Local models in the ramified case
II. Splitting models

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Contents

1 Introduction

The present paper is a continuation of our previous paper [PR]. The aim of both papers is to come to grips with the local models of Shimura varieties in the case where the group G defining the Shimura variety splits over a ramified extension of $\mathbf{Q}_p$. As was first pointed out in [P], in this case the "naive" local models of [RZ] are not flat in general. One wants to define closed subschemes of the naive local models which are flat and to understand the structure of their special fibers. In this paper, we continue our study of local models for PEL Shimura varieties in this "ramified case". Two typical cases in which ramification occurs for a PEL Shimura variety are the following:

(i) $G_{\mathbf{Q}_p}$ is of the form $G_{\mathbf{Q}_p} = R_{F/\mathbf{Q}_p}G'$, where G' is a quasi-split group over F which splits over an unramified extension of F. And $F/\mathbf{Q}_p$ is a ramified extension.

(ii) $G_{\mathbf{Q}_p}$ is the group of unitary similitudes corresponding to a ramified quadratic extension of $\mathbf{Q}_p$.

The case (ii) was first addressed in [P]; unfortunately, a number of conjectures formulated there have to be resolved in order to understand this case, even for the simplest kind of level structures (see Remark 14.2), and more general level structures pose additional problems. Here we will be concerned with the case (i). Although we will only consider the cases where $G' = \mathrm{GL}_d$ or $G' = \mathrm{GSp}_{2g}$, our method applies more generally, comp. section 14. Loosely speaking, the method developed here allows us to deal with ramification caused by restriction of scalars. In this introduction we will concentrate on the case $G' = \mathrm{GL}_d$ which brings out better the outlines of our approach. We intend to return to case (ii) in subsequent work.

Let F_0 be the complete discretely valued field with ring of integers $\mathcal{O}_{F_0}$ and perfect residue field. Let F be a totally ramified extension of degree e contained in a fixed separable closure F_0^{sep} of F_0. Let $\mathcal{O}_F$ be the ring of integers of F and π a uniformizer which is the root of an Eisenstein polynomial $Q(T) \in \mathcal{O}_{F_0}[T]$. Let K be the Galois hull of F in F_0^{sep}, with ring of integers $\mathcal{O}_K$ and residue field k'.

Let V be an F-vector space of dimension d. Fix an F-basis $e_1, \ldots, e_d$ of V and let Λ_i, $0 \leq i \leq d-1$ be the $\mathcal{O}_F$-lattice in V spanned by $\pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_d$. For a subset $I = \{i_0 < \ldots < i_{m-1}\} \subset \{0, \ldots, d-1\}$ we obtain a periodic lattice chain Λ_I in V which is given by all multiples of the lattices Λ_i with $i \in I$.

Choose for each embedding $\varphi : F \to F_0^{\mathrm{sep}}$ an integer r_φ with $0 \leq r_\varphi \leq d$. Set $r = \Sigma r_\varphi$. Then the naive local model $M_I^{\mathrm{naive}} = M^{\mathrm{naive}}(\mathcal{O}_F, \Lambda_I, \mathbf{r})$ associated to the lattice chain Λ_I and to $\mathbf{r} = (r_\varphi)$ parametrizes points $\mathcal{F}_j$ in the Grassmannian of subspaces of rank r of $\Lambda_{i_j,S} = \Lambda_{i_j} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ which are $\mathcal{O}_F$-stable and on which the representation of $\mathcal{O}_F$ is

prescribed in terms of $\mathbf{r}$, and which are compatible with varying $j = 0, \dots, m-1$. It is a projective scheme defined over Spec $\mathcal{O}_E$, where $E = E(V, \mathbf{r})$ is the reflex field. Let k be the residue field of $\mathcal{O}_E$. The special fiber of M^{naive} can be considered as a closed subscheme of the affine partial flag variety $\mathbf{Fl}_I = \mathrm{GL}_d(k[[\Pi]])/P_I$,

$$i : M_I^{\text{naive}} \otimes_{\mathcal{O}_E} k \hookrightarrow \mathbf{Fl}_I \quad .$$

We choose an ordering of the embeddings φ. The basic new ingredient of the present paper is the splitting model $\mathcal{M}_I = \mathcal{M}(\mathcal{O}_F, \Lambda_I, \mathbf{r})$. This is a projective scheme over Spec $\mathcal{O}_K$ which to a $\mathcal{O}_K$-scheme S associates the set of commutative diagrams of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-morphisms resp. -inclusions,

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^e & \to & \mathcal{F}_1^e & \to & \cdots & \to & \mathcal{F}_{m-1}^e & \to & \mathcal{F}_0^e \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^{e-1} & \to & \mathcal{F}_1^{e-1} & \to & \cdots & \to & \mathcal{F}_{m-1}^{e-1} & \to & \mathcal{F}_0^{e-1} \\
\cup & & \cup & & & & \cup & & \cup \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^1 & \to & \mathcal{F}_1^1 & \to & \cdots & \to & \mathcal{F}_{m-1}^1 & \to & \mathcal{F}_0^1
\end{array}$$

which satisfy the following conditions:

a) $\mathcal{F}_k^j$ is locally on S an $\mathcal{O}_S$-direct summand of

$$\Lambda_{i_k,S} \quad \text{of rank } \sum_{l=1}^{j} r_l \quad .$$

b) For each $a \in \mathcal{O}_F$ and $j = 1, \dots, e$

$$(a \otimes 1 - 1 \otimes \varphi_j(a))(\mathcal{F}_k^j) \subset \mathcal{F}_k^{j-1} \quad .$$

Here we have set $\mathcal{F}_k^0 = (0)$.

We obtain an $\mathcal{O}_K$-morphism

$$\pi_I : \mathcal{M}_I \longrightarrow M_I^{\text{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \quad ,$$

given by $\{\mathcal{F}_k^j\}_{k,j} \longmapsto \{\mathcal{F}_k^e\}_k$.

Our crucial observation is that $\mathcal{M}_I$ can be identified with a twisted product of unramified local models for GL_d over $\mathcal{O}_K$, $M_I^l = M(\mathcal{O}_K, \Lambda_I \otimes_{\mathcal{O}_F, \varphi_l} \mathcal{O}_K, r_l)$, for $l = 1, \dots, e$:

$$\mathcal{M}_I = M_I^1 \tilde{\times} \dots \tilde{\times} M_I^e \quad .$$

Let us define the canonical local model M_I^{can} as the scheme-theoretic image of the composed morphism

$$\mathcal{M}_I \xrightarrow{\pi_I} M_I^{\text{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \longrightarrow M_I^{\text{naive}} \quad .$$

We may then state the following result:

Theorem A: *a) M_I^{can} is the flat closure of the generic fiber $M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E$ in M_I^{naive}, and it coincides with the local model M_I^{loc} defined in [PR], §8. Its special fiber is reduced, and all its irreducible components are normal and with rational singularities.*

b) The special fiber $M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} k$ is the union of the Schubert strata in $\mathbf{Fl}_I$ for all $w \in \tilde{W}_I \setminus \tilde{W}/\tilde{W}_I$ in the μ-admissible set ([KR]), for $\mu = \omega_{r_1} + \ldots + \omega_{r_e}$,

$$M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} k = \bigcup_{w \in \mathrm{Adm}_I(\mu)} \mathcal{O}_w \quad .$$

Here $\tilde{W}$ denotes the extended affine Weyl group for GL_d and $\tilde{W}_I$ the parabolic subgroup corresponding to I.

The basic ingredients of the proof of Theorem A are the presentation of the splitting model as a twisted direct product of unramified local models and the results of Görtz [G1] on these unramified local models. We also need Görtz's Lifting Theorem [G3]. When the integers r_φ differ by at most one amongst each other, we conjecture that $M_I^{\mathrm{can}} = M_I^{\mathrm{naive}}$, comp. [PR]. Similarly, it seems reasonable to expect in the symplectic case that the canonical local model coincides with the naive local model, i.e., that the naive local model is flat in this case, comp. [G3]. Theorem A seems to indicate that M_I^{can} is the "correct" way to extend its generic fiber into an integral model. Even though M_I^{can} does not represent a good moduli problem, the geometric points of $M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} k$ can be described as a subset of $M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k$ and M_I^{can} satisfies a maximal property with respect to the morphism from $\mathcal{M}_I$ to M_I^{naive}. The situation is therefore quite similar to the solution of an orbit problem by its coarse moduli space.

Our second use of the presentation of $\mathcal{M}_I$ as a twisted direct product of unramified local models concerns the calculation of the complex of nearby cycles of M_I^{can}. Let us assume that the residue field k of $\mathcal{O}_E$ is finite. Let us denote by

$$R\Psi_K^{M_I^{\mathrm{can}}} = R\Psi(M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} \mathcal{O}_K/\mathcal{O}_K)\mathbf{Q}_\ell[d]\left(\tfrac{d}{2}\right)$$

the adjusted complex of nearby cycles, where d denotes the relative dimension of M_I^{can}. This is a perverse $\overline{\mathbf{Q}}_\ell$-sheaf of weight zero on $M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} \overline{k}$, which we may regard as a P_I-equivariant perverse $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{Fl}_I \otimes_k \overline{k}$, equipped with an action by $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$.

Theorem B: *There is an isomorphism of perverse $\overline{\mathbf{Q}}_\ell$-sheaves with $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action*

$$R\Psi_K^{M_I^{\mathrm{can}}} = R\Psi_K^{M_I^1} * \ldots * R\Psi_K^{M_I^e} \quad .$$

Here on the right hand side there appears the convolution in the sense of Lusztig of the adjusted complexes of nearby cycles of the unramified local models M_I^j, $j = 1, \ldots, e$. The latter perverse sheaves on $\mathbf{Fl}_I \otimes_k \overline{k}$ are known due to the solution of the Kottwitz conjecture by Haines and Ngo [HN1].

In section 14 we extend our construction of the splitting model to the general ramified PEL case. The splitting model comes equipped with a morphism to the naive PEL local

model of [RZ]. The scheme-theoretic image of this morphism is a closed subscheme of the naive local model; as our examples indicate, it is reasonable to expect that in many cases (see §14 for details) this is the "canonical" flat local model. However, as A. Genestier pointed out to us, this is not true when the group is even orthogonal. Also, this is not true in general in the case of a unitary group corresponding to a ramified quadratic extension. However, we believe that even in these cases the methods of the present paper will turn out to be useful. In the last section we briefly indicate how to construct integral models of the relevant moduli spaces of abelian varieties with additional structure, defined by the splitting local models and canonical local models.

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2 General notations

Most of the time, we will follow the notations and assumptions of [R-P], §2. In particular, F_0 is a complete discretely valued field with ring of integers $\mathcal{O}_{F_0}$, uniformizer π_0 and perfect residue field. We fix a separable closure F_0^{sep} of F_0. Let F be a totally ramified separable extension of degree e of F_0 with ring of integers $\mathcal{O}_F$. Let π be a uniformizer of $\mathcal{O}_F$ which is a root of the Eisenstein polynomial

$$Q(T) = T^e + \sum_{k=0}^{e-1} b_k T^k, \quad b_0 \in \pi_0 \cdot \mathcal{O}_{F_0}^{\times}, \ b_k \in (\pi_0). \tag{2.1}$$

Let us denote by K the Galois hull of F in F_0^{sep} and let $\mathcal{O}_K$ be the ring of integers of K; denote by k' the residue field of $\mathcal{O}_K$. Let us choose an ordering of the embeddings $\phi : F \to F_0^{\mathrm{sep}}$ and for $i \in \{1, \cdots, e\}$ let us set $a_i = \phi_i(\pi)$. We have

$$\mathcal{O}_{F_0}[T]/(Q(T)) \simeq \mathcal{O}_F \tag{2.2}$$

given by $T \mapsto \pi$. For $i = 1, \ldots, e$, we set

$$Q^i(T) = \prod_{j=i}^{e}(T - a_j), \quad Q_i(T) = \prod_{j=1}^{i-1}(T - a_j) \in \mathcal{O}_K[T], \quad \mathcal{O}_K^{(i)} = \mathcal{O}_K[T]/(Q^i(T)) \ , \tag{2.3}$$

so that $Q^1(T) = Q(T)$, and $Q_i(T)Q^i(T) = Q(T)$. There are natural surjective $\mathcal{O}_K$-algebra homomorphisms

$$\phi^i : \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \simeq \mathcal{O}_K[T]/(Q(T)) \to \mathcal{O}_K[T]/(Q^i(T)) = \mathcal{O}_K^{(i)}$$

obtained by sending $\pi \otimes 1$ to T.

There are exact sequences

$$\mathcal{O}_K[T]/(Q(T)) \overset{Q_i(T)}{\to} \mathcal{O}_K[T]/(Q(T)) \overset{Q^i(T)}{\to} \mathcal{O}_K[T]/(Q(T)) \ ,$$

$$\mathcal{O}_K[T]/(Q(T)) \overset{Q^i(T)}{\to} \mathcal{O}_K[T]/(Q(T)) \overset{Q_i(T)}{\to} \mathcal{O}_K[T]/(Q(T)) \ ,$$

with the image and the kernel of each morphism $\mathcal{O}_K$-free. We conclude that if S is an $\mathcal{O}_K$-scheme, there are functorial isomorphisms

$$\mathcal{O}_K^{(i)} \otimes_{\mathcal{O}_K} \mathcal{O}_S \simeq \mathrm{Im}(Q_i(T) \mid \mathcal{O}_S[T]/(Q(T))) = \ker(Q^i(T) \mid \mathcal{O}_S[T]/(Q(T))) \ , \tag{2.4}$$

the first one obtained by multiplying with $Q_i(T)$.

Part II

3 The "naive" local models for $G = \mathrm{Res}_{F/F_0}\mathrm{GL}_d$

Now let V be an F-vector space of dimension d. Fix an F-basis $e_1, \dots e_d$ of V and let Λ_i, $0 \le i \le d-1$, be the free $\mathcal{O}_F$–module of rank d with basis $e_1^i := \pi^{-1}e_1, \dots, e_i^i := \pi^{-1}e_i$, $e_{i+1}^i := e_{i+1}, \dots, e_d^i := e_d$. Let us choose a subset $I = \{i_0 < \cdots < i_{m-1}\} \subset \{0, \dots, d-1\}$ and consider the $\mathcal{O}_F$-lattice chain Λ_I in V which is given by all multiples of the lattices Λ_i with $i \in I$.

Let us choose for each embedding $\varphi : F \to F_0^{\mathrm{sep}}$ an integer r_φ with $0 \le r_\varphi \le d$. Set $r = \sum_\varphi r_\phi$. Associated to these data we have the *reflex field* E, a finite extension of F_0 contained in F_0^{sep} with

$$\mathrm{Gal}(F_0^{\mathrm{sep}}/E) = \{\sigma \in \mathrm{Gal}(F_0^{\mathrm{sep}}/F_0);\ r_{\sigma\varphi} = r_\varphi,\ \forall\varphi\} \quad . \tag{3.1}$$

We also have a cocharacter $\mu : \mathbf{G}_m/F_0^{\mathrm{sep}} \to (\mathrm{Res}_{F/F_0}\mathrm{GL}_d)/F_0^{\mathrm{sep}}$ given by $(1^{r_\phi}, 0^{d-r_\phi})_\phi$. The conjugacy class of μ is defined over the reflex field E. Let $\mathcal{O}_E$ be the ring of integers in E and k its residue field.

The "naive" local model $M_I^{\mathrm{naive}} = M(\mathcal{O}_F, \Lambda_I, \mathbf{r})$ of [RZ] for $G = \mathrm{Res}_{F/F_0}\mathrm{GL}_d$, the cocharacter μ and the lattice chain Λ_I, is the $\mathcal{O}_E$–scheme which represents the following functor: To each $\mathcal{O}_E$–scheme S, we associate the set $M_I^{\mathrm{naive}}(S)$ of collections $\{\mathcal{F}_k\}_k$ of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules of $\Lambda_{i_k,S} := \Lambda_{i_k} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ which fit into a commutative diagram

$$\begin{array}{ccccccccc} \Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_0 & \to & \mathcal{F}_1 & \to & \cdots & \to & \mathcal{F}_{m-1} & \to & \mathcal{F}_{0,S} \ , \end{array}$$

(with the morphisms $\Lambda_{i_k,S} \to \Lambda_{i_{k+1},S}$ of the first row induced by the lattice inclusions $\Lambda_{i_k} \to \Lambda_{i_{k+1}}$). We require the following conditions:

i) $\mathcal{F}_k$ is Zariski locally on S a $\mathcal{O}_S$-direct summand of $\Lambda_{i_k,S}$ of rank r,

ii) for $a \in \mathcal{O}_F$, we have

$$\det(a \mid \mathcal{F}_k) = \prod_\phi \phi(a)^{r_\phi} \ ,$$

where this last identity is meant as an identity of polynomial functions on $\mathcal{O}_F$ (comp. [K], [RZ]).

It is clear that this functor is represented by a projective scheme over $\mathrm{Spec}\,\mathcal{O}_E$.

Consider the group scheme $\mathcal{G}_I$ over $\operatorname{Spec}\mathcal{O}_{F_0}$

$$\mathcal{G}_I := \underline{\mathrm{Aut}}_{\mathcal{O}_F}(\Lambda_I) \tag{3.2}$$

with S-valued points the $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-automorphisms of the lattice chain $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$. A simple extension of the arguments of [RZ] Appendix (see also [P]) shows that $\mathcal{G}_I$ is smooth over $\operatorname{Spec}\mathcal{O}_{F_0}$, comp. Remark 3.1 below. Often we will use the base change of $\mathcal{G}_I$ to $\operatorname{Spec}\mathcal{O}_E$, which we will denote by the same symbol.

Remark 3.1 The arguments of [RZ] carry over with essentially no changes to the following situation. Let $(F_0, \mathcal{O}_{F_0}, \pi_0)$ as in section 2. Let $\mathcal{O}$ be an $\mathcal{O}_{F_0}$-order in a semi-simple F_0-algebra $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} F_0$. For the purposes of the present paper we may assume that $\mathcal{O}$ is commutative, i.e. $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} F_0$ is a product of field extensions of F_0. Let $\Pi \in \mathcal{O}$ be an element with $\pi_0 \in (\Pi)$. Let V be a finite-dimensional F_0-vector space which is an $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} F_0$-module. An $\mathcal{O}$-lattice in V is a $\mathcal{O}_{F_0}$-lattice in V stable under $\mathcal{O}$. A $(\mathcal{O}, \Pi)$-*periodic lattice chain* is a chain of inclusions of $\mathcal{O}$-lattices in V,

$$\subset \Lambda_{i-1} \subset \Lambda_i \subset \dots \quad , \; i \in \mathbf{Z} \;\; ,$$

such that

(i) $\exists r : \; \Lambda_{i-r} = \Pi\Lambda_i \;\; , \;\; \forall i \in \mathbf{Z}$.

(ii) Λ_i/Λ_{i-1} is a free $\mathcal{O}/\Pi\mathcal{O}$-module $\forall i \in \mathbf{Z}$.

Let us fix a $(\mathcal{O}, \Pi)$-periodic lattice chain $\mathcal{L}$. Let S be a $\mathcal{O}_{F_0}$-scheme such that π_0 is locally nilpotent on S. A *chain of $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-modules of type $(\mathcal{L})$ on S* is given by a chain of $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-module homomorphisms,

$$\dots \xrightarrow{\varrho} M_{i-1} \xrightarrow{\varrho} M_i \xrightarrow{\varrho} \dots$$

such that the following conditions are satisfied.

(i) $\varrho^r = \Pi$.

(ii) Locally on S there exist isomorphisms of $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-modules,

$$M_i \simeq \Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \;\; , \;\; M_i/\varrho(M_{i-1}) \simeq \Lambda_i/\Lambda_{i-1} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \;\; .$$

The proof of Prop. A.4 of loc. cit. shows then that any chain $\{M_i\}$ of $\mathcal{O} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-modules of type $(\mathcal{L})$ on S is locally on S isomorphic to $\mathcal{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$, and that the functor on (Sch/S),

$$S' \longmapsto \mathrm{Aut}(\{M_i \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}\})$$

is representable by a smooth group scheme over S.

