Highlights of Peter Scholze's Contributions

MICHAEL RAPOPORT

Peter Scholze (Universität Bonn) was given the 2014 Clay Research Award for "his many and significant contributions to arithmetic algebraic geometry, particularly in the development and applications of the theory of perfectoid spaces". Here is a list of highlights of Scholze's contributions to date.

• *A new proof of the Local Langlands Correspondence* (LLC): Scholze's main contribution to this circle of ideas is the determination of the *semi-simple* sheaf of nearby-cycles in certain geometric situations. The hypotheses of Scholze's formula are satisfied in the case of Shimura varieties studied by Kottwitz, Harris and Taylor, and which are used as the basic geometric input of the proof of Harris/Taylor and Henniart of the local Langlands correspondence for the general linear group of a local field of characteristic zero. Based on this geometric theorem, Scholze gave a completely different proof of LLC that does not use the cumbersome reduction to characteristic p and avoids complicated ad hoc methods that lie behind the *numerical Langlands correspondence* of Henniart.

We state Scholze's theorem in the case when the local field is $\mathbb{Q}_p$. Let τ be an element of the Weil group $W_{\mathbb{Q}_p}$ projecting to the rth power of the Frobenius element. Let $h \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}_p))$ have values in $\mathbb{Q}$. Scholze defines a function $\beta \mapsto \varphi_{\tau,h}(\beta)$ on $\mathrm{GL}_n(\mathbb{Q}_{p^r})$ by associating to β the alternating trace of $\tau \times h^\vee$ on the ℓ-adic sheaf of nearby cycles of a certain moduli space of p-divisible groups that depends on β. What is essential here is the fact that the function $\varphi_{\tau,h}$ is defined in a geometric way. It lies in $C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_{p^r}))$ and takes values in $\mathbb{Q}$ independent of ℓ.

Theorem 1. *(i) For any irreducible smooth representation* π *of* $\mathrm{GL}_n(\mathbb{Q}_p)$*, there exists a unique* n*-dimensional representation* $\mathrm{rec}(\pi)$ *of* $W_{\mathbb{Q}_p}$ *such that for all* τ *and* h *as above,*

$$\mathrm{trace}(f_{\tau,h}|\pi) = \mathrm{trace}(\tau|\mathrm{rec}(\pi)) \cdot \mathrm{trace}(h|\pi).$$

Here $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_p))$ *is any function matching* $\varphi_{\tau,h}$ *in the sense of local harmonic analysis.*

(ii) The map $\pi \mapsto \mathrm{rec}(\pi)$ *induces a bijection between the set of isomorphism classes of supercuspidal irreducible smooth representations of* $\mathrm{GL}_n(F)$ *and the set of isomorphism classes of irreducible* n*-dimensional representations of* W_F.

(iii) If π *is supercuspidal, then, up to Tate twist,* $\mathrm{rec}(\pi)$ *is in Langlands correspondence with* π.

Here (i) asserts that the geometrically defined identity appearing in it sets up a map between irreducible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ and n-dimensional representations of the Weil group $W_{\mathbb{Q}_p}$; (ii) asserts that this sets up a bijection between the two sets in question; and (iii) asserts that, for supercuspidal representations, this bijection can be characterized à la Langlands through L-functions and ε-factors.

• *The theory of perfectoid spaces:* This is a general method to reduce problems of algebraic geometry over fields of mixed characteristic, like $\mathbb{Q}_p$, to fields of positive characteristic, like $\mathbb{F}_p((t))$, which are often easier to solve, mainly due to the power of the Frobenius map in characteristic p. One application of this theory is a general form of Faltings' *almost purity theorem*. Rather than explaining this method, we will give here only a sampling of the many applications of this theory. Detailed accounts of this theory have been given by Scholze himself (in his original paper in Pub. math. IHES, in his paper for the *Current developments in Mathematics* series, and his ICM 2014 paper, and his video talk at the ICM in Seoul; also an exposition by J.-M. Fontaine in the Bourbaki seminar is devoted to this topic, as well as a short note by B. Bhatt in the *What is . . .?* series of the *Notices of the AMS*).

• *A proof of the Weight Monodromy Conjecture (WMC) in many cases:* As a first application of perfectoid spaces, Scholze proved the WMC in many cases. The conjecture explains, through the monodromy operator, the lack of Frobenius purity in the cohomology of smooth projective algebraic varieties over fields of mixed characteristic with bad reduction modulo p. This conjecture, due to Deligne in 1970, is *the* major open problem in the etale cohomology of algebraic varieties, and Scholze's theorem is the first significant advance in over 30 years. His theorem is as follows.

Theorem 2. *Let X be a geometrically connected projective smooth variety over a finite extension of $\mathbb{Q}_p$ which is a set-theoretic complete intersection in projective space. Then the WMC holds for X.*

Scholze's proof uses perfectoid spaces to reduce the assertion to the equi-characteristic analogue which is due to Deligne.

• *p-adic Hodge theory for rigid-analytic spaces:* This is a second application of perfectoid spaces, and generalizes and streamlines the p-adic Hodge theory for schemes, due to Fontaine, Messing, Faltings, Kato and Tsuji, and Beilinson and Niziol. Tate asked more than 40 years ago whether such a theory exists. The next theorem (which incorporates improvements due to Gabber and Conrad on Scholze's original theorem) gives a confirmation of one of Tate's conjectures; it is the rigid-analytic analogue of theorems of Poincaré, Hodge and deRham for complex-analytic manifolds.

