy) = b(x,y)c^\iota$ and $(y,x) = (x,y)^\iota$, where $b \mapsto b^\iota$ is the main involution on B'_p . Write*

$$(x,y) = (x,y)_0\delta + (x,y)_1\delta\Pi,$$

where $(x,y)_0$ and $(x,y)_1 \in \mathbb{Q}_{p^2}$. Then,

$$(\ ,\ )_1 : U \times U \longrightarrow \mathbb{Q}_{p^2}$$

is a symplectic $\mathbb{Q}_{p^2}$ -bilinear form on the $\mathbb{Q}_{p^2}$ -vector space U such that

$$(*) \qquad (\Pi x, y)_1 = (x, \Pi y)_1^\sigma,$$

and

$$(x,y)_0 = -(x, \Pi y)_1.$$

The map $(\ ,\ ) \mapsto (\ ,\ )_1$ yields a bijection between the space of B'_p -Hermitian forms on U and the space of symplectic forms satisfying (*). Moreover,

$$\begin{aligned} G'(\mathbb{Q}_p) &= \{\ g \in GL(\mathcal{L}) \mid < gx, gy >= \nu(g) < x,y > \ \text{ and } Fg = gF\} \\ &\simeq \{\ g \in GL_{B'}(\mathcal{L}_0) \mid (gx, gy) = \nu(g)(x,y)\ \}. \end{aligned}$$

Proof. We just check the behavior of Π . We have

$$\begin{aligned} (4.22) \qquad (\Pi x, y) &= \Pi(x,y)_0\delta + \Pi(x,y)_1\delta\Pi \\ &= -p(x,y)_1^\sigma\delta - (x,y)_0^\sigma\delta\Pi \end{aligned}$$

and

$$(4.23) \qquad (x, \Pi y) = -(x,y)_0\delta\Pi - p(x,y)_1\delta \quad .$$

Thus

$$(4.24) \qquad (\Pi x, y)_1 = -(x,y)_0^\sigma = (x, \Pi y)_1^\sigma,$$

as required. It is at this point that the factor of δ is required in the formulas. $\square$

Let $\mathcal{O}' = \mathcal{O}_{B'_p} = \mathbb{Z}_{p^2} + \Pi\mathbb{Z}_{p^2}$ be the maximal order in B'_p . If L is an admissible lattice such that $p^{-1}F^2L = L$, then the fixed point set L_0 of $p^{-1}F^2$ is naturally an $\mathcal{O}'$ -lattice in the B'_p -vector space $\mathcal{L}_0$, and $\dim_{\mathbb{F}_{p^2}} L_0/\Pi L_0 = 2$. Conversely, given any $\mathcal{O}'$ -lattice Λ with this last property, we set $F = \sigma \otimes \Pi$ on $L = L(\Lambda) := W \otimes_{\mathbb{Z}_{p^2}} \Lambda$. Then since $\Pi^2 = p$ on Λ , we have $L \supset FL \supset pL$ and $\dim_{\mathbb{F}} L/FL = 2$, i.e., L is admissible, and $p^{-1}F^2L = L$. The following is easily checked, using the formulas of Lemma 4.17.

Lemma 4.18. *(i) Suppose that* $L \in X_0$ *is superspecial with* $L = L^\perp$ *, and let* $e_1, \ldots, e_4$ *be a standard basis as in Lemma 4.11. Then* $e'_1 = \delta^{-1} e_1$ *and* $e'_2 = \delta^{-1} e_2$ *is an* $\mathcal{O}'$ *-basis for* L_0 *, and the matrix for the* B'_p *-Hermitian form on* $\mathcal{L}_0$ *is* $((e'_i, e'_j))_{i,j} = 1_2$ *.*
(ii) Suppose that $\tilde{L} \in \tilde{X}$ *is distinguished and that* $F\tilde{L} = p\tilde{L}^\perp$ *, and let* $e_1, \ldots, e_4$ *be a standard basis as in Lemma 4.14. Then* $e'_1 = \delta^{-1} e_1$ *and* $e'_2 = -\delta^{-1} e_2$ *form an* $\mathcal{O}'$ *-basis for* $\tilde{L}_0$ *and*

$$((e'_i, e'_j))_{i,j} = \begin{pmatrix} & \Pi \\ -\Pi & \end{pmatrix} .$$

Thus, in classical language, cf. [**29**], [**8**], the superspecial lattices correspond to local components of the principal genus of quaternion Hermitian lattices, while the distinguished lattices correspond to local components of a non-principal genus of such lattices.

In less classical language we may describe our results in terms of the Bruhat-Tits building of the adjoint group G'_{ad} over $\mathbb{Q}_p$, comp. also [**11**]. The building $\mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p)$ is a tree and may be identified with the fixed points

$$\mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p) = \mathcal{B}(G'_{\mathrm{ad}}, \mathcal{K})^F \quad .$$

The special vertices in $\mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p)$ correspond to the equivalence classes of lattices $L \subset \mathcal{L}$ which are F-invariant. Here two lattices L_1 and L_2 are equivalent if L_1 is homothetic to L_2 or to $L_2^\perp$. Hence the special vertices are in one-to-one correspondence with the distinguished lattices $\tilde{L}$ which are standard and with $F\tilde{L} = p\tilde{L}^\perp$. The non-special vertices in $\mathcal{B}(G'_{ad}, \mathbb{Q}_p)$ correspond to the edges in $\mathcal{B}(G'_{\mathrm{ad}}, \mathcal{K})$ whose vertices are interchanged by F. Equivalently, they correspond to pairs $\{L, FL\}$ of lattices in X_0 which are standard. We thus obtain bijections

$$\tilde{X} \leftrightarrow \mathbb{Z} \times \{\text{special vertices in } \mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p)\}$$

and

$$X_0 \leftrightarrow \mathbb{Z} \times \{\text{non-special vertices in } \mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p)\} \ .$$

These bijections are $G'(\mathbb{Q}_p)$-equivariant, where $g \in G'(\mathbb{Q}_p)$ acts on the $\mathbb{Z}$-component on the right via $n \mapsto n + \mathrm{ord}(\nu(g))$. The action of F on the left corresponds to the translation $n \mapsto n+1$ on the first factor and the trivial action on the second factor on the right. Furthermore, a lattice $L \in X_0$ and $\tilde{L} \in \tilde{X}$ are incident (i.e. $L \in \mathbb{P}_{\tilde{L}}$) if and only if the corresponding vertices of $\mathcal{B}(G'_{\mathrm{ad}}, \mathbb{Q}_p)$ lie on one and the same edge.

In these terms the stabilizer K^{d} of a distinguished lattice $\tilde{L} \in \tilde{X}$ is a special maximal compact subgroup of $G'(\mathbb{Q}_p)$, and the stabilizer K^{ss} of a superspecial lattice $L \in X_0$ is a non-special maximal compact subgroup of $G'(\mathbb{Q}_p)$.

Remark 4.19 We return, for a moment, to the global situation, and recall that $\tilde{X}$ is the set of distinguished lattices in $\mathcal{L}$. As observed in Remark 4.10, our calculations 'show' that the supersingular locus $\mathcal{M}^{\mathrm{ss}}$ is a union of rational curves and that the irreducible components are in bijection with the set

$$G'(\mathbb{Q})\backslash\Big(\tilde{X} \times G(\mathbb{A}_f^p)/K^p\Big) \simeq G'(\mathbb{Q})\backslash\Big(G'(\mathbb{Q}_p)/K_p^{\mathrm{d}} \times G(\mathbb{A}_f^p)/K^p\Big), \tag{4.25}$$

where K_p^{d} is the stabilizer in $G'(\mathbb{Q}_p)$ of a fixed distinguished lattice $\tilde{L} \in \tilde{X}$. These curves cross, $p+1$ at a time, at the superspecial points, and there are p^2+1 such crossing points on each component. The set of all crossing points is in bijection with the set

$$G'(\mathbb{Q})\backslash\Big(X_0 \times G(K_f^p)/K^p\Big) \simeq G'(\mathbb{Q})\backslash\Big(G'(\mathbb{Q}_p)/K_p^{\mathrm{ss}} \times G(\mathbb{A}_f^p)/K^p\Big), \tag{4.26}$$

where K_p^{ss} is the stabilizer in $G'(\mathbb{Q}_p)$ of a fixed superspecial lattice $L \in X_0$.

We finally observe two consequences of our description of $\mathcal{M}^{\mathrm{ss}}$.

Fix a factorization $D(B) = D_1 D_2$, and let $K = \prod_\ell K_\ell$ be the compact open subgroup of $G(\mathbb{A}_f)$ with local factors

$$K_\ell = \begin{cases} K_\ell^{\mathrm{ss}} & \text{if } \ell \mid D_1, \\ K_\ell^{\mathrm{d}} & \text{if } \ell \mid D_2, \\ K_\ell^0 & \text{if } \ell \nmid D(B). \end{cases}$$

Here, we have fixed a maximal order R in B, and for $\ell \nmid D(B)$, we fix an isomorphism $M_2(B_\ell) \simeq M_4(\mathbb{Q}_\ell)$ such that $M_2(R_\ell) \simeq M_4(\mathbb{Z}_\ell)$. Then let $K_\ell^0 = G(\mathbb{Q}_\ell) \cap M_4(\mathbb{Z}_\ell)$. Thus, for $\ell \mid D_1$ (resp. $\ell \mid D_2$), K_ℓ is the stabilizer of a Hermitian $\mathcal{O}_{B_\ell}$-lattice of principal (resp. non-principal) type, and, for $\ell \nmid D(B)$, K_ℓ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_\ell)$. Note that, in contrast to the general assumptions above, K is not neat. Still, for a fixed prime $p \nmid D(B)$, one can consider the coarse moduli space $\mathcal{M}_K$ (the quotient by a finite group of one of the schemes considered above) and its points over $\mathbb{F}$. Let $B^{(p)}$ denote the definite quaternion algebra with $D(B^{(p)}) = D(B)p$. Then, by (4.25), the components of the supersingular locus in the fiber of $\mathcal{M}_K$ at p correspond to the classes of maximal Hermitian lattices in the genus of type (D_1, pD_2) for $B^{(p)}$. An explicit formula for this number $H(D_1, pD_2)$ was found by Hashimoto and Ibukiyama [**9**]. In the case $D(B) = 1$, so that $B = M_2(\mathbb{Q})$, the abelian varieties parameterized by $\mathcal{M}_K(\mathbb{F})$ have the form $A \simeq A_0^4$, where A_0 is a principally polarized abelian surface. Thus, in this case, $\mathcal{M}_K \simeq A_{2,1}$, and the description of the supersingular locus reduces to some of the information given by Katsura and Oort [**12**], Theorem 5.7, and Ibukiyama, Katsura and Oort [**10**]. In particular, the number of irreducible components of the supersingular locus is $H(1,p)$.

As another example, fix a square free positive integer D and distinct primes p_1 and p_2 relatively prime to D. Consider indefinite quaternion algebras B_1 and B_2 over $\mathbb{Q}$ with discriminants $D(B_1) = Dp_1$ and $D(B_2) = Dp_2$. Let G_1 and G_2 be the associated groups, via (1.3). As in (4.5), let G_1' be the twist of G_1 at p_2 and let G_2' be the twist of G_2 at p_1. These groups are both associated to the definite quaternion algebra $B_1^{(p_2)} \simeq B_2^{(p_1)}$, and are isomorphic. Fix an isomorphism $G_1' \simeq G_2'$ and compatible isomorphisms

$$G_1(\mathbb{A}_f^{p_1p_2}) \simeq G_1'(\mathbb{A}_f^{p_1p_2}) \simeq G_2'(\mathbb{A}_f^{p_1p_2}) \simeq G_2(\mathbb{A}_f^{p_1p_2}),$$

and let $K^{p_1p_2} = K_1^{p_1p_2} = K_2^{p_1p_2}$ be a sufficiently small compact open subgroup. Also let

$$\begin{aligned} K_{1,p_1} &= K_{p_1}^{*_1}, && \text{for } *_1 = \mathrm{d} \text{ or ss}, \\ K_{1,p_2} &= K_{p_2}^{0}, \\ K_{2,p_1} &= K_{p_1}^{0}, \\ K_{2,p_2} &= K_{p_2}^{*_2}, && \text{for } *_1 = \mathrm{d} \text{ or ss}, \end{aligned}$$

where the notation is as above. Let

$$\begin{aligned} K_1^{*_1} &= K^{p_1p_2} K_{1,p_1} K_{1,p_2}, \\ K_2^{*_2} &= K^{p_1p_2} K_{2,p_1} K_{2,p_2}. \end{aligned}$$

Let $\mathcal{M}_1^{*_1} = \mathcal{M}_{K_1^{*_1}}$ and $\mathcal{M}_2^{*_2} = \mathcal{M}_{K_2^{*_2}}$ be the corresponding moduli schemes, defined over $\mathbb{Z}_{(p_2)}$ and $\mathbb{Z}_{(p_1)}$ respectively.

Then, using (4.25) and (4.26), there are (non-canonical but equivariant) bijections between various sets of irreducible components or crossing points as follows:

$$\text{Components}\Big((\mathcal{M}_2^{\mathrm{d}} \times \mathbb{F}_{p_1})^{\text{s.s.}}\Big) \simeq \text{Components}\Big((\mathcal{M}_1^{\mathrm{d}} \times \mathbb{F}_{p_2})^{\text{s.s.}}\Big),$$

$$\text{Components}\Big((\mathcal{M}_2^{\mathrm{ss}} \times \mathbb{F}_{p_1})^{\text{s.s.}}\Big) \simeq \text{Crossing points}\Big((\mathcal{M}_1^{\mathrm{d}} \times \mathbb{F}_{p_2})^{\text{s.s.}}\Big),$$

$$\text{Crossing points}\Big((\mathcal{M}_2^{\mathrm{d}} \times \mathbb{F}_{p_1})^{\text{s.s.}}\Big) \simeq \text{Components}\Big((\mathcal{M}_1^{\mathrm{ss}} \times \mathbb{F}_{p_2})^{\text{s.s.}}\Big),$$

$$\text{Crossing points}\Big((\mathcal{M}_2^{\mathrm{ss}} \times \mathbb{F}_{p_1})^{\text{s.s.}}\Big) \simeq \text{Crossing points}\Big((\mathcal{M}_1^{\mathrm{ss}} \times \mathbb{F}_{p_2})^{\text{s.s.}}\Big).$$

Here we have written $(\mathcal{M}_1^{\mathrm{ss}} \times \mathbb{F}_{p_2})^{\text{s.s.}}$ for the supersingular locus of the fiber over p_2 of $\mathcal{M}_1^{*_1}$, where $*_1 = \mathrm{ss}$, for example. These results are in the spirit of those of Ribet [**25**], [**26**], who considers components and their crossing points for the fibers of Shimura curves and modular curves at primes of bad reduction.

§5. Endomorphism algebras and points of proper intersection.

In this section, we consider the points of intersection of the special cycles in the supersingular locus, using the information obtained in section 4 about the structure of this locus. In particular, in the decomposition

$$\mathcal{Z}(d_1,\omega_1)\times_{\mathcal{M}}\dots\times_{\mathcal{M}}\mathcal{Z}(d_r,\omega_r)=\coprod_{\substack{T\in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{\geq 0}\\ \mathrm{diag}(T)=(d_1,\dots,d_r)}}\mathcal{Z}(T,\omega)$$

of (3.6), we fix a matrix T and we obtain a criterion, in terms of T, for $\mathcal{Z}(T,\omega)$ to consist of isolated points. We also show that, even when $\det(T)\neq 0$, there can be components of the supersingular locus in the image of $\mathcal{Z}(T,\omega)$ in $\mathcal{M}^{\rm ss}$.

We retain the notation of sections 2–4, and we begin by obtaining information about the endomorphism rings of various types of admissible lattices.

For an admissible lattice L, let $\mathcal{O}_L=\mathrm{End}_W(L,F)$ be the $\mathbb{Z}_p$-algebra of W-linear endomorphisms of L which commute with F. Note that $\mathrm{End}_W(L,F)$ is an order in the $\mathbb{Q}_p$-algebra $\mathrm{End}_{\mathcal{K}}(\mathcal{L},F)=C'_p=C'\otimes_{\mathbb{Q}}\mathbb{Q}_p\simeq M_2(B'_p)$. Also, observe that $\mathrm{End}_W(L,F)=\mathrm{End}_W(F^jL,F)$ for any $j\in\mathbb{Z}$. If $L=c\cdot L^\perp$ is special, we have

$$(F^jL)=p^jc\cdot(F^jL)^\perp. \tag{5.1}$$

Thus, to determine $\mathcal{O}_L$ for $L\in X_0$ we may assume $L=L^\perp$.

By Lemma 4.18, we immediately have the following.

Lemma 5.1. *For any superspecial lattice* $L\in X_0$, *(resp. any distinguished lattice* $\tilde L\in\tilde X$ *)* $\mathrm{End}_W(L,F)$ *(resp.* $\mathrm{End}_W(\tilde L,F)$ *) is a maximal order in* C'_p.

In either case, this order is isomorphic to $M_2(\mathcal{O}')$, where $\mathcal{O}'=\mathbb{Z}_{p^2}+\Pi\mathbb{Z}_{p^2}$, as in section 4. The map $M_2(\mathcal{O}')\longrightarrow M_2(\mathbb{F}_{p^2})$ given by reduction modulo Π can be described as follows. Consider the case of $L\in X_0$. As in section 4, let L_0 be the fixed points of $p^{-1}F^2$ on L. Then define

$$\mathrm{red}_L:\mathrm{End}_W(L,F)\longrightarrow \mathrm{End}_{\mathbb{F}_{p^2}}(L_0/FL_0)\simeq M_2(\mathbb{F}_{p^2}) \tag{5.2}$$

as the composition

$$\mathrm{End}_W(L,F)\xrightarrow{\sim}\mathrm{End}_{\mathbb{Z}_{p^2}}(L_0,F)\longrightarrow \mathrm{End}_{\mathbb{F}_{p^2}}(L_0/FL_0). \tag{5.3}$$

This map is surjective. The surjective reduction map for $\tilde L\in\tilde X$

$$\mathrm{red}_{\tilde L}:\mathrm{End}_W(\tilde L,F)\longrightarrow \mathrm{End}_{\mathbb{F}_{p^2}}(\tilde L_0/F\tilde L_0)\simeq M_2(\mathbb{F}_{p^2}) \tag{5.4}$$

is defined analogously. Note that $\tilde{L}/F\tilde{L} \simeq \tilde{L}_0/F\tilde{L}_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$, and that the endomorphism $\bar{\alpha}$ induced on $\tilde{L}/F\tilde{L}$ by $\alpha \in \mathrm{End}_W(\tilde{L}, F)$ is $\mathrm{red}_{\tilde{L}}(\alpha) \otimes 1$.