4 Affine flag varieties for GL_d

If R is a k-algebra, a lattice in $R((\Pi))^d$ is by definition a sub-$R[[\Pi]]$-module $\mathcal{L}$ of $R((\Pi))^d$ which is locally on $\operatorname{Spec} R$ free of rank d and such that $\mathcal{L} \otimes_{R[[\Pi]]} R((\Pi)) = R((\Pi))^d$. (Here $R[[\Pi]]$, resp. $R((\Pi))$ denotes the power series ring, resp. Laurent power series ring in

the indeterminate Π over R). Equivalently, a lattice is a sub-$R[[\Pi]]$-module $\mathcal{L}$ of $R((\Pi))^d$ such that $\Pi^N R[[\Pi]]^d \subset \mathcal{L} \subset \Pi^{-N} R[[\Pi]]^d$ for some N and such that $\Pi^{-N} R[[\pi]]^d/\mathcal{L}$ is a locally free R-module.

Recall that the affine Grassmannian $\mathbf{Gr}$ over k associated to GL_d is the Ind-scheme over $\operatorname{Spec} k$ which represents the functor on k-algebras which to a k-algebra R associates the set of lattices $\mathcal{L}$ in $R((\Pi))^d$. The affine Grassmannian can be identified with the fpqc quotient $\mathrm{GL}_d(k((\Pi)))/\mathrm{GL}_d(k[[\Pi]])$ where $\mathrm{GL}_d(k((\Pi)))$, resp. $\mathrm{GL}_d(k[[\Pi]])$ is the Ind-group scheme, resp. group scheme over $\operatorname{Spec} k$ whose R-rational points is $\mathrm{GL}_d(R((\Pi)))$, resp. $\mathrm{GL}_d(R[[\Pi]])$.

For each $i \in \{0, \dots, d-1\}$, we will denote by $\tilde{\Lambda}_i$ the $k[[\Pi]]$-lattice in

$$k((\Pi))^d = k((\Pi))\tilde{e}_1 \oplus \cdots \oplus k((\Pi))\tilde{e}_d$$

which is generated by $\Pi^{-1}\tilde{e}_1, \dots, \Pi^{-1}\tilde{e}_i, \tilde{e}_{i+1}, \dots, \tilde{e}_d$.

Denote by P_I, resp. P_I', the parahoric subgroup scheme of $\mathrm{GL}_d(k((\Pi)))$, resp. $\mathrm{SL}_d(k((\Pi)))$, whose k-valued points stabilize the lattice chain

$$\tilde{\Lambda}_{i_0} \subset \tilde{\Lambda}_{i_1} \subset \cdots \subset \tilde{\Lambda}_{i_{m-1}} \subset \Pi^{-1}\tilde{\Lambda}_{i_0} \ . \tag{4.1}$$

If $I = \{0, \dots, d-1\}$, then P_I, resp. P_I', is an Iwahori subgroup scheme of $\mathrm{GL}_d(k((\Pi)))$, resp. $\mathrm{SL}_d(k((\Pi)))$.

For every nonempty subset $I = \{i_0 < \cdots < i_{m-1}\} \subset \{0, \dots, d-1\}$, we have the partial affine flag variety $\mathbf{Fl}_I$ whose R-rational points parametrize lattice chains in $R((\Pi))^d$

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{m-1} \subset \Pi^{-1}\mathcal{L}_0 \tag{4.2}$$

with $\mathcal{L}_{k+1}/\mathcal{L}_k$, resp. $\Pi^{-1}\mathcal{L}_0/\mathcal{L}_{m-1}$ locally free R-modules of rank $i_{k+1} - i_k$ for $k = 0 \dots m-2$, resp. $(d + i_0) - i_{m-1}$.

The affine Grassmannian variety corresponds to the choice $I = \{0\}$, while the full affine flag variety corresponds to $I = \{0, \dots, d-1\}$. The Ind-group scheme $\mathrm{GL}_d(k((\Pi)))$ acts on the partial affine flag variety $\mathbf{Fl}_I$ and we can identify $\mathbf{Fl}_I$ ($\mathrm{GL}_d(k((\Pi)))$-equivariantly) with the fpqc quotient

$$\mathbf{Fl}_I = \mathrm{GL}_d((k((\Pi)))/P_I \ . \tag{4.3}$$

Given $r \in \mathbf{Z}$, we may also consider the special partial affine flag variety $\mathbf{Fl}_I^r$ whose R-rational points parametrize lattice chains in $R((\Pi))^d$

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{m-1} \subset \Pi^{-1}\mathcal{L}_0 \tag{4.4}$$

such that:

i) $\mathcal{L}_{k+1}/\mathcal{L}_k$, resp. $\Pi^{-1}\mathcal{L}_0/\mathcal{L}_{m-1}$ are locally free R-modules of rank $i_{k+1} - i_k$ for $k = 0 \dots m-2$, resp. $(d + i_0) - i_{m-1}$,

ii) $\wedge^d \mathcal{L}_0 = \Pi^r R[[\Pi]]^d$ (as a submodule of $\wedge^d R((\Pi))^d = R((\Pi))$).

The special affine flag varieties $\mathbf{Fl}_I^r$ for various r are all isomorphic to the fpqc quotient

$$\mathrm{SL}_d(k((\Pi)))/P_I'$$

(as abstract Ind-schemes but not $\mathrm{SL}_d(k((\Pi)))$-equivariantly, unless $r = -i_0$). For $I = \{0\}$, we obtain the special affine Grassmannian

$$\mathbf{Gr}^r \;\simeq\; \mathrm{SL}_d(k((\Pi)))/\mathrm{SL}_d(k[[\Pi]])\ .$$

Now fix an identification $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k = k[[\Pi]]/(\Pi^e)$ and $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k$-isomorphisms

$$\Lambda_{i_k} \otimes_{\mathcal{O}_{F_0}} k \simeq \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$$

which induce a $k[[\Pi]]/(\Pi^e)$-module chain isomorphism

$$\Lambda_I \otimes_{\mathcal{O}_{F_0}} k \simeq \tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)\ .$$

Let R be a k-algebra. For an R-valued point $\{\mathcal{F}_k\}_k$ of M_I^{naive}, we have

$$\mathcal{F}_k \subset \Lambda_{i_k} \otimes_{\mathcal{O}_{F_0}} R = \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e)\ . \tag{4.5}$$

Let $\mathcal{L}_k \subset \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]$ be the inverse image of $\mathcal{F}_k$ under the canonical projection

$$\tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]] \to \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e)\ .$$

so that we have

$$\Pi^e \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]] \subset \mathcal{L}_k \subset \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]\ .$$

Then $\{\mathcal{L}_k\}_k$ gives an R-valued point of $\mathbf{Fl}_I$. In this way, we obtain a morphism

$$i : M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k \longrightarrow \mathbf{Fl}_I \tag{4.6}$$

which is a closed immersion (of Ind-schemes).

5 The splitting model for $G = \mathrm{Res}_{F/F_0}\mathrm{GL}_d$

Fix $I = \{i_0 < i_1 < \ldots < i_{m-1}\} \subset \{0, \ldots, d-1\}$. Consider the functor $\mathcal{M}_I = \mathcal{M}(\mathcal{O}_F, \Lambda_I, \mathbf{r})$ on $(\mathrm{Schemes}/\mathrm{Spec}\,\mathcal{O}_K)$ which to a $\mathcal{O}_K$-scheme S associates the set $\mathcal{M}_I(S)$ of collections $\{\mathcal{F}_k^j\}_{j,k}$ of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules of $\Lambda_{i_k,S}$ which fit into a commutative diagram

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \stackrel{\pi}{\to} & \Lambda_{i_0,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^e & \to & \mathcal{F}_1^e & \to & \cdots & \to & \mathcal{F}_{m-1}^e & \to & \mathcal{F}_0^e \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^{e-1} & \to & \mathcal{F}_1^{e-1} & \to & \cdots & \to & \mathcal{F}_{m-1}^{e-1} & \to & \mathcal{F}_0^{e-1} \\
\cup & & \cup & & & & \cup & & \cup \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^1 & \to & \mathcal{F}_1^1 & \to & \cdots & \to & \mathcal{F}_{m-1}^1 & \to & \mathcal{F}_0^1
\end{array}$$

and are such that:

a) $\mathcal{F}_k^j$ is Zariski locally on S a $\mathcal{O}_S$-direct summand of $\Lambda_{i_k,S}$ of rank $\sum_{l=1}^j r_l$.

b) For each $a \in \mathcal{O}_F$ and $j = 1, \ldots, e$,

$$(a \otimes 1 - 1 \otimes \phi_j(a))(\mathcal{F}_k^j) \subset \mathcal{F}_k^{j-1}$$

where the tensor products $a \otimes 1$, $1 \otimes \phi_j(a)$ are in $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and where, for each k, we set $\mathcal{F}_k^0 = (0)$.

The functor $\mathcal{M}_I$ is obviously represented by a projective scheme over $\operatorname{Spec} \mathcal{O}_K$. Notice that there is an $\mathcal{O}_K$-morphism

$$\pi_I : \mathcal{M}_I \to M_I^{\text{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \tag{5.1}$$

given by $\{\mathcal{F}_k^j\}_{k,j} \mapsto \{\mathcal{F}_k^e\}_k$. Indeed, if $\mathcal{F}_k = \mathcal{F}_k^e$ supports a filtration $\{\mathcal{F}_k^j\}_j$ with the above properties, then the characteristic polynomial of the action of $a \in \mathcal{O}_F$ on $\mathcal{F}_k$ is

$$\prod_{l=1}^{e} (T - \phi_l(a))^{r_l} \tag{5.2}$$

and therefore $\mathcal{F}_k$ satisfies the condition ii) in the definition of M_I^{naive}.

Proposition 5.1 *The morphism π_I induces an isomorphism*

$$\pi_I \otimes_{\mathcal{O}_K} K : \mathcal{M}_I \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} M_I^{\text{naive}} \otimes_{\mathcal{O}_E} K$$

on the generic fibers.

Proof. To each S-valued point of M_I^{naive} with S a K-scheme, given by $\{\mathcal{F}_k\}_k$, we can associate an S-valued point of $\mathcal{M}_I$ by considering, for each k, the filtration $\{\mathcal{F}_k^l\}_l$ associated to the grading on the $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} K$-module $\mathcal{F}_k = \mathcal{F}_k^e$ given using the decomposition

$$\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} K \simeq \oplus_{l=1}^e K, \quad a \otimes b \mapsto (b\phi_l(a))_{l=1,\ldots,e} \ .$$

This gives a morphism inverse to $\pi_I \otimes_{\mathcal{O}_K} K$. □

For each $l = 1, \ldots, e$, $k = 0, \ldots, m-1$, set $\Xi_{i_k}^l = \Lambda_{i_k} \otimes_{\mathcal{O}_F, \phi_l} \mathcal{O}_K$ (an $\mathcal{O}_K$-lattice in $V \otimes_{F,\phi_l} K$). Denote by Ξ_I^l the $\mathcal{O}_K$-lattice chain in $V \otimes_{F,\phi_l} K$ given by the lattices $\{a_l^n \Xi_{i_k}^l\}_{k,n \in \mathbf{Z}}$. An "essential" part of the lattice chain Ξ_I^l is

$$\Xi_{i_0}^l \subset \Xi_{i_1}^l \subset \cdots \subset \Xi_{i_{m-1}}^l \subset a_l^{-1} \Xi_{i_0}^l \ ,$$

in the sense that each successive link $\Xi_i^l \subset \Xi_{i'}^l$ in the total lattice chain Ξ_I^l is a multiple of one of the links in the part above.

Let $\mathcal{G}_I^l$ be the group scheme over $\operatorname{Spec} \mathcal{O}_K$ whose S-points are the $\mathcal{O}_S$-automorphisms of the chain $\Xi_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$ (once again, a simple extension of [RZ] Prop. A.4 shows that this is a smooth group scheme, comp. Remark 3.1).

Now if S a scheme over $\operatorname{Spec}\mathcal{O}_K$, we obtain from $\Lambda_{I,S}$ a $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-lattice chain $\Lambda_{I,S}^l$ by extending scalars via

$$\phi^l \otimes_{\mathcal{O}_K} \mathcal{O}_S : \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \simeq \mathcal{O}_S[T]/(Q(T)) \to \mathcal{O}_S[T]/(Q^l(T)) = \mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S \ .$$

An argument as in the proof of (2.4), shows that we have functorial isomorphisms of chains of $\mathcal{O}_S$-modules

$$\Xi_{I,S}^l = \Lambda_I \otimes_{\mathcal{O}_F,\phi_l} \mathcal{O}_S \simeq \ker(\pi - a_l \mid \Lambda_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S) \tag{5.3}$$

obtained by sending the element $\lambda \otimes 1$ of $\Lambda_I \otimes_{\mathcal{O}_F,\phi_l} \mathcal{O}_S$ to the image of $Q^{l+1}(\pi) \cdot (\lambda \otimes 1)$ in $\Lambda_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$.

Denote by $\mathcal{G}_I^{(l)}$ the group scheme over $\operatorname{Spec}\mathcal{O}_K$ whose S-points are the $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} O_S$-automorphisms of the chain $\Lambda_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$ (once again we can see that this is a smooth group scheme, comp. Remark 3.1). The isomorphism (5.3) induces a group scheme homomorphism

$$\mathcal{G}_I^{(l)} \to \mathcal{G}_I^l \ . \tag{5.4}$$

Now suppose that $\{\mathcal{F}_k^j\}_{j,k}$ is an S-valued point of $\mathcal{M}_I$. For $l = 1, \dots, e$, let us set

$$\Psi_{i_k,S}^l = \ker(Q^l(\pi) \mid \Lambda_{i_k,S}/\mathcal{F}_k^{l-1}) \ ;$$

this is an $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module. We also set

$$\Upsilon_{i_k,S}^l := \ker(\pi - a_l \mid \Lambda_{i_k,S}/\mathcal{F}_k^{l-1}) = \ker(\pi - a_l \mid \Psi_{i_k,S}^l) \ .$$

We have $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module, resp. $\mathcal{O}_S$-module, homomorphisms

$$\Psi_{i_k,S}^l \to \Psi_{i_{k+1},S}^l, \qquad \Psi_{i_{m-1},S}^l \xrightarrow{T} \Psi_{i_0,S}^l$$

resp.

$$\Upsilon_{i_k,S}^l \to \Upsilon_{i_{k+1},S}^l, \qquad \Upsilon_{i_{m-1},S}^l \xrightarrow{a_l} \Upsilon_{i_0,S}^l$$

induced by the $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-module homomorphisms

$$\Lambda_{i_k,S}/\mathcal{F}_k^{l-1} \to \Lambda_{i_{k+1},S}/\mathcal{F}_{k+1}^{l-1}, \qquad \Lambda_{i_{m-1},S}/\mathcal{F}_{m-1}^{l-1} \xrightarrow{\pi} \Lambda_{i_0,S}/\mathcal{F}_0^{l-1}$$

by taking the kernel of $Q^l(\pi)$, resp. of $\pi - \phi_l(\pi) = \pi - a_l$.

Proposition 5.2 *a) The formation of $\Psi_{i_k,S}^l$, resp. of $\Upsilon_{i_k,S}^l$, from $\{\mathcal{F}_k^j\}_{j,k}$ commutes with base change.*

b) The $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module $\Psi_{i_k,S}^l$, resp. the $\mathcal{O}_S$-module $\Upsilon_{i_k,S}^l$, is locally on S free of rank d.

c) The chain $\Psi_{i_k,S}^l$ of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules given by

$$\cdots \to \Psi_{i_0,S}^l \to \cdots \to \Psi_{i_{m-1},S}^l \xrightarrow{\pi} \Psi_{i_0,S}^l \to \cdots$$

is Zariski locally on S isomorphic to the chain of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $\Lambda_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$. Similarly, the chain $\Upsilon_{I,S}^l$ of $\mathcal{O}_S$-modules on S given by

$$\cdots \to \Upsilon_{i_0,S}^l \to \cdots \to \Upsilon_{i_{m-1},S}^l \xrightarrow{a_l} \Upsilon_{i_0,S}^l \to \cdots$$

is Zariski locally on S isomorphic to the chain of $\mathcal{O}_S$-modules $\Xi_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$.

PROOF. The statements for the modules $\Upsilon_{i_k,S}^l$ follow from the corresponding statements for the modules $\Psi_{i_k,S}^l$. Indeed, we can see this fact using the functorial isomorphisms (5.3) and the fact that

$$\Upsilon_{i_k,S}^l = \ker(\pi - a_l \mid \Psi_{i_k,S}^l) \ .$$

Write $Q^l(T)^{-1}(\mathcal{F}_k^{l-1})$ for the inverse image of $\mathcal{F}_k^{l-1} \subset \Lambda_{i_k,S}$ under $\Lambda_{i_k,S} \to \Lambda_{i_k,S}$ given by multiplication by $Q^l(T)$. Notice that since $Q_l(T)(\mathcal{F}_k^{l-1}) = (0)$, by (2.4) we have $\mathcal{F}_k^{l-1} \subset Q^l(T)(\Lambda_{i_k,S})$. Hence, there is an exact sequence

$$0 \to \ker(Q^l(T) \mid \Lambda_{i_k,S}) \to Q^l(T)^{-1}(\mathcal{F}_k^{l-1}) \to \mathcal{F}_k^{l-1} \to 0 \ .$$

By (2.4), $\ker(Q^l(T) \mid \Lambda_{i_k,S}) \simeq \Lambda_{i_k,S}/Q_l(T)\Lambda_{i_k,S}$. Hence, $Q^l(T)^{-1}(\mathcal{F}_k^{l-1})$ is a locally free $\mathcal{O}_S$-module of rank $d(e-l+1) + \sum_{i=1}^{l-1} r_i$ whose formation commutes with base change. The exact sequence

$$0 \to \Lambda_{i_k,S}/Q^l(T)^{-1}(\mathcal{F}_k^{l-1}) \xrightarrow{Q^l(T)} \Lambda_{i_k,S}/\mathcal{F}_k^{l-1} \to \Lambda_{i_k,S}/Q^l(T)\Lambda_{i_k,S} \to 0$$

now implies that $\Lambda_{i_k,S}/Q^l(T)^{-1}(\mathcal{F}_k^{l-1})$ is also $\mathcal{O}_S$-locally free. Hence, $Q^l(T)^{-1}(\mathcal{F}_k^{l-1}) \subset \Lambda_{i_k,S}$ is locally an $\mathcal{O}_S$-direct summand. Now

$$0 \to Q^l(T)^{-1}(\mathcal{F}_k^{l-1})/\mathcal{F}_k^{l-1} \to \Lambda_{k,S}/\mathcal{F}_k^{l-1} \to \Lambda_{i_k,S}/Q^l(T)^{-1}(\mathcal{F}_k^{l-1}) \to 0$$

implies that $\Psi_{i_k,S}^l = Q^l(T)^{-1}(\mathcal{F}_k^{l-1})/\mathcal{F}_k^{l-1}$ is a $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module which is locally free of rank $d(e-l+1)$ as an $\mathcal{O}_S$-module and that its formation commutes with base change in S. To show that $\Psi_{i_k,S}^l$ is locally on S a free $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module it is enough to show this for $S = \operatorname{Spec} L$, L a field. This is easy to see if L is an extension of K. If L is an extension of k', then $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} L = L[\pi]/(\pi^{e-l+1})$. In this case, there is a $L[\pi]/(\pi^e)$-basis $f_1, \ldots, f_d$ of $\Lambda_{i_k} \otimes_{\mathcal{O}_K} L$ and $l-1 \geq s_1 \geq \cdots \geq s_d \geq 0$ such that

$$\mathcal{F}_k^{l-1} = L[\pi]/(\pi^e) \cdot \pi^{e-s_1} f_1 \oplus \cdots \oplus L[\pi]/(\pi^e) \cdot \pi^{e-s_d} f_d \ . \tag{5.5}$$

Then

$$\Psi_{i_k,S}^l = \frac{L[\pi]/(\pi^e) \cdot \pi^{l-1-s_1} f_1 \oplus \cdots \oplus L[\pi]/(\pi^e) \cdot \pi^{l-1-s_d} f_d}{L[\pi]/(\pi^e) \cdot \pi^{e-s_1} f_1 \oplus \cdots \oplus L[\pi]/(\pi^e) \cdot \pi^{e-s_d} f_d} \ , \tag{5.6}$$

which is freely generated over $L[\pi]/(\pi^{e-l+1})$ by the classes of $\pi^{l-1-s_1} f_1, \ldots, \pi^{l-1-s_d} f_d$.