Theorem 3. *Let C be an algebraically closed complete p-adic field, and let X be a proper smooth rigid-analytic space over C. Let*

$$h^{ij} = \dim H^j(X, \Omega^i_X), \quad h^n_{\mathrm{dR}} = \dim H^n_{\mathrm{dR}}(X), \text{ and } h^n_{\mathrm{et}} = \dim H^n_{\mathrm{et}}(X, \mathbb{Q}_\ell).$$

Then

$$h^n_{\mathrm{et}} = h^n_{\mathrm{dR}} = \sum_{i+j=n} h^{ij}.$$

• *Moduli of p-divisible groups:* This is a third application of perfectoid spaces. Scholze and J. Weinstein develop a theory of universal coverings of p-divisible groups, with a number of striking applications. Here is a sample application of this theory. It is the analogue of Riemann's classification of abelian varieties over $\mathbb{C}$ by their first singular homology, together with the Hodge filtration.

Theorem 4. *Let C be an algebraically closed complete p-adic field, and O_C its ring of integers. There is an equivalence of categories*

$$\{\text{ p-divisible groups over } O_C\} \xrightarrow{\sim} \{\text{ pairs } (\Lambda, W), \text{ where } \Lambda \text{ is a finite free } \mathbb{Z}_p\text{-module, and } W \subset \Lambda \otimes C \text{ is a } C\text{-subvectorspace}\}.$$

• *Torsion in the cohomology of symmetric spaces* : This is a fourth application of perfectoid spaces. Scholze shows that one can associate a Galois representation to any system of Hecke eigenvalues appearing in the p-power torsion cohomology of GL_n over a totally real or CM field, with matching Frobenius eigenvalues. This confirms conjectures of Grunewald, Ash and others that were open for 40 years. Here is a more precise version.

Theorem 5. *Let G be the restriction of scalars of GL_n from a totally real or CM field F down to $\mathbb{Q}$. Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup, with corresponding locally symmetric space Y_K. Then for any system of Hecke eigenvalues ψ appearing in $H^i(Y_K, \bar{\mathbb{F}}_p)$, there exists a continuous semisimple Galois representation*

$$\rho_\psi : \mathrm{Gal}_F \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$$

characterized by the property that for all but finitely many 'ramified' places v of F, the characteristic polynomial of $\rho_\psi(\mathrm{Frob}_v)$ is described by the Hecke eigenvalues ψ.

The proof proceeds by realizing ψ as a *boundary contribution* to the cohomology of a Shimura variety (a device invented by Harris/Lan/Taylor/Thorne in the context of *rational cohomology*), and uses perfectoid spaces obtained in the limit as the p-component of the level structure shrinks to zero.

Here is a cohomological vanishing theorem that results from this theory, and which proves a conjecture of Calegari and Emerton.

Theorem 6. *Let* $\mathrm{Sh}_K = \mathrm{Sh}_{K^pK_p}$ *be a Shimura variety, assumed compact for convenience. Let*

$$H^i(K^pK_p, \mathbb{F}_p) = H^i(\mathrm{Sh}_{K^pK_p}, \mathbb{F}_p) \quad \text{and} \quad H^i(K^p, \mathbb{F}_p) = \varinjlim_{K_p} H^i(K^pK_p, \mathbb{F}_p)$$

Then

$$H^i(K^p, \mathbb{F}_p) = (0), \qquad i > \dim \mathrm{Sh}_K.$$

It is in a weak sense an analogue of the theorem of Borel, Casselman, Garland, Prasad, Wallach, Zuckerman that states that $H^i(K^pK_p, \mathbb{Q})$ is *banal* for $i > 2\dim \mathrm{Sh}_K - \mathrm{rank}_{\mathbb{R}}(G)$.

• *The pro-etale site*: Grothendieck had explained 50 years ago that the étale cohomology of non-torsion sheaves is essentially trivial and that, in order to obtain interesting cohomology groups in characteristic zero, one had to first take the cohomology of torsion sheaves and then pass to the limit. Here Grothendieck was following Weil, who treated the case of abelian varieties through their Tate modules which arise by considering the ℓ^n-torsion points and passing to the limit. Scholze introduces a new topology on the category of schemes which has the virtue that sheaves like $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$ behave in just the way we expect from classical topology and, in joint work with B. Bhatt also introduces a fundamental group of schemes which is reasonable even for highly singular schemes. To make the latter statement precise, call a topological group G a *Noohi group* if G is complete, and admits a basis of open neighborhoods of the unit element given by open subgroups (examples of Noohi groups are locally pro-finite groups and discrete groups).

Theorem 7. *Let X be a connected scheme whose underlying topological space is locally noetherian. For any geometric point x of X, there is a Noohi group* $\pi_{\mathrm{proet}}(X,x)$ *such that local systems in* $\mathbb{Q}_\ell$*-vector spaces on X are equivalent to continuous representations of* $\pi_{\mathrm{proet}}(X,x)$ *on finite-dimensional* $\mathbb{Q}_\ell$*-vector spaces.*

This enumeration is not complete. It leaves out many further results and concepts that Scholze introduced in the last four years. His ICM 2014 paper gives an excellent overview of these; of course, that overview does not contain those results that Scholze has obtained in the last few months; to get an idea of some of these, in particular of his theory of *diamonds*, one may consult the video of his Berkeley course in the fall of 2014 that is available on the homepage of the MSRI.