Next, suppose that $L \in X \setminus X_0$, and let $\tilde{L}$ be the unique distinguished lattice associated to L by Proposition 4.8. Recall that $F\tilde{L} = FL + VL$. In particular, for every element $\alpha \in \mathrm{End}_W(L, F)$, $\alpha F\tilde{L} \subset F\tilde{L}$, so that $\alpha\tilde{L} \subset \tilde{L}$, and there is a natural homomorphism which is injective,

$$\mathrm{End}_W(L, F) \hookrightarrow \mathrm{End}_W(\tilde{L}, F). \tag{5.5}$$

On the other hand, there is a unique line $\ell \subset \tilde{L}/F\tilde{L}$ such that $L = L(\ell)$ is the inverse image of ℓ in $\tilde{L}$.

Lemma 5.2. *Let $L \in X \setminus X_0$. With the notations introduced above,*

$$\mathrm{End}_W(L, F) = \{\ \alpha \in \mathrm{End}_W(\tilde{L}, F) \mid \bar{\alpha}(\ell) \subset \ell\ \}.$$

Here $\bar{\alpha}$ is the endomorphism of $\tilde{L}/F\tilde{L}$ induced by α. In fact, there are two possibilities.
(i) If $\ell \in \mathbb{P}_{\tilde{L}}(\mathbb{F}) - \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^4})$, then

$$\mathrm{End}_W(L, F) = (\mathrm{red}_{\tilde{L}})^{-1}(\mathbb{F}_{p^2} \cdot 1_2).$$

(ii) If $\ell \in \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^4}) - \mathbb{P}_{\tilde{L}}(\mathbb{F}_{p^2})$, then

$$\mathrm{End}_W(L, F) = (\mathrm{red}_{\tilde{L}})^{-1}(\mathbb{F}_{p^4}),$$

for some embedding $\mathbb{F}_{p^4} \hookrightarrow M_2(\mathbb{F}_{p^2})$.

Proof. As remarked above, the automorphism of $\tilde{L}/F\tilde{L} = \tilde{L}_0/F\tilde{L}_0 \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$ induced by $p^{-1}F^2$ is just $1 \otimes \sigma^2$. Since, for any $\alpha \in \mathrm{End}_W(\tilde{L}, F)$, $\bar{\alpha}$ commutes with this automorphism, $\bar{\alpha}(\ell) \subset \ell$ implies that $\bar{\alpha}(\sigma^2(\ell)) \subset \sigma^2(\ell)$. Since a non-scalar endomorphism can have at most two eigenlines, $\bar{\alpha}(\ell) \subset \ell$ and $\sigma^4(\ell) \neq \ell$ implies that $\bar{\alpha} = a \cdot 1_2$, for $a \in \mathbb{F}_{p^2}$. If $\sigma^4(\ell) = \ell$ but $\sigma^2(\ell) \neq \ell$, and if $\bar{\alpha}$ is not a scalar endomorphism, then ℓ and $\sigma^2(\ell)$ are the distinct eigenlines of $\bar{\alpha}$. Then $\mathbb{F}_{p^2}[\bar{\alpha}] \simeq \mathbb{F}_{p^4}$, and any endomorphism $\bar{\beta}$, with $\beta \in \mathrm{End}_W(L, F)$ must lie in this subfield of $M_2(\mathbb{F}_{p^2})$. $\square$

Note that the lattices in (ii) of lemma 5.2 are characterized intrinsically by the condition that $F^4L = p^2L$ but $F^2L \neq pL$. We let $X_{(ii)}$ be the set of lattices appearing in (ii) and $X_{(i)} = X \setminus X_{(ii)} \setminus X_0$ the set appearing in (i). Recall that $\mathcal{O}_L = \mathrm{End}_W(L, F)$ and $\mathcal{O}_{\tilde{L}} = \mathrm{End}_W(\tilde{L}, F)$. Then

$$\begin{aligned}
&\mathrm{red}_L(\mathcal{O}_L) = \mathrm{red}_L(\mathrm{End}_W(L, F)) \simeq M_2(\mathbb{F}_{p^2}) && \text{if } L \in X_0 \\
&\mathrm{red}_{\tilde{L}}(\mathcal{O}_L) = \mathrm{red}_{\tilde{L}}(\mathrm{End}_W(L, F)) \simeq \mathbb{F}_{p^4} && \text{if } L \in X_{(ii)} \\
&\mathrm{red}_{\tilde{L}}(\mathcal{O}_L) = \mathrm{red}_{\tilde{L}}(\mathrm{End}_W(L, F)) \simeq \mathbb{F}_{p^2} && \text{if } L \in X_{(i)}.
\end{aligned} \tag{5.6}$$

In particular, the endomorphism algebras of all the L's with $L \in X_{(i)}$ and with a given associated $\tilde{L}$ coincide. Any endomorphism of one such L preserves all lattices $L' \in X$ in the image of $\mathbb{P}_{\tilde{L}}$.

Recall that $C'_p = \mathrm{End}_{\mathcal{K}}(\mathcal{L}, F)$, and let

$$V'_p = \{\ x \in C'_p \mid x^* = x,\ \text{and } \mathrm{tr}^0(x) = 0\ \}. \tag{5.7}$$

Note that $\mathcal{O}_L$ and $\mathcal{O}_{\tilde{L}}$ are invariant under the involution $*$ on C'_p. Indeed, let $L^\perp = cL$ and $x \in \mathcal{O}_L$. Then

$$x^*(L^\perp) \subset L^\perp \quad , \quad \text{i.e. } x^*(L) \subset L \quad .$$

Similarly, if $F\tilde{L} = c\tilde{L}^\perp$ and $x \in \mathcal{O}_{\tilde{L}}$, then $x^*(\tilde{L}^\perp) \subset \tilde{L}^\perp$, i.e. $x^*(F\tilde{L}) \subset F\tilde{L}$, i.e., $x^*(\tilde{L}) \subset \tilde{L}$, since x^* commutes with F.

For $L \in X$ and for $\tilde{L} \in \tilde{X}$, let

$$N_L = \mathrm{End}_W(L, F) \cap V'_p \qquad \text{and} \qquad N_{\tilde{L}} = \mathrm{End}_W(\tilde{L}, F) \cap V'_p. \tag{5.8}$$

These are $\mathbb{Z}_p$-lattices in V'_p on which the quadratic form given by squaring, $x^2 = q(x) \cdot id$ is valued in $\mathbb{Z}_p$.

We now describe the reduction maps for distinguished and for superspecial lattices. We start with the case of distinguished lattices.

Lemma 5.3. *Let $\tilde{L} \in \tilde{X}$, and put $\mathfrak{n}_{\tilde{L}} = \mathrm{red}_{\tilde{L}}(N_{\tilde{L}})$. Then $\mathfrak{n}_{\tilde{L}}$ is equal to*

$$\{x = a \cdot 1_2;\ a \in \mathbb{F}_{p^2},\ a^\sigma = -a\}$$

and the $\mathbb{F}_p$-valued quadratic form q on $\mathfrak{n}_{\tilde{L}}$ is given by $x^2 = q(x) \cdot 1_2$, i.e. $q(x) = -a \cdot a^\sigma$. In particular, q does not represent 1 and hence the Clifford algebra $C(\mathfrak{n}_{\tilde{L}})$ is isomorphic to $\mathbb{F}_{p^2}$. The following diagram is commutative

$$\begin{array}{ccc} N_{\tilde{L}} & \xrightarrow{q} & \mathbb{Z}_p \\ {\scriptstyle \mathrm{red}_{\tilde{L}}}\downarrow & & \downarrow \\ \mathfrak{n}_{\tilde{L}} & \xrightarrow{q} & \mathbb{F}_p \end{array} \quad .$$

Proof. Replacing $\tilde{L}$ by $F^j\tilde{L}$ we may assume that $\tilde{L}$ is standard with $F\tilde{L} = p \cdot \tilde{L}^\perp$. The symplectic form $<\ ,\ >$ on $\mathcal{L}$ induces a nondegenerate alternating pairing

$$<\ ,\ >: \tilde{L}/F\tilde{L} \times \tilde{L}/F\tilde{L} \longrightarrow \mathbb{F} \quad . \tag{5.9}$$

This pairing descends to a non-degenerate alternating $\mathbb{F}_{p^2}$-bilinear pairing on $\tilde{L}_0/F\tilde{L}_0$ with values in $\mathbb{F}_{p^2}$. The induced involution on $\mathrm{End}_{\mathbb{F}_{p^2}}(\tilde{L}_0/F\tilde{L}_0)$ is compatible with the reduction map,

$$\mathrm{red}_{\tilde{L}}(x^*) = \mathrm{red}_{\tilde{L}}(x)^* \quad , \quad x \in \mathcal{O}_L \quad .$$

Now any endomorphism $\overline{x}$ of the 2-dimensional symplectic vector space $\tilde{L}_0/F\tilde{L}_0$ over $\mathbb{F}_{p^2}$ with $\overline{x}^* = \overline{x}$ is a scalar. Hence for $x \in N_{\tilde{L}}$ we get

$$\mathrm{red}(x) = a \cdot 1_2 \quad , \quad a \in \mathbb{F}_{p^2} \quad .$$

But $x \in N_{\tilde{L}}$ acts on $\tilde{L}_0/p\tilde{L}_0$ preserving the subspace $F\tilde{L}_0/p\tilde{L}_0$. Since x commutes with F, it acts on the subspace as $a^\sigma \cdot 1_2$. The condition $\mathrm{tr}^0(x) = 0$ implies therefore that $a = -a^\sigma$. Therefore we have proved that $\mathfrak{n}_{\tilde{L}}$ is contained in the subspace above. It is easy to see that we have in fact an equality. The remaining assertions are obvious. $\square$

Next we consider the case of superspecial lattices.

Lemma 5.4. *Let $L \in X_0$ and put $\mathfrak{n}_L = \mathrm{red}_L(N_L)$.*
(i) $\mathfrak{n}_L$ is isomorphic to

$$\begin{aligned} &\{\ x \in M_2(\mathbb{F}_{p^2}) \mid {}^t x^\sigma = x \ \textit{and}\ tr(x) = 0\ \} \\ =&\{\ x = \begin{pmatrix} a & b \\ b^\sigma & -a \end{pmatrix} \mid a \in \mathbb{F}_p,\ b \in \mathbb{F}_{p^2}\ \}. \end{aligned}$$

The $\mathbb{F}_p$-valued quadratic form q on $\mathfrak{n}_L$ is given by $x^2 = q(x) \cdot 1_2$, i.e., $q(x) = -(a^2 + bb^\sigma)$.
(ii) Let $C(\mathfrak{n}_L)$ be the Clifford algebra of the three dimensional quadratic space $\mathfrak{n}_L$. Then the natural map $C(\mathfrak{n}_L) \xrightarrow{\sim} M_2(\mathbb{F}_{p^2})$ is an isomorphism.
(iii) The following diagram is commutative.

$$\begin{array}{ccc} N_L & \xrightarrow{q} & \mathbb{Z}_p \\ {\scriptstyle \mathrm{red}_L}\downarrow & & \downarrow \\ \mathfrak{n}_L & \xrightarrow{q} & \mathbb{F}_p \end{array} \quad .$$

Proof. Replacing L by $F^j L$ we may assume $L^\perp = L$. On L_0/FL_0 we have the non-degenerate anti-hermitian form

$$(5.10) \qquad (\ ,\ ) : L_0/FL_0 \times L_0/FL_0 \longrightarrow \mathbb{F}_{p^2}$$

induced by the formula

$$(5.11) \qquad (v, w) = \langle \tilde{v}, F\tilde{w} \rangle \mod p \quad ,$$

where $\tilde{v}$ and $\tilde{w}$ are representatives of v and w in L_0. We may find a basis of L_0/FL_0 such that the induced involution on $M_2(\mathbb{F}_{p^2})$ is given by $x \mapsto {}^t x^\sigma$. Now the lemma is proved in a way similar to Lemma 5.3. above. $\square$

We now return to the points of intersection of the special cycles in the supersingular locus. Let $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})$ with $\det T \neq 0$ and $\omega \subset V(\mathbb{A}_f^p)^4$ with corresponding special cycle $\mathcal{Z}(T,\omega)$. Let $\xi \in \mathcal{Z}(T,\omega)$ correspond to the collection $(A_\xi, \iota, \lambda, \bar{\eta}^p; \mathbf{j})$. By Corollary 4.3, the point corresponding to the collection $(A_\xi, \iota, \lambda, \bar{\eta}^p)$ lies in $\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$. Thus, $\mathrm{End}^0(A_\xi, \iota)^{\mathrm{op}} = C_\xi^0 = C' \simeq M_2(B')$, where B' is the definite quaternion algebra over $\mathbb{Q}$ with discriminant $D(B)p$. The last isomorphism here can be chosen so that the Rosati involution corresponds to the involution $x \mapsto x' = {}^t x^\iota$ of $M_2(B')$. Then, as in (3.8.vi),

$$V_\xi^0 = V' \simeq \{x \in M_2(B') \mid x' = x \text{ and } \mathrm{tr}(x) = 0 \}. \tag{5.12}$$

The components $j_1, \dots, j_4$ of $\mathbf{j}$ lie in V', and therefore, in particular, we must have $T > 0$ if $\mathcal{Z}(T,\omega)$ is to be non-empty.

Let L be the contravariant Dieudonné module of the formal group $A_0(p)$, where we write $A_\xi(p) \simeq A_0(p)^4$, as in section 4. By choosing an isogeny of ξ with the chosen base point ξ_o we obtain, as in section 4, an identification $\mathcal{L} = L \otimes_W \mathcal{K}$ of its isocrystal with that of the base point. Then $L \in X$, and there is a natural algebra homomorphism

$$C_\xi \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{End}(A_\xi, \iota)^{\mathrm{op}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow \mathrm{End}_W(L, F) = \mathcal{O}_L \subset C_p'. \tag{5.13}$$

Let $N_L = \mathrm{End}_W(L,F) \cap V_p'$, as in (5.8) above. The collections of endomorphisms $\mathbf{j}$ induce collections of elements of $\mathrm{End}_W(L,F)$ and of V_p', which we will denote by the same letters. Let M be the $\mathbb{Z}_p$-submodule of N_L spanned by the components j_1, j_2, j_3, j_4 of $\mathbf{j}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} \{j_1, \dots, j_4\} & \subset & C_\xi & \longrightarrow & \mathrm{End}_W(L,F) & \supset & N_L & \supset & M \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ V' & \subset & C' & \longrightarrow & C_p' & \supset & V_p' & & \end{array} \tag{5.14}$$

Recall that $T = T_\xi \in \mathrm{Sym}_4(\mathbb{Z}_{(p)}) \subset \mathrm{Sym}_4(\mathbb{Z}_p)$ is the matrix of inner products of the elements $j_1, \dots, j_4$ with respect to the quadratic form on V_p'. Thus we have the following basic observation:

Lemma 5.5. *At a point of intersection $\xi \in \mathcal{Z}(T,\omega) \cap \mathcal{M}^{\mathrm{ss}}(\mathbb{F})$ with corresponding lattice $L \in X$, the matrix $T_\xi = T$ is represented by the lattice $N_L = \mathrm{End}_W(L,F) \cap$*

V'_p in the quadratic space V'_p. In fact, T is the matrix for the restriction of the quadratic form on N_L to the sublattice M spanned by $j_1, \dots, j_4$.

Suppose $L \in X \setminus X_0$ with associated distinguished lattice $\tilde{L}$. Recall that $\mathcal{O}_L \subset \mathcal{O}_{\tilde{L}}$ and let $\mathcal{O}_M$ be the $\mathbb{Z}_p$-subalgebra of $\mathcal{O}_{\tilde{L}} = \mathrm{End}_W(\tilde{L}, F)$ generated by $j_1, \dots, j_4$, i.e., by M. Also let $C(M)$ be the Clifford algebra of M. Let $\mathfrak{n}_L = \mathrm{red}_{\tilde{L}}(N_L)$ and let $\mathfrak{m} = \mathfrak{m}_{\tilde{L}} = \mathrm{red}_{\tilde{L}}(M)$, so that

$$\begin{array}{ccccc} M & \subset & N_L & \subset & N_{\tilde{L}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{m}_{\tilde{L}} & \subset & \mathfrak{n}_L & \subset & \mathfrak{n}_{\tilde{L}}. \end{array} \tag{5.15}$$

Lemma 5.6. *Suppose that $L \in X \setminus X_0$ with associated distinguished lattice $\tilde{L}$.*
(i) The natural map $C(M) \to \mathcal{O}_M$ is an isomorphism.
(ii) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_M & \subset & \mathcal{O}_{\tilde{L}} \\ \downarrow & & \downarrow \\ \mathrm{red}_{\tilde{L}}(\mathcal{O}_M) & & \mathrm{red}_{\tilde{L}}(\mathcal{O}_{\tilde{L}}) \simeq M_2(\mathbb{F}_{p^2}) \\ || & & \cup \\ C(\mathfrak{m}_{\tilde{L}}) & \hookrightarrow & C(\mathfrak{n}_{\tilde{L}}) = \mathbb{F}_{p^2} \cdot 1_2, \end{array}$$

Proof. The inclusion of $C(\mathfrak{m}_{\tilde{L}})$ into $C(\mathfrak{n}_{\tilde{L}})$ is induced by the inclusion of quadratic spaces $\mathfrak{m} \subset \mathfrak{n}_{\tilde{L}}$. We obviously have a commutative diagram with surjective vertical arrows,

$$\begin{array}{ccc} C(M) & \longrightarrow & \mathcal{O}_M \\ \downarrow & & \downarrow \\ C(\mathfrak{m}) & \longrightarrow & \mathrm{red}_{\tilde{L}}(\mathcal{O}_M). \end{array} \tag{5.16}$$

But the upper horizontal arrow is surjective since both algebras are generated by M. This proves that the lower horizontal arrow is surjective. By the statement at the beginning it is also injective which proves the equality sign at the south-west corner of the diagram in (ii). The rest of the Lemma follows from Lemma 5.3. $\square$

Next let us consider the case when $L \in X_0$. We use somewhat similar notation: let $\mathfrak{n}_L = \mathrm{red}_L(N_L)$ and $\mathfrak{m} = \mathfrak{m}_L = \mathrm{red}_L(\mathcal{O}_M)$.