It remains to show (c) for $\Psi_{i_k,S}^l$, i.e that the chain $\Psi_{I,S}^l$ is Zariski locally isomorphic to the chain $\Lambda_I^l \otimes_{\mathcal{O}_K} \mathcal{O}_S$. Given (b), an extension of the arguments in the proof of [RZ] Prop. A 4, p. 133 shows that it will be enough to prove that the cokernels of

$$\Psi_{i_k,S}^l \to \Psi_{i_{k+1},S}^l \ , \quad \Psi_{i_{m-1},S}^l \xrightarrow{\pi} \Psi_{i_0,S}^l$$

are Zariski locally on S isomorphic to the $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $(\Lambda_{i_k}^l/\Lambda_{i_{k+1}}^l) \otimes_{\mathcal{O}_K} \mathcal{O}_S$ and $(\Lambda_{i_0}^l/T\Lambda_{i_{m-1}}^l) \otimes_{\mathcal{O}_K} \mathcal{O}_S$ respectively, comp. Remark 3.1. In what follows, we will only deal with the case of $\Psi_{i_k,S}^l \to \Psi_{i_{k+1},S}^l$, the case of $\Psi_{i_{m-1},S}^l \xrightarrow{\pi} \Psi_{i_0,S}^l$ being similar. Notice that $\Lambda_{i_k}^l/\Lambda_{i_{k+1}}^l$ is a module over $\mathcal{O}_K^{(l)}/T\mathcal{O}_K^{(l)} = \mathcal{O}_K[T]/(T, Q^l(T)) \simeq \mathcal{O}_K/(\varpi^{e-l+1})$ with ϖ a uniformizer of $\mathcal{O}_K$. For simplicity of notation, set

$$R_l = \mathcal{O}_K/(\varpi^{e-l+1}) \ .$$

The cokernel of $\Psi_{i_k,S}^l \to \Psi_{i_{k+1},S}^l$ is a module over $R_1 \otimes_{\mathcal{O}_K} \mathcal{O}_S$. Hence, it is enough to assume that $S = \operatorname{Spec} R$ is an affine R_1-scheme and prove the result in this case. In fact, since $\mathcal{M}_I$ is a Noetherian scheme, we can also assume that R is Noetherian. We can lift the R_1-chain $\Lambda_{I,R_1} := \Lambda_I \otimes_{\mathcal{O}_{F_0}} R_1$ to a chain of $R_1[[T]]$-free modules $\tilde{\Lambda}_I$

$$\tilde{\Lambda}_{i_0} \subset \tilde{\Lambda}_{i_1} \subset \cdots \subset \tilde{\Lambda}_{i_{m-1}} \subset T^{-1}\tilde{\Lambda}_{i_0}$$

which are all $R_1[[T]]$-submodules of $R_1((T))^d$ such that there is an isomorphism of $R_1[T]/(Q(T)) = \mathcal{O}_K \otimes_{\mathcal{O}_{F_0}} R_1$-chains

$$\tilde{\Lambda}_I \otimes_{R_1[[T]]} R_1[[T]]/(Q(T)) \simeq \Lambda_{I,R_1} \ .$$

Now notice that since each a_j is nilpotent in R_1, the elements $T - a_j$ of $R_1[[T]]$ are invertible in $R_1((T))$ and hence the inverse $Q^l(T)^{-1}$ makes sense in $R_1((T))$. The diagram corresponding to the R-valued point $\{\mathcal{F}_k^j\}_{j,k}$ of $\mathcal{M}_I$ now provides us with a diagram of $R[[T]]$-lattices in $R((T))^d$:

$$\begin{array}{ccccccccc}
\tilde{\Lambda}_{i_0,R} & \subset & \tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & \tilde{\Lambda}_{i_{m-1},R} & \subset & T^{-1}\tilde{\Lambda}_{i_0,R} \\
\cup & & \cup & & & & \cup & & \cup \\
Q^l(T)^{-1}\mathcal{L}_0^{l-1} & \subset & Q^l(T)^{-1}\mathcal{L}_1^{l-1} & \subset & \cdots & \subset & Q^l(T)^{-1}\mathcal{L}_{m-1}^{l-1} & \subset & T^{-1}Q^l(T)^{-1}\mathcal{L}_0^{l-1} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{L}_0^{l-1} & \subset & \mathcal{L}_1^{l-1} & \subset & \cdots & \subset & \mathcal{L}_{m-1}^{l-1} & \subset & T^{-1}\mathcal{L}_0^{l-1} \\
\cup & & \cup & & & & \cup & & \cup \\
Q(T)\tilde{\Lambda}_{i_0,R} & \subset & Q(T)\tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & Q(T)\tilde{\Lambda}_{i_{m-1},R} & \subset & T^{-1}Q(T)\tilde{\Lambda}_{i_0,R} \ ,
\end{array}$$

where $\tilde{\Lambda}_{i_k,R} = \tilde{\Lambda}_{i_k} \otimes_{R_1[[T]]} R[[T]]$, and $\mathcal{L}_k^{l-1}$, resp. $Q^l(T)^{-1}\mathcal{L}_k^{l-1}$, is the inverse image of $\mathcal{F}_k^{l-1}$, resp. $Q^l(T)^{-1}(\mathcal{F}_k^{l-1})$ under the surjection

$$\tilde{\Lambda}_{i_k,R} \to \tilde{\Lambda}_{i_k,R} \otimes_{R[[T]]} R[[T]]/(Q(T)) \simeq \Lambda_{i_k,R} \ .$$

Each quotient created by the inclusion of any two modules in this diagram is a finitely generated locally free R-module. In particular

$$Q^l(T)^{-1}\mathcal{L}_{k+1}^{l-1}/Q^l(T)^{-1}\mathcal{L}_k^{l-1}$$

is annihilated by T and is R-locally free of rank equal to the rank of $\tilde{\Lambda}_{i_{k+1},S}/\tilde{\Lambda}_{i_k,S}$. It now follows that the cokernel of

$$\Psi_{i_k,S}^l = Q^l(T)^{-1}\mathcal{L}_k^{l-1}/\mathcal{L}_k^{l-1} \to Q^l(T)^{-1}\mathcal{L}_{k+1}^{l-1}/\mathcal{L}_{k+1}^{l-1} = \Psi_{i_{k+1},S}^l$$

is isomorphic to

$$\left(Q^l(T)^{-1}\mathcal{L}^{l-1}_{k+1}/Q^l(T)^{-1}\mathcal{L}^{l-1}_k\right)\otimes_{R_1} R_l$$

and therefore it is a locally free $R\otimes_{R_1} R_l$-module of the expected rank. This concludes the proof. □

Now let $M^l_I := M(\mathcal{O}_K, \Xi^l_I, r_l)$ be the ("unramified") local model over $\operatorname{Spec}\mathcal{O}_K$ for $G = \mathrm{GL}_d/K = \mathrm{GL}(V\otimes_{F,\phi_l} K)$, μ given by $(1^{r_l}, 0^{d-r_l})$, and the lattice chain Ξ^l_I ([RZ]). By definition, $M^l_I = M(\mathcal{O}_K, \Xi^l_I, r_l)$ is the projective scheme over $\mathcal{O}_K$ which classifies collections $\{\mathcal{F}_k\}_k$ of $\mathcal{O}_S$-submodules of $\Xi^l_{i_k,S} := \Xi^l_{i_k}\otimes_{\mathcal{O}_K}\mathcal{O}_S$ which fit into a commutative diagram

$$\begin{array}{ccccccccc}
\Xi^l_{i_0,S} & \to & \Xi^l_{i_1,S} & \to & \cdots & \to & \Xi^l_{i_{m-1},S} & \xrightarrow{a_l} & \Xi^l_{i_0,S}\\
\cup & & \cup & & & & \cup & & \cup\\
\mathcal{F}_0 & \to & \mathcal{F}_1 & \to & \cdots & \to & \mathcal{F}_{m-1} & \to & \mathcal{F}_0
\end{array}$$

and are such that $\mathcal{F}_k$ is Zariski locally on S a $\mathcal{O}_S$-direct summand of $\Xi^l_{i_k,S}$ of rank r_k.

Let us denote by $\widetilde{\mathcal{M}}_I$ the scheme over $\operatorname{Spec}\mathcal{O}_K$ whose S-points correspond to pairs

$$\widetilde{\mathcal{M}}_I(S) := (\{\mathcal{F}^l_I\}_l\ ,\ \{\sigma^l_I\})^e_{l=2}\ ,$$

where $\{\mathcal{F}^l_k\}_{l,k}$ is an S-valued point of $\mathcal{M}_I$ and for $l = 2,\dots,e$,

$$\sigma^l_I : \Psi^l_{I,S}\xrightarrow{\sim}\Lambda^l_{I,S}$$

is an isomorphism of chains of $\mathcal{O}^{(l)}_K\otimes_{\mathcal{O}_K}\mathcal{O}_S$-modules. The natural projection morphism

$$q_I : \widetilde{\mathcal{M}}_I\to\mathcal{M}_I$$

is a torsor for the smooth group scheme $\prod^e_{l=2}\mathcal{G}^{(l)}_I$ by the action

$$(g^l)^e_{l=2}\cdot\left(\{\mathcal{F}^l_k\}_{l,k}\ ,\ \{\sigma^l_I\}^e_{l=2}\right) = \left(\{\mathcal{F}^l_k\}_{l,k}\ ,\ \{g^l\cdot\sigma^l_I\}^e_{l=2}\right)\ . \tag{5.7}$$

Notice that an isomorphism σ^l_I as above, in view of (5.3), induces an isomorphism of chains of $\mathcal{O}_S$-modules

$$\tau^l_I : \Upsilon^l_{I,S}\xrightarrow{\sim}\Xi^l_{I,S}\ , l = 2,\dots,e\ .$$

For $l = 1$, $\Psi^1_{i_k,S} = \Lambda_{i_k,S}$ and (5.3) gives a canonical isomorphism

$$v_I : \Upsilon^1_{I,S}\xrightarrow{\sim}\Xi^1_{I,S}\ .$$

Now if $\{\mathcal{F}^l_k\}_{l,k}$ is an S-valued point of $\mathcal{M}_I$, then since $(\pi - a_l)\mathcal{F}^l_k\subset\mathcal{F}^{l-1}_k$ we can consider $\mathcal{F}^l_k/\mathcal{F}^{l-1}_k$ as an $\mathcal{O}_S$-submodule of $\Upsilon^l_{i_k,S} = \ker(\pi - a_l \mid \Psi^l_{i_k,S})$. Consequently, if $(\{\mathcal{F}^l_k\}_{l,k}\ ,\{\ \sigma^l_I\}^e_{l=2})$ is an S-valued point of $\widetilde{\mathcal{M}}_I$, then we can consider

$$\tau^l_{i_k}(\mathcal{F}^l_k/\mathcal{F}^{l-1}_k)\subset\Xi^l_{i_k,S}\ . \tag{5.8}$$

For $l = 2, \ldots, e$, the $\mathcal{O}_S$-modules $\tau^l_{i_k}(\mathcal{F}^l_k/\mathcal{F}^{l-1}_k)$ are locally direct summands of $\Xi^l_{i_k,S}$ and they provide us with an S-valued point of the "unramified" local model M^l_I. For $l = 1$, the $\mathcal{O}_S$-modules $v_{i_k}(\mathcal{F}^1_k)$ are locally direct summands of $\Xi^1_{i_k,S}$ and provide us with an S-valued point of the local model M^1_I. We conclude that there is a morphism of $\mathcal{O}_K$-schemes

$$p_I : \widetilde{\mathcal{M}}_I \ \to \ \prod_{l=1}^{e} M^l_I$$

given by

$$(\{\mathcal{F}^l_k\}_{l,k} \ , \ \{\sigma^l_I\}^e_{l=2}) \mapsto \Big(\{v_k(\mathcal{F}^1_k)\}_k, \{\sigma^2_k(\mathcal{F}^l_k/\mathcal{F}^1_k)\}_k, \ldots, \{\sigma^e_k(\mathcal{F}^e_k/\mathcal{F}^{e-1}_k)\}_k\Big) \ .$$

It is easy to see that the morphism p_I is also a $\prod^e_{l=2} \mathcal{G}^{(l)}_I$-torsor. Note that the corresponding $\prod^e_{l=2} \mathcal{G}^{(l)}_I$-action on $\widetilde{\mathcal{M}}_I$ is different from the action which produces the torsor $q_I : \widetilde{\mathcal{M}}_I \to \mathcal{M}_I$:

$$(g^l)^e_{l=2} \cdot \Big(\{\mathcal{F}^l_k\}_{l,k} \ , \ \{\sigma^l_I\}^e_{l=2}\Big) = \Big(\{(\sigma^l_I)^{-1} \cdot g^l \cdot \sigma^l_I(\mathcal{F}^l_k)\}_{l,k} \ , \ \{g^l \cdot \sigma^l_I\}^e_{l=2}\Big) \ . \tag{5.9}$$

In short, we have obtained a diagram of morphisms of schemes over $\operatorname{Spec} \mathcal{O}_K$:

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}_I & & \\ & p_I \swarrow & & \searrow q_I & \\ \prod^e_{l=1} M^l_I & & & & \mathcal{M}_I \ \stackrel{\pi_I}{\to} \ M^{\mathrm{naive}}_I \otimes_{\mathcal{O}_E} \mathcal{O}_K \end{array} \tag{5.10}$$

in which both of the slanted arrows are torsors for the smooth group scheme $\prod^e_{l=2} \mathcal{G}^{(l)}_I$. This diagram allows us to think of the splitting model as a twisted product of the "unramified" local models M^l_I. In the next sections, we will see that the special fiber of this diagram coincides with a certain geometric convolution diagram ([Lu], [H-N]). By the main result of [G1] the schemes M^l_I are flat over $\operatorname{Spec} \mathcal{O}_K$. The existence of such a diagram of torsors for a smooth group scheme therefore implies:

Theorem 5.3 *The scheme $\mathcal{M}_I$ is flat over* $\operatorname{Spec} \mathcal{O}_K$. □

6 Local models and affine flag varieties

We continue with the notation of the previous sections. Recall that there is a closed immersion of Ind-schemes

$$i : M^{\mathrm{naive}}_I \otimes_{\mathcal{O}_E} k \longrightarrow \mathbf{Fl}_I$$

which is described in §4. This immersion is equivariant for the action of $P_I \subset \mathrm{GL}_d(k((\Pi)))$ in the following sense. The special fiber $\bar{\mathcal{G}}_I := \mathcal{G}_I \otimes_{\mathcal{O}_E} k$ of the group scheme $\mathcal{G}_I$ defined in §3 acts on $M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k$. The isomorphism $\Lambda_I \otimes_{\mathcal{O}_{F_0}} k \simeq \tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$ allows us to identify $\bar{\mathcal{G}}_I$ with the group scheme giving the $k[[\Pi]]/(\Pi^e)$-automorphisms of the chain $\tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$. The immersion i is P_I-equivariant in the sense that the action of P_I on $\mathbf{Fl}_I$ stabilizes the image of i, the action on this image factors through the natural group scheme homomorphism $P_I \to \bar{\mathcal{G}}_I$ and i is $\bar{\mathcal{G}}_I$-equivariant. As a result, the image of i is a (finite) union of P_I-orbits in $\mathbf{Fl}_I = \mathrm{GL}_d(k((\Pi)))/P_I$. In fact, if R is a k-algebra, the R-rational points of the image of i correspond to the lattice chains

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{m-1} \subset \Pi^{-1}\mathcal{L}_0$$

which fit into a diagram

$$\begin{array}{ccccccccc}
\tilde{\Lambda}_{i_0,R} & \subset & \tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & \tilde{\Lambda}_{i_{m-1},R} & \subset & \Pi^{-1}\tilde{\Lambda}_{i_0,R} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{L}_0 & \subset & \mathcal{L}_1 & \subset & \cdots & \subset & \mathcal{L}_{m-1} & \subset & \Pi^{-1}\mathcal{L}_0 \\
\cup & & \cup & & & & \cup & & \cup \\
\Pi^e\tilde{\Lambda}_{i_0,R} & \subset & \Pi^e\tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & \Pi^e\tilde{\Lambda}_{i_{m-1},R} & \subset & \Pi^{e-1}\tilde{\Lambda}_{i_0,R}
\end{array}$$

and are such that $\mathcal{L}_k/\Pi^e\tilde{\Lambda}_{i_k,R}$, and $\tilde{\Lambda}_{i_k,R}/\mathcal{L}_k$ are R-locally free of rank r, resp. $de - r$.

Similarly, the special fiber $M_I^l \otimes_{\mathcal{O}_K} k'$ of the unramified local model M_I^l can be considered as a closed subscheme of the affine flag variety $\mathbf{Fl}_I \otimes_k k'$ via a natural closed immersion

$$i^l : M_I^l \otimes_{\mathcal{O}_K} k' \ \to \ \mathbf{Fl}_I \otimes_k k' \ .$$

In fact, by [G1], $M_I^l \otimes_{\mathcal{O}_K} k'$ can be identified with the scheme-theoretic union of a finite number of Schubert varieties in $\mathbf{Fl}_I$ and is reduced (see [G1]). This union is stable under the action of P_I.

Suppose now that R is a k'-algebra and that $\{\mathcal{F}_k^j\}_{j,k}$ gives a $\mathrm{Spec}\, R$-valued point of $\mathcal{M}_I \otimes_{\mathcal{O}_K} k'$. For $j = 1, \ldots, e$, let

$$\mathcal{L}_k^j \subset \Lambda_k \otimes_{k[[\Pi]]} R[[\Pi]]$$

be the inverse image of $\mathcal{F}_k^j \subset \Lambda_{i_k} \otimes_{\mathcal{O}_{F_0}} R \simeq \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e)$ under

$$\tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]] \to \tilde{\Lambda}_{i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e) \ .$$

We obtain a $R[[\Pi]]$-lattice chain $\mathcal{L}_I^j$

$$\mathcal{L}_0^j \subset \mathcal{L}_1^j \subset \cdots \subset \mathcal{L}_{m-1}^j \subset \Pi^{-1}\mathcal{L}_0^j$$

which provides us with a $\mathrm{Spec}\, R$-valued point of the affine flag variety $\mathbf{Fl}_I$. In this way we obtain morphisms of Ind-schemes

$$F^j : \mathcal{M}_I \otimes_{\mathcal{O}_K} k' \ \to \ \mathbf{Fl}_I \otimes_k k' \tag{6.1}$$

and

$$F = (F^j)_j : \mathcal{M}_I \otimes_{\mathcal{O}_K} k' \to \prod_{j=1}^{e} \mathbf{Fl}_I \otimes_k k' \ . \tag{6.2}$$

The morphism F is a closed immersion. Actually, the R-rational points of $\mathcal{M}_I \otimes_{\mathcal{O}_K} k'$ correspond to collections of lattice chains for $j = 1, \dots, e$,

$$\mathcal{L}_0^j \subset \mathcal{L}_1^j \subset \cdots \subset \mathcal{L}_{m-1}^j \subset \Pi^{-1}\mathcal{L}_0^j$$

which fit into a diagram

$$\begin{array}{ccccccccc}
\tilde{\Lambda}_{i_0,R} & \subset & \tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & \tilde{\Lambda}_{i_{m-1},R} & \subset & \Pi^{-1}\tilde{\Lambda}_{i_0,R} \\
\cup & & \cup & & & & \cup & & \cup \\
\Pi^{1-e}\mathcal{L}_0^1 & \subset & \Pi^{1-e}\mathcal{L}_1^1 & \subset & \cdots & \subset & \Pi^{1-e}\mathcal{L}_{m-1}^1 & \subset & \Pi^{-e}\mathcal{L}_0^1 \\
\cup & & \cup & & & & \cup & & \cup \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{L}_0^e & \subset & \mathcal{L}_1^e & \subset & \cdots & \subset & \mathcal{L}_{m-1}^e & \subset & \Pi^{-1}\mathcal{L}_0^e \\
\cup & & \cup & & & & \cup & & \cup \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{L}_0^1 & \subset & \mathcal{L}_1^1 & \subset & \cdots & \subset & \mathcal{L}_{m-1}^1 & \subset & \Pi^{-1}\mathcal{L}_0^1 \\
\cup & & \cup & & & & \cup & & \cup \\
\Pi^e\tilde{\Lambda}_{i_0,R} & \subset & \Pi^e\tilde{\Lambda}_{i_1,R} & \subset & \cdots & \subset & \Pi^e\tilde{\Lambda}_{i_{m-1},R} & \subset & \Pi^{e-1}\tilde{\Lambda}_{i_0,R},
\end{array}$$

and are such that $\mathcal{L}_k^j/\mathcal{L}_k^{j-1}$, and $\Pi^{j-e}\mathcal{L}_k^j/\Pi^{j+1-e}\mathcal{L}_k^{j+1}$ are R-locally free of rank r_j for $j = 2, \dots, e$, while $\mathcal{L}_k^1/\Pi^e\tilde{\Lambda}_{i_k,R}$ and $\tilde{\Lambda}_{i_k,R}/\Pi^{1-e}\mathcal{L}_k^1$ are R-locally free of rank r_1.