The same arguments yield:

Lemma 5.7. *Suppose that $L \in X_0$ is superspecial. There is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{O}_M & \subset & \mathcal{O}_L \\ \downarrow & & \downarrow \\ \mathrm{red}_L(\mathcal{O}_M) & \subset & \mathrm{red}_L(\mathcal{O}_L) \simeq M_2(\mathbb{F}_{p^2}) \\ || & & || \\ C(\mathfrak{m}_L) & & C(\mathfrak{n}_L) \end{array} .$$

Our next task will be to show that the matrix $T \bmod p$ in $M_4(\mathbb{F}_p)$ controls the size of $\mathfrak{m}$. More precisely, we now list the possibilities for $\mathfrak{m}$ in the non-superspecial and the superspecial case separately.

Lemma 5.8. *Let $L \in X \setminus X_0$ with associated $\tilde{L} \in \tilde{X}$. The possibilities for $\mathfrak{m} = \mathfrak{m}_{\tilde{L}}$ are the following:*

(i) *If $\dim_{\mathbb{F}_p} \mathfrak{m} = 1$, then T has rank 1 modulo p and does not represent 1.*
(ii) *If $\mathfrak{m} = 0$, then $p \mid T$.*

Proof. The first alternative corresponds to the case where $\mathfrak{m} = \mathfrak{n}_{\tilde{L}}$, by Lemma 5.3. The assertion now follows from Lemma 5.5. □

Lemma 5.9. *Let $L \in X_0$. The possibilities for $\mathfrak{m} = \mathfrak{m}_L$ and $C(\mathfrak{m}) \subset M_2(\mathbb{F}_{p^2})$ are the following:*

(i) *The rank of $T \bmod p$ is 3, or equivalently $\dim \mathfrak{m} = 3$. Then $C(\mathfrak{m}) \simeq M_2(\mathbb{F}_{p^2})$.*
(ii) *The rank of $T \bmod p$ is 2. Then $\dim \mathfrak{m} = 2$ and $C(\mathfrak{m}) \simeq M_2(\mathbb{F}_p)$.*
(iii) *The rank of $T \bmod p$ is 1 and $\dim \mathfrak{m} = 2$. Then $\mathfrak{m}$ is of the form $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{r}$ where $\mathfrak{r}$ is the radical and $\dim \mathfrak{m}_0 = \dim \mathfrak{r} = 1$. In this case*

$$C(\mathfrak{m}) \simeq C(\mathfrak{m}_0)^{\sim}[\epsilon]/(\epsilon^2)$$

where

$$C(\mathfrak{m}_0) \simeq \begin{cases} \mathbb{F}_p \oplus \mathbb{F}_p \\ \mathbb{F}_{p^2} \end{cases},$$

and the element ϵ acts on $C(\mathfrak{m}_0)$ by the nontrivial automorphism of order 2.
(iv) *The rank of $T \bmod p$ is 1 and $\dim \mathfrak{m} = 1$. Then*

$$C(\mathfrak{m}) \simeq \begin{cases} \mathbb{F}_p \oplus \mathbb{F}_p \\ \mathbb{F}_{p^2} \end{cases}.$$

(v) *$T \equiv 0 \bmod p$ and $\dim \mathfrak{m} = 1$. Then*

$$C(\mathfrak{m}) = \wedge(\mathfrak{m}) \simeq \mathbb{F}_p[\epsilon]/(\epsilon^2) .$$

(vi) *$\mathfrak{m} = 0$. Then $T \equiv 0 \bmod p$ and $C(\mathfrak{m}) = \mathbb{F}_p$ is in the center of $M_2(\mathbb{F}_{p^2})$.*

In cases (iii) resp. (iv), $\mathfrak{m}_0$ resp. $\mathfrak{m}$ is a nondegenerate line, so that the quadratic form on it is isomorphic to either x^2 or ax^2, with $a \in \mathbb{F}_p^\times \backslash \mathbb{F}_p^{\times,2}$, yielding a Clifford algebra $\mathbb{F}_p \oplus \mathbb{F}_p$ or $\mathbb{F}_{p^2}$.

Lemma 5.10. *In cases (iii) and (iv) above, when an $\mathbb{F}_{p^2}$ arises in the Clifford algebra $C(\mathfrak{m})$, this $\mathbb{F}_{p^2}$ is not central in $M_2(\mathbb{F}_{p^2})$.*

Proof. Choose $x \in \mathfrak{m}$ spanning a nondegenerate line for which $C(\mathbb{F}_p x) \simeq \mathbb{F}_{p^2}$. Then x is an endomorphism of the 2-dimensional $\mathbb{F}_{p^2}$-vector space $\tilde{L}_0/F\tilde{L}_0$ with $\mathrm{tr}(x) = 0$, and with $x^2 = q(x) \cdot id$ where $q(x) \notin \mathbb{F}_p^{\times,2}$. This last condition is equivalent to our hypothesis on the Clifford algebra. Thus, x has two distinct eigenvalues $\pm\sqrt{q(x)}$ on $\tilde{L}_0/F\tilde{L}_0$, and hence does not lie in the center. □

We can now describe the intersections of our special cycle with the supersingular locus.

Theorem 5.11. *Suppose that $\xi \in \mathcal{Z}(T,\omega)$ with image in the supersingular locus $\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$ with corresponding $L \in X$.*
(i) The rank of $T = T_\xi$ modulo p is at most 3.
(ii) If T_ξ represents 1, then $L \in X_0$ and ξ is a point of proper intersection.
(iii) If $L \in X \backslash X_0$, with associated distinguished lattice $\tilde{L}$, the whole distinguished $\mathbb{P}_{\tilde{L}}$ associated to $\tilde{L}$ in the supersingular locus, and passing through ξ, occurs in $\mathcal{Z}(T,\omega)$. In particular, ξ is not a point of proper intersection.

Proof. The reduction of $T = T_\xi$ modulo p is the matrix for the quadratic form on the images of $j_1, \dots, j_4$ in $\mathfrak{m} = \mathfrak{m}_L$, resp. $\mathfrak{m} = \mathfrak{m}_{\tilde{L}}$, and $\mathfrak{m}$ has dimension at most 3. This proves (i). If $L \in X \setminus X_0$, then $T = T_\xi$ does not represent 1, by Lemma 5.8. Furthermore in this case by Lemma 5.3.

$$C(\mathfrak{m}) \subset \mathbb{F}_{p^2} \cdot 1 \subset \mathrm{red}_{\tilde{L}}(\mathcal{O}_{L'}) \quad ,$$

for any $L' \in \mathbb{P}_{\tilde{L}}$, which is not superspecial. This implies $M \subset \mathcal{O}_M \subset \mathcal{O}_{L'}$. If now $L' \in \mathbb{P}_{\tilde{L}}$ is superspecial it follows that $M \subset \mathcal{O}_M \subset \mathcal{O}_{L'}$ by specialization. □

It remains to consider the cases where T does not represent 1. We first treat the case when $p \mid T$.

Theorem 5.12. *Suppose that $p \mid T$ and that $\xi \in \mathcal{Z}(T,\omega)$ has image in $\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$ with corresponding $L \in X_0$. Then ξ is not a point of proper intersection. More precisely:*
(i) If $\mathfrak{m} = \mathrm{red}_L(M) = 0$ then each of the $p+1$ distinguished $\mathbb{P}^1$'s through $pr(\xi) \in$

$\mathcal{M}^{\rm ss}$ occurs in the image of $\mathcal{Z}(T,\omega)$, i.e., for every distinguished $\tilde{L}$ with $\tilde{L} \supset L \supset F\tilde{L}$, we have $M \subset \mathrm{End}_W(\tilde{L}, F)$; furthermore $\mathrm{red}_{\tilde{L}}(M) = 0$.
(ii) If $\mathfrak{m} = \mathrm{red}_L(M)$ is a null line in $\mathfrak{n}_L$, then there is a unique distinguished $\tilde{L}$ with $\tilde{L} \supset L \supset F\tilde{L}$ and with $M \subset \mathrm{End}_W(\tilde{L}, F)$; furthermore $\mathrm{red}_{\tilde{L}}(M) = 0$. Hence there is a unique distinguished $\mathbb{P}^1$ passing through $pr(\xi) \in \mathcal{M}^{\rm ss}$ and contained in the image of $\mathcal{Z}(T,\omega)$.

Proof. We may assume that $L^\perp = L$. First suppose that $\mathfrak{m}$ is a null line, and choose $x_0 \in M$ such that $\bar{x}_0 = \mathrm{red}_L(x_0)$ spans $\mathfrak{m} = \mathrm{red}_L(M)$. The endomorphism $\bar{x}_0$ of the two dimensional vector space L_0/FL_0 satisfies $\bar{x}_0^2 = 0$ but $\bar{x}_0 \neq 0$. Thus $\mathrm{im}(\bar{x}_0)$ is a line in L_0/FL_0.

Lemma 5.13. *Assume that $L = L^\perp$. The lattice $\tilde{L}$ defined by $F\tilde{L} = x_0(L) + FL$ lies in $\tilde{X}$ and $L \in \mathbb{P}_{\tilde{L}}$. Moreover, $M \subset \mathrm{End}_W(\tilde{L}, F)$, and $\mathrm{red}_{\tilde{L}}(M) = 0$.*

Proof. Since x_0 commutes with F, we clearly have $F(F\tilde{L}) = x_0(FL) + F^2L = x_0(FL) + pL \subset F\tilde{L}$. Similarly one sees that $V(F\tilde{L}) \subset F\tilde{L}$, i.e., $pF\tilde{L} \subset F(F\tilde{L})$, hence $F\tilde{L}$ is admissible. Note that $(F\tilde{L})^\perp \supset L^\perp = L \supset F\tilde{L}$ with $[(F\tilde{L})^\perp : L] = [L : F\tilde{L}] = 1$.

To show that $F\tilde{L}$ is distinguished, it will suffice to prove that $F(F\tilde{L}) \subset p(F\tilde{L})^\perp$, i.e., that $< F^2\tilde{L}, F\tilde{L} > \subset pW$. But

$$\begin{aligned} < F^2\tilde{L}, F\tilde{L} > &= < x_0(FL) + pL, x_0(L) + FL > \\ &\subset < x_0(FL), x_0(L) > + pW \\ &= < FL, x_0^2(L) > + pW \\ &\subset < FL, FL > + pW \\ &\subset \ pW. \end{aligned} \tag{5.17}$$

Here we have used the fact that $x_0^* = x_0$ and that $\bar{x}_0^2 = 0$, i.e., that $x_0^2(L) \subset FL$. We conclude that $F\tilde{L} \in \tilde{X}$ and hence also $\tilde{L} \in \tilde{X}$.

Next, we must show that every element of M preserves $\tilde{L}$ or, equivalently, $F\tilde{M}$. In fact, we show that $M \cdot \tilde{L} \subset F\tilde{L}$, so that $\mathrm{red}_{\tilde{L}}(M) = 0$. First consider the reduction sequence

$$0 \longrightarrow M_0 \longrightarrow M \overset{\mathrm{red}_L}{\longrightarrow} \mathbb{F}_p \cdot \bar{x}_0 \longrightarrow 0, \tag{5.18}$$

where

$$M_0 = \{\ y \in M \mid y(L) \subset FL\ \}. \tag{5.19}$$

It suffices to prove the inclusions $x_0(\tilde{L}) \subset F\tilde{L}$ and $y(\tilde{L}) \subset F\tilde{L}$ for all $y \in M_0$. Recall that, for $x \in M$, $x^2 = q(x) \cdot id$. Since $p \mid T_\xi$, the resulting quadratic form on M/pM is identically zero, and so $C(M/pM) = \wedge(M/pM)$. In particular, for any x_1 and $x_2 \in M$, $x_1x_2 \equiv -x_2x_1 \mod p$, i.e.,

$$x_1x_2(L) \subset x_2x_1(L) + pL. \tag{5.20}$$

Now, for $y \in M_0$,

$$\begin{aligned} y(F\tilde{L}) &= yx_0(L) + y(FL) \\ &\subset x_0y(L) + pL + F(y(L)) \\ &\subset x_0(FL) + F^2L \\ &\subset F(x_0(L) + FL) = F^2\tilde{L}. \end{aligned} \tag{5.21}$$

Next, observe that $x_0^2 = q(x_0)\cdot id$ and $q(x_0) \equiv 0 \mod p$ implies that $x_0^2(L) \subset pL$, not just FL. Thus

$$\begin{aligned} x_0(F\tilde{L}) &= x_0^2(L) + Fx_0(L) \\ &\subset pL + Fx_0(L) \\ &= F(FL + x_0(L)) = F^2\tilde{L}. \end{aligned} \tag{5.22}$$

This completes the proof of the Lemma. $\square$

To finish the proof of (ii), we show that the distinguished lattice $F\tilde{L}$ constructed in Lemma 5.13 is unique. Note that $\ker(\bar{x}_0) = \operatorname{im}(\bar{x}_0)$. If $\tilde{L}' = W \cdot u + FL$ is another distinguished lattice, whose image $\ell' = \tilde{L}'/FL$ is distinct from $\ker(\bar{x}_0)$, then

$$\bar{x}_0(\ell') = \operatorname{im}(\bar{x}_0) \neq \ell', \tag{5.23}$$

so that $\tilde{L}'$ is not preserved by x_0.

Now suppose that $\mathrm{red}_L(M) = 0$, i.e., that $M \cdot L \subset FL$. Let $F\tilde{L} = W \cdot u + FL$ be any distinguished lattice with $L \supset F\tilde{L} \supset FL$. We want to show that, for any $x \in M$, $x(\tilde{L}) \subset F\tilde{L} = p\tilde{L}^\perp$ or, equivalently $x(F\tilde{L}) \subset F^2\tilde{L} = p \cdot (F\tilde{L})^\perp$. But now

$$\begin{aligned} < x(F\tilde{L}), F\tilde{L} > &= < Wx(u) + Fx(L), Wu + FL > \\ &\subset \; W < x(u), u > + pW. \end{aligned} \tag{5.24}$$

But now, since $x^* = x$,

$$< x(u), u > = < u, x(u) > = \; - < x(u), u > \tag{5.25}$$

so that $< x(u), u >= 0$. Thus $< x(F\tilde{L}), F\tilde{L} > \subset pW$, i.e., $x(F\tilde{L}) \subset pF(\tilde{L})^\perp = F^2\tilde{L}$, as required. This concludes the proof of Theorem 5.12. $\square$ $\square$

We now turn to the case when $p \nmid T$ but T does not represent 1.

Theorem 5.14. *Suppose that* $p \nmid T$ *and that* T *does not represent* 1 *. Let* $\xi \in \mathcal{Z}(T,\omega)$ *with* $pr(\xi) \in \mathcal{M}^{ss}(\mathbb{F})$ *and with corresponding* $L \in X_0$ *. Then* ξ *is not a point of proper intersection. More precisely:*
(i) If $\dim_{\mathbb{F}_p} \mathfrak{m} = 1$ *and* $\mathfrak{m}$ *does not represent* 1 *, then precisely two of the* $p+1$ *distinguished* $\mathbb{P}_{\tilde{L}}$ *'s through* $pr(\xi)$ *occur in the image of* $\mathcal{Z}(T,\omega)$ *. These are the only two distinguished lattices* $\tilde{L}_1$ *and* $\tilde{L}_2$ *with* $\tilde{L}_i \supset L \supset F\tilde{L}_i$ *and with* $M \subset \mathrm{End}_W(\tilde{L}_i, F)$ $(i=1,2)$ *. Furthermore* $\mathrm{red}_{\tilde{L}_i}(M) \neq (0)$ *,* $i=1,2$ *.*
(ii) If $\dim_{\mathbb{F}_p} \mathfrak{m} = 2$ *and* $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{r}$ *as in Lemma 5.9, (iii), where* $\mathfrak{m}$ *does not represent* 1 *, then precisely one of the* $p+1$ *distinguished* $\mathbb{P}_{\tilde{L}}$ *'s through* $pr(\xi)$ *occurs in* $pr(\mathcal{Z}(T,\omega))$ *. It is the only distinguished lattice* $\tilde{L}_1$ *with* $\tilde{L}_1 \supset L \supset F\tilde{L}_1$ *and with* $M \subset \mathrm{End}_W(\tilde{L}_1, F)$ *. Furthermore* $\mathrm{red}_{\tilde{L}_1}(M) \neq (0)$ *.*

Proof. We may again assume $L^\perp = L$. We first consider case (i). Choose $x_0 \in M$ such that $\overline{x}_0 = \mathrm{red}_L(x_0)$ spans $\mathfrak{m} = \mathrm{red}_L(M)$. Then $\overline{x}_0$ is an automorphism of L_0/FL_0 with

$$\overline{x}_0^2 = \varepsilon \cdot 1 \quad ,$$

where $\varepsilon \in \mathbb{F}_p^\times \setminus \mathbb{F}_p^{\times,2}$. Furthermore, by Lemma 5.8., $\overline{x}_0$ is not central. Hence $\overline{x}_0$ has two distinct eigenvalues $\varepsilon_1 = \sqrt{\varepsilon}$ and $\varepsilon_2 = -\sqrt{\varepsilon}$ in $\mathbb{F}_{p^2}$. Let E_1 and E_2 be the corresponding eigenspaces and let $F\tilde{L}_1$ and $F\tilde{L}_2$ be the corresponding lattices in $\mathcal{L}$,

$$FL \subset F\tilde{L}_i \subset L \quad , \quad i = 1,2 \quad .$$

Since $\overline{x}_0$ commutes with F and V , the eigenspaces E_1 and E_2 are preserved by F and V , hence $F\tilde{L}_1$ and $F\tilde{L}_2$ are admissible. To see that $F\tilde{L}_1$ and $F\tilde{L}_2$ are distinguished, it suffices to see that $F\tilde{L}_i \subset p \cdot \tilde{L}_i^\perp$, i.e., that

$$\langle F\tilde{L}_i, \tilde{L}_i \rangle \subset p \cdot W \quad , \quad i = 1,2 \quad .$$

Equivalently we have to see that the eigenspaces E_1 and E_2 are isotropic with respect to the antihermitian form (5.11) on L_0/FL_0 . If $v \in E_i$, then $\overline{x}_0 v = \varepsilon_i \cdot v$ and

$$\varepsilon \cdot (v,v) = (\varepsilon v, v) = (\overline{x}_0^2 v, v) = (\overline{x}_0 v, \overline{x}_0 v) = (\varepsilon_i v, \varepsilon_i v) = -\varepsilon \cdot (v,v).$$

It follows that $F\tilde{L}_1$ and $F\tilde{L}_2$ are distinguished.

The lattices $\tilde{L}_i = F^{-1}(F\tilde{L}_i) \in \tilde{X}$ for $i=1$ and 2 are the distinguished lattices appearing in the statement of (i). We have $\mathrm{red}_{\tilde{L}_i}(M) \neq 0$ since $\overline{x}_0$ induces an automorphism of the eigenspace E_i . On the other hand any $y \in M$ with $\mathrm{red}_L(y) = 0$, i.e., with $y(L) \subset FL$, also satisfies $y(\tilde{L}_i) \subset L \subset \tilde{L}_i$. It follows that $M \subset \mathrm{End}_W(\tilde{L}_i, F)$, hence (i).

Now we consider case (ii). Let $\overline{x}_0 \in \mathfrak{m}_0$ with $\overline{x}_0^2 = \varepsilon \cdot 1$, for $\varepsilon \in \mathbb{F}_p^\times \setminus \mathbb{F}_p^{\times,2}$, and let $\overline{y}_0$ be a generator of the radical $\mathfrak{r}$. Then $\overline{x}_0 \overline{y}_0 = -\overline{y}_0 \overline{x}_0$. Therefore $\overline{y}_0$ maps the eigenspace E_1 of $\overline{x}_0$ in L_0/FL_0 to the eigenspace E_2 and the eigenspace E_2 to E_1. Since $\overline{y}_0^2 = 0$, but $\overline{y}_0 \neq 0$, precisely one of the two eigenspaces is annihilated by $\overline{y}_0$. The corresponding lattice is distinguished and yields as in case (i) the lattice $\tilde{L}_1$ appearing in the statement of (ii). $\square$

Corollary 5.15. *Let $\xi \in \mathcal{Z}(T,\omega) \subset \mathcal{Z}(d_1,\omega_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(d_r,\omega_r)$, where $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})_{>0}$ with $n_1 + \cdots + n_r = 4$, cf. (3.1) and (3.6). Then ξ is a point of proper intersection if and only if its fundamental matrix $T = T_\xi$ is non-singular and represents 1 over $\mathbb{Z}_p$. In this case ξ is supersingular and superspecial.*

A topic we have not touched upon in the present paper is to describe the shape of the intersection of our cycles in the case of improper intersection, or, equivalently, to describe, for $T \in \mathrm{Sym}_4(\mathbb{Q})_{>0}$, the cycle $\mathcal{Z}(T,\omega)$ when its dimension is positive. We refer to the companion paper [**21**] to the present one for more information on this topic.

§6. Intersection multiplicities.

In this section we consider the intersection multiplicity at a point of proper intersection. More precisely we return to the setup of the third section, i.e., we fix a decomposition $4 = n_1 + \cdots + n_r$, where $n_i \geq 1$ for all i, elements $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})_{>0}$ and $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$ giving rise to special cycles $\mathcal{Z}(d_1, \omega_1), \ldots, \mathcal{Z}(d_r, \omega_r)$. We fix a point $\xi \in \mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r)$ with $\det(T_\xi) \neq 0$ and where $T = T_\xi$ represents 1 over $\mathbb{Z}_p$. Let ξ correspond to $(A, \iota, \lambda, \overline{\eta}^p; \mathbf{j}_1, \ldots, \mathbf{j}_r)$. Since $\det(T) \neq 0$ and since T represents 1 over $\mathbb{Z}_p$, the associated Dieudonné module L is superspecial and corresponds to a formal group $\mathcal{A}$ of dimension 2 and height 4, with a collection of endomorphisms $\mathbf{j} = (j_1, \ldots, j_4)$ spanning a $\mathbb{Z}_p$-submodule M of rank 4 in $\mathrm{End}_W(L, F)$. By changing the trivialization of the rational Dieudonné module we may assume that $L = L^\perp$, i.e., that $\mathcal{A}$ is equipped with a principal quasi-polarization $\lambda_{\mathcal{A}}$. An immediate application of the theorem of Serre and Tate shows that the formal completions at ξ of $\mathcal{M}$ and of its closed subschemes may be interpreted as versal deformation spaces, i.e., with obvious notation,

$$\hat{\mathcal{M}}_\xi = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}) \tag{6.1}$$

$$\hat{\mathcal{Z}}(d_i, \omega_i)_\xi = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}; \mathbf{j}_i) \tag{6.2}$$

$$\begin{aligned} \hat{\mathcal{Z}}(T, \omega)_\xi &= \big(\mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r)\big)\hat{}_\xi \\ &= \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}; \mathbf{j}) = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}; M). \end{aligned} \tag{6.3}$$

Here $\omega = \omega_1 \times \cdots \times \omega_r$. Recall that any $x \in M$ satisfies $x^* = x$ and $tr(x) = 0$, that the quadratic form on M is given by $x^2 = q(x) \cdot id$, and that T is the matrix for that quadratic form with respect to the basis $j_1, \ldots, j_4$. Since T represents 1 over $\mathbb{Z}_p$, there exists an $x_0 \in M$ such that $x_0^2 = id$. The quadratic lattice M can be written as $M = \mathbb{Z}_p \cdot x_0 + M_0$, where $M_0 = x_0^\perp$. Moreover, if $x \in M_0$, then $xx_0 = -x_0 x$, since the subalgebra of $\mathrm{End}_W(L, F)$ generated by M is the image of $C(M)$, the Clifford algebra of M.

The idempotents $e_1 = \frac{1}{2}(1 + x_0)$, $e_2 = \frac{1}{2}(1 - x_0)$ – recall that $p \neq 2$ – give a splitting $\mathcal{A} \simeq \mathcal{A}_1 \times \mathcal{A}_2$, with $\mathcal{A}_1 = e_1 \mathcal{A}$ and $\mathcal{A}_2 = e_2 \mathcal{A}$ of dimension 1 and height 2. If $x \in M_0$, $xe_1 = e_2 x$, and so M_0 can be viewed as a submodule of $Hom(\mathcal{A}_1, \mathcal{A}_2)$. Let $L_i = e_i L$ be the Dieudonné module of $\mathcal{A}_i$. Then $L = L_1 \oplus L_2$. Furthermore L_1 and L_2 are paired trivially under the symplectic pairing on L. Indeed, if $v_1 \in L_1$ and $v_2 \in L_2$, we have

$$< v_1, v_2 > = < e_1 v_1, e_2 v_2 > = < v_1, e_1 e_2 v_2 > = 0 \quad ,$$

since $e_1^* = e_1$. It follows that $< \,,\, >$ induces a perfect symplectic pairing $< \,,\, >_i$ on L_i, i.e., $\mathcal{A}_1$ and $\mathcal{A}_2$ are equipped with principal quasi-polarizations. Since a principal quasi-polarization on a p-divisible formal group of dimension 1 and height 2 deforms automatically we obtain a natural identification

$$\mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}; M) = \mathrm{Def}(\mathcal{A}_1, \mathcal{A}_2; M_0) \quad .$$

The length $e(\xi)$ of the local Artin ring appearing on the right was determined by Gross and Keating in section 5 of [**7**]. Since we have assumed in all of the above that $p \neq 2$, we may as well continue to make this assumption, although Gross and Keating do not. Choose a basis $\psi_1,\ \psi_2,\ \psi_3$ for M_0 such that

$$q(u_1\psi_1 + u_2\psi_2 + u_3\psi_3) = \epsilon_1 p^{a_1} u_1^2 + \epsilon_2 p^{a_2} u_2^2 + \epsilon_3 p^{a_3} u_3^2,$$

with $0 \leq a_1 \leq a_2 \leq a_3$. Thus, over $\mathbb{Z}_p$, T is equivalent to the diagonal matrix $\mathrm{diag}(1, \epsilon_1 p^{a_1}, \epsilon_2 p^{a_2}, \epsilon_3 p^{a_3})$.

Proposition 6.1. (Gross, Keating, [**7**], Proposition 5.4) *If $a_1 + a_2$ is even, then $e(\xi) = e_p(T_\xi)$ is equal to:*

$$\sum_{i=0}^{a_1-1} (i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-2)/2} (a_1+1)(2a_1+a_2+a_3-4i)p^i$$
$$+ \frac{1}{2}(a_1+1)(a_3-a_2+1)p^{(a_1+a_2)/2}.$$

If $a_1 + a_2$ is odd, then $e(\xi) = e_p(T_\xi)$ is equal to:

$$\sum_{i=0}^{a_1-1} (i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-1)/2} (a_1+1)(2a_1+a_2+a_3-4i)p^i.$$

Corollary 6.2. *The cycles $\mathcal{Z}(d_1, \omega_1), \ldots, \mathcal{Z}(d_r, \omega_r)$ intersect transversally at the point ξ if and only if* $\mathrm{ord}(\det T_\xi) = 1$.

Proof. The above formulas show that $e(\xi) = 1$ if and only if $\mathrm{ord}(\det T_\xi) = 1$. If this is the case, the cycles $\mathcal{Z}(d_i, \omega_i)$ have to be irreducible and reduced locally at ξ, and the intersection multiplicity in the sense of Serre, which is bounded by the length, is equal to 1. In this case the cycles $\mathcal{Z}(d_i, \omega_i)$ are all regular at ξ and their tangent spaces give a direct sum decomposition of the tangent space of $\mathcal{M}$ at ξ. For all this, cf. [**5**], Prop. 8.2, and Example 8.2.1.

At this point we have completely answered the question a) at the end of section 3. What is not clear is whether the length $e(\xi)$ is indeed the intersection multiplicity of $\mathcal{Z}(d_1, \omega_1), \ldots, \mathcal{Z}(d_r, \omega_r)$ at ξ. This is the content of the question b) of section 3.

Conjecture 6.3. *Let ξ be an isolated intersection point of $\mathcal{Z}(d_1, \omega_1), \ldots, \mathcal{Z}(d_r, \omega_r)$. Then*

$$(\mathcal{O}_{\mathcal{Z}(d_1,\omega_1)} \overset{\mathbb{L}}{\otimes} \cdots \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\mathcal{Z}(d_r,\omega_r)})_\xi = (\mathcal{O}_{\mathcal{Z}(d_1,\omega_1)} \otimes \cdots \otimes \mathcal{O}_{\mathcal{Z}(d_r,\omega_r)})_\xi \quad ,$$

hence $e(\xi)$ *is the intersection multiplicity of* $\mathcal{Z}(d_1,\omega_1),\ldots,\mathcal{Z}(d_r,\omega_r)$ *at* ξ *.*

We stress that this conjecture is reasonable only because $\mathcal{M}$ is smooth over $\operatorname{Spec}\mathbb{Z}_{(p)}$. Indeed, Genestier [**6**] has shown that in the Drinfeld-Cherednik situation of bad reduction the analogues of the special cycles considered here may have embedded components. On the other hand, assume in our situation that $\mathcal{Z}(d_i,\omega_i)$ is an intersection of n_i divisors in $\mathcal{M}$. Then if ξ is an isolated intersection point of $\mathcal{Z}(d_1,\omega_1),\ldots,\mathcal{Z}(d_r,\omega_r)$ it follows that each partial intersection $\mathcal{Z}(\underline{d}_{i_1},\omega_{i_1})\cap\ldots\cap\mathcal{Z}(d_{i_s},\omega_{i_s})$ ($1\leq i_1\leq\ldots\leq i_s\leq r$) is locally at ξ a complete intersection. Hence it also follows that the length $e(\xi)$ is the intersection multiplicity of $\mathcal{Z}(d_1,\omega_1),\ldots,\mathcal{Z}(d_r,\omega_r)$ at ξ, and the above conjecture holds true.

Remark 6.4. Assume that $\xi\in\mathcal{Z}(d_1,\omega_1)\cap\cdots\cap\mathcal{Z}(d_r,\omega_r)$ is a point with fundamental matrix $T=T_\xi$ which is non-singular and represents over $\mathbb{Z}_p$ a unit $\varepsilon\in\mathbb{Z}_p^\times\setminus\mathbb{Z}_p^{\times,2}$. Therefore there exists $x_1\in M$ such that $x_1^2=\varepsilon\cdot\mathrm{id}$. Hence we obtain an action of $\mathbb{Z}_{p^2}=\mathbb{Z}_p[\sqrt{\varepsilon}]$ on $\mathcal{A}$,

$$\alpha:\mathbb{Z}_{p^2}\longrightarrow\operatorname{End}(\mathcal{A})\quad .$$

We may write M in the form $M=\mathbb{Z}_p\cdot x_1+M_1$, where $M_1=x_1^\perp$. For $x\in M_1$ we have $xx_1=-x_1x$. Comparing with the definitions in the companion paper to this one, we see that $(\mathcal{A},\alpha)$ is precisely one of the formal groups with $\mathbb{Z}_{p^2}$-action considered there [] and that the elements of M_1 are *special endomorphisms* in the sense of that paper. In particular, the formal completion of $\mathcal{Z}(d_1,\omega_1)\cap\cdots\cap\mathcal{Z}(d_r,\omega_r)$ at ξ coincides with the formal completion of the corresponding subvariety of the Hilbert-Blumenthal surface considered in [**21**].

§7. The total contribution of isolated points.

In this section we will consider the total contribution of the points of proper intersection of our special cycles. Using our previous results and a counting argument, we are able to give an explicit formula.

We return to the global situation of sections 1 and 2 and fix data as follows. We assume as always that $p \nmid 2D(B)$ and that $K = K^p \cdot K_p$ where K_p is the standard maximal compact subgroup (see the end of section 4), and where K^p is neat. We then have the moduli scheme $\mathcal{M} = \mathcal{M}_{K^p}$ which is smooth over $\operatorname{Spec} \mathbb{Z}_{(p)}$. As in section 3, we fix $n_1, \ldots , n_r$ with $1 \leq n_i \leq 4$ and with $n_1 + \ldots + n_r = 4$. For $i = 1, \ldots , r$, choose positive definite matrices $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Z}_{(p)})_{>0}$ and K^p-invariant open compact subsets $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$. We then have the cycles $\mathcal{Z}(d_i, \omega_i)$, $i = 1, \ldots , r$. We then define the contribution of the points of proper intersection to the intersection number of $\mathcal{Z}(d_1, \omega_1), \ldots , \mathcal{Z}(d_r, \omega_r)$ to be

$$(7.1) \qquad < \mathcal{Z}(d_1, \omega_1), \ldots , \mathcal{Z}(d_r, \omega_r) >_p^{\text{proper}} := \sum_{\xi} e(\xi) \quad .$$

Here the sum runs over the points of proper intersection ξ in $\mathcal{Z}(d_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(d_r, \omega_r)$, and $e(\xi)$ denotes the length of the local ring at ξ, as described in section 6. Note that, if Conjecture 6.3 were known to hold, this is also the local intersection multiplicity at ξ.

In the special case $r = 1$, we let $d_1 = T$, and we have the cycle $\mathcal{Z}(T, \omega)$, whose image in $\mathcal{M}$ lies in the supersingular locus $\mathcal{M}^{ss}$. Then $\mathcal{Z}(T, \omega)$ is a collection of isolated points if and only if T represents 1 over $\mathbb{Z}_p$ (Corollary 5.15). In this case we use the notation

$$(7.2) \qquad < \mathcal{Z}(T, \omega) >_p = \sum_{\xi \in \mathcal{Z}(T, \omega)} e(\xi) \ .$$

In general, by (3.6) and the analysis of the previous sections, we may write

$$(7.3) \qquad < \mathcal{Z}(d_1, \omega_1), \ldots , \mathcal{Z}(d_r, \omega_r) >_p^{\text{proper}} = \sum_T < \mathcal{Z}(T, \omega) >_p,$$

where the summation is over $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ which are nonsingular, represent 1 over $\mathbb{Z}_p$, and have diagonal blocks $d_1, \ldots , d_r$:

$$T = \begin{pmatrix} d_1 & & \cdots & \\ & d_2 & \cdots & \\ \vdots & \vdots & \ddots & \\ & & \cdots & d_r \end{pmatrix}.$$

We will now give more explicit expressions for the above entities. For this it will suffice to give an expression for (7.2). But the results of section 6 show that the intersection multiplicities $e(\xi)$ in the sum of (7.2) only depend on T and even only on its $\mathbb{Z}_p$-equivalence class. As in Proposition 6.1, we denote this integer by $e_p(T)$ and thus may write

$$< \mathcal{Z}(T,\omega) >_p = e_p(T) \cdot |\mathcal{Z}(T,\omega)(\mathbb{F})| \ . \tag{7.4}$$

It remains to determine the cardinality of $\mathcal{Z}(T,\omega)(\mathbb{F})$.