In what follows, for simplicity, we will use a bar to denote the special fiber of a scheme (or of a morphism of schemes) over $\operatorname{Spec}\mathcal{O}_K$ or over $\operatorname{Spec}\mathcal{O}_E$.

We will see that the special fiber $\overline{\mathcal{M}_I}$ can be naturally identified with the geometric convolution of the reduced subschemes $\overline{M}_I^l$, $l = 1, \dots, e$, of the affine flag variety $\mathbf{Fl}_I \otimes_k k'$. More precisely, we will see below that the special fiber of the diagram (5.10) relates to a convolution diagram for the P_I-equivariant subschemes $\overline{M}_I^l$, $l = 1, \dots, e$, defined as by Lusztig, Ginzburg etc. ([Lu]):

$$\begin{array}{ccccc}
 & & U & & \\
 & p_1 \swarrow & & \searrow p_2 & \\
\overline{M}_I^1 \times \cdots \times \overline{M}_I^e & & \overline{M}_I^1 \tilde{\times} \cdots \tilde{\times} \overline{M}_I^e & \xrightarrow{p_3} & \overline{M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K} \subset \mathbf{Fl}_I \otimes_k k' \ .
\end{array} \tag{6.3}$$

Let us explain how the diagram (6.3) is obtained (e.g [Lu]). For simplicity of notation, we set $G = \mathrm{GL}_d(k((\Pi)))$ and let $\pi : G \to \mathbf{Fl}_I = G/P_I$ be the natural quotient morphism (of Ind-schemes). We also set $Z_l = \overline{M}_I^l \subset \mathbf{Fl}_I \otimes_k k'$ and denote by $\tilde{Z}_l$ the inverse image

of Z_l under $\pi \otimes_k k'$. Often we will omit from the notation the base change from k to k'; this should not cause any confusion. Now set

$$U = \tilde{Z}_1 \times \cdots \times \tilde{Z}_{e-1} \times Z_e \subset G \times \cdots \times G \times G/P_I$$

and let

$$p_1 : U \ \rightarrow \ Z_1 \times \cdots \times Z_{e-1} \times Z_e$$

be the coordinate-wise projection $(g_1, \cdots, g_{e-1}, g_e P_I) \mapsto (g_1 P_I, \ldots, g_{e-1} P_I, g_e P_I)$. The morphism p_1 is a $(P_I)^{e-1}$-torsor for the action given by

$$(v_1, \ldots, v_{e-1}) \cdot (g_1, \ldots, g_{e-1}, g_e P_I) = (g_1 v_1^{-1}, \ldots, g_{e-1} v_{e-1}^{-1}, g_e P_I) \ . \tag{6.4}$$

The convolution $Z_1 \tilde{\times} \cdots \tilde{\times} Z_e$ is defined as the quotient of U by the free action of $(P_I)^{e-1}$ given by

$$(v_1, .., v_{e-1}) * (g_1, .., g_{e-1}, g_e P_I) = (g_1 v_1^{-1}, v_1 g_2 v_2^{-1}, .., v_{e-2} g_{e-1} v_{e-1}^{-1}, v_{e-1} g_e P_I) \ . \tag{6.5}$$

We denote by

$$p_2 : U \rightarrow Z_1 \tilde{\times} \cdots \tilde{\times} Z_e$$

the quotient morphism.

Finally, the morphism $\tilde{Z}_1 \times \cdots \times \tilde{Z}_{e-1} \times Z_e \rightarrow G/P_I$ given by $(g_1, \ldots, g_{e-1}, g_e P_I) \mapsto g_1 g_2 \cdots g_e P_I$ factors through the quotient to give

$$p_3 \ : \ Z_1 \tilde{\times} \cdots \tilde{\times} Z_e \rightarrow G/P_I \ .$$

Let us now explain how the above convolution diagram (6.3) relates to the diagram (5.10): There is an isomorphism $Z_1 \tilde{\times} \cdots \tilde{\times} Z_e \simeq \overline{\mathcal{M}}_I$ given by

$$(g_1, \ldots, g_e) \mapsto (\Pi^{e-1} g_1 \cdot \tilde{\Lambda}_I, \Pi^{e-2} g_1 g_2 \cdot \tilde{\Lambda}_I, \ldots, (g_1 g_2 \cdots g_e) \cdot \tilde{\Lambda}_I) \ .$$

In fact, an R-valued point $(g_1, \ldots, g_{e-1}, g_e P_I)$ of $\tilde{Z}_1 \times \cdots \times \tilde{Z}_{e-1} \times Z_e$ determines a pair consisting of a point

$$(\mathcal{L}_I^1, \mathcal{L}_I^2, \ldots, \mathcal{L}_I^e) = (\Pi^{e-1} g_1 \cdot \tilde{\Lambda}_{I,R}, \Pi^{e-2} g_1 g_2 \cdot \tilde{\Lambda}_{I,R}, \ldots, (g_1 g_2 \cdots g_e) \cdot \tilde{\Lambda}_{I,R})$$

of $\overline{\mathcal{M}}_I$ and a collection, for $j = 2, \ldots, e$, of isomorphisms of chains

$$\sigma_I^j \ : \ \Pi^{-e+j-1} \mathcal{L}_I^{j-1} / \mathcal{L}_I^{j-1} \ \simeq \ \tilde{\Lambda}_{I,R} / \Pi^{e-j+1} \tilde{\Lambda}_{I,R} \simeq \Lambda_{I,R}^j \ .$$

The isomorphisms σ_I^j are given via the inverses of the maps given by the action of $g_1 \cdots g_{j-1}$

$$\tilde{\Lambda}_{I,R} \rightarrow g_1 \cdots g_{j-1} \cdot \tilde{\Lambda}_{I,R} = \Pi^{-e+j-1} \mathcal{L}_I^{j-1} \ .$$

The pair $((\mathcal{L}_I^j)_j, \ (\sigma_I^j))_{j=2}^e$ corresponds to a point in the special fiber $\widetilde{\overline{\mathcal{M}}}_I$. Hence, we obtain a morphism

$$u : U \to \overline{\widetilde{\mathcal{M}}}_I \tag{6.6}$$

and (after the identification $Z_1 \tilde{\times} \cdots \tilde{\times} Z_e = \overline{M}_I^1 \tilde{\times} \cdots \tilde{\times} \overline{M}_I^e \simeq \overline{\mathcal{M}}_I$) a diagram

$$\begin{array}{ccc} & U & \\ & \downarrow u & \\ & \overline{\widetilde{\mathcal{M}}}_I & \\ \overline{p}_I \swarrow & & \searrow \overline{q}_I \\ \overline{M}_I^1 \times \cdots \times \overline{M}_I^e & & \overline{M}_I^1 \tilde{\times} \cdots \tilde{\times} \overline{M}_I^e \end{array} \tag{6.7}$$

$$\overset{p_3}{\to} \overline{M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K} \subset \mathbf{Fl}_I \otimes_k k' \ .$$

It is easy to see that we have $p_1 = u \cdot \overline{p}_I$ and $p_2 = u \cdot \overline{q}_I$.

There is a natural surjective group scheme homomorphism

$$(P_I)^{e-1} \to \prod_{l=2}^{e} \mathrm{Aut}_k(\tilde{\Lambda}_I / \Pi^{e-l+1} \tilde{\Lambda}_I) = \prod_{l=2}^{e} \overline{\mathcal{G}}_I^{(l)} \ ; \tag{6.8}$$

Denote its kernel by $\mathcal{K}$. Then the morphism u realizes $\overline{\widetilde{\mathcal{M}}_I}$ as the quotient of U by the action of $\mathcal{K} \subset (P_I)^{e-1}$ given by (6.4). Then, the torsor $\overline{p}_I$ is identified with the $\prod_{l=2}^{e} \overline{\mathcal{G}}_I^{(l)}$-torsor obtained from p_1 by taking the quotient by $\mathcal{K}$. Similarly, and at the same time, the morphism u realizes $\overline{\widetilde{\mathcal{M}}_I}$ as the quotient of U by the action of $\mathcal{K} \subset (P_I)^{e-1}$ given by (6.5). Then, the torsor $\overline{q}_I$ is identified with the $\prod_{l=2}^{e} \overline{\mathcal{G}}_I^{(l)}$-torsor obtained from p_2 by taking the quotient.

7 The canonical local model for $G = \mathrm{Res}_{F/F_0} \mathrm{GL}_d$

We continue with the assumptions and the notation of the previous sections.

Definition 7.1 The canonical model $M_I^{\mathrm{can}} := M^{\mathrm{can}}(\mathcal{O}_F, \Lambda_I, \mathbf{r})$ for the group $G = \mathrm{Res}_{F/F_0} GL_d$, the coweight μ given by $\mathbf{r}$, and the lattice chain Λ_I, is the scheme theoretic image of the morphism

$$\pi_I' : \mathcal{M}_I \to M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \to M_I^{\mathrm{naive}}$$

which is obtained by composing the morphism π_I with the base change morphism.

Since π_I' is proper, the canonical local model M_I^{can} is a closed subscheme of the naive local model M_I^{naive}. Using Proposition 5.1 we see that M_I^{can} and M_I^{naive} have the same generic fiber. The scheme M_I^{can} is flat over $\mathrm{Spec}\, \mathcal{O}_E$ since, by Theorem 5.3, $\mathcal{M}_I$ is flat over $\mathrm{Spec}\, \mathcal{O}_K$. Therefore, M_I^{can} is the (flat) scheme theoretic closure of the generic fiber $M_I \otimes_{\mathcal{O}_E} E$ in M_I^{naive}.

In [PR] §8, we have defined the *local model* M_I^{loc} for $\mathcal{O}_F$, Λ_I and $\mathbf{r}$ as follows. For every $k \in \{0, \dots, m-1\}$, we consider the standard (naive) local model $M^{\mathrm{naive}}(\Lambda_{i_k}) := M^{\mathrm{naive}}(\mathcal{O}_F, \Lambda_{i_k}, \mathbf{r})$ associated to the lattice Λ_{i_k} (and (F, V, μ)). There is a morphism

$$\pi_{i_k} : M_I^{\mathrm{naive}} \to M^{\mathrm{naive}}(\Lambda_{i_k}) \ , \tag{7.1}$$

obtained by $\{\mathcal{F}_i\}_{i=0}^{m-1} \mapsto \mathcal{F}_k$. In [PR], we set

$$M_I^{\mathrm{loc}} := \bigcap_{i_k \in I} \pi_{i_k}^{-1}(M^{\mathrm{loc}}(\Lambda_{i_k})) \tag{7.2}$$

(scheme theoretic intersection in M_I^{naive}) where $M^{\mathrm{loc}}(\Lambda_{i_k}) \subset M^{\mathrm{naive}}(\Lambda_{i_k})$ are the (flat) local models of EL-type which were studied in [PR]. By the above remarks, we have

$$M^{\mathrm{loc}}(\Lambda_{i_k}) = M_{\{i_k\}}^{\mathrm{can}}.$$

The recent results of Görtz imply now the following theorem.

Theorem 7.2
(a) M_I^{loc} is flat over $\mathcal{O}_E$ and hence $M_I^{\mathrm{can}} = M_I^{\mathrm{loc}}$.
(b) The special fiber $M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} k$ is reduced; its irreducible components are normal and with rational singularities.

Remark 7.3 (a) The flatness property in (a) above was conjectured in [PR], §8.

(b) Denote by μ_i the miniscule coweight $(1^{r_i}, 0^{d-r_i})$ of GL_d. By [G1], the special fiber $\overline{M}_I^l$ can be identified with the union of Schubert cells $\bigcup_{w \in \mathrm{Adm}_I(\mu_l)} \mathcal{O}_w$ of the partial affine flag variety $\mathbf{Fl}_I \otimes_k k'$. Here $\mathrm{Adm}_I(\mu_l)$ denotes the μ_l-admissible set inside $\tilde{W}_I \backslash \tilde{W} / \tilde{W}_I$. Here $\tilde{W}$ denotes the extended affine Weyl group of $\mathrm{GL}_d(k((\Pi)))$ and $\tilde{W}_I$ the subgroup of $\tilde{W}$ which corresponds to the parahoric subgroup P_I; see [KR]. By Theorem 7.2 (b) and the discussion in §6, $\overline{M}_I^{\mathrm{can}}$ can be identified with the reduced image of the convolution morphism

$$\left(\bigcup_{w \in \mathrm{Adm}_I(\mu_1)} \mathcal{O}_w \right) \tilde{\times} \cdots \tilde{\times} \left(\bigcup_{w \in \mathrm{Adm}_I(\mu_e)} \mathcal{O}_w \right) \to \mathbf{Fl}_I \otimes_k k' \ .$$

This image is equal to the union $\bigcup_{w \in \mathrm{Adm}_I(\mu)} \mathcal{O}_w$ with $\mu = \mu_1 + \cdots + \mu_e$.

Proof. Notice that each morphism π_{i_k} induces an isomorphism between the generic fibers:

$$\pi_{i_k} \otimes_{\mathcal{O}_E} E : M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} M^{\mathrm{naive}}(\Lambda_{i_k}) \otimes_{\mathcal{O}_E} E \ .$$

Therefore, $M_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} E = M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E = M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} E$. Since M_I^{can} is the scheme theoretic closure of its generic fiber in M_I^{naive}, we obtain

$$M_I^{\mathrm{can}} \subset M_I^{\mathrm{loc}} \subset M_I^{\mathrm{naive}} \tag{7.3}$$

where the inclusions are inclusions of closed subschemes. In what follows, for simplicity, we will use a bar to denote the special fiber of a scheme or a morphism of schemes over $\operatorname{Spec} \mathcal{O}_E$. Definition 7.2 implies

$$\overline{M_I^{\mathrm{loc}}} = \bigcap_{i_k \in I} \bar{\pi}_{i_k}^{-1}(\overline{M^{\mathrm{loc}}(\Lambda_{i_k})})$$

As we have seen above, $\overline{M_I^{\mathrm{naive}}}$, resp. $\overline{M^{\mathrm{naive}}(\Lambda_{i_k})}$, can be identified with a closed subscheme of the affine flag, resp. affine Grassmannian, variety for GL_d over k. The morphisms $\bar{\pi}_{i_k}$ can then be identified with the restrictions of natural (smooth) projection morphisms from the affine flag variety to the affine Grassmannian. By [PR], the special fibers $\overline{M^{\mathrm{loc}}(\Lambda_{i_k})}$ are reduced and they are identified with Schubert varieties in the affine Grassmannian; therefore the inverse images are also (reduced) Schubert varieties in the affine flag variety. By [G1] (see also [Fa]) all Schubert varieties in the affine flag variety are normal, simultaneously Frobenius split, and with rational singularities; therefore arbitrary intersections of Schubert varieties in the affine flag variety are also reduced unions of Schubert varieties. We conclude that $\overline{M_I^{\mathrm{loc}}}$ is reduced and that its irreducible components are normal with rational singularities. Therefore, to show that M_I^{loc} is flat and hence that $M_I^{\mathrm{can}} = M_I^{\mathrm{loc}}$, it will be enough to show that the generic points of the irreducible components of $\overline{M_I^{\mathrm{loc}}}$ lift to characteristic zero. This statement has recently been shown by Görtz ([G3]) by using results of Haines and Ngô ([HN2]). Hence part (a) follows. Part (b) now follows from (a) and the above description of the special fiber $\overline{M}_I^{\mathrm{loc}}$. □

Part III

8 The "naive" local models for $G = \mathrm{Res}_{F/F_0}\mathrm{GSp}_{2g}$

We continue with the notation of §2. Let $(V, \{\ ,\ \})$ be the standard symplectic vector space over F of dimension $2g$ with basis $e_1, \ldots, e_g, f_1, \ldots, f_g$, i.e

$$\{e_i, e_j\} = \{f_i, f_j\} = 0, \quad \{e_i, f_j\} = \delta_{ij}\ . \tag{8.1}$$

Let $< v, w >= \mathrm{Tr}_{F/F_0}\{v, w\}$. Then, since F/F_0 is separable, $<\ ,\ >$ is a non-degenerate alternating form on V with values in F_0 which, for all $a \in F$, satisfies

$$< av, w >=< v, aw >\ . \tag{8.2}$$

If Λ is an $\mathcal{O}_F$-lattice in V, we set

$$\Lambda^* := \{v \in V \mid \{v, \lambda\} \in \mathcal{O}_F, \text{ for all } \lambda \in \Lambda\}\ ,$$

$$\hat{\Lambda} := \{v \in V \mid < v, \lambda > \in \mathcal{O}_{F_0}, \text{ for all } \lambda \in \Lambda\}\ ,$$

for the dual ("complementary") $\mathcal{O}_F$-lattices with respect to the forms $\{\ ,\ \}$ and $<\ ,\ >$ respectively.

Now let δ be an $\mathcal{O}_F$-generator of the inverse different $\mathcal{D}_{F/F_0}^{-1}$ (if F/F_0 is tamely ramified, we can take $\delta = \pi^{1-e}$). Set

$$\Lambda_0 = \mathrm{Span}_{\mathcal{O}_F}\{e_1, \ldots, e_g, \delta f_1, \ldots, \delta f_g\} \subset V.$$

Then the $\mathcal{O}_F$-lattice Λ_0 is self-dual with respect to the form $< \ , \ >$, i.e

$$\hat{\Lambda}_0 = \Lambda_0 \ . \tag{8.3}$$

Indeed, $< e_i, a\delta f_j >= 0$ if $i \neq j$, and $< e_i, a\delta f_i >= \mathrm{Tr}_{F/F_0}(a\delta)$; this is in $\mathcal{O}_{F_0}$ exactly when a is in $\mathcal{O}_F$.