As before, let B' be the definite quaternion algebra with discriminant $D(B)p$, let $C' = M_2(B')$, and let $V' = \{x \in C' \mid x' = x \text{ and } tr(x) = 0 \}$. Let G' be as in (4.5). Recall that we also have fixed an isomorphism $G'(\mathbb{A}_f^p) \simeq G(\mathbb{A}_f^p)$, and a base point $\xi_o = (A_o, \iota_o, \lambda_o, \overline{\eta}_o^p) \in \mathcal{M}^{\rm ss}(\mathbb{F})$ such that the associated Dieudonné module $L_o \in X$ is superspecial, with stabilizer K'_p in $G'(\mathbb{Q}_p)$. Then, under the parametrization (4.7), the set of superspecial points in $\mathcal{M}^{\rm ss}(\mathbb{F})$ corresponds to the double coset space

$$G'(\mathbb{Q})\backslash\Big(G'(\mathbb{Q}_p)/K'_p \times G(\mathbb{A}_f^p)/K^p\Big), \tag{7.5}$$

cf. Corollary 4.15. For a superspecial point $(A, \iota, \lambda, \bar{\eta}^p)$ of $\mathcal{M}^{\rm ss}(\mathbb{F})$, the choice an isogeny $\gamma : (A, \iota) \to (A_o, \iota_o)$ compatible with the polarizations determines a pair $(g^p, g_p) \in G'(\mathbb{Q}_p)/K'_p \times G(\mathbb{A}_f^p)/K^p$, and the passage to $G'(\mathbb{Q})$-orbits removes the dependence on the choice of γ.

The choice of an isogeny γ also yields an identification of the space $\mathrm{End}^0(A,\iota)^{\rm op}$ with $\mathrm{End}^0(A_o,\iota_o)^{\rm op} = C'$, and of the space of special endomorphisms of (A,ι,λ) with $V'(\mathbb{Q})$. Let $\Omega'_T(\mathbb{Q}) \subset V'(\mathbb{Q})^4$ be the fibre over T of the map defined by the quadratic form on $V'(\mathbb{Q})$,

$$V'(\mathbb{Q})^4 \longrightarrow \mathrm{Sym}_4(\mathbb{Q}) \ . \tag{7.6}$$

Returning to the set $\mathcal{Z}(T,\omega)(\mathbb{F})$, we consider the map

$$\mathcal{Z}(T,\omega)(\mathbb{F}) \hookrightarrow G'(\mathbb{Q})\backslash\Big(\Omega'_T(\mathbb{Q}) \times G'(\mathbb{Q}_p)/K'_p \times G(\mathbb{A}_f^p)/K^p\Big) \tag{7.7}$$

defined as follows. To a point $\xi = (A,\iota,\lambda,\overline{\eta}^p;\mathbf{j}) \in \mathcal{Z}(T,\omega)(\mathbb{F})$, and a choice of isogeny $\gamma : (A,\iota) \to (A_o,\iota_o)$, there is an associated triple $(\gamma_*\mathbf{j}, g^p, g_p)$, where $\gamma_*\mathbf{j} \in V'(\mathbb{Q})^4$ is the 4-tuple of endomorphisms determined by $\mathbf{j}$ and γ. Again, the passage to $G'(\mathbb{Q})$-orbits removes the dependence on the choice of γ.

It is not difficult to describe the image of $\mathcal{Z}(T,\omega)(\mathbb{F})$. For $\mathbf{y} \in \Omega'_T(\mathbb{Q})$, the triple $(\mathbf{y}, g^p, g_p)$ lies in the image if and only if

(i) The images of the components of $\mathbf{y}$ under the inclusion $V' \hookrightarrow \mathrm{End}_W(\mathcal{L}, F)$ preserve the lattice $g_p\overline{L}$, and

(ii) The image of $\mathbf{y}$ under η^p lies in $g^p \cdot \omega$.

We note that the condition (i) is equivalent to the assertion that the components of the 4-tuple $g_p^{-1}\mathbf{y}$ lie in

$$V'(\mathbb{Z}_p) = V'(\mathbb{Q}_p) \cap \mathrm{End}_W(L, F) = N_L \ . \tag{7.8}$$

We let φ'_p be the characteristic function of $V'(\mathbb{Z}_p)^4$, let $\varphi_f^p = \mathrm{char}(\omega)$ be the characteristic function of ω, and set $\varphi'_f = \varphi'_p \otimes \varphi_f^p$. Then $\varphi'_f \in S(V'(\mathbb{A}_f)^4)^{K'}$. Conditions (i) and (ii) can then be summarized as follows.

Lemma 7.1. *The $G'(\mathbb{Q})$-orbit of the triple $(\mathbf{y}, g^p, g_p)$ lies in the image of $\mathcal{Z}(T,\omega)(\mathbb{F})$ if and only if $\varphi'_f(g^{-1}\mathbf{y}) \neq 0$, where $g = (g^p, g_p) \in G'(\mathbb{A}_f)$.*

Note that the function $(\mathbf{y}, g) \mapsto \varphi'_f(g^{-1}\mathbf{y})$ is invariant under the diagonal action of $G'(\mathbb{Q})$ on the left and under the action of $K' = K'_p K^p$ and of $Z'(\mathbb{A}_f)$ on the right. The total contribution of the superspecial points may be expressed as an integral.

Theorem 7.2. *Let $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ be non-singular and such that T represents 1 over $\mathbb{Z}_p$. Let $\omega \subset V(\mathbb{A}_f^p)^4$ be K^p-invariant open and compact, and let $K' = K'_p K^p \subset G'(\mathbb{A}_f)$. Let $\mathrm{pr}(K')$ be the image of K' in $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f) \simeq SO(V')(\mathbb{A}_f)$. Then*

$$< \mathcal{Z}(T,\omega) >_p = e_p(T) \cdot \mathrm{vol}(\mathrm{pr}(K'))^{-1} \cdot I_{T,f}(\varphi'_f) \ .$$

Here $\varphi'_f = \varphi'_p \otimes \varphi_f^p \in S(V(\mathbb{A}_f)^4)$ as above, and $I_{T,f}(\varphi'_f)$ denotes the theta integral

$$I_{T,f}(\varphi'_f) = \int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \sum_{\mathbf{y} \in \Omega'_T(\mathbb{Q})} \varphi'_f(g^{-1}\mathbf{y})\, dg \ .$$

The measure dg is induced by an arbitrary Haar measure on $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)$ and the atomic measure on $Z'(\mathbb{Q})\backslash G'(\mathbb{Q})$. The coefficient $e_p(T)$ is given by the formulas in Proposition 6.1. The identity of the Theorem remains valid if T is nonsingular but not positive definite, since, in that case, T is not represented by V', and hence both sides of the identity vanish.

Proof. By Lemma 7.1, we see that

$$|\mathcal{Z}(T,\omega)(\mathbb{F})| = \sum_{G'(\mathbb{Q})\backslash\left(\Omega'_T(\mathbb{Q}) \times G'(\mathbb{A}_f)/K'Z'(\mathbb{A}_f)\right)} \varphi'_f(g^{-1}\mathbf{y}). \tag{7.9}$$

On the other hand, since $\mathrm{pr}(K')$ is neat, the stabilizer in $Z'(\mathbb{Q})\backslash G'(\mathbb{Q})$ of a coset $gK'Z'(\mathbb{A}_f)/Z'(\mathbb{A}_f)$ is trivial. Thus, we have

$$|\mathcal{Z}(T,\omega)(\mathbb{F})| = \mathrm{vol}(\mathrm{pr}(K'))^{-1} \int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \sum_{\mathbf{y} \in \Omega'_T(\mathbb{Q})} \varphi'_f(g^{-1}\mathbf{y})\, dg, \tag{7.10}$$

for a measure as described in the Theorem. In combination with (7.4), this gives the claimed expression. $\square$

Corollary 7.3. *In the situation of the beginning of this section,*

$$< \mathcal{Z}(d_1,\omega_1),\ldots,\mathcal{Z}(d_r,\omega_r) >_p^{\text{proper}} = \sum_T e_p(T)\ \mathrm{vol}(\mathrm{pr}(K'))^{-1}\cdot I_{T,f}(\varphi'_f)\ .$$

where T runs over all $T\in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ which represent 1 over $\mathbb{Z}_p$ and have diagonal blocks $d_1,\ldots,d_r$. The function $\varphi'_f=\varphi'_p\otimes\varphi^p_f\in S(V'(\mathbb{A}_f)^4)$ is defined by

$$\varphi'_p = \mathrm{char}\ V'(\mathbb{Z}_p)^4$$
$$\varphi^p_f = \mathrm{char}(\omega_1\times\ldots\times\omega_r).$$

Remark 7.4. Formula (7.10) expresses the quantity $|\mathcal{Z}(T,\omega)(\mathbb{F})|$ as a product of orbital integrals. More precisely, note that the components of $\mathbf{y}\in\Omega'_T(\mathbb{Q})$ span a 4-dimensional subspace of the 5-dimensional space V'. Since G' acts on V' via its projection to $SO(V')$, the stabilizer of $\mathbf{y}$ in $G'(\mathbb{Q})$ is precisely $Z'(\mathbb{Q})$, the kernel of this projection. Since $G'(\mathbb{Q})$ acts transitively on $\Omega'_T(\mathbb{Q})$, we can unfold to obtain:

(7.11)
$$\begin{aligned}|\mathcal{Z}(T,\omega)(\mathbb{F})| &= \mathrm{vol}(K')^{-1}\int_{Z'(\mathbb{Q})\backslash G'(\mathbb{A}_f)}\varphi'_f(g^{-1}\mathbf{y})\ dg\\ &= \mathrm{vol}(K')^{-1}\mathrm{vol}(Z'(\mathbb{Q})\backslash Z'(\mathbb{A}_f))\,O_{\mathbf{y}}(\varphi^p)\,O_{\mathbf{y}}(\varphi'_p),\end{aligned}$$

for orbital integrals

$$O_{\mathbf{y}}(\varphi^p)=\int_{Z(\mathbb{A}^p_f)\backslash G(\mathbb{A}^p_f)}\varphi^p_f(g^{-1}\mathbf{y})\,dg, \tag{7.12}$$

and

$$O_{\mathbf{y}}(\varphi'_p)=\int_{Z'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)}\varphi'_p(g^{-1}\mathbf{y})\,dg. \tag{7.13}$$

In our main theorem (in section 9), we will identify the right hand sides of the formulas of Theorem 7.2 and Corollary 7.3 as special values of derivatives of Fourier coefficients of certain Eisenstein series. In the next section we will explain more precisely the Eisenstein series in question.

§8. Fourier coefficients of Siegel Eisenstein series.

In this section, we recall, from [**19**], the construction of certain incoherent Siegel Eisenstein series and the structure of the Fourier coefficients of their derivative at $s = 0$, the center of symmetry. To be more precise, these Eisenstein series occur on the metaplectic cover of the symplectic group of rank 4 over $\mathbb{Q}$, and have an odd functional equation. Their Fourier coefficients are parameterized by rational symmetric matrices $T \in Sym_4(\mathbb{Q})$. In [**19**], a formula was given for the derivative at $s = 0$ of such a coefficient, when $\det(T) \neq 0$.

We retain the notation of section 1, and we refer to sections 1 – 6 of [**19**] for more details. Thus B is an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D(B)$, $C = M_2(B)$, V is given by (1.1), and G is given by (1.3), etc.. In particular, V is a five-dimensional quadratic space over $\mathbb{Q}$ with signature $(3, 2)$. Let $\chi = \chi_V$ be the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ attached to V: $\chi(x) = (x, \det(V))_{\mathbb{A}}$, where $(\ ,\ )_{\mathbb{A}}$ is the global Hilbert symbol. Note that $\chi_\infty(-1) = 1$.

Let W be a symplectic vector space of dimension 8 over $\mathbb{Q}$, with a fixed symplectic basis $e_1, \ldots, e_4, e'_1, \ldots, e'_4$, and let $H_{\mathbb{A}}$ be the metaplectic extension of $Sp(W_{\mathbb{A}})$, with Siegel parabolic $P_{\mathbb{A}}$. For $s \in \mathbb{C}$ and for χ as above, let $I_4(s, \chi)$ be the global degenerate principal series representation of $H_{\mathbb{A}}$. As explained in [**19**], (2.9), the representation $I_4(0, \chi)$ has a direct sum decomposition into two types of irreducible representations. One of these types are the irreducible summands, like $\Pi_4(V)$, associated to five-dimensional quadratic spaces with character χ_V. The other type are the irreducible summands associated to **incoherent collections,** in the sense of section 2 of [**19**]. One such summand is $\Pi_4(\mathcal{C})$, associated to the incoherent collection $\mathcal{C}$, defined as follows. For any finite prime ℓ, $\mathcal{C}_\ell = V_\ell$, while $\mathcal{C}_\infty = V'_\infty$, where V'_∞ is the quadratic space over $\mathbb{R}$ of signature $(5, 0)$. There is a surjective map

$$\lambda_f : S((\mathcal{C}_{\mathbb{A}_f})^4) = S(V(\mathbb{A}_f)^4) \longrightarrow \Pi_4(\mathcal{C})_f \subset I_4(0, \chi)_f. \tag{8.1}$$

A section $\Phi(s) \in I_4(s, \chi)$ is **standard** if its restriction to the standard maximal compact subgroup K_H in $H_{\mathbb{A}}$ is independent of s. For $\varphi_f \in S(V(\mathbb{A}_f)^4)$, let $\Phi_f(s)$ be the standard section of $I_4(s, \chi)_f$ such that $\Phi_f(0) = \lambda(\varphi_f)$. Let $\Phi(s) = \Phi_\infty^{\frac{5}{2}}(s) \otimes \Phi_f(s)$, where $\Phi_\infty^{\frac{5}{2}}(s)$ is the standard section of $I_4(s, \chi)_\infty$ whose restriction to K_{H_∞} is the character $\det^{\frac{5}{2}}$. Then $\Phi(s)$ is an incoherent section with $\Phi(0) \in \Pi_4(\mathcal{C})$. The incoherent Eisenstein series

$$E(h, s, \Phi) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash H_{\mathbb{Q}}} \Phi(\gamma h, s) \tag{8.2}$$

converges for $\mathrm{Re}(s) > \frac{5}{2}$, and its analytic continuation vanishes at the point $s = 0$, [**19**]. There is a Fourier expansion

$$E(h,s,\Phi) = \sum_{T \in Sym_4(\mathbb{Q})} E_T(h,s,\Phi), \tag{8.3}$$

with respect to the unipotent radical of P. When $\Phi(s) = \otimes_\ell \Phi_\ell(s)$ is a factorizable section, and when $\det(T) \neq 0$, there is a product formula

$$E_T(h,s,\Phi) = \prod_{\ell \leq \infty} W_{T,\ell}(h_\ell, s, \Phi_\ell), \tag{8.4}$$

where $W_{T,\ell}(h_\ell, s, \Phi_\ell)$ is the local generalized Whittaker integral, cf. section 4 of [**19**]. For fixed h, T, and Φ, there is a finite set of places S such that, [**19**], Proposition 4.1,

$$\prod_{\ell \notin S} W_{T,\ell}(h_\ell, s, \Phi_\ell) = \zeta^S(2s+4)^{-1}\zeta^S(2s+2)^{-1}, \tag{8.5}$$

and hence

$$E_T(h,s,\Phi) = \zeta^S(2s+4)^{-1}\zeta^S(2s+2)^{-1} \cdot \prod_{\ell \in S} W_{T,\ell}(h_\ell, s, \Phi_\ell). \tag{8.6}$$

Since $\det(T) \neq 0$, the factors $W_{T,\ell}(h_\ell, s, \Phi_\ell)$ have an entire analytic continuation.

Fix T with $\det(T) \neq 0$. Since $E_T(h,0,\Phi) = 0$, at least one of the factors in the product formula (8.6) vanishes at $s = 0$. In particular, by Proposition 1.4 of [**19**], the factor at ℓ vanishes whenever the five-dimensional quadratic space $\mathcal{C}_\ell$ does not represent T. Let $\mathrm{Diff}(T,\mathcal{C})_f$ be the set of finite places at which $\mathcal{C}_\ell$ fails to represent T, and let

$$\mathrm{Diff}(T,\mathcal{C}) = \begin{cases} \mathrm{Diff}(T,\mathcal{C})_f \cup \{\infty\} & \text{if } sig(T) = (3,1) \text{ or } (1,3) \\ \mathrm{Diff}(T,\mathcal{C})_f & \text{otherwise.} \end{cases} \tag{8.7}$$

By Corollary 5.3 of [**19**], $|\mathrm{Diff}(T,\mathcal{C})|$ is odd; and, by Corollary 5.4 of loc. cit.,

$$\operatorname*{ord}_{s=0}\ E_T(h,s,\Phi) \geq |\mathrm{Diff}(T,\mathcal{C})|. \tag{8.8}$$

Thus, the only nonsingular T for which $E'_T(h,0,\Phi)$ can be nonzero are those for which $|\mathrm{Diff}(T,\mathcal{C})| = 1$. We will relate the value $E'_T(h,0,\Phi)$ for $\mathrm{Diff}(T,\mathcal{C}) = \{p\}$ to the numbers $< \mathcal{Z}(T,\omega) >_p$ in the previous section.

Let us fix a finite prime p. We wish to give a formula for $E'_T(h,0,\Phi)$ if $T \in \mathrm{Sym}_4(\mathbb{Q})$ is nonsingular with $\mathrm{Diff}(T,\mathcal{C}) = \{p\}$. Let B' be the definite quaternion algebra over $\mathbb{Q}$ which is ramified at p and whose invariants coincide with those of B at all finite primes other than p. Let $C' = M_2(B')$, and let

$$V' = \{\ x \in M_2(B') \mid x' = x \text{ and } tr(x) = 0\ \}, \tag{8.9}$$

with quadratic form defined by squaring, as in section 1. Let $G' = GSpin(V')$ be defined by the analogue of (1.3). Note that there is an exact sequence

$$1 \longrightarrow Z' \longrightarrow G' \longrightarrow SO(V') \longrightarrow 1 \tag{8.10}$$

of algebraic groups over $\mathbb{Q}$, where Z' is the center of G'.

We fix identifications $B'(\mathbb{A}_f^p) = B(\mathbb{A}_f^p)$, and hence $V'(\mathbb{A}_f^p) = V(\mathbb{A}_f^p)$, and $G'(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$. We also assume that $\varphi_f \in S(V(\mathbb{A}_f)^4)$ is factorizable, so that $\varphi_f = \varphi_p \otimes \varphi_f^p$, and we can view φ_f^p as a Schwartz function on $V'(\mathbb{A}_f^p)^4$. Recall that there is a surjective map

$$\lambda'_f : S(V'(\mathbb{A}_f)^4) \longrightarrow \Pi_4(V')_f \subset I_4(0,\chi)_f. \tag{8.11}$$

Recall, [**31**], [**19**], that the local degenerate principal series representation $I_{4,p}(0,\chi_p)$ has a direct sum decomposition with irreducible factors

$$I_{4,p}(0,\chi_p) = R_4(V_p) \oplus R_4(V'_p). \tag{8.12}$$

Let $T \in \mathrm{Sym}_4(\mathbb{Q})$ be nonsingular with $\mathrm{Diff}(T,\mathcal{C}) = \{p\}$. Then the linear functional

$$W_{T,p}(h,0,\cdot) : I_{4,p}(0,\chi_p) \longrightarrow \mathbb{C} \tag{8.13}$$

vanishes identically on $R_4(V_p) = R_4(\mathcal{C}_p)$, and does not vanish identically on the summand $R_4(V'_p)$, [**19**], Proposition 1.4. We choose a standard section $\Phi'_p(s)$, with $\Phi'_p(0) \in R_4(V'_p)$, and such that for a given $h \in H_{\mathbb{Q}_p}$,

$$W_{T,p}(h,0,\Phi'_p) \neq 0. \tag{8.14}$$

Let $\varphi'_p \in S((V'_p)^4)$ be a Schwartz function whose image $\lambda_p(\varphi'_p)$ in $I_{4,p}(0,\chi_p)$ is $\Phi'_p(0)$. Note that V' is positive definite, and let $\varphi'_\infty \in S((V'_\infty)^4)$ be the Gaussian, $\varphi'_\infty(x) = \exp\left(-\pi tr(q(x))\right)$. Finally, let $\varphi'_f = \varphi'_p \otimes \varphi_f^p$ so that

$$\varphi' = \varphi'_\infty \otimes \varphi'_f = \varphi'_\infty \otimes \varphi'_p \otimes \varphi_f^p \in S(V'(\mathbb{A})^4). \tag{8.15}$$

Recall that the metaplectic group $H_{\mathbb{A}}$ acts on the space $S(V'(\mathbb{A})^4)$ via the Weil representation $\omega = \omega_\psi$, defined using our fixed additive character ψ of $\mathbb{A}/\mathbb{Q}$. For $g \in G'(\mathbb{A})$ and $h \in H_{\mathbb{A}}$, let

$$\theta(g,h,\varphi') = \sum_{\mathbf{y} \in V'(\mathbb{Q})^4} \big(\omega(h)\varphi'\big)(g^{-1}\mathbf{y}) \tag{8.16}$$

be the theta function attached to φ', and let

$$I(h,\varphi') = \frac{1}{2}\int_{G'(\mathbb{Q})Z(\mathbb{A})\backslash G'(\mathbb{A})} \theta(g,h,\mathrm{ev}(\varphi'))\,dg, \tag{8.17}$$

for the Tamagawa measure dg on $Z(\mathbb{A})\backslash G'(\mathbb{A})$, and where $\mathrm{ev}(\varphi')$ denotes the projection of φ' to the subspace of functions all of whose local components are even, cf. [**19**], (7.19). Note that $\theta(g,h,\varphi')$ can be defined by the same formula for $g \in O(V')(\mathbb{A})$, and that

$$I(h,\varphi') = \int_{O(V')(\mathbb{Q})\backslash O(V')(\mathbb{A})} \theta(g,h,\varphi')\,dg, \tag{8.18}$$

where $\mathrm{vol}(O(V')(\mathbb{Q})\backslash O(V')(\mathbb{A}),dg) = 1$.

For $g \in G'(\mathbb{A})$ and $h \in H_\infty$, let

$$\theta_T(g,h;\varphi') = \sum_{\mathbf{y}\in\Omega'_T(\mathbb{Q})} \big(\omega(h)\varphi'\big)(g^{-1}\mathbf{y}), \tag{8.19}$$

and

$$\theta_{T,f}(g,\varphi'_f) = \sum_{\mathbf{y}\in\Omega'_T(\mathbb{Q})} \varphi'_f(g^{-1}\mathbf{y}), \tag{8.20}$$

where $\theta_{T,f}(g,\varphi'_f)$ depends only on g_f. Recall that $\varphi' = \varphi'_\infty \otimes \varphi'_f$, with

$$\varphi'_\infty(x) = e^{-\pi tr((x,x))}, \tag{8.21}$$

the Gaussian attached to V'. This function is invariant under $G'(\mathbb{R})$ and, for $\mathbf{y} \in \Omega'_T(\mathbb{Q})$, it has the value

$$\varphi'_\infty(\mathbf{y}) = e^{-2\pi tr(T)}. \tag{8.22}$$

Therefore, for $h \in H_\infty$, we have

$$\begin{aligned}\theta_T(h,g;\varphi') &= \sum_{\mathbf{y}\in\Omega'_T(\mathbb{Q})} \big(\omega(h)\varphi'\big)(g^{-1}\mathbf{y}) \\ &= \big(\omega(h)\varphi'_\infty\big)(\mathbf{y}_0)\cdot\theta_{T,f}(g,\varphi'_f),\end{aligned} \tag{8.23}$$

where $\mathbf{y}_0$ is any fixed element of $\Omega'_T(\mathbb{Q})$.

For $h \in H_\infty$, and for $\mathbf{y}_0 \in \Omega'_T(\mathbb{Q})$, set

$$W_T^{\frac{5}{2}}(h) := \big(\omega(h)\varphi'_\infty\big)(\mathbf{y}_0). \tag{8.24}$$

More explicitly, as in (11.74) of [**19**], if h has Iwasawa decomposition $h=(n(b)m(a)k,t)\in Sp_4(\mathbb{R})\times\mathbb{C}^1\simeq Mp_4(W_\infty)$, for $b\in Sym_4(\mathbb{R})$, $a\in GL_4(\mathbb{R})^+$, and $k\in K_{H_\infty}$, then

$$\begin{aligned}W_T^{\frac{5}{2}}(h) &= t\cdot\det(a)^{\frac{5}{2}}\, e(tr(Tb))\, e^{-\pi tr({}^t aTa)}\,\det(k)^{\frac{5}{2}}\\ &= t\cdot\det(a)^{\frac{5}{2}}\, e(tr(T\tau))\,\det(k)^{\frac{5}{2}},\end{aligned}\tag{8.25}$$

where $\tau=b+ia^ta$.

Recalling that $Z'(\mathbb{R})\backslash G'(\mathbb{R})\simeq SO(V')(\mathbb{R})$ is compact, we have the following formula for the T-th Fourier coefficient of the theta integral:

(8.26)

$$\begin{aligned}&2\,I_T(h,\varphi')\\ &=\int_{G'(\mathbb{Q})Z'(\mathbb{A})\backslash G'(\mathbb{A})}\theta_T(g,h;\mathrm{ev}(\varphi'))\,dg\\ &=W_T^{\frac{5}{2}}(h)\cdot\int_{G'(\mathbb{Q})Z'(\mathbb{A})\backslash G'(\mathbb{A})}\theta_{T,f}(g,\mathrm{ev}(\varphi'_f))\,dg\\ &=W_T^{\frac{5}{2}}(h)\cdot\mathrm{vol}(SO(V')(\mathbb{R}),d_\infty g)\cdot\int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)}\theta_{T,f}(g,\mathrm{ev}(\varphi'_f))\,d_f g,\end{aligned}$$

where $d_f g$ is the measure arising from the counting measure on $Z'(\mathbb{Q})\backslash G'(\mathbb{Q})$ and the Haar measure on $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)\simeq SO(V')(\mathbb{A}_f)$ coming from some choice of a gauge form μ on $SO(V')$. Also $d_\infty g$ is the Haar measure on $SO(V')(\mathbb{R})$ induced by μ.

With the notation just described, and for $h\in H_\infty$, Corollary 6.3 of [**19**] specializes to

$$E'_T(h,0,\Phi)=\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}\cdot 2I_T(h,\varphi'),\tag{8.27}$$

if T is nonsingular with $\mathrm{Diff}(T,\mathcal{C})=\{p\}$.

Substituting the expression (8.26) for the Fourier coefficient of the theta integral found above, we obtain:

Proposition 8.1. *Suppose that* $\Phi(s)=\Phi_\infty^{\frac{5}{2}}(s)\otimes\Phi_f(s)$ *with* $\Phi_f(0)=\lambda_f(\varphi_f)$, *is an incoherent standard section. For* $h\in H_\infty$, *and for each* $T\in Sym_4(\mathbb{Q})$ *with* $\det(T)\neq 0$ *and Diff*$(T,\mathcal{C})=\{p\}$, *choose* φ'_p *and* $\Phi'_p(s)$, *such that* $W_{T,p}(e,0,\Phi'_p)\neq 0$. *Then*

$$E'_T(h,0,\Phi)=\mathrm{vol}(SO(V')(\mathbb{R}))\cdot W_T^{5/2}(h)\cdot\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}\cdot I_{T,f}(\varphi'_f).$$

Here

$$I_{T,f}(\varphi'_f) = \int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \sum_{\mathbf{y}\in\Omega'_T(\mathbb{Q})} \mathrm{ev}(\varphi'_f)(g^{-1}\mathbf{y})\ d_f g$$
$$= \int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \theta_{T,f}(g, \mathrm{ev}(\varphi'_f))\ d_f g,$$

and the measures are as described after (8.26) above.

If the function φ'_f is locally even, then the integral

$$I_{T,f}(\varphi'_f) = \int_{G'(\mathbb{Q})Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \theta_{T,f}(g, \varphi'_f)\ d_f g \tag{8.28}$$

occurs in Theorem 7.2, where the measure arises from an arbitrary Haar measure on $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)$, and the quantity

$$\mathrm{vol}(\mathrm{pr}(K'))^{-1} I_{T,f}(\varphi'_f) \tag{8.29}$$

is independent of the choice. Therefore, we can obtain the expression

$$\begin{aligned} E'_T(h,0,\Phi) = \mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K')) \cdot W_T^{5/2}(h) \cdot \\ \cdot \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)} \cdot \mathrm{vol}(\mathrm{pr}(K'))^{-1} I_{T,f}(\varphi'_f), \end{aligned} \tag{8.30}$$

where the factor $\mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K'))$ is computed using the Tamagawa measure on $SO(V')(\mathbb{A})$. Hence, since $\mathrm{pr}(K')$ is neat,

$$\mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K')) = 2|SO(V')(\mathbb{A}) : SO(V')(\mathbb{Q})SO(V')(\mathbb{R})\mathrm{pr}(K')|^{-1},$$

and the quantities in (8.30) separated by a dot do not depend on any choice of measure.

§9. The main theorem.

In this section we assemble the results of previous sections and state our main results.

We begin by further specializing the formula of Proposition 8.1. Specifically, we need more information about the factor

$$\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}. \tag{9.1}$$

Fix the prime p with $p\nmid 2D(B)$, and assume that φ_p is the characteristic function of $V(\mathbb{Z}_p)^4$. Recall that $\Phi_p(s)$ is the standard section with $\Phi_p(0)=\lambda_p(\varphi_p)$. Also, let φ'_p be the characteristic function of the lattice $V'(\mathbb{Z}_p)^4$, and let $\Phi'_p(s)$ be the standard section with $\Phi'_p(0)=\lambda'_p(\varphi'_p)$.

Recall that a nonsingular $T\in \mathrm{Sym}_4(\mathbb{Q}_p)$ is represented by precisely one of the quadratic spaces $V(\mathbb{Q}_p)$ and $V'(\mathbb{Q}_p)$, [**19**], Proposition 1.3.

Proposition 9.1. *Suppose that $\varphi_p,\varphi'_p,\Phi_p,\Phi'_p$ are as above, and that $T\in\mathrm{Sym}_4(\mathbb{Q}_p)$ with $\det(T)\ne 0$.*
(i) If $W'_{T,p}(e,0,\Phi_p)\ne 0$, then $T\in\mathrm{Sym}_4(\mathbb{Z}_p)$.
(ii) If $T\in\mathrm{Sym}_4(\mathbb{Z}_p)$ and if T is represented by $V'(\mathbb{Q}_p)$, then $W_{T,p}(e,0,\Phi'_p)\ne 0$.
(iii) If $T\in\mathrm{Sym}_4(\mathbb{Z}_p)$ is represented by $V'(\mathbb{Q}_p)$, and if T represents 1, then

$$\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}=\frac{1}{2}\log p\cdot(p^2+1)(p-1)\cdot e_p(T),$$

where $e_p(T)$ is the local intersection multiplicity given in Proposition 6.1.

The proof will be given in section 10.

A subset $\omega\subset V(\mathbb{A}_f^p)^n$ is said to be **locally centrally symmetric** if it is invariant under the action of the group $\mu_2(\mathbb{A}_f^p)$. The characteristic function $\varphi_\omega\in S(V(\mathbb{A}_f^p)^n)$ of such a set is locally even, as in (8.17), i.e. $\varphi_\omega=\mathrm{ev}(\varphi_\omega)$. The function $\varphi'_f=\varphi'_p\otimes\varphi_\omega\in S(V'(\mathbb{A}_f)^n)$ is then locally even as well, so that the expression (8.30) holds for the derivative of the Fourier coefficients of the associated Eisenstein series.

Our first main result is the following.

Theorem 9.2. *Assume that $p\nmid 2D(B)$ and that $\varphi_p,\varphi'_p,\Phi_p,\Phi'_p$ are as above. Let $\omega\subset V(\mathbb{A}_f^p)^4$ be a locally centrally symmetric K^p-invariant compact open*

subset. Let $\Phi(s) = \Phi_\infty(s) \otimes \Phi_p(s) \otimes \Phi_f^p(s)$ *be the standard section corresponding to* $\varphi = \varphi_\infty \otimes \varphi_p \otimes \varphi_f^p \in S(V^{(p)}(\mathbb{A})^4)$ *with* $\varphi_f^p = \mathrm{char}(\omega)$ *, cf Lemma 7.1. Suppose that* $T \in \mathrm{Sym}_4(\mathbb{Q})$ *with* $\det(T) \neq 0$ *and with* $\mathrm{Diff}(T, \mathcal{C}) = \{p\}$ *.*
(i) If $T \notin \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ *, then* $\mathcal{Z}(T,\omega) = \emptyset$ *,* $< \mathcal{Z}(T,\omega) >_p = 0$ *, and*

$$E'_T(h, 0, \Phi) = 0.$$

(ii) If $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ *represents 1 over* $\mathbb{Z}_p$ *, then* $\mathcal{Z}(T,\omega)$ *is zero dimensional, and, for* $h \in H_\infty$ *,*

$$E'_T(h, 0, \Phi) = \frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R})) \cdot W_T^{5/2}(h) \cdot \mathrm{vol}(\mathrm{pr}(K)) \cdot \log p < \mathcal{Z}(T,\omega) >_p \quad .$$

Note that, if $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ does not represent 1, then $\mathcal{Z}(T,\omega)$ contains components of the supersingular locus (Corollary 5.15 and Theorems 5.12 and 5.14). In this case, we do not have a formula for $< \mathcal{Z}(T,\omega) >_p$.

In Theorem 9.2, the chosen gauge form μ on $SO(V') = Z'\backslash G'$ determines the Haar measure on $SO(V')(\mathbb{R})$ used to compute $\mathrm{vol}(SO(V')(\mathbb{R}))$. The corresponding gauge form on the inner twist $SO(V) = Z\backslash G$ determines the measure on $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)$ used to compute $\mathrm{vol}(\mathrm{pr}(K))$. Note that the product $\mathrm{vol}(SO(V')(\mathbb{R})\mathrm{vol}(\mathrm{pr}(K))$ is independent of the choice of μ.

Proof of Theorem 9.2. Beginning with formula (8.30), and using (iii) of Proposition 9.1 and Theorem 7.2, we have

$$\begin{aligned} E'_T(h,0,\Phi) &= \mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K')) \cdot W_T^{5/2}(h) \cdot \\ &\qquad \cdot \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)} \cdot \mathrm{vol}(\mathrm{pr}(K'))^{-1} I_{T,f}(\varphi'_f) \qquad (9.2) \\ &= \mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K')) \cdot W_T^{5/2}(h) \cdot \\ &\qquad \cdot \frac{1}{2}\log p \cdot (p^2+1)(p-1) \cdot e_p(T) \cdot \mathrm{vol}(\mathrm{pr}(K'))^{-1} I_{T,f}(\varphi'_f) \\ &= \mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K')) \cdot W_T^{5/2}(h) \cdot \\ &\qquad \cdot \frac{1}{2}\log p \cdot (p^2+1)(p-1) \cdot < \mathcal{Z}(T,\omega) >_p . \end{aligned}$$

To finish the proof, we simply note the following relation between volumes.