For $0 \leq r \leq g$, let

$$\Lambda_r = \mathrm{Span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \ldots, \pi^{-1}e_r, e_{r+1}, \ldots, e_g, \delta f_1, \ldots, \delta f_g\} \ .$$

We obtain a chain of inclusions of $\mathcal{O}_F$-lattices

$$\pi\Lambda_0 \subset \hat{\Lambda}_r \subset \hat{\Lambda}_0 = \Lambda_0 \subset \Lambda_r \subset \pi^{-1}\Lambda_0 \ . \tag{8.4}$$

In fact, we have

$$\hat{\Lambda}_r = \mathrm{Span}_{\mathcal{O}_F}\{e_1, \ldots, e_g, \pi\delta f_1, \ldots, \pi\delta f_r, \delta f_{r+1}, \ldots, \delta f_g\} \ .$$

We can extend $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_g$ to a complete $\mathcal{O}_F$-lattice chain $\{\Lambda_i\}_{i\in\mathbf{Z}}$ in V by setting

$$\Lambda_i = \pi^{-k}\Lambda_j, \text{ for } i = 2gk + j,\ 0 \leq j \leq g,$$
$$\Lambda_i = \pi^{-k}\hat{\Lambda}_{-j}, \text{ for } i = 2gk + j,\ -g \leq j < 0,$$

The essential part of this $\mathcal{O}_F$-lattice chain is

$$\hat{\Lambda}_g \subset \cdots \subset \hat{\Lambda}_1 \subset \hat{\Lambda}_0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_g = \pi^{-1}\hat{\Lambda}_g \ .$$

The lattice chain $\{\Lambda_i\}_{i\in\mathbf{Z}}$ is "self-dual" (for every i there is a j such that $\hat{\Lambda}_i = \Lambda_j$, in fact we have $\hat{\Lambda}_i = \Lambda_{-i}$) and "complete" (for every i, $\dim_k(\Lambda_{i+1}/\Lambda_i) = 1$). We will sometimes write

$$< \ , \ >_{\pm i} : \ \Lambda_{\pm i} \times \Lambda_{\mp i} \ \rightarrow \ \mathcal{O}_{F_0} \tag{8.5}$$

for the corresponding perfect form. These sets of forms are alternating in the sense that

$$< v, w >_{\pm i} = - < w, v >_{\mp i} \ .$$

Now fix a subset $I = \{i_0 < \cdots < i_{m-1}\} \subset \{0, 1, \ldots, g\}$ and consider the self-dual periodic $\mathcal{O}_F$-lattice chain Λ_I given by taking all lattices of the form $\pi^n\Lambda_{i_k}$, $\pi^n\hat{\Lambda}_{i_k}$ for $n \in \mathbf{Z}$, $k = 0, \ldots, m-1$. An essential part of the lattice chain Λ_I is

$$\hat{\Lambda}_{i_{m-1}} \subset \cdots \subset \hat{\Lambda}_{i_0} \subset \Lambda_{i_0} \subset \cdots \subset \Lambda_{i_{m-1}} \subset \pi^{-1}\hat{\Lambda}_{i_{m-1}} \ .$$

The standard ("naive") local model N_I^{naive} associated by Rapoport-Zink [RZ] to the reductive group $G = \mathrm{Res}_{F/F_0}\mathrm{GSp}(V, < \ , \ >)$, the cocharacter μ given by $\{(1^g, 0^g)\}_\phi$ and

the parahoric subgroup which is the stabilizer of the $\mathcal{O}_F$-lattice chain Λ_I, is by definition the $\mathcal{O}_{F_0}$-scheme representing the following functor on (Schemes/$\mathcal{O}_{F_0}$):

For every $\mathcal{O}_{F_0}$-scheme S, $N_I^{\mathrm{naive}}(S)$ is the set of collections $\{\mathcal{F}_{i_k}, \mathcal{F}_{-i_k}\}_{k=0,\dots,m-1}$ of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules of $\Lambda_{i_k,S}$, resp. $\hat{\Lambda}_{i_k,S}$ which fit into a commutative diagram

$$\begin{array}{ccccccccccccc} \hat{\Lambda}_{i_{m-1},S} & \to & \cdots & \to & \hat{\Lambda}_{i_0,S} & \to & \Lambda_{i_0,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \stackrel{\pi}{\to} & \hat{\Lambda}_{i_{m-1},S} \\ \cup & & & & \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_{-i_{m-1}} & \to & \cdots & \to & \mathcal{F}_{-i_0} & \to & \mathcal{F}_{i_0} & \to & \cdots & \to & \mathcal{F}_{i_{m-1}} & \to & \mathcal{F}_{-i_{m-1}} \end{array}$$

and are such that:

a) $\mathcal{F}_{i_k}$, resp. $\mathcal{F}_{-i_k}$, is Zariski locally on S a $\mathcal{O}_S$-direct summand of $\Lambda_{i_k,S}$, resp. $\hat{\Lambda}_{i_k,S}$, of rank eg,

b) the compositions $\mathcal{F}_{-i_k} \subset \hat{\Lambda}_{i_k,S} \to \hat{\mathcal{F}}_{i_k}$, $\mathcal{F}_{i_k} \subset \Lambda_{i_k,S} = \hat{\hat{\Lambda}}_{i_k,S} \to \hat{\mathcal{F}}_{-i_k}$, where $\hat{\mathcal{F}}_{\pm i_k} = Hom_{\mathcal{O}_S}(\mathcal{F}_{\pm i_k}, \mathcal{O}_S)$ and the second maps are the duals of the inclusions $\mathcal{F}_{i_k} \subset \Lambda_{i_k,S}$, resp. $\mathcal{F}_{-i_k} \subset \hat{\Lambda}_{i_k,S}$, are the zero maps.

c) For every $a \in \mathcal{O}_F$, and $k = 0, \dots, m-1$, we have

$$\det(a \mid \mathcal{F}_{\pm i_k}) = \prod_{\phi} \phi(a)^g$$

where again this identity is meant as an identity of polynomial functions on $\mathcal{O}_F$.

Remark 8.1 For $\mathcal{F} \subset \Lambda_{i_k,S}$, we set $\mathcal{F}^\perp := \ker(\hat{\Lambda}_{i_k,S} \to \hat{\mathcal{F}}) \subset \hat{\Lambda}_{i_k,S}$. For $\mathcal{G} \subset \hat{\Lambda}_{i_k,S}$ we set $\mathcal{G}^\perp := \ker(\Lambda_{i_k,S} = \hat{\hat{\Lambda}}_{i_k,S} \to \hat{\mathcal{G}}) \subset \Lambda_{i_k,S}$. If $\mathcal{F}$, resp. $\mathcal{G}$ are locally $\mathcal{O}_S$-direct summands of $\Lambda_{i_k,S}$, resp. $\hat{\Lambda}_{i_k,S}$, then $\mathcal{F}^\perp$, resp. $\mathcal{G}^\perp$ are locally $\mathcal{O}_S$-direct summands of $\hat{\Lambda}_{i_k,S}$, resp. $\Lambda_{i_k,S}$. Condition (b) implies that

$$\mathcal{F}_{-i_k} \subset \ker(\hat{\Lambda}_{i_k,S} \to \hat{\mathcal{F}}_{i_k}) = (\mathcal{F}_{i_k})^\perp \ , \quad \mathcal{F}_{i_k} \subset \ker(\Lambda_{i_k,S} \to \hat{\mathcal{F}}_{-i_k}) = (\mathcal{F}_{-i_k})^\perp \ .$$

Since by (a), $\mathcal{F}_{\pm i_k}$, $(\mathcal{F}_{\pm i_k})^\perp$ all have rank eg, we obtain $\mathcal{F}_{-i_k} = (\mathcal{F}_{i_k})^\perp$, $\mathcal{F}_{i_k} = (\mathcal{F}_{-i_k})^\perp$.

Hence, $N_I^{\mathrm{naive}}(S)$ is in bijection with the set of collections $\{\mathcal{F}_k\}_k$ of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules $\mathcal{F}_k \subset \Lambda_{i_k,S}$, which are, Zariski locally on S, $\mathcal{O}_S$-direct summands of $\Lambda_{i_k,S}$ of rank eg and which satisfy:

i) For every $a \in \mathcal{O}_F$, $k = 0, \cdots, m-1$,

$$\det(a \mid \mathcal{F}_k) = \prod_{\phi} \phi(a)^g$$

(as always this identity is meant as an identity of polynomial functions on $\mathcal{O}_F$),

ii) The inclusions $\mathcal{F}_k \subset \Lambda_{i_k,S}$, $\mathcal{F}_k^\perp \subset \hat{\Lambda}_{i_k,S}$ fit into a commutative diagram

$$\begin{array}{ccccccccccccc} \hat{\Lambda}_{i_{m-1},S} & \to & \cdots & \to & \hat{\Lambda}_{i_0,S} & \to & \Lambda_{i_0,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \stackrel{\pi}{\to} & \hat{\Lambda}_{i_{m-1},S} \\ \cup & & & & \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_{m-1}^\perp & \to & \cdots & \to & \mathcal{F}_0^\perp & \to & \mathcal{F}_0 & \to & \cdots & \to & \mathcal{F}_{m-1} & \to & \mathcal{F}_{m-1}^\perp \ . \end{array}$$

9 The splitting model for $G = \mathrm{Res}_{F/F_0}\mathrm{GSp}_{2g}$

We continue with the notation of the previous section. Consider the functor $\mathcal{N}_I = \mathcal{N}(\mathcal{O}_F, \Lambda_I, g)$ on (Schemes/Spec $\mathcal{O}_K$) which to a $\mathcal{O}_K$-scheme S associates the set of collections $\{\mathcal{F}^j_{i_k}, \mathcal{F}^j_{-i_k}\}_{k,j=1,\dots,e}$ of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules of $\Lambda_{i_k,S}$, resp. $\hat{\Lambda}_{i_k,S}$ which fit into a commutative diagram

$$\begin{array}{ccccccccccccc}
\hat{\Lambda}_{i_{m-1},S} & \to & \cdots & \to & \hat{\Lambda}_{i_0,S} & \to & \Lambda_{i_0,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \stackrel{\pi}{\to} & \hat{\Lambda}_{i_{m-1},S} \\
\cup & & & & \cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}^e_{-i_{m-1}} & \to & \cdots & \to & \mathcal{F}^e_{-i_0} & \to & \mathcal{F}^e_{i_0} & \to & \cdots & \to & \mathcal{F}^e_{i_{m-1}} & \to & \mathcal{F}^e_{-i_{m-1}} \\
\cup & & & & \cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}^{e-1}_{-i_{m-1}} & \to & \cdots & \to & \mathcal{F}^{e-1}_{-i_0} & \to & \mathcal{F}^{e-1}_{i_0} & \to & \cdots & \to & \mathcal{F}^{e-1}_{i_{m-1}} & \to & \mathcal{F}^{e-1}_{-i_{m-1}} \\
\cup & & & & \cup & & \cup & & & & \cup & & \cup \\
\vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & & & \cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}^1_{-i_{m-1}} & \to & \cdots & \to & \mathcal{F}^1_{-i_0} & \to & \mathcal{F}^1_{i_0} & \to & \cdots & \to & \mathcal{F}^1_{i_{m-1}} & \to & \mathcal{F}^1_{-i_{m-1}}
\end{array}$$

and are such that:

a) $\mathcal{F}^j_{i_k}$, resp. $\mathcal{F}^j_{-i_k}$, is Zariski locally on S a $\mathcal{O}_S$-direct summand of $\Lambda_{i_k,S}$, resp. $\hat{\Lambda}_{i_k,S}$, of rank jg and satisfies, for all $a \in \mathcal{O}_F$,

$$(a \otimes 1 - 1 \otimes \phi_j(a))\mathcal{F}^j_{\pm i_k} \subset \mathcal{F}^{j-1}_{\pm i_k} \ .$$

b) the compositions

$$\mathcal{F}^j_{-i_k} \subset \hat{\Lambda}_{i_k,S} \to \hat{\mathcal{F}}^j_{i_k} \ , \quad \mathcal{F}^j_{i_k} \subset \Lambda_{i_k,S} = \hat{\hat{\Lambda}}_{i_k,S} \to \hat{\mathcal{F}}^j_{-i_k} \ ,$$

are the zero maps.

By Remark 8.1, we see that the above conditions imply that for every k, $\mathcal{F}^e_{-i_k} = (\mathcal{F}^e_{i_k})^\perp$, $\mathcal{F}^e_{i_k} = (\mathcal{F}^e_{-i_k})^\perp$. Hence, we obtain chains

$$(0) \subset \mathcal{F}^1_{-i_k} \subset \cdots \subset \mathcal{F}^e_{-i_k} = (\mathcal{F}^e_{i_k})^\perp \subset \cdots \subset (\mathcal{F}^1_{i_k})^\perp \subset \hat{\Lambda}_{i_k,S} \ ,$$

$$(0) \subset \mathcal{F}^1_{i_k} \subset \cdots \subset \mathcal{F}^e_{i_k} = (\mathcal{F}^e_{-i_k})^\perp \subset \cdots \subset (\mathcal{F}^1_{-i_k})^\perp \subset \Lambda_{i_k,S} \ .$$

c) In addition to (a) and (b), we require that, for all $j = 1, \dots, e-1$, and every $a \in \mathcal{O}_F$,

$$\prod_{j+1 \le q \le e} (a \otimes 1 - 1 \otimes \phi_q(a)) \ (\mathcal{F}^j_{-i_k})^\perp \subset \mathcal{F}^j_{i_k} \ ,$$

$$\prod_{j+1 \le q \le e} (a \otimes 1 - 1 \otimes \phi_q(a)) \ (\mathcal{F}^j_{i_k})^\perp \subset \mathcal{F}^j_{-i_k} \ .$$

Obviously the functor $\mathcal{N}_I$ is represented by a projective scheme over $\operatorname{Spec}\mathcal{O}_K$ which we will also denote by $\mathcal{N}_I$. As in the case of $G = \operatorname{Res}_{F/F_0}\mathrm{GL}_d$, there is a projective morphism

$$\pi_I : \mathcal{N}_I \to N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \tag{9.1}$$

given by $\{\mathcal{F}^j_{\pm i_k}\}_{j,k} \mapsto \{\mathcal{F}^e_{\pm i_k}\}_k$. A construction similar to the one in the proof of Proposition 5.1 shows that, also in this case, π_I induces an isomorphism

$$\pi_I \otimes_{\mathcal{O}_K} K : \mathcal{N}_I \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} K \tag{9.2}$$

on the generic fibers.

Now suppose that $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ is an S-valued point of $\mathcal{N}_I$. As in the case of $G = \operatorname{Res}_{F/F_0}\mathrm{GL}_d$, for $l = 1, \ldots, e$, let us set

$$\Psi^l_{\pm i_k,S} = \ker(Q^l(\pi) \mid \Lambda_{\pm i_k,S}/\mathcal{F}^{l-1}_{\pm i_k}) \ ;$$

this is an $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module. We also set

$$\Upsilon^l_{\pm i_k,S} := \ker(\pi - a_l \mid \Lambda_{i_k,S}/\mathcal{F}^{l-1}_{\pm i_k}) = \ker(\pi - a_l \mid \Psi^l_{\pm i_k,S}) \ .$$

The proof of Proposition 5.2 implies that the $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module $\Psi^l_{\pm i_k,S}$ is, locally on S, free of rank $2g$ and that its formation commutes with base change in S. Similarly, $\Upsilon^l_{\pm i_k,S}$ is a locally free $\mathcal{O}_S$-module of rank $2g$ whose formation commutes with base change in S.

Lemma 9.1 *Suppose that $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ is an S-valued point of $\mathcal{N}_I$. Then for $l = 1, \ldots, e$, $k = 0, \ldots, m-1$, we have*

$$Q^l(\pi)^{-1}(\mathcal{F}^{l-1}_{\pm i_k}) = (\mathcal{F}^{l-1}_{\mp i_k})^\perp$$

where the left hand side is the inverse image of the submodule $\mathcal{F}^{l-1}_{\pm i_k} \subset \Lambda_{\pm i_k,S}$ under $\Lambda_{\pm i_k,S} \to \Lambda_{\pm i_k,S}$ given by multiplication by $Q^l(\pi)$.

Proof. The proof of Proposition 5.2 (b) shows that $Q^l(\pi)^{-1}(\mathcal{F}^l_{\pm i_k}) \subset \Lambda_{\pm i_k,S}$ is locally an $\mathcal{O}_S$-direct summand of rank $g(2e - l + 1)$. Observe that the condition (c) in the definition of the splitting model $\mathcal{N}_I$ translates to

$$(\mathcal{F}^{l-1}_{\mp i_k})^\perp \subset Q^l(\pi)^{-1}(\mathcal{F}^{l-1}_{\pm i_k}) \ .$$

Now $(\mathcal{F}^{l-1}_{\mp i_k})^\perp$ and $Q^l(\pi)^{-1}(\mathcal{F}^{l-1}_{\pm i_k})$ have the same $\mathcal{O}_S$-rank and they are both locally $\mathcal{O}_S$-direct summands of $\Lambda_{\pm i_k,S}$. Hence, they are equal. □

Suppose that $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ is an S-valued point of $\mathcal{N}_I$. Lemma 9.1 implies that

$$\Psi^l_{\pm i_k,S} = (\mathcal{F}^{l-1}_{\pm i_k,S})^\perp/\mathcal{F}^{l-1}_{\pm i_k,S} \ . \tag{9.3}$$

Therefore, there are perfect $\mathcal{O}_S$-bilinear forms

$$< \ , \ >^l_{\pm i_k} : \ \Psi^l_{\pm i_k,S} \times \Psi^l_{\mp i_k,S} \ \rightarrow \ \mathcal{O}_S \tag{9.4}$$

induced by the forms $< \ , \ >_{\pm i_k}$. These satisfy

$$< v, w >^l_{\pm i_k} = - < w, v >^l_{\mp i_k}, \ \text{ and } \ < av, w >^l_{\pm i_k} = < v, aw >^l_{\pm i_k} \ ,$$

for all $a \in \mathcal{O}_K^{(l)}$ (i.e the pairings respect the action of $\mathcal{O}_K^{(l)}$).

For $l = 1, \ldots, e$, consider the chain of free $\mathcal{O}_K^{(l)}$-modules Λ^l_I obtained from the free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K$-module chain $\Lambda_{I,\mathcal{O}_K} := \Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K$ by extending scalars via

$$\phi^l : \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \ \rightarrow \ \mathcal{O}_K[T]/(Q^l(T)) = \mathcal{O}_K^{(l)} \ .$$

We will define perfect $\mathcal{O}_K$-bilinear alternating pairings

$$< \ , \ >^l_{\pm i_k} : \ \Lambda^l_{\pm i_k} \times \Lambda^l_{\mp i_k} \ \rightarrow \ \mathcal{O}_K$$

which respect the action of $\mathcal{O}_K^{(l)}$ as follows: Using (2.4), we see that there are canonical isomorphisms

$$\Lambda^l_{\pm i_k} \ \simeq \ \mathrm{Im}(\ Q_l(T) \mid \Lambda_{\pm i_k,\mathcal{O}_K}) = \ker(Q^l(T) \mid \Lambda_{\pm i_k,\mathcal{O}_K}) \ . \tag{9.5}$$

Suppose that $v \in \Lambda^l_{\pm i_k}$, $w \in \Lambda^l_{\mp i_k}$. Via (9.5) we can identify v with an element of $\ker(Q^l(T) \mid \Lambda_{\pm i_k,\mathcal{O}_K}) \subset \Lambda_{\pm i_k,\mathcal{O}_K}$ and choose $\tilde{w} \in \Lambda_{\mp i_k,\mathcal{O}_K}$ such that

$$Q_l(T) \cdot \tilde{w} = w \ .$$

We set

$$< v, w >^l_{\pm i_k} = < v, \tilde{w} >_{\pm i_k} \ . \tag{9.6}$$

It is easy to see that this is independent of the choice of $\tilde{w}$. It provides us with perfect $\mathcal{O}_K$-bilinear forms which respect the action of $\mathcal{O}_K^{(l)}$ and satisfy

$$< w, v >^l_{\mp i_k} = - < v, w >^l_{\pm i_k}$$

(i.e they are alternating).

Let us set V^l for the $K[T]/(Q^l(T))$-module obtained from the $F \otimes_{F_0} K$-module $V \otimes_{F_0} K$ by extending scalars via $\phi^l \otimes_{\mathcal{O}_K} K \ : \ F \otimes_{F_0} K \rightarrow K[T]/(Q^l(T))$. Then, for all $k = 0, \ldots, m-1$, $\Lambda^l_{\pm i_k} \subset V^l$ and the pairings $< \ , \ >^l_{\pm i_k}$ are all restrictions of a single perfect K-bilinear alternating pairing

$$< \ , \ >^l : \ V^l \times V^l \ \rightarrow \ K$$

which respects the action of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} K = K[T]/(Q^l(T))$. It is easy to see that, under this form, $\Lambda^l_{-i_k}$ is dual ("complementary") to $\Lambda^l_{i_k}$. In this sense, the chain Λ^l_I is a periodic self-dual chain of free $\mathcal{O}_K^{(l)}$-modules in V^l.

Consider the chain of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $\Psi^l_{I,S}$:

$$\cdots \to \Psi^l_{-i_{m-1},S} \to \cdots \to \Psi^l_{-i_0,S} \to \Psi^l_{i_0,S} \to \cdots \to \Psi^l_{i_{m-1},S} \xrightarrow{\pi} \Psi^l_{-i_{m-1},S} \to \cdots \tag{9.7}$$

over S with the morphisms induced by the commutative diagram in the definition of $\mathcal{N}_I$, and with the bilinear forms (9.4).

Proposition 9.2 *a) The pairings (9.4) provide the chain $\Psi^l_{I,S}$ with the structure of a polarized chain of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $\Psi^l_{I,S}$ of type (Λ^l_I) in the sense of [RZ] Def. 3.14, p. 75 ($\mathcal{O}_K^{(l)}$ is not a maximal order in $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} K$, however the definition still makes sense).*

b) Zariski locally on S, the polarized chain of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $\Psi^l_{I,S}$ is (symplectically) isomorphic to the polarized chain of $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-modules $\Lambda^l_{I,S} := \Lambda^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$ which is obtained from the $\mathcal{O}_K^{(l)}$-chain Λ^l_I.

Proof. To show (a) we have to show that the chain $\Psi^l_{I,S}$ satisfies the conditions of [RZ] Def. 3.14 p. 75 (see also Def. 3.6 and Cor. 3.7). Assuming (a), part (b) of the proposition follows from a simple extension of [RZ] Prop. A 21 to the case at hand.