Lemma 9.3. *Recall that* $K_p = GL_2(\mathcal{O}_{B_p}) \cap G(\mathbb{Q}_p)$ *and* $K'_p = GL_2(\mathcal{O}_{B'_p}) \cap G'(\mathbb{Q}_p)$ *. Then, for the Haar measures on* $Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)$ *,* $Z'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)$ *,* $Z(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$ *, and* $Z(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ *determined by the fixed gauge form* μ *and the corresponding* *form on the inner twist,*

$$\frac{\mathrm{vol}(\mathrm{pr}(K))}{\mathrm{vol}(\mathrm{pr}(K'))} = \frac{\mathrm{vol}(\mathrm{pr}(K_p))}{\mathrm{vol}(\mathrm{pr}(K'_p))} = (p^2+1)(p-1).$$

This finishes the proof of Theorem 9.2 $\square$

Proof of Lemma 9.3, following Kottwitz [**16**]. We may replace G/Z and G'/Z' by their simply connected coverings $\tilde{G}$ resp. $\tilde{G}'$ and $\mathrm{pr}(K_p)$ and $\mathrm{pr}(K'_p)$ by their inverse images $\tilde{K}_p$ resp. $\tilde{K}'_p$. We use on $\tilde{G}(\mathbb{Q}_p)$ resp. $\tilde{G}'(\mathbb{Q}_p)$ the Haar measure induced by a top differential form on the $\mathbb{Z}_p$-form of $\tilde{G}$ resp. $\tilde{G}'$ corresponding to an Iwahori subgroup $\tilde{I}_p \subset \tilde{K}_p$ resp. $\tilde{I}'_p \subset \tilde{K}'_p$. These measures are compatible, cf. [**16**], p. 632. The volumes of $\tilde{I}_p$ and $\tilde{I}'_p$ are related as follows. Choose as in [**16**] a maximal split torus S in $\tilde{G}$ and a maximal torus S_1 containing S which splits over an unramified extension. We also denote by S_1 the canonical $\mathbb{Z}_p$-form of S_1. Choose S', S'_1 of the same sort for $\tilde{G}'$. Then

$$\frac{\mathrm{vol}(\tilde{I}_p)}{\mathrm{vol}(\tilde{I}'_p)} = \frac{S_1(\mathbb{F}_p)}{S'_1(\mathbb{F}_p)} = \frac{(p-1)^2}{p^2-1} \quad ,$$

since in the case at hand $S_1 \cong \mathbb{G}_m^2$ and $S'_1 \cong \mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\mathbb{G}_m$. The result follows since

$$|\tilde{K}_p/\tilde{I}_p| = 1 + 2p + 2p^2 + 2p^3 + p^4 \quad , \quad |\tilde{K}'_p/\tilde{I}'_p| = p+1 \quad ,$$

hence

$$\frac{\mathrm{vol}(\mathrm{pr}(K_p))}{\mathrm{vol}(\mathrm{pr}(K'_p))} = \frac{\mathrm{vol}(\tilde{I}_p)}{\mathrm{vol}(\tilde{I}'_p)} \cdot \frac{|\tilde{K}_p/\tilde{I}_p|}{|\tilde{K}'_p/\tilde{I}'_p|} = \frac{(p-1)^2}{p^2-1} \cdot \frac{p^4-1}{p-1} \quad . \qquad \square$$

We next formulate the corresponding result for the intersection of special cycles.

For $n_1, \ldots, n_r$ with $1 \le n_i \le 4$ and with $n_1 + \ldots + n_r = 4$, let $d_i \in \mathrm{Sym}_{n_i}(\mathbb{Z}_{(p)})_{>0}$ and fix locally centrally symmetric K^p-invariant open compact subsets $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$. Let

$$W = W_1 + \ldots + W_r \tag{9.3}$$

be a decomposition of W into symplectic subspaces of dimensions $2n_i$, compatible with the fixed symplectic basis, and let

$$\iota : H_{1,\mathbb{A}} \times \ldots \times H_{r,\mathbb{A}} \longrightarrow H_{\mathbb{A}} \tag{9.4}$$

be the corresponding homomorphism of metaplectic groups, covering the embedding

$$\iota : Sp(W_{1,\mathbb{A}}) \times \ldots \times Sp(W_{r,\mathbb{A}}) \hookrightarrow Sp(W_{\mathbb{A}}). \tag{9.5}$$

Restricting to the archimedean place, for $(h_1, \ldots, h_n) \in H_{1,\infty} \times \ldots \times H_{r,\infty}$, we have

$$W_T^{\frac{5}{2}}(\iota(h_1, \ldots, h_r)) = W_{d_1}^{\frac{5}{2}}(h_1) \ldots W_{d_r}^{\frac{5}{2}}(h_r), \tag{9.6}$$

where T has diagonal blocks $d_1, \ldots, d_r$. Thus, by (7.3), we obtain:

Corollary 9.4. *With the above notations,*

$$\sum_T E'(\iota(h_1,\dots,h_r),0,\Phi) = \frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R}))\cdot W_{d_1}^{\frac{5}{2}}(h_1)\dots W_{d_r}^{\frac{5}{2}}(h_r)$$
$$\times\ \mathrm{vol}(\mathrm{pr}(K))\cdot\log p < \mathcal{Z}(d_1,\omega_1),\dots,\mathcal{Z}(d_r,\omega_r) >_p^{\mathrm{proper}},$$

where the intersection number on the right side is defined by (7.3), and the summation runs over $T \in Sym_4(\mathbb{Z}_{(p)})_{>0}$ *such that* $Diff(T,\mathcal{C}) = \{p\}$, $diag(T) = (d_1,\dots,d_r)$, *and* T *represents* 1 *over* $\mathbb{Z}_p$. *Also,* Φ *is determined as in Theorem 9.2 with* $\omega = \omega_1\times\cdots\times\omega_r$.

Of course, the left side of the expression of Corollary 9.4 is part of the $(d_1,\dots,d_r)$-th Fourier coefficient of the pullback

$$F(h_1,\dots,h_r;\Phi) := E'(\iota(h_1,\dots,h_r),0,\Phi), \tag{9.7}$$

cf. [**19**], (6.13). This result gives an analogue of the results of [**19**].

§10. Representation densities.

In this section, we give the proof of Propositions 9.1, which is based on a formula of Kitaoka, [**14**], for representation densities. We then describe a conjectural generalization of Kitaoka's formula. In this section, for $x \in \mathbb{Q}_p^\times$, $\chi(x) = (x,p)_p$.

We begin by recalling the well known relation between the values of the function $W_{T,p}(e,s,\Phi_p)$, at integer values of s and classical representation densities.

For a suitable choice of basis for $V(\mathbb{Z}_p)$ the quadratic form q has matrix

$$S = S_0 = \begin{pmatrix} 1 & & \\ & & \frac{1}{2}\cdot 1_2 \\ & \frac{1}{2}\cdot 1_2 & \end{pmatrix}. \tag{10.1}$$

For $r \geq 0$, let

$$S_r = \begin{pmatrix} S_0 & & \\ & & \frac{1}{2}\cdot 1_r \\ & \frac{1}{2}\cdot 1_r & \end{pmatrix}. \tag{10.2}$$

For nonsingular matrix $T \in Sym_4(\mathbb{Z}_p)$, let
(10.3)
$$\alpha_p(S_r, T) = \lim_{t\to\infty} p^{-t(10+8r)} \#\{\ x \in M_{5+2r,4}(\mathbb{Z}/p^t\mathbb{Z}) \ \mid\ S_r[x] - T \in p^t Sym_4(\mathbb{Z}_p)\ \}$$

be the classical representation density [**15**], p.98. This quantity depends only on the $GL_4(\mathbb{Z}_p)$-equivalence class of T, so we assume that

$$T = \mathrm{diag}(\epsilon_0 p^{a_0}, \epsilon_1 p^{a_1}, \epsilon_2 p^{a_2}, \epsilon_3 p^{a_3}), \tag{10.4}$$

with $\epsilon_i \in \mathbb{Z}_p^\times$ and $0 \leq a_0 \leq a_1 \leq a_2 \leq a_3$. Then, as explained in Corollary A.1.5 of [**19**], $W_{T,p}(e,r,\Phi_p) = 0$ if $T \in \mathrm{Sym}_4(\mathbb{Q}_p) \setminus \mathrm{Sym}_4(\mathbb{Z}_p)$, and

$$W_{T,p}(e,r,\Phi_p) = \alpha_p(S_r, T) \tag{10.5}$$

if $T \in \mathrm{Sym}_4(\mathbb{Z}_p)$, since the factor $\gamma_p(V_p)$ in loc.cit. is 1 in our present case. Recall – see [**14**], Lemma 9 and the discussion on pp. 450–453, for example – that $\alpha_p(S_r,T)$ is a rational function of $X = p^{-r}$, i.e. there is a rational function $A_{S,T}(X)$ such that

$$\alpha_p(S_r, T) = A_{S,T}(p^{-r}) \quad . \tag{10.6}$$

We therefore have

$$W'_{T,p}(e,0,\Phi_p) = -\log(p)\cdot \frac{\partial}{\partial X}\{A_{S,T}(X)\}\big|_{X=1}. \tag{10.7}$$

At this point we have proved part (i) of Proposition 9.1.

Similarly, let φ'_p be the characteristic function of the lattice $V'(\mathbb{Z}_p)^4 = V^{(p)}(\mathbb{Z}_p)^4$ and let $\Phi'_p(s)$ be the corresponding standard section. Again, for a suitable choice of basis for $V'(\mathbb{Z}_p)$, the quadratic form on $V'(\mathbb{Z}_p)$ has matrix

$$S' = S'_0 = \mathrm{diag}(1,1,\beta,p,-p\beta), \tag{10.8}$$

where $\beta \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^{\times,2}$. Again, the factor $\gamma_p(V'_p) = 1$, and so

$$W_{T,p}(e,0,\Phi'_p) = p^{-4}\cdot \alpha_p(S'_0,T). \tag{10.9}$$

The following two results imply parts (ii) and (iii) of Proposition 9.1.

Proposition 10.1. *Suppose that $T \in \mathrm{Sym}_4(\mathbb{Z}_p)$ is not represented by $V(\mathbb{Q}_p)$ and that T represents 1. Let $e_p(T)$ be the local intersection multiplicity, given by the formulas of Proposition 6.1. Then,*

$$\begin{aligned} W'_{T,p}(e,0,\Phi_p) &= -\log(p)\cdot \frac{\partial}{\partial X}\{A_{S,T}(X)\}\bigg|_{X=1} \\ &= \log p\cdot(1-p^{-4})(1-p^{-2})\cdot e_p(T). \end{aligned}$$

Proposition 10.2. *Suppose that $T \in \mathrm{Sym}_4(\mathbb{Z}_p)$ with $\det(T)\neq 0$ represents 1. Then*

$$W_{T,p}(e,0,\Phi'_p) = p^{-4}\cdot\alpha_p(S'_0,T) = \begin{cases} p^{-4}(1-p^{-2})2(p+1) & \text{if } V'(\mathbb{Q}_p) \text{ represents } T \\ 0 & \text{otherwise.}\end{cases}$$

Of course, we would like to have analogous information about $W'_{T,p}(e,0,\Phi_p)$ and $W_{T,p}(e,0,\Phi'_p)$ for all T. At first, we simply restrict to the case where $p\nmid T$, so that, we may assume that $a_0 = 0$, i.e.,

$$T = \mathrm{diag}(\epsilon_0,\epsilon_1 p^{a_1},\epsilon_2 p^{a_2},\epsilon_3 p^{a_3}). \tag{10.10}$$

Note that $S \simeq 1_5$. Then, by the standard reduction formula, [**13**], p.149,

$$\alpha_p(S_r,T) = \alpha_p(S_r,\epsilon_0)\alpha_p(\tilde{S}_r,\tilde{T}), \tag{10.11}$$

where $\tilde{S}_r$ is obtained by adding a split space of dimension $2r$ to

$$\tilde{S} = \mathrm{diag}(1,1,1,\epsilon_0) \tag{10.12}$$

and

$$\tilde{T} = \mathrm{diag}(\epsilon_1 p^{a_1}, \epsilon_2 p^{a_2}, \epsilon_3 p^{a_3}). \tag{10.13}$$

Note that

$$\alpha_p(S_r, \epsilon_0) = (1 + \chi(\epsilon_0)p^{-2-r}) = (1 + \chi(\epsilon_0)p^{-2}X), \tag{10.14}$$

where $X = p^{-r}$, [**32**].

Now suppose that $\chi(\epsilon_0) = 1$, i.e., that T represents 1. Let H_{2m} be the split quadratic form of rank $2m$ over $\mathbb{Z}_p$, so that

$$H_{2m} = \begin{pmatrix} & 1_m \\ 1_m & \end{pmatrix}. \tag{10.15}$$

Then $\tilde{S}_r$ is isomorphic to the split space H_{2r+4}, and Kitaoka gives an explicit formula for the representation density $\alpha_p(H_{2m}, \tilde{T})$ for any ternary form $\tilde{T}$, [**14**]. His formulas, in the cases $a_1 - a_2$ even and $a_1 - a_2$ odd, are given as a sum of five double sums! These can be simplified to yield the following expressions:

Proposition 10.3. (Kitaoka, [**14**]) *Let* $X = p^{-r}$, *and let*

$$\tilde{T} = diag(\epsilon_1 p^{a_1}, \epsilon_2 p^{a_2}, \epsilon_3 p^{a_3}),$$

with $0 \le a_1 \le a_2 \le a_3$.

Let

$$\chi(\tilde{T}) = \begin{cases} 1 & \text{if } a_1 \equiv a_2 \equiv a_3 \mod (2), \\ \chi(-\epsilon_1\epsilon_2) & \text{if } a_1 \equiv a_2 \not\equiv a_3 \mod (2), \\ \chi(-\epsilon_1\epsilon_3) & \text{if } a_1 \not\equiv a_2 \not\equiv a_3 \mod (2), \\ \chi(-\epsilon_2\epsilon_3) & \text{if } a_1 \not\equiv a_2 \equiv a_3 \mod (2). \end{cases}$$

(i) If $a_1 \equiv a_2 \mod (2)$, *then*

$$\begin{aligned} \frac{\alpha_p(H_{2r+4}, \tilde{T})}{(1-p^{-2}X)(1-p^{-2}X^2)} &= \\ & \sum_{\ell=0}^{\frac{a_1+a_2}{2}-1} p^{\ell} \Big(\sum_{k=0}^{\min(a_1,\ell)} X^{2\ell-k} + \chi(\tilde{T}) X^{a_1+a_2+a_3+k-2\ell} \Big) \\ & \qquad + p^{\frac{a_1+a_2}{2}} X^{a_2} \Big(\sum_{k=0}^{a_1} X^k \Big) \Big(\sum_{j=0}^{a_3-a_2} (\epsilon X)^j \Big), \end{aligned}$$

where $\epsilon = \chi(-\epsilon_1\epsilon_2)$.

(ii) If $a_1 \not\equiv a_2 \mod (2)$, *then*

$$\frac{\alpha_p(H_{2r+4},\tilde{T})}{(1-p^{-2}X)(1-p^{-2}X^2)} =$$

$$\sum_{\ell=0}^{\frac{a_1+a_2-1}{2}} p^\ell \Big(\sum_{k=0}^{\min(a_1,\ell)} X^{2\ell-k} + \chi(\tilde{T}) X^{a_1+a_2+a_3+k-2\ell} \Big).$$

Note that these expressions exhibit the functional equation of the local degenerate Whittaker function under $X \mapsto X^{-1}$. Evaluating at $X=1$ and taking (10.11) and (10.14) into account, we obtain:

Corollary 10.4. *Suppose that* T *represents* 1. *(i) If* $a_1 \equiv a_2 \mod (2)$, *then*

$$\frac{\alpha_p(S,T)}{(1-p^{-2})(1-p^{-4})} = (1+\chi(\tilde{T})) \sum_{\ell=0}^{\frac{a_1+a_2}{2}-1} \big(min(a_1,\ell)+1\big)\, p^\ell$$
$$+\, p^{\frac{a_1+a_2}{2}} (a_1+1) \Big(\sum_{j=0}^{a_3-a_2} \epsilon^j \Big),$$

where $\epsilon = \chi(-\epsilon_1\epsilon_2)$.

(ii) If $a_1 \not\equiv a_2 \mod (2)$, *then*

$$\frac{\alpha_p(S,T)}{(1-p^{-2})(1-p^{-4})} = (1+\chi(\tilde{T})) \sum_{\ell=0}^{\frac{a_1+a_2-1}{2}} \big(min(a_1,\ell)+1\big) p^\ell.$$

In case (ii), this quantity vanishes if and only if $\chi(\tilde{T}) = -1$. *In case (i), if* $a_2 \equiv a_3 \mod (2)$, *then* $\chi(\tilde{T}) = 1$ *and there are an odd number of terms in the last sum, so that the whole expression is nonzero. If* $a_2 \not\equiv a_3 \mod (2)$, *then* $\chi(\tilde{T}) = \chi(-\epsilon_1\epsilon_2) = \epsilon$, *so that the whole expression vanishes if and only if* $\chi(\tilde{T}) = -1$.

Proposition 10.5. *Suppose that* T *represents* 1. *Also suppose that* $\chi(\tilde{T}) = -1$, *so that* T *is not represented by* S, *i.e., by* $V(\mathbb{Q}_p)$ *(i) If* $a_1 \equiv a_2 \mod (2)$, *then*

$$\frac{\partial}{\partial X}\Big\{ \frac{A_{S,T}(X)}{(1-p^{-2}X^2)(1-p^{-4}X^2)} \Big\}\Big|_{X=1}$$
$$= -\sum_{\ell=0}^{\frac{a_1+a_2}{2}-1} p^\ell \Big(\sum_{k=0}^{\min(a_1,\ell)} (a_1+a_2+a_3+2k-4\ell) \Big)$$
$$-\, p^{\frac{a_1+a_2}{2}} (a_1+1) \big(\frac{a_3-a_2+1}{2} \big).$$

(ii) If $a_1 \not\equiv a_2 \mod (2)$, *then*

$$\frac{\partial}{\partial X}\left\{\frac{A_{S,T}(X)}{(1-p^{-2}X)(1-p^{-4}X^2)}\right\}\Big|_{X=1}$$
$$= -\sum_{\ell=0}^{\frac{a_1+a_2-1}{2}} p^{\ell} \sum_{k=0}^{\min(a_1,\ell)} (a_1+a_2+a_3+2k-4\ell).$$

After a short manipulation, these expressions coincide, up to sign, with those given in Proposition 6.1 for the local intersection multiplicity $e_p(T)$!

Corollary 10.6. *Suppose that* $p \nmid T$ *and* ϵ_0 *is a square, i.e., that* T *represents* 1 *over* $\mathbb{Z}_p$. *Also suppose that* T *is not represented by* S. *Then*

$$\frac{\partial}{\partial X}\{A_{S,T}(X)\}\Big|_{X=1} = -(1-p^{-2})(1-p^{-4})\,e_p(T),$$

where $e_p(T)$ *is as in Proposition 6.1.*

This completes the proof of Proposition 10.1.

Proof of Proposition 10.2. We apply the reduction formula to obtain:

$$\alpha_p(S'_r,T) = \alpha_p(S'_r,\epsilon_0)\alpha_p(\tilde{S}'_r,\tilde{T}), \tag{10.15}$$

where $\tilde{S}_r$ is obtained by adding a split space of dimension $2r$ to

$$\tilde{S}' = \mathrm{diag}(1,\epsilon_0\beta,p,-p\beta) \tag{10.16}$$

and $\tilde{T}$ is as in (10.12).