Now the only condition in loc. cit. that does not follow immediately from the definitions is the requirement (corresponding to condition (2) of Def. 3.6) that Zariski locally on S the quotient of two successive modules in the chain $\Psi^l_{I,S}$ is $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-isomorphic to the quotient of the two corresponding successive modules of the chain $\Lambda^l_{I,S}$. This can be shown exactly as the corresponding statement in the proof of Proposition 5.2. □

For $l = 1, \ldots, e$, there is a natural isomorphism

$$V \otimes_{F,\phi_l} K \simeq \mathrm{Im}(Q^{l+1}(T) \mid V^l) = \ker(T - a_l \mid V^l) \ .$$

A construction analogous to (9.6) allows us to define a perfect K-bilinear alternating form

$$< \ , \ >_l : \ V \otimes_{F,\phi_l} K \times V \otimes_{F,\phi_l} K \ \to \ K \ .$$

Now set

$$\Xi^l_{\pm i_k} = \Lambda_{\pm i_k} \otimes_{\mathcal{O}_F,\phi_l} \mathcal{O}_K \ \simeq \ \mathrm{Im}(Q^{l+1}(T) \mid \Lambda^l_{\pm i_k}) = \ker(T - a_l \mid \Lambda^l_{\pm i_k}) \ . \tag{9.8}$$

Once again, we can see that

$$< \ , \ >_{l,\pm i_k} : \ \Xi^l_{\pm i_k} \times \Xi^l_{\mp i_k} \to \mathcal{O}_K \tag{9.9}$$

defined by restricting $< \ , \ >_l$ to the lattices $\Xi_{\pm i_k}$, $\Xi_{\mp i_k}$ give a system of perfect $\mathcal{O}_K$-bilinear alternating forms. By construction, we have

$$< v, w >_{l,\pm i_k} = < v, \tilde{w} >^l_{\pm i_k}, \tag{9.10}$$

where we regard v as an element of $\ker(T - a_l \mid \Lambda^l_{\pm i_k})$ and where $\tilde{w} \in \Lambda^l_{\mp i_k}$ satisfies $Q^{l+1}(T) \cdot \tilde{w} = w$.

Hence, for each $l = 1, \ldots, e$, we obtain a self-dual $\mathcal{O}_K$-lattice chain Ξ^l_I in the K-vector space $V \otimes_{F,\phi_l} K$ by using the lattices $\Xi^l_{\pm i_k} = \Lambda_{\pm i_k} \otimes_{\mathcal{O}_F,\phi_l} \mathcal{O}_K$. The essential part of this chain is:

$$\Xi^l_{-i_{m-1}} \subset \cdots \subset \Xi^l_{-i_0} \subset \Xi^l_{i_0} \subset \cdots \subset \Xi^l_{i_{m-1}} \subset a_l^{-1}\Xi^l_{i_{m-1}} \ .$$

Now denote by N^l_I the "unramified" local model $N^l_I := N(\mathcal{O}_K, \Xi^l_I, <\ ,\ >)$ over $\operatorname{Spec} \mathcal{O}_K$ defined in [RZ] for $G = \mathrm{GSp}(V \otimes_{F,\phi_l} K, <, >_l)$ (a group over K), the cocharacter μ given by $(1^g, 0^g)$ and the self-dual lattice chain Ξ^l_I. By definition, N^l_I is the projective scheme over $\operatorname{Spec} \mathcal{O}_K$ which classifies collections $\{\mathcal{F}_{\pm i_k}\}_k$ of $\mathcal{O}_S$-submodules $\mathcal{F}_{\pm i_k} \subset \Xi^l_{\pm i_k,S} := \Xi^l_{\pm i_k} \otimes_{\mathcal{O}_K} \mathcal{O}_S$ which fit into a commutative diagram

$$\begin{array}{ccccccccccccc}
\Xi^l_{-i_{m-1},S} & \to & \cdots & \to & \Xi^l_{-i_0,S} & \to & \Xi^l_{i_0,S} & \to & \cdots & \to & \Xi^l_{i_{m-1}} & \stackrel{a_l}{\to} & \Xi^l_{i_{m-1}} \\
\cup & & & & \cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_{-i_{m-1}} & \to & \cdots & \to & \mathcal{F}_{-i_0} & \to & \mathcal{F}_{i_0} & \to & \cdots & \to & \mathcal{F}_{i_{m-1}} & \to & \mathcal{F}_{-i_{m-1}}
\end{array}$$

where $\mathcal{F}_{\pm i_k}$ are Zariski locally $\mathcal{O}_S$-direct summands of $\Xi^l_{i_k,S}$ of rank g and which satisfy

$$\mathcal{F}_{-i_k} = \mathcal{F}^{\perp}_{i_k}, \quad \mathcal{F}_{i_k} = \mathcal{F}^{\perp}_{-i_k} \ .$$

Now let us denote by $\mathcal{H}^{(l)}_I$, resp. $\mathcal{H}^l_I$, the group scheme over $\operatorname{Spec} \mathcal{O}_K$ whose S-points are the $\mathcal{O}^{(l)}_K \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module, resp. $\mathcal{O}_S$-module, automorphisms of the polarized chain $\Lambda^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$, resp. $\Xi^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$, which respect the forms $<\ ,\ >^l_{\pm i_k}$, resp. $<\ ,\ >_{l,\pm i_k}$, up to a similitude which is the same for all indices k. These groups are extensions of the multiplicative group by the group scheme of symplectic $\mathcal{O}^{(l)}_K \otimes_{\mathcal{O}_K} \mathcal{O}_S$-module, resp. $\mathcal{O}_S$-module, automorphisms of the polarized chains $\Lambda^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$, resp. $\Xi^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$. An argument as in the proof of [RZ] Prop. A.21 shows that the latter group schemes are smooth over $\operatorname{Spec} \mathcal{O}_K$. Therefore, $\mathcal{H}^{(l)}_I$ and $\mathcal{H}^l_I$ are also smooth group schemes over $\operatorname{Spec} \mathcal{O}_K$.

Now for an S-valued point of $\mathcal{N}_I$ given by $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ and $l = 1, \ldots, e$, $k = 0, \ldots, m-1$, we set

$$\Upsilon^l_{\pm i_k,S} = \ker(\pi - a_l \mid \Psi^l_{\pm i_k,S}) = \ker(\pi - a_l \mid (\mathcal{F}^{l-1}_{\mp i_k,S})^{\perp}/\mathcal{F}^{l-1}_{\pm i_k,S}) \ .$$

Notice that there is a canonical $\mathcal{O}_S$-homomorphism

$$\mathrm{Im}(Q^{l+1}(\pi) \mid \Psi^l_{\pm i_k,S}) \to \ker(\pi - a_l \mid \Psi^l_{\pm i_k,S}) = \Upsilon^l_{\pm i_k,S} \ .$$

It follows from Proposition 9.2 and (9.8) that this is an isomorphism.

A construction as in (9.10) now allows us to use this isomorphism and the forms $<\ ,\ >^l_{\pm i_k} \colon \Psi^l_{\pm i_k,S} \times \Psi^l_{\mp i_k,S} \ \to \ \mathcal{O}_S$ to derive $\mathcal{O}_S$-bilinear alternating forms

$$<\ ,\ >_{l,\pm i_k} \colon \Upsilon^l_{\pm i_k,S} \times \Upsilon^l_{\mp i_k,S} \ \to \ \mathcal{O}_S \ .$$

Proposition 9.3 *The $\mathcal{O}_S$-modules $\Upsilon^l_{\pm i_k,S}$ are locally free of rank $2g$ and the $\mathcal{O}_S$-bilinear alternating forms*

$$< \ , \ >_{l,\pm i_k} \ : \ \Upsilon^l_{\pm i_k,S} \times \Upsilon^l_{\mp i_k,S} \ \to \ \mathcal{O}_S \ .$$

are perfect. Furthermore, the resulting chain of $\mathcal{O}_S$-modules over S

$$(9.11) \quad \cdots \to \Upsilon^l_{-i_{m-1},S} \to \cdots \to \Upsilon^l_{-i_0,S} \to \Upsilon^l_{i_0,S} \to \cdots \to \Upsilon^l_{i_{m-1},S} \stackrel{a_l}{\to} \Upsilon^l_{-i_{m-1},S} \to \cdots$$

is a polarized chain of type (Ξ^l_I) and is, Zariski locally on S, (symplectically) isomorphic to the polarized chain of $\mathcal{O}_S$-modules $\Xi^l_I \otimes_{\mathcal{O}_K} \mathcal{O}_S$ obtained from the self-dual lattice chain Ξ^l_I.

Proof. This follows from Proposition 9.2 and the above discussion. □

Let $\widetilde{\mathcal{N}_I}$ denote the scheme over $\operatorname{Spec} \mathcal{O}_K$ whose S-points correspond to pairs

$$\widetilde{\mathcal{N}_I}(S) := \left(\{\mathcal{F}^j_{\pm i_k}\}_{j,k} \ , \ \{\sigma^j_I\} \right)_{j=2}^{e}$$

where $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ is an S-valued point of $\mathcal{N}_I$ and for $l = 2, \ldots, e$,

$$\sigma^l_I \ : \ \Psi^l_{I,S} \ \stackrel{\sim}{\to} \ \Lambda^l_{I,S}$$

is a symplectic (up to similitude) isomorphism of polarized $\mathcal{O}_K^{(l)} \otimes_{\mathcal{O}_K} \mathcal{O}_S$-chains. The natural projection morphism $q_I : \widetilde{\mathcal{N}_I} \to \mathcal{N}_I$ is a torsor for $\prod_{l=2}^e \mathcal{H}_I^{(l)}$.

Notice that an isomorphism σ^l_I as above, induces a symplectic (up to a similitude) isomorphism of chains of $\mathcal{O}_S$-modules

$$\tau^l_I : \Upsilon^l_{I,S} = \ker(\pi - a_l \mid \Psi^l_{I,S}) \stackrel{\sim}{\to} \ \ker(\pi - a_l \mid \Lambda^l_{I,S}) \simeq \Xi^l_{I,S} \ .$$

Similarly, for $l = 1$, $\Psi^1_{\pm i_k,S} = \Lambda_{\pm i_k,S}$, and we obtain a canonical symplectic isomorphism

$$v_I : \Upsilon^1_{I,S} \stackrel{\sim}{\to} \ \Xi^1_{I,S} \ .$$

Now if $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ is an S-valued point of $\mathcal{N}_I$, then since

$$\mathcal{F}^{l-1}_{\pm i_k} \subset \mathcal{F}^l_{\pm i_k} \subset (\mathcal{F}^{l-1}_{\mp i_k})^\perp \ , \quad (\pi - a_l)\mathcal{F}^l_{\pm i_k} \subset \mathcal{F}^{l-1}_{\pm i_k} \ ,$$

we can consider $\mathcal{F}^l_{\pm i_k}/\mathcal{F}^{l-1}_{\pm i_k}$ as an $\mathcal{O}_S$-submodule of $\Upsilon^l_{I,S} = \ker(\pi - a_l \mid (\mathcal{F}^{l-1}_{\mp i_k})^\perp/\mathcal{F}^{l-1}_{\pm i_k})$. In fact, $\mathcal{F}^l_{\pm i_k}/\mathcal{F}^{l-1}_{\pm i_k}$ is locally a direct $\mathcal{O}_S$-summand and

$$\left(\mathcal{F}^l_{\pm i_k}/\mathcal{F}^{l-1}_{\pm i_k}\right)^\perp = \mathcal{F}^l_{\mp i_k}/\mathcal{F}^{l-1}_{\mp i_k}$$

under the "derived" forms $< \ , \ >_{l,\pm i_k}$: $\Upsilon^l_{\pm i_k,S} \times \Upsilon^l_{\mp i_k,S} \to \mathcal{O}_S$. Therefore,

$$(9.12) \quad v_{\pm i_k}(\mathcal{F}^1_{\pm i_k}) \ \subset \ \Xi^1_{\pm i_k} \otimes_{\mathcal{O}_K} \mathcal{O}_S \ , \quad \text{resp. } \sigma^l_{\pm i_k}(\mathcal{F}^l_{\pm i_k}/\mathcal{F}^{l-1}_{\pm i_k}) \ \subset \ \Xi^l_{\pm i_k} \otimes_{\mathcal{O}_K} \mathcal{O}_S$$

provide us with S-valued points of the "unramified" local models N_I^1, resp. N_I^l for $l = 2, \ldots, e$. As in the case of $\mathrm{Res}_{F/F_0}\mathrm{GL}_d$ we obtain a morphism of schemes

$$p_I : \widetilde{\mathcal{N}_I} \ \to \ \prod_{j=1}^{e} N_I^l \ .$$

This is again a torsor for the smooth group scheme $\prod_{l=2}^{e} \mathcal{H}_I^{(l)}$. Again, as in the case of $\mathrm{Res}_{F/F_0}\mathrm{GL}_d$, we have obtained a diagram of morphisms of schemes over $\mathrm{Spec}\,\mathcal{O}_K$:

$$\begin{array}{ccc} & \widetilde{\mathcal{N}_I} & \\ & & \\ p_I \swarrow & & \searrow q_I \\ & & \\ \prod_{l=1}^{e} N_I^l & & \mathcal{N}_I \quad \stackrel{\pi_I}{\to} \quad N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \end{array} \tag{9.13}$$

in which both of the slanted arrows are torsors for the smooth group scheme $\prod_{l=2}^{e} \mathcal{H}_I^{(l)}$. Once again, since by [G2] the schemes N_I^l are flat over $\mathrm{Spec}\,\mathcal{O}_K$, the existence of such a diagram implies:

Theorem 9.4 *The scheme $\mathcal{N}_I$ is flat over* $\mathrm{Spec}\,\mathcal{O}_K$. □

10 Affine flag varieties for the symplectic group

In this section, we will use the notations and terminology of §4. Let us consider

$$\tilde{\Lambda}_0 = k[[\Pi]]^{2g} = k[[\Pi]]\tilde{e}_1 \oplus \cdots \oplus k[[\Pi]]\tilde{e}_g \oplus k[[\Pi]]\tilde{f}_1 \oplus \cdots \oplus k[[\Pi]]\tilde{f}_g$$

with the $k[[\Pi]]$-bilinear alternating form $< \ , \ >: \tilde{\Lambda}_0 \times \tilde{\Lambda}_0 \to k[[\Pi]]$ given by

$$< \tilde{e}_i, \tilde{e}_j >= 0 \ , \quad < \tilde{f}_i, \tilde{f}_j >= 0 \ , \quad < \tilde{e}_i, \tilde{f}_j >= \delta_{ij} \ .$$

For $0 \leq r \leq g$, we introduce the $k[[\Pi]]$-lattices $\tilde{\Lambda}_r$ in $\tilde{\Lambda}_0 \otimes_{k[[\Pi]]} k((\Pi))$ by

$$\tilde{\Lambda}_r = \mathrm{Span}_{k[[\Pi]]}\{\Pi^{-1}\tilde{e}_1, \cdots, \Pi^{-1}\tilde{e}_r, \tilde{e}_{r+1}, \cdots, \tilde{e}_g, \tilde{f}_1, \cdots, \tilde{f}_g\} \ .$$

Set $\tilde{\Lambda}_{-r} = \widehat{\tilde{\Lambda}_r} := \{v \in k((\Pi))^{2g} \mid \ < v, w >\in k[[\Pi]], \text{ for all } w \in \tilde{\Lambda}_r\}$. It is easy to see that

$$\tilde{\Lambda}_{-r} = \mathrm{Span}_{k[[\Pi]]}\{\tilde{e}_1, \cdots, \tilde{e}_g, \Pi\tilde{f}_1, \cdots, \Pi\tilde{f}_r, \tilde{f}_{r+1}, \cdots, \tilde{f}_g\} \ .$$

Now consider a subset $I = \{i_0 < \cdots < i_{m-1}\} \subset \{0, \ldots, g\}$. From this we obtain the lattice chain $\tilde{\Lambda}_I$ in $k((\Pi))^{2g}$

$$\tilde{\Lambda}_{-i_{m-1}} \subset \cdots \subset \tilde{\Lambda}_{-i_0} \subset \tilde{\Lambda}_{i_0} \subset \cdots \subset \tilde{\Lambda}_{i_{m-1}} \subset \Pi^{-1}\Lambda_{-i_{m-1}} \ . \tag{10.1}$$

By adding all the multiples $\Pi^m\Lambda_{\pm i_k}$, $m \in \mathbf{Z}$, to the above lattice chain, we obtain the corresponding periodic lattice chain. In what follows, we will sometimes use the same symbol to denote both a lattice chain and its corresponding periodic lattice chain. This

should not cause any confusion. By definition, a lattice chain (which is not necessarily periodic) is self-dual if the dual of every lattice in the chain appears in the corresponding periodic lattice chain. It is clear that $\tilde{\Lambda}_I$ is a self-dual lattice chain.

The partial affine flag variety $\mathbf{SFl}_I$ associated to the symplectic similitude group GSp_{2g} and the subset I is the ind-scheme over k which represents the functor which to a k-algebra R associates the set of self dual $R[[\Pi]]$-lattice chains

$$\mathcal{L}_{-i_{m-1}} \subset \cdots \subset \mathcal{L}_{-i_0} \subset \mathcal{L}_{i_0} \subset \cdots \subset \mathcal{L}_{i_{m-1}} \subset \Pi^{-1}\mathcal{L}_{-i_{m-1}} \tag{10.2}$$

in $R((\Pi))^{2g} = k((\Pi))^{2g} \otimes_k R$, such that each successive quotient of the above chain is a locally free R-module of rank equal to the k-dimension of the corresponding quotient in (10.1). The Ind-group scheme $\mathrm{GSp}_{2g}(k((\Pi)))$ acts on $\mathbf{SFl}_I$ and we can identify ($\mathrm{GSp}_{2g}(k((\Pi)))$-equivariantly) $\mathbf{SFl}_I$ with the fpqc quotient

$$\mathbf{SFl}_I = \mathrm{GSp}_{2g}(k((\Pi)))/P_I$$

where P_I is the parahoric subgroup scheme of $\mathrm{GSp}_{2g}(k((\Pi)))$ whose k-valued points stabilize the lattice chain $\tilde{\Lambda}_I$ of (10.1).

Fix an integer r. We may also consider the partial affine flag variety $\mathbf{SFl}_I^r$ associated to the symplectic group Sp_{2g} and the subset I. This is the ind-scheme over k which represents the functor which to a k-algebra R associates the set of self dual $R[[\Pi]]$-lattice chains

$$\mathcal{L}_{-i_{m-1}} \subset \cdots \subset \mathcal{L}_{-i_0} \subset \mathcal{L}_{i_0} \subset \cdots \subset \mathcal{L}_{i_{m-1}} \subset \Pi^{-1}\mathcal{L}_{-i_{m-1}} \tag{10.3}$$

in $R((\Pi))^{2g} = k((\Pi))^{2g} \otimes_k R$, such that

i) each successive quotient of the above chain is a locally free R-module of rank equal to the k-dimension of the corresponding quotient in (10.1),

ii) we have $\widehat{\mathcal{L}}_{i_0} = \Pi^r \mathcal{L}_{-i_0}$.

The Ind-group scheme $\mathrm{Sp}_{2g}(k((\Pi)))$ acts on $\mathbf{SFl}_I^r$. Sending the lattice chain $\mathcal{L}_I$ to $\Pi^m \mathcal{L}_I$ gives an $\mathrm{Sp}_{2g}(k((\Pi)))$-equivariant isomorphism

$$\mathbf{SFl}_I^r \ \tilde{\to}\ \mathbf{SFl}_I^{r-2m} \ .$$

The Ind-schemes $\mathbf{SFl}_I^r$ are all closed Ind-subschemes of $\mathbf{SFl}_I$. In fact, $\mathbf{SFl}_I^r$ for different r are all isomorphic as Ind-schemes (but not necessarily $\mathrm{Sp}_{2g}(k((\Pi)))$-equivariantly).

11 Local models and symplectic affine flag varieties

Let us identify $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k$ and $k[[\Pi]]/(\Pi^e)$ via the isomorphism $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k \simeq k[[\Pi]]/(\Pi^e)$ given by $\pi \otimes 1 \mapsto \Pi$. Consider the $k[[\Pi]]/(\Pi^e)$-isomorphism $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} k \simeq \tilde{\Lambda}_0 \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$ given by $e_i \mapsto \tilde{e}_i$, $\delta f_j \mapsto \tilde{f}_j$. This isomorphism is compatible with the symplectic forms on both sides. In fact, there are obvious similar isomorphisms

$$\Lambda_i \otimes_{\mathcal{O}_{F_0}} k \simeq \tilde{\Lambda}_i \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e) \tag{11.1}$$

which induce a (symplectic) isomorphism between the polarized $k[[\Pi]]/(\Pi^e)$-chains $\Lambda_I \otimes_{\mathcal{O}_{F_0}} k$ and $\tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$.

Suppose that $\{\mathcal{F}_{\pm i_k}\}_k$ corresponds to an $\operatorname{Spec} R$-valued point of the special fiber $N_I^{\text{naive}} \otimes_{\mathcal{O}_{F_0}} k$ of the naive local model. Set $\tilde{\Lambda}_{\pm i_k,R} = \tilde{\Lambda}_{\pm i_k} \otimes_{k[[\Pi]]} R[[\Pi]]$. Let

$$\mathcal{L}_{\pm i_k} \subset \tilde{\Lambda}_{\pm i_k,R} \tag{11.2}$$

be the inverse image of $\mathcal{F}_{\pm i_k} \subset \Lambda_{\pm i_k} \otimes_{\mathcal{O}_{F_0}} R \simeq \tilde{\Lambda}_{\pm i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e)$ under

$$\tilde{\Lambda}_{\pm i_k,R} \to \tilde{\Lambda}_{\pm i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e) \ .$$

We obtain an $R[[\Pi]]$-lattice chain $\mathcal{L}_I$

$$\mathcal{L}_{-i_{m-1}} \subset \cdots \subset \mathcal{L}_{-i_0} \subset \mathcal{L}_{i_0} \subset \cdots \subset \mathcal{L}_{i_{m-1}} \subset \Pi^{-1}\mathcal{L}_{-i_{m-1}}$$

which satisfies property (i) of the definition of the (partial) symplectic affine flag variety. We claim that $\widehat{\mathcal{L}}_{i_k} = \Pi^{-e}\mathcal{L}_{-i_k}$. This will establish that the chain above is self-dual and satisfies property (ii) with $r = -e$. Now we have

$$\mathcal{L}_{-i_k} \subset \tilde{\Lambda}_{-i_k,R} = \widehat{\tilde{\Lambda}}_{i_k,R} \subset \widehat{\mathcal{L}}_{i_k} \ .$$

Here the quotients $\tilde{\Lambda}_{-i_k,R}/\mathcal{L}_{-i_k}$ and therefore $\widehat{\mathcal{L}}_{i_k}/\tilde{\Lambda}_{-i_k,R}$ are R-locally free of rank eg. Hence, $\widehat{\mathcal{L}}_{i_k}/\mathcal{L}_{-i_k}$ is R-locally free of rank $2eg$; this is the same as the R-rank of $\Pi^{-e}\mathcal{L}_{-i_k}/\mathcal{L}_{-i_k}$. By our definitions, $< \mathcal{L}_{-i_k}, \mathcal{L}_{i_k} > \subset \Pi^e R[[\Pi]]$ and so $\Pi^{-e}\mathcal{L}_{-i_k} \subset \widehat{\mathcal{L}}_{i_k}$. Since the formation of $\widehat{\mathcal{L}}_{-i_k}$ and $\Pi^{-e}\mathcal{L}_{-i_k}$ from the $R[[\Pi]]$-lattice $\mathcal{L}_{-i_k}$ commutes with base change we obtain that $\Pi^{-e}\mathcal{L}_{-i_k} = \widehat{\mathcal{L}}_{i_k}$. Therefore, the $R[[\Pi]]$-lattice chain $\mathcal{L}_I$ gives an R-valued point of $\mathbf{SFl}_I^{-e} \subset \mathbf{SFl}_I$.

We have therefore obtained a morphism

$$i : N_I^{\text{naive}} \otimes_{\mathcal{O}_{F_0}} k \to \mathbf{SFl}_I^{-e} \subset \mathbf{SFl}_I \tag{11.3}$$

which is a closed immersion of Ind-schemes.

Similarly, the special fiber $N_I^l \otimes_{\mathcal{O}_K} k'$ of the "unramified" local model N_I^l can be considered as a closed subscheme of the symplectic affine flag variety $\mathbf{SFl}_I \otimes_k k'$. In fact, by [G2], $N_I^l \otimes_{\mathcal{O}_K} k'$ is reduced and can be identified with the scheme-theoretic union of a finite number of Schubert varieties in $\mathbf{SFl}_I \otimes_k k'$.

Recall that $\mathcal{H}_I$ is the group scheme over $\operatorname{Spec} \mathcal{O}_{F_0}$ whose S-valued points give the symplectic automorphisms up to similitude of the polarized chain $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$. The above symplectic isomorphism between the polarized $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k = k[[\Pi]]/(\Pi^e)$-chains $\Lambda_I \otimes_{\mathcal{O}_{F_0}} k$ and $\tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$ allows us to identify the special fiber $\overline{\mathcal{H}}_I$ with the group scheme giving the symplectic similitude automorphisms of $\tilde{\Lambda}_I \otimes_{k[[\Pi]]} k[[\Pi]]/(\Pi^e)$. This is a factor group of the parahoric group scheme P_I giving the symplectic similitude isomorphisms of $\tilde{\Lambda}_I$. The closed immersion i is equivariant for the action of P_I in the sense that the action of $\mathcal{H}_I$ stabilizes the image of i, that the action on this image factors through $P_I \to \overline{\mathcal{H}}_I$ and that i is $\overline{\mathcal{H}}_I$-equivariant.

Suppose now that $\{\mathcal{F}^j_{\pm i_k}\}_{j,k}$ corresponds to an $\operatorname{Spec} R$-valued point of the special fiber $\mathcal{N}_I \otimes_{\mathcal{O}_K} k'$ of the splitting model. For $j = 1, \dots, e$ let

$$\mathcal{L}^j_{\pm i_k} \subset \tilde{\Lambda}_{\pm i_k, R} \tag{11.4}$$

be the inverse image of $\mathcal{F}^j_{\pm i_k} \subset \Lambda_{\pm i_k} \otimes_{\mathcal{O}_{F_0}} R \simeq \tilde{\Lambda}_{\pm i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e)$ under

$$\tilde{\Lambda}_{\pm i_k, R} \to \tilde{\Lambda}_{\pm i_k} \otimes_{k[[\Pi]]} R[[\Pi]]/(\Pi^e) \ .$$

As above, we obtain an $R[[\Pi]]$-lattice chain $\mathcal{L}^j_I$

$$\mathcal{L}^j_{-i_{m-1}} \subset \cdots \subset \mathcal{L}^j_{-i_0} \subset \mathcal{L}^j_{i_0} \subset \cdots \subset \mathcal{L}^j_{i_{m-1}} \subset \Pi^{-1}\mathcal{L}^j_{-i_{m-1}} \ .$$

Using a similar argument as above, one can see that it satisfies properties (i) and (ii) of the definition with $r = -2e + j$ and therefore gives an R-valued point of the symplectic affine flag variety $\mathbf{SFl}_I^{-2e+j}$. We obtain morphisms:

$$F^j : \mathcal{N}_I \otimes_{\mathcal{O}_K} k' \ \to \ \mathbf{SFl}_I^{-2e+j} \otimes_k k' \subset \mathbf{SFl}_I \otimes_k k' \ . \tag{11.5}$$

and

$$F = (F^j)_j \ : \ \mathcal{N}_I \otimes_{\mathcal{O}_K} k' \ \to \ \prod_{j=1}^{e} \mathbf{SFl}_I^{-2e+j} \otimes_k k' \subset \prod_{j=1}^{e} \mathbf{SFl}_I \otimes_k k' \ . \tag{11.6}$$

The morphism F is a closed immersion. Exactly as in the case of $\operatorname{Res}_{F/F_0}\mathrm{GL}_d$ we can see that the special fiber $\overline{\mathcal{N}_I} := \mathcal{N}_I \otimes_{\mathcal{O}_K} k'$ can be naturally identified with the geometric convolution of the reduced subschemes $N_I^j \otimes_{\mathcal{O}_K} k'$ of the symplectic affine flag variety $\mathbf{SFl}_I \otimes_k k'$. Similarly, the special fiber of the diagram (9.13) relates to the convolution diagram for these subschemes (the analogue of (6.3)) in exactly the same fashion as it was explained in §6 for $G = \operatorname{Res}_{F/F_0}\mathrm{GL}_d$.

12 The canonical local model for $G = \operatorname{Res}_{F/F_0}\mathrm{GSp}_{2g}$

Definition 12.1 The canonical local model $N_I^{\mathrm{can}} := N^{\mathrm{can}}(\mathcal{O}_F, \Lambda_I, <\ ,\ >)$ for the group $G = \operatorname{Res}_{F/F_0}\mathrm{GSp}_{2g}$ and the self-dual lattice chain Λ_I is the scheme theoretic image of the morphism

$$\pi'_I \ : \ \mathcal{N}_I \to N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \to N_I^{\mathrm{naive}}$$

which is obtained by composing the morphism π_I with the base change morphism.

Since π'_I is proper, the canonical local model N_I^{can} is a closed subscheme of the naive local model N_I^{naive}. Since $\pi_I \otimes_{\mathcal{O}_K} K : \mathcal{N}_I \otimes_{\mathcal{O}_{F_0}} K \to N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} K$ is an isomorphism, N_I^{can} and N_I^{naive} have the same generic fiber. The scheme N_I^{can} is flat over $\operatorname{Spec}\mathcal{O}_{F_0}$ since $\mathcal{N}_I$ is flat over $\operatorname{Spec}\mathcal{O}_K$. Therefore, N_I^{can} is the (flat) scheme theoretic closure of the generic fiber $N_I^{\mathrm{naive}} \otimes_{\mathcal{O}_{F_0}} F_0$ in N_I^{naive}.

Suppose now that $I = \{0\}$ or that $I = \{g\}$. In this case, the self-dual lattice chain Λ_I consists of $\{\pi^k\Lambda_0\}_{k\in\mathbf{Z}}$, resp. $\{\pi^k\Lambda_g\}_{k\in\mathbf{Z}}$ (we have $\hat{\Lambda}_0 = \Lambda_0$, $\hat{\Lambda}_g = \pi\Lambda_g$) and the subgroup of $G(F_0) = \mathrm{GSp}_{2g}(F)$ which stabilizes Λ_I is a special maximal parahoric

subgroup. Then it follows that the unramified local models N_I^l are smooth Lagrangian Grassmannians over $\operatorname{Spec}\mathcal{O}_K$. Hence, we deduce that $\prod_{l=1}^e N_I^l$ is irreducible and smooth over $\operatorname{Spec}\mathcal{O}_K$. Since $\prod_{l=2}^e \mathcal{H}_I^{(l)}$ is a smooth group scheme with geometrically connected fibers, we conclude, using the diagram (9.13), that, in this case, $\mathcal{N}_I$ is also irreducible and smooth over $\operatorname{Spec}\mathcal{O}_K$; therefore the special fiber $\mathcal{N}_I \otimes_{\mathcal{O}_K} k'$ is irreducible. As a result, the special fiber $N_I^{\mathrm{can}} \otimes_{\mathcal{O}_{F_0}} k$ of the canonical local model N_I^{can} is irreducible. More generally, suppose that $I = \{i_0\}$ consists of one index only. This is the case in which the subgroup of $G(F_0) = \mathrm{GSp}_{2g}(F)$ which stabilizes Λ_I is a maximal parahoric subgroup. Then by [G2], the geometric special fibers of the unramified local models N_I^l are irreducible. As above, we conclude that the special fiber $\overline{N}_I^{\mathrm{can}} = N_I^{\mathrm{can}} \otimes_{\mathcal{O}_{F_0}} k$ is once again irreducible. In fact, we can then show more:

Theorem 12.2 *Suppose that $I = \{i_0\}$ consists of one index only. Then:*

(i) N_I^{can} is normal and Cohen-Macaulay.

(ii) The special fiber $\overline{N}_I^{\mathrm{can}}$ is integral and normal with rational singularities. It can be identified with the Schubert variety $\overline{\mathcal{O}}_{e\mu_1}$ in $\mathbf{SFl}_I$, where μ_1 is the coweight $(1^g, 0^g)$ of GSp_{2g}.

Proof. This follows closely the arguments in [PR], proofs of Propositions 5.2–5.3 (see also loc. cit. Remark 5.5). For simplicity of notation, we will drop the subscript I and write N instead of N^{naive}. We will also use a bar to denote the special fiber of a scheme over $\mathcal{O}_K$ or over $\mathcal{O}_{F_0}$, depending on the context. Consider the proper morphism

$$\pi \ : \ \mathcal{N} \to N \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \ .$$

Let $N' = \operatorname{Spec}(\pi_*(\mathcal{O}_{\mathcal{N}}))$ and consider the scheme-theoretic image $\pi(\mathcal{N}) \subset N \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K$. Since $\mathcal{N}$ is flat over $\operatorname{Spec}\mathcal{O}_K$ the same is true for $\pi(\mathcal{N})$. Let ϖ be a uniformizer of $\mathcal{O}_K$. The cohomology exact sequence obtained by applying π_* to

$$0 \to \mathcal{O}_{\mathcal{N}} \xrightarrow{\varpi} \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\overline{\mathcal{N}}} \to 0$$

gives an injective homomorphism

$$\mathcal{O}_{N'}/\varpi\mathcal{O}_{N'} \to \overline{\pi}_*(\mathcal{O}_{\overline{\mathcal{N}}}) \ .$$

This fits in a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\pi(\mathcal{N})}/\varpi\mathcal{O}_{\pi(\mathcal{N})} & \to & \mathcal{O}_{\overline{\pi}(\overline{\mathcal{N}})} \\ \downarrow & & \downarrow \\ \mathcal{O}_{N'}/\varpi\mathcal{O}_{N'} & \to & \overline{\pi}_*(\mathcal{O}_{\overline{\mathcal{N}}}) \end{array} \ . \tag{12.1}$$

By the definition of the scheme theoretic image the upper horizontal homomorphism is surjective. Since by the discussion before the statement of the theorem, $\overline{\mathcal{N}}$ is reduced and irreducible, the same is true for the scheme-theoretic image $\overline{\pi}(\overline{\mathcal{N}}) \subset \overline{N}$. Let μ_1 be the miniscule coweight $(1^g, 0^g)$ of GSp_{2g}. The special fibers $\overline{N}_I^l$ of the corresponding unramified models can be identified with the Schubert variety $\overline{\mathcal{O}}_{\mu_1}$ in the affine partial

flag variety $\mathbf{SFl}_I \otimes_k k'$ (see [G2]). By §11, §6 the morphism $\overline{\pi} : \overline{\mathcal{N}} \to \overline{\pi}(\overline{\mathcal{N}}) \subset \overline{N}$ can be identified with the convolution morphism

$$m_{(\mu_1,\ldots,\mu_1)} : \overline{\mathcal{O}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \overline{\mathcal{O}}_{\mu_1} \ \to \ \overline{\mathcal{O}}_{e\mu_1} \subset \mathbf{SFl}_I \otimes_k k'$$

This morphism is birational on its image. The scheme $\overline{\pi}(\overline{\mathcal{N}})$ can be identified with the Schubert variety $\overline{\mathcal{O}}_{e\mu_1}$ in $\mathbf{SFl}_I \otimes_k k'$; it is therefore normal with rational singularities ([Fa], [G2]). Since $\overline{\pi}$ is proper, the natural morphism $\mathrm{Spec}\,(\overline{\pi}_*(\overline{\mathcal{N}})) \to \overline{\pi}(\overline{\mathcal{N}})$ is finite, and now since by the above $\overline{\pi}(\overline{\mathcal{N}})$ is normal, $\mathrm{Spec}\,(\overline{\pi}_*(\overline{\mathcal{N}})) \to \overline{\pi}(\overline{\mathcal{N}})$ is actually an isomorphism. We conclude that in the diagram (12.1) above, the right vertical homomorphism is an isomorphism. An argument as in [PR] proof of Proposition 5.2 now implies that the homomorphisms $\mathcal{O}_{\pi(\mathcal{N})}/\varpi\mathcal{O}_{\pi(\mathcal{N})} \to \mathcal{O}_{\overline{\pi}(\overline{\mathcal{N}})}$ and $\mathcal{O}_{N'}/\varpi\mathcal{O}_{N'} \to \overline{\pi}_*(\mathcal{O}_{\overline{\mathcal{N}}})$ which appear in (12.1) are also isomorphisms. Therefore, the special fibers of N' and $\pi(\mathcal{N})$ coincide and they are both equal to $\overline{\pi}(\overline{\mathcal{N}})$ which by the above is integral, normal and with rational singularities. In fact, we can see as in loc. cit. that $N' = \pi(\mathcal{N})$ and that $\pi(\mathcal{N})$ is normal and Cohen-Macaulay. To deduce the claims of the theorem for N^{can} we can now proceed along the lines of [PR], proof of Proposition 5.3: Recall that the canonical local model is the scheme-theoretic image of the morphism

$$\pi' : \mathcal{N} \to N \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K \to N \ ,$$

i.e $N^{\mathrm{can}} = \pi'(\mathcal{N})$. An argument as in loc. cit. now shows that

$$N^{\mathrm{can}} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K = \pi(\mathcal{N}) \ , \quad \pi(\mathcal{N})/\mathrm{Gal}(K/F_0) = N^{\mathrm{can}} \ ,$$

and the desired statements for N^{can} follow (see loc. cit. for more details). □

Remark 12.3 It follows that $\overline{N}_I^{\mathrm{can}} = \overline{\mathcal{O}}_{e\mu_1}$ is the union of all the Schubert strata (cells) in $\mathbf{SFl}_I$ which correspond to double cosets in the extended affine Weyl group which, in the Bruhat order, are $\leq$ to the coset given by the coweight $\mu = e\mu_1$. The set of these cosets is exactly the μ-admissible set as defined in ([KR]).

We now consider general index sets I. For $I = \{i_0, \ldots, i_{m-1}\} \subset \{0, \ldots, g\}$, and $i_k \in I$, we can consider the morphism

$$\pi_{i_k} : N_I^{\mathrm{naive}} \ \to \ N_{\{i_k\}}^{\mathrm{naive}}$$

obtained by $\{\mathcal{F}_{\pm i_n}\}_{n=0}^{m-1} \mapsto \mathcal{F}_{\pm i_k}$. As in the case of $G = \mathrm{Res}_{F/F_0}\mathrm{GL}_d$ (see §7 and [PR], §8), we can consider the scheme theoretic intersection in N_I^{naive},

$$N_I^{\mathrm{loc}} := \bigcap_{i_k \in I} \pi_{i_k}^{-1}(N_{\{i_k\}}^{\mathrm{can}}) \ . \tag{12.2}$$

Theorem 12.4 *(a)* $N_I^{\mathrm{can}} = N_I^{\mathrm{loc}}$.
(b) The special fiber $N_I^{\mathrm{can}} \otimes_{\mathcal{O}_{F_0}} k$ *is reduced and its irreducible components are normal with rational singularities. It can be identified with the union in* $\mathbf{SFl}_I$ *of the Schubert cells* $\mathcal{O}_w$ *with* w *in the* $e\mu_1$*-admissible set in* $\tilde{W}_I \backslash \tilde{W} / \tilde{W}_I$.

Here $\tilde{W}$ denotes the extended affine Weyl group of $\mathrm{GSp}_{2g}(k[[\Pi]])$ and $\tilde{W}_I$ the subgroup corresponding to the parahoric subgroup P_I.

PROOF. We consider the chain of closed embeddings of $\mathcal{O}_{F_0}$-schemes with identical generic fibers,

$$N_I^{\mathrm{can}} \subset N_I^{\mathrm{loc}} \subset N_I^{\mathrm{naive}} \quad .$$

By [G3], Prop. 6.1 all generic points of the special fiber of N_I^{naive} can be lifted to the generic fiber. In other words, the above inclusions induce bijections on the underlying topological spaces. On the other hand, by Theorem 12.2 the special fiber of $N_{\{i_k\}}^{\mathrm{can}}$ is reduced and hence may be identified with a Schubert variety in a symplectic Grassmannian. Now the same argument as in the proof of Theorem 7.2 implies that the special fiber of (12.2) is reduced with all its irreducible components normal and with rational singularities. It follows that N_I^{loc} is flat over Spec $\mathcal{O}_{F_0}$ and hence $N_I^{\mathrm{can}} = N_I^{\mathrm{loc}}$. The last statement of (b) follows as Remark 7.3, (b) from Section 11. □

Remark 12.5 It seems plausible to expect that $N_I^{\mathrm{can}} = N_I^{\mathrm{naive}}$, i.e. that N_I^{naive} is flat over Spec $\mathcal{O}_{F_0}$, comp. [G3].