If ϵ_0 is a square, then

$$\alpha_p(\tilde{S}',\epsilon_0) = (1-\chi(-1)p^{-1}),$$

[**32**]. On the other hand, $\tilde{S}'$ is just the norm form on the maximal order of the division quaternion algebra over $\mathbb{Q}_p$.

Lemma 10.7.
$$\alpha_p(\tilde{S}',\tilde{T}) = 2(1+\chi(-1)p^{-1})(p+1).$$

Proof. Let $\mathbb{B}$ be the division quaternion algebra over $\mathbb{Q}_p$, and let R be its maximal order. Then, for a suitable $\mathbb{Z}_p$-basis, $\tilde{S}'$ is the matrix for the quadratic form Q given by the reduced norm on R. Let

$$A_{p^r}(T) = \#\{x \in (R/p^rR)^3 \mid Q[x] \equiv \tilde{T} \mod p^r\},$$

so that

$$\alpha_p(\tilde{S}', \tilde{T}) = \lim_{r\to\infty} p^{-6r} A_{p^r}(T).$$

Choose a uniformizer $\pi \in R$ such that $\pi^2 = -p$, and hence $Q[\pi x] = pQ[x]$. Note that $x \in R$ if and only if $Q[x] \in \mathbb{Z}_p$. Thus there is a bijection

$$\{x \in (R/p^r R)^3 \mid Q[x] \equiv p\tilde{T} \mod p^r\} \xrightarrow{\sim} \\ \{y \in (R/p^{r-1}\pi R)^3 \mid Q[y] \equiv \tilde{T} \mod p^{r-1}\},$$

given by $x \mapsto \pi^{-1}x$. Since $|R/\pi R| = p^2$, we have

$$A_{p^r}(p\tilde{T}) = p^6 A_{p^{r-1}}(\tilde{T}),$$

and hence

$$\alpha_p(\tilde{S}', p\tilde{T}) = \alpha_p(\tilde{S}', \tilde{T}).$$

Thus, we may replace $\tilde{T}$ by $T' = \mathrm{diag}(\epsilon_1, \epsilon_2 p^{a_2-a_1}, \epsilon_3 p^{a_3-a_1})$. Here ϵ_1 can be taken to be equal to either 1 or β. Using reduction, we have

$$\alpha_p(\tilde{S}', T') = \alpha_p(\tilde{S}', \epsilon_1)\alpha_p(S'', T''),$$

where

$$S'' = \mathrm{diag}(\epsilon_1\beta, p, -\beta p), \qquad \text{and} \qquad T'' = \mathrm{diag}(\epsilon_2 p^{a_2-a_1}, \epsilon_3 p^{a_3-a_1}).$$

By Theorem 3.1 of [**32**],

$$\alpha_p(\tilde{S}', \epsilon_1) = (1 + \chi(-1)p^{-1}).$$

If $\epsilon_1 = 1$, the form S'' is just the norm form on the trace zero elements in R, while, if $\epsilon_1 = \beta$, then S'' is isomorphic to β times this norm form. Since $\alpha_p(\beta S'', \beta T'') = \alpha_p(S'', T'')$, Proposition 8.6 of [**19**] yields

$$\alpha_p(S'', T'') = \begin{cases} 2(p+1) & \text{if } T'' \text{ is anisotropic,} \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 10.8. The value $\alpha_p(\tilde{S}', \tilde{T})$ is given erroneously by Gross and Keating as $2 \cdot (1 + p^{-1}) \cdot (p+1)$, [**7**], Proposition 6.10.

Thus

$$\begin{aligned} \alpha_p(S', T) &= \alpha_p(S', 1)\alpha_p(\tilde{S}', \tilde{T}) \\ &= 2(1 - p^{-2})(p+1), \end{aligned}$$

as claimed in Proposition 10.2. □ □

Notes on Clifford algebras

A.1. Let (V,q) be a non-degenerate quadratic space of dimension 5 over a field F of characteristic not 2. Let $C(V)$ be its Clifford algebra, with its 2-grading

$$C(V) = C^+(V) \oplus C^-(V) \quad .$$

The **Clifford involution** $c \mapsto c'$ of $C(V)$ is the unique involution which acts by the identity map on $V \subset C^-(V)$. Thus

$$(v_1 \cdots v_r)' = v_r' \cdots v_1' \quad .$$

If $v_1, \ldots, v_5$ is a basis for V, then the element $\delta = v_1 \cdots v_5$ lies in the center of $C(V)$ and satisfies

$$\delta' = \delta \quad .$$

Let

$$G = G\,\mathrm{Spin}(V) = \{g \in C^+(V)^\times \mid gVg^{-1} = V, \text{ and } gg' = \nu(g)\}$$

which may be considered as an algebraic group over $\mathrm{Spec}\, F$.

A.2. In this section suppose that F is algebraically closed and choose a Witt decomposition of the quadratic space V,

$$V = V_+ \oplus V_0 \oplus V_-$$

where $\dim V_\pm = 2$ and $V_\pm$ are maximal isotropic subspaces of V. Let $v_0 \in V_0$ be a basis vector with $q(v_0) = 1$. We recall the Spin representation of G. We use the identifications of representations of $C(V)$,

$$\begin{aligned} C(V)/C(V)C(V_-)_{>0} &= C(V_+ \oplus V_0) = \\ &= C(V_+)(1+v_0) \oplus C(V_+)(1-v_0) \quad . \end{aligned}$$

As $C(V)^+$-modules the last two modules are isomorphic. Either one of them defines the Spin representation W of G. Its dimension is 4.

Fix an isomorphism $\Lambda^2 V_+ = F$ and let

$$\lambda : W \to F$$

be the linear functional obtained by composing this isomorphism with the projection of $C(V_+) = \Lambda(V_+)$ onto $\Lambda^2 V_+$. We obtain an alternating F-form on W by

$$< x, y >= \lambda(x'y) \quad .$$

Lemma. *For* $c \in C(V)$, *and for* x *and* $y \in W$,

$$< \sigma(c)x, y >=< x, \sigma(c')y > .$$

In particular, for $g \in G = GSpin(V)$,

$$< \sigma(g)x, \sigma(g)y >= \nu(g) < x, y > .$$

Here $\sigma(g)$ *denotes the spin representation action of* g *on* W , *and* $\nu : G \longrightarrow F^\times$, $\nu(g) = gg'$ *is the restriction to* G *of the spinor norm on* $C(V)$.

Proof. Choose a basis $e_0,\ e_1,\ v_0,\ f_0,\ f_1$ for V such that the matrix for the quadratic form is

$$\begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 1 \\ & & 1 & & \\ 1 & 0 & & & \\ 0 & 1 & & & \end{pmatrix}.$$

In $C(V)$, $v_0^2 = 1$, $e_0 f_0 + f_0 e_0 = 1$, $e_1 f_1 + f_1 e_1 = 1$, $e_0^2 = 0$, $v_0(1+v_0) = (1+v_0)$, etc. The spin representation $W = C(V_+)(1+v_0)$ has basis $(1+v_0)$, $e_0(1+v_0)$, $e_0 e_1(1+v_0)$, and $e_1(1+v_0)$. We take λ to be the coefficient of $e_0 e_1(1+v_0)$ and the symplectic form has matrix

$$J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}.$$

It is easy to check that

$$\sigma(e_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma(v_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\sigma(f_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\sigma(f_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\sigma(c)$ is any of these matrices, then $J^t\sigma(c)J^{-1} = \sigma(c)$, and hence, for any $c \in C(V)$, $J^t\sigma(c)J^{-1} = \sigma(c')$, as claimed. $\square$

Corollary. $\sigma : G = GSpin(V) \xrightarrow{\sim} GSp(W)$.

A.3. In this section F is again arbitrary, of characteristic not 2.

Lemma. *Let (V,q) be a non-degenerate quadratic space of dimension 5. The subspace $\delta \cdot V \subset C^+(V)$ is characterized as:*

$$\delta \cdot V = \{\ x \in C^+(V) \mid x' = x \textit{ and } tr(x) = 0\ \}.$$

Proof. Recall that $\delta \in C^-(V)$ is central in $C(V)$ and satisfies $\delta' = \delta$. It is, thus, clear that $x = \delta v$ satisfies $x' = x$. On the other hand, $x^2 = q(v)\delta^2 = a$ lies in F, the center of $C^+(V)$. In addition, if $x \neq 0$, then x cannot lie in the center of $C^+(V)$, since, if it did, then $v = \delta^{-1}x$ would lie in the center of $C(V)$, and this is not the case. If $a = 0$, so that $x^2 = 0$, the condition $tr(x) = 0$ is immediate. If $x^2 = a \neq 0$, choose $u \in V$ with $q(u) \neq 0$ but with $(u,v) = 0$, and set $y = \delta u$. Then $xy = -yx$, and so, over an algebraic closure of F, left multiplication by y gives an isomorphism between the $\pm\sqrt{a}$ eigenspaces of x, and thus these spaces have the same dimension and $tr(x) = 0$. This proves that δV is contained in the space on the right hand side. The converse inclusion will be proved further down. $\square$

Let B be a quaternion algebra over F with main involution ι, and let $C = M_2(B)$ with involution $x \mapsto x' = {}^t x^\iota$. Let

$$\begin{aligned} V_B &= \{\ x \in C \ \mid\ x' = x \text{ and } tr(x) = 0\ \} \\ &= \{\ x = \begin{pmatrix} a & b \\ b^\iota & -a \end{pmatrix} \ \mid\ a \in F,\ b \in B\ \}. \end{aligned}$$

Note that

$$xx' = x^2 = \begin{pmatrix} a^2 + \nu(b) & \\ & a^2 + \nu(b) \end{pmatrix},$$

so that the inclusion $V_B \hookrightarrow M_2(B)$ induces a homomorphism

$$C(V_B, q_B) \longrightarrow M_2(B),$$

where the quadratic form on V_B is $q_B(x) = xx'$. The diagram

$$\begin{array}{ccc} C(V_B, q_B) & \longrightarrow & M_2(B) \\ \downarrow ' & & \downarrow ' \\ C(V_B, q_B) & \longrightarrow & M_2(B) \end{array}$$

commutes, and induces an isomorphism $C^+(V_B, q_B) \xrightarrow{\sim} M_2(B)$, compatible with the involutions.

Conversely, let V be a nondegenerate quadratic space of dimension 5. The Clifford involution induces an isomorphism $C^+(V) \simeq C^+(V)^{\mathrm{op}}$, hence $C^+(V)$ is of the form

$$C^+(V) \simeq M_2(B) \quad ,$$

for a quaternion algebra B over F . We may choose the isomorphism compatible with the involutions $x \mapsto x'$. This map then carries δV into V_B . For dimension reasons we obtain an isometry,

$$(V, \delta^2 \cdot q_V) \simeq (V_B, q_B) \quad .$$

This also concludes the proof of the lemma above.

Corollary.

$$G = \{g \in C^+(V)^\times \mid gg' = \nu(g)\} \quad .$$

A.4. Any involution of the central simple algebra $C = M_n(B)$, has the form $x \mapsto hx'h^{-1}$ where $x' = {}^t x^\iota$, where $h \in GL_n(B)$ with $h' = \pm h$. If $h' = h$, we say that the involution is of *main type*, while, if $h' = -h$, we say that h is of *nebentype*. As observed above, the Clifford involution on $M_2(B)$ is of main type.

Let E be a central simple algebra over F , with $\dim_F E = 16^2$, and with a nontrivial involution $x \mapsto x^\eta$ whose restriction to F is trivial. Then there is a quaternion algebra B over F and an isomorphism $E \simeq M_8(B)$. For a quaternion algebra B_1 over F , let $C_1 = M_2(B_1)$ and let $x \mapsto x^{\eta_1}$ be an involution of C_1 whose restriction to F is trivial. Suppose that there is a (unitary) homomorphism

$$i_1 : C_1 = M_2(B_1) \hookrightarrow E = M_8(B)$$

such that

$$i_1(c)^\eta = i_1(c^{\eta_1}).$$

Let

$$C_2 = Cent_E(i_1(C_1))$$

be the centralizer of the image of C_1 and let $i_2 : C_2 \hookrightarrow E$ be the natural inclusion. Then $C_2 \simeq M_2(B_2)$, where

$$B_1 \otimes B_2 \simeq M_2(B),$$

and we have an isomorphism

$$i = i_1 \otimes i_2 : C_1 \otimes C_2 \xrightarrow{\sim} E$$

such that

$$i(c_1 \otimes c_2)^\eta = i(c_1^{\eta_1} \otimes c_2^{\eta_2}),$$

for an involution η_2 of C_2.

Proposition. *The types of the involutions $\eta = \eta_1 \otimes \eta_2$ are:*

$$\textit{main} = \begin{cases} \textit{main} \otimes \textit{neben} \\ \textit{neben} \otimes \textit{main} \end{cases}$$

and

$$\textit{neben} = \begin{cases} \textit{main} \otimes \textit{main} \\ \textit{neben} \otimes \textit{neben}. \end{cases}$$

Proof. We can assume that F is algebraically closed. Then, on $B = B_1 = B_2 = M_2(F)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

as usual, and a main involution on $C_1 = C_2 = M_2(B)$ or on $E = M_8(B)$, is given by $x \mapsto {}^t x^\iota$. On $M_n(B) \simeq M_{2n}(F)$, this amounts to applying transpose on the matrix of the 2×2 blocks and then applying ι blockwise. We denote this type of transpose by $x \mapsto {}^t x$ and write $x \mapsto {}^T x$ for the usual transpose on $M_{2n}(F)$. Let $\tau = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in B = M_2(F)$, so that $\tau^\iota = -\tau$, and, for $x \in B$,

$$\tau x^\iota \tau^{-1} = {}^T x.$$

Setting

$$h = h_n = \mathrm{diag}(\tau, \dots, \tau)$$

in $M_n(B)$, we have involutions of nebentype

$$x \mapsto h\, {}^t x^\iota h^{-1}$$

which, on $M_n(B) \simeq M_{2n}(F)$ are just given by $x \mapsto {}^T x$, the *usual transpose*, rather than the blockwise transpose.

Now consider the explicit isomorphism

$$\begin{aligned} i : M_4(F) \otimes M_2(B) &\xrightarrow{\sim} M_8(B) \\ (a_{ij}) \otimes y &\mapsto (a_{ij} y). \end{aligned}$$

Applying the involution of main type on $M_8(B)$, we have

$$ {}^t i(x \otimes y)^\iota = i({}^T x \otimes {}^t y^\iota). $$

Similarly, applying the involution of nebentype on $M_8(B)$, we have

$$ {}^T i(x \otimes y) = i({}^T x \otimes h\, {}^t y^\iota h^{-1}) = i({}^T x \otimes {}^T y). $$

Every involution on E compatible with the isomorphism $i : C_1 \otimes C_2 \xrightarrow{\sim} E$ is conjugate to one of these two by an element of the form $g = i(g_1 \otimes g_2)$, with ${}^t g^\iota = \pm g$. Note that

$$ {}^t g^\iota = g \iff \begin{cases} {}^T g_1 = g_1 \text{ and } {}^t g_2^\iota = g_2 \\ {}^T g_1 = -g_1 \text{ and } {}^t g_2^\iota = -g_2, \end{cases} $$

and

$$ {}^t g^\iota = -g \iff \begin{cases} {}^T g_1 = g_1 \text{ and } {}^t g_2^\iota = -g_2 \\ {}^T g_1 = -g_1 \text{ and } {}^t g_2^\iota = g_2. \end{cases} $$

Also observe that the involution

$$ x \mapsto g_1 {}^T x g_1^{-1} = g_1 h\, {}^t x^\iota h^{-1} g_1^{-1} $$

is of main type if ${}^T g_1 = -g_1$ and of nebentype if ${}^T g_1 = g_1$, since

$$ {}^T g_1 = h\, {}^t g_1 h^{-1} = \pm g_1 \iff \pm\, {}^t (g_1 h)^\iota = -g_1 h. $$

□

A.5. Let B be a quaternion algebra over $\mathbb{R}$.

Lemma. *For $\tau \in B^\times$ with $\tau^\iota = \pm\tau$, the involution $x \mapsto x^* = \tau x^\iota \tau^{-1}$ on B is positive if and only if:*

$$ \begin{cases} \tau^\iota = -\tau \text{ and } \tau^2 < 0 & \text{if } B = M_2(\mathbb{R}) \\ \tau^\iota = \tau & \text{if } B = H \text{ is division.} \end{cases} $$

In particular, if $B = H$, then $x^* = x^\iota$ is the unique positive involution on B.

Proof. Take $\tau \in B^\times$ such that $\tau^\iota = -\tau$ and $\tau^2 < 0$. Note that the condition on τ^2 is automatic when $B = H$. Choose $\eta \in B^\times$ such that $\eta\tau = -\tau\eta$ and

$\eta^\iota = -\eta$. Then every element $x \in B$ can be written uniquely in the form $x = a + b\eta$ with a and $b \in \mathbb{R}(\tau) \simeq \mathbb{C}$. Then

$$x^* = \tau(a + b\eta)^\iota \tau^{-1} = a^\iota - \eta^\iota b^\iota$$

and

$$\begin{aligned} tr(xx^*) &= tr((a + b\eta)(a^\iota - \eta^\iota b^\iota)) \\ &= tr(aa^\iota + b\eta a^\iota - a\eta^\iota b^\iota - b\eta\eta^\iota b^\iota) \\ &= 2(aa^\iota + bb^\iota \eta^2). \end{aligned}$$

If $B = M_2(\mathbb{R})$, then $\eta^2 > 0$, and this quantity is positive, while, if $B = H$, then $\eta^2 < 0$, and this quantity can be negative. Note that, when $B = M_2(\mathbb{R})$, then an involution defined by a τ with $\tau^2 > 0$ cannot be positive. $\square$

A.6. Let $B = M_2(\mathbb{R})$ and let $C = M_2(B)$ with involution $x' = {}^t x^\iota$ as above and let

$$V = \{x \in C;\ x' = x \text{ and } \mathrm{tr}(x) = 0\} \quad .$$

Then the signature of V for the form q_B of A.3 is (3,2).

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