Let $I = \{0\}$. The conjecture above may be reduced to a question on a certain space of matrices. Let

$$P = \{ \quad A = \begin{pmatrix} a & b \\ 0 & {}^t a \end{pmatrix} \in M_{2ge} \ ; \ a, b \in M_{ge} \ , \ \ {}^t b = -b \ ,$$
$$\mathrm{char}_a(T) = (\prod_{i=1}^{e} (T - a_i))^g \ , Q(A) = 0 \ \ \} \ .$$

The question is whether P is flat over Spec $\mathcal{O}_{F_0}$.

The relation to the previous conjecture is given by the following diagram analogous to [PR], (1.3),

$$N_{\{0\}}^{\mathrm{naive}} \xleftarrow{\pi} \tilde{N}_{\{0\}}^{\mathrm{naive}} \xrightarrow{\phi} P \quad .$$

Here

$$\tilde{N}_{\{0\}}^{\mathrm{naive}}(S) = \{(\mathcal{F} \subset \Lambda_{0,S}, \alpha)\} \quad ,$$

where $\mathcal{F}$ defines a point of $N_{\{0\}}^{\mathrm{naive}}(S)$ and where α is a symplectic automorphism of $\Lambda_{0,S}$ which carries $\mathcal{F}$ into the Lagrangian subspace $\mathcal{F}_0$ of Λ_0 generated over $\mathcal{O}_F$ by $e_1, \ldots, e_g$. Then π is a torsor under the Siegel parabolic in $\mathrm{Sp}_{2ge} \simeq \mathrm{Sp}(\Lambda_0, <\ ,\ >)$ and ϕ is a smooth morphism, given by

$$\phi((\mathcal{F}, \alpha)) = \alpha^{-1} \cdot \pi \cdot \alpha \quad ,$$

which we express as a matrix in terms of the $\mathcal{O}_S$-basis $e_1, \ldots, e_g, \pi e_1, \ldots, \pi e_g, \ldots, \pi^{e-1} e_1, \ldots, \pi^{e-1} e_g, \delta f_1, \ldots, \delta f_g, \pi \delta f_1, \ldots, \pi \delta f_g, \ldots, \pi^{e-1} \delta f_1, \ldots, \pi^{e-1} \delta f_g$ of $\Lambda_{0,S}$.

Part IV

13 Nearby cycles

In this section, we will assume that the residue field k of $\mathcal{O}_E$ is finite. Our aim is to describe the sheaves of nearby cycles for the local models $M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} \mathcal{O}_K$ and $N_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} \mathcal{O}_K$ as convolutions of the sheaves of nearby cycles associated to the "unramified" local models M_I^j and N_I^j respectively (see below for a precise statement). For simplicity, we will restrict our discussion mostly to the case of $G = \mathrm{Res}_{F/F_0}\mathrm{GL}_d$, i.e to the models M_I^{can}; the case of $G = \mathrm{Res}_{F/F_0}\mathrm{GSp}_{2g}$ is similar.

Fix a prime number ℓ which is invertible in $\mathcal{O}_E$ and a square root of the cardinality $|k|$ in $\overline{\mathbf{Q}}_\ell$. Let $\mathcal{O}$ be a discrete valuation ring which is a finite flat extension of $\mathcal{O}_E$ with fraction field L contained in F_0^{sep}. If X is a scheme of finite type over $\mathrm{Spec}\,\mathcal{O}$ with constant relative dimension d denote by

$$R\Psi_L^X = R\Psi_{\mathcal{O}}^X \overline{\mathbf{Q}}_\ell[d]\left(\tfrac{d}{2}\right)$$

the (adjusted) complex of nearby cycles of X over $\mathrm{Spec}\,\mathcal{O}$. This is an element in the derived category of complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on the geometric special fiber $X \otimes_{\mathcal{O}} \bar{k}$ with bounded constructible cohomology sheaves and continuous $\mathrm{Gal}(F_0^{\mathrm{sep}}/L)$-action which lifts (i.e is compatible with) the action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/L)$ on $X \otimes_{\mathcal{O}} \bar{k}$ through the Galois group of the residue field of $\mathcal{O}$. By [I], Theorem 4.2 and Cor. 4.5, $R\Psi_L^X$ is a perverse $\overline{\mathbf{Q}}_\ell$-sheaf of weight zero on $X \otimes_{\mathcal{O}} \bar{k}$.

For simplicity, if X is a scheme over $\mathcal{O}_E$, we will write $R\Psi_K^X$ instead of $R\Psi_K^{X \otimes_{\mathcal{O}_E} \mathcal{O}_K}$ for the (adjusted) complex of nearby cycles of $X \otimes_{\mathcal{O}_E} \mathcal{O}_K$ over $\mathcal{O}_K$. Again, this is a perverse $\overline{\mathbf{Q}}_\ell$-sheaf of weight zero on $X \otimes_{\mathcal{O}_E} \bar{k}$ with an action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$; it is isomorphic to the complex of $\overline{\mathbf{Q}}_\ell$-sheaves $R\Psi_E^X$ with the $\mathrm{Gal}(F_0^{\mathrm{sep}}/E)$-action restricted to the subgroup $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$.

By §6 and Remark 7.3 part (b), the special fiber $\overline{M}_I^{\mathrm{can}}$ can be naturally identified with a reduced finite union of Schubert varieties in the partial affine flag variety $\mathbf{Fl}_I$. On the other hand, for each $j = 1, \ldots, e$, the special fiber $\overline{M}_I^j$ of the unramified local model M_I^j over $\mathrm{Spec}\,\mathcal{O}_K$ can also be identified with a finite union of Schubert varieties in $\mathbf{Fl}_I \otimes_k k'$. In this way, we can regard

$$R\Psi_K^{M_I^j}, \quad R\Psi_K^{M_I^{\mathrm{can}}}$$

as perverse $\overline{\mathbf{Q}}_\ell$-sheaves of weight zero on $\mathbf{Fl}_I \otimes_k \bar{k}$ with compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-actions which are P_I-equivariant. By Remark 7.3 (b), these perverse sheaves are supported on the union of Schubert cells corresponding to the μ_j-admissible, resp. μ-admissible cosets, where $\mu = \mu_1 + \cdots + \mu_e$.

For each $j = 1, \ldots, e$, we now let Φ_j be a P_I-equivariant perverse $\overline{\mathbf{Q}}_\ell$-sheaf on $\overline{M}_I^j$ with compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action. The convolution construction of Ginzburg, Lusztig, etc. (see for example [Lu]) allows us to construct an element

$$\Phi_1 \star \cdots \star \Phi_e$$

in the derived category of complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on $\mathbf{Fl}_I \otimes_k \bar{k}$ supported on $\overline{M}_I^{\mathrm{can}}$ with bounded constructible cohomology sheaves and compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action. (In what follows, for simplicity of notation, we will use a bar to denote the geometric special fiber over $\bar{k}$ and omit the base change from the notation). The construction proceeds as follows ([Lu] 1.2 and 1.3). Consider the diagram obtained by the convolution diagram (6.3) by base changing from k' to $\bar{k}$:

$$\begin{array}{ccccc} & & U_{\bar{k}} & & \\ & p_1 \swarrow & & \searrow p_2 & \\ \overline{M}_I^1 \times \cdots \times \overline{M}_I^e & & \overline{M}_I^1 \tilde{\times} \cdots \tilde{\times} \overline{M}_I^e & \stackrel{p_3}{\to} & \overline{M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K} \subset \mathbf{Fl}_I \otimes_k \bar{k} \ . \end{array} \tag{13.1}$$

Consider the pull back of the exterior tensor product $p_1^*(\Phi_1 \boxtimes \cdots \boxtimes \Phi_e)$; since p_1 is a smooth morphism, this is a perverse $\overline{\mathbf{Q}}_\ell$-sheaf up to a shift by the relative dimension of p_1. By its definition, $p_1^*(\Phi_1 \boxtimes \cdots \boxtimes \Phi_e)$ is equivariant for the action (6.4); however, since the complexes of sheaves Φ_j are P_I-equivariant, it is also equivariant for the action (6.5). Recall that p_2 is a P_I-torsor for the action (6.5) (which is actually locally trivial in the Zariski topology). Therefore, by descent (see also [BBD] Theorem 4.2.5), there is a perverse $\overline{\mathbf{Q}}_\ell$-sheaf with compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action

$$\Phi_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \Phi_e$$

on $\overline{M}_I^1 \tilde{\times} \cdots \tilde{\times} \overline{M}_I^e$, which is unique up to unique isomorphism, such that

$$p_2^*(\Phi_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \Phi_e) = p_1^*(\Phi_1 \boxtimes \cdots \boxtimes \Phi_e) \ .$$

We now set

$$\Phi_1 \star \cdots \star \Phi_e := Rp_{3*}(\Phi_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \Phi_e) \ .$$

Theorem 13.1

(a) The sheaf $R\Psi_K^{M_I^1} \star \cdots \star R\Psi_K^{M_I^e}$ on $\mathbf{Fl}_I \otimes_k \bar{k}$ is perverse of weight zero.

(b) There is an isomorphism of perverse $\overline{\mathbf{Q}}_\ell$-sheaves with $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action

$$R\Psi_K^{M_I^{\mathrm{can}}} \simeq R\Psi_K^{M_I^1} \star \cdots \star R\Psi_K^{M_I^e}$$

on $\mathbf{Fl}_I \otimes_k \bar{k}$.

Proof. Recall the diagram (5.10)

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}_I & & \\ & p_I \swarrow & & \searrow q_I & \\ \prod_{l=1}^e M_I^l & & \mathcal{M}_I & \stackrel{\pi_I}{\to} & M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \ . \end{array} \tag{13.2}$$

By the Künneth formula [I], Theorem 4.7, we have an isomorphism of perverse $\overline{\mathbf{Q}}_\ell$-sheaves with compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action on the geometric special fiber of $M_I^1 \times \cdots \times M_I^e$,

$$R\Psi_K^{M_I^1 \times \cdots \times M_I^e} \;\simeq\; R\Psi_K^{M_I^1} \boxtimes \cdots \boxtimes R\Psi_K^{M_I^e} \;. \tag{13.3}$$

This induces an isomorphism between the pull-backs

$$\overline{p}_I^*(R\Psi_K^{M_I^1 \times \cdots \times M_I^e}) \;\simeq\; \overline{p}_I^*(R\Psi_K^{M_I^1} \boxtimes \cdots \boxtimes R\Psi_K^{M_I^e}) \;. \tag{13.4}$$

From the definitions, and using the comparisons of the special fiber of the diagram (5.10) with the convolution diagram (6.3) explained at the end of §6, we obtain an isomorphism

$$\overline{p}_I^*(R\Psi_K^{M_I^1} \boxtimes \cdots \boxtimes R\Psi_K^{M_I^e}) \;\simeq\; \overline{q}_I^*(R\Psi_K^{M_I^1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} R\Psi_K^{M_I^e}) \;. \tag{13.5}$$

Since both p_I and q_I are smooth, p_I^* and q_I^* commute with the nearby cycle functor. Therefore, we obtain an isomorphism

$$\overline{p}_I^*(R\Psi_K^{M_I^1} \boxtimes \cdots \boxtimes R\Psi_K^{M_I^e}) \;\simeq\; R\Psi_K^{\tilde{\mathcal{M}}} \;\simeq\; \overline{q}_I^*(R\Psi_K^{\mathcal{M}})$$

which by [BBD] Theorem 4.2.5 and (13.5) gives an isomorphism of perverse $\overline{\mathbf{Q}}_\ell$-sheaves of weight zero with compatible $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action

$$R\Psi_K^{\mathcal{M}} \;\simeq\; R\Psi_K^{M_I^1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} R\Psi_K^{M_I^e} \;. \tag{13.6}$$

We now notice that since $\pi_I : \mathcal{M} \to M_I^{\mathrm{can}} \otimes_{\mathcal{O}_E} \mathcal{O}_K \subset M_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K$ is proper and since π_I induces an isomorphism on the generic fibers, there is a canonical isomorphism

$$R\overline{\pi_I}_*(R\Psi_K^{\mathcal{M}}) \;\simeq\; R\Psi_K^{M_I^{\mathrm{can}}} \;. \tag{13.7}$$

Hence, by (13.6) there is an isomorphism

$$R\Psi_K^{M_I^{\mathrm{can}}} \;\simeq\; R\overline{\pi_I}_*(R\Psi_K^{M_I^1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} R\Psi_K^{M_I^e}) = R\Psi_K^{M_I^1} \star \cdots \star R\Psi_K^{M_I^e} \tag{13.8}$$

with the last equality given by the identification of p_3 with $\overline{\pi}_I$. This establishes both parts (a) and (b) of the Theorem. □

We note that the factors $R\Psi_K^{M_I^l}$ are known perverse sheaves thanks to the result of Haines and Ngo regarding the unramified case [HN1].

Remark 13.2 The same arguments applied to the local models N_I^{can} for the group $G = \mathrm{Res}_{F/F_0}\mathrm{GSp}_{2g}$ show that

$$R\Psi_K^{N_I^{\mathrm{can}}} \simeq R\Psi_K^{N_I^1} \star \cdots \star R\Psi_K^{N_I^e} \tag{13.9}$$

as perverse $\overline{\mathbf{Q}}_\ell$-sheaves with $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-action.

Part V

14 Splitting and local models in the general PEL case

In this section, we explain the construction of splitting models in the general (ramified) PEL case. As we shall see this also suggests a general construction of local models. We take $F_0 = \mathbf{Q}_p$ in the notation used elsewhere in this paper. Specifically, we will use the following notation (following closely [RZ], see 1.38):

- F a finite direct product of finite field extensions of $\mathbf{Q}_p$,
- B a finite central algebra over F,
- V a finite dimensional (left) B-module,
- $(\ ,\ )$ a nondegenerate alternating $\mathbf{Q}_p$-bilinear form on V,
- $b \mapsto b^*$ an involution on B which satisfies $(bv, w) = (v, b^*w),\ \ v,\, w \in V$,
- $\mathcal{O}_B$ a maximal order of B invariant under $*$.

If W is a right B-module, we define a left B-module on W by restriction of scalars $* : B \to B^{\mathrm{opp}}$. With this convention the dual vector space $V^* = \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p)$ is a left B-module and the form $(\ ,\ )$ induces an isomorphism of B-modules

$$\psi : V \to V^* \ .$$

In the same way, for an $\mathcal{O}_B$-lattice Λ in V, the $\mathbf{Z}_p$-module $\Lambda^* = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{Z}_p)$ becomes a left $\mathcal{O}_B$-module. The image of Λ^* under the map

$$\Lambda^* \to V^* \overset{\psi^{-1}}{\simeq} V$$

is the "dual" lattice $\hat{\Lambda}$ of $\Lambda \subset V$ with respect to $(\ ,\ )$. The form $(\ ,\ )$ induces a perfect bilinear pairing

$$(\ ,\ ) : \Lambda \times \hat{\Lambda} \to \mathbf{Z}_p \ .$$

Let F_1 be the $\mathbf{Q}_p$-algebra which consists of the $*$-invariant elements of F. For simplicity we will assume that F_1 is a field; the local models in the general case are products of local models for cases in which F_1 is a field. We will denote by τ the automorphism of F obtained by restricting the involution $*$. There are three cases:

(I) $F = F_1 \times F_1$ and $\tau(a_1, a_2) = (a_2, a_1)$,
(II) $F = F_1$,
(III) F is a quadratic field extension of F_1.

The existence of the $*$-linear form $(\ ,\ )$ implies that, even in case I, V is a free F-module; we will denote its rank by d.

Let G be the algebraic group over $\mathbf{Q}_p$, whose points with values in a $\mathbf{Q}_p$-algebra R are given by:

$$G(R) = \{g \in \mathrm{GL}_B(V \otimes_{\mathbf{Q}_p} R) \mid (gv, gw) = c(g)(v, w), c(g) \in R\} \ .$$

Let us fix in addition

- a cocharacter $\mu : \mathbf{G}_{mN} \to G_N$ defined over the finite extension N of $\mathbf{Q}_p$, given up to conjugation.

We assume that the corresponding eigenspace decomposition of $V \otimes_{\mathbf{Q}_p} N$ is given by

$$V \otimes_{\mathbf{Q}_p} N = V_0 \oplus V_1$$

(i.e the only weights are 0 and 1) and that the composition $c \circ \mu : \mathbf{G}_{mN} \to \mathbf{G}_{mN}$ is the identity. This implies that both V_0 and V_1 are totally isotropic for the form on $V \otimes_{\mathbf{Q}_p} N$ obtained by $(\ ,\ )$ by extending scalars (by [RZ] Definition 3.18 and 3.19 (b) these conditions correspond to the situation describing moduli of p-divisible groups). Notice that this implies that the pairing $(\ ,\ )$ induces an isomorphism

$$V_0 \simeq V_1^* = \mathrm{Hom}_N(V_1, N) \tag{14.1}$$

where V_1^* becomes a left B-module as above, by first regarding it naturally as a right B-module and then composing with the involution $* : B \to B^{\mathrm{opp}}$. As usual let E be the field of definition of the conjugacy class of μ. We shall also fix

- $\mathcal{L}$ a selfdual periodic multichain of $\mathcal{O}_B$-lattices in V ([RZ] Definition 3.13).

Recall that "selfdual" means that if Λ is in $\mathcal{L}$ then the dual lattice $\hat{\Lambda}$ is also in $\mathcal{L}$. As in loc. cit. we can consider $\mathcal{L}$ as a category with morphisms given by inclusions of lattices.

Now let Φ be the set of $\mathbf{Q}_p$-algebra homomorphisms of F in $\bar{\mathbf{Q}}_p$. For $a \in F$ let

$$\det(T \cdot I - a \mid V_1) = \prod_{\phi \in \Phi} (T - \phi(a))^{r_\phi}$$

so that the cocharacter

$$\mu_{\bar{\mathbf{Q}}_p} : \mathbf{G}_{m\bar{\mathbf{Q}}_p} \to G_{\bar{\mathbf{Q}}_p} \subset \mathrm{GL}_B(V \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p) \subset \mathrm{GL}_F(V \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p) = \prod_{\phi \in \Phi} \mathrm{GL}(V \otimes_{F,\phi} \bar{\mathbf{Q}}_p)$$

is given, up to conjugation, by $\{(1^{r_\phi}, 0^{d-r_\phi})\}_{\phi \in \Phi}$ with d the F-rank of V. We can think of the automorphism τ of F as giving a permutation of Φ by $\phi \mapsto \phi \cdot \tau$. For every $\phi \in \Phi$ we have

$$r_\phi + r_{\phi \cdot \tau} = d\ . \tag{14.2}$$

Indeed, by (14.1), the sum $r_\phi + r_{\phi \cdot \tau}$ is the multiplicity of the eigenvalue $\phi(a)$ for the action of $a \in F$ on $V \otimes_{\mathbf{Q}_p} \bar{\mathbf{Q}}_p$. This is equal to d since V is F-free of rank d.

Set $m = [F_1 : \mathbf{Q}_p]$ and let n be the $\mathbf{Q}_p$-dimension of F. We choose an ordering of the $\mathbf{Q}_p$-algebra homomorphisms $\phi_i : F \to \bar{\mathbf{Q}}_p$, $1 \le i \le n$, which in the case that $F \neq F_1$ has the property that any two embeddings ϕ, ϕ' with the same restriction to F_1 are successive. Denote by K the Galois closure of F in $\bar{\mathbf{Q}}_p$. Then $E \subset K$.

Suppose now that S is an $\mathcal{O}_K$-scheme. In what follows undecorated tensor products are meant to be over $\mathbf{Z}_p$. If b is a unit of B which normalizes $\mathcal{O}_B$ and $\Lambda \in \mathcal{L}$ then by the definitions $b\Lambda \in \mathcal{L}$. For such a b, conjugation by b^{-1} defines an isomorphism $\mathcal{O}_B \to \mathcal{O}_B$, $x \mapsto b^{-1}xb$. If M is an $\mathcal{O}_B \otimes \mathcal{O}_S$-module we denote by M^b the $\mathcal{O}_B \otimes \mathcal{O}_S$-module obtained by restriction of scalars with respect to this isomorphism. Left multiplication by b induces a $\mathcal{O}_B \otimes \mathcal{O}_S$-linear homomorphism $b : M^b \to M$.

Let us now define a functor $\mathcal{M}$ on the category of $\mathcal{O}_N$-schemes.

Definition 14.1 A point of $\mathcal{M}$ with values in an $\mathcal{O}_N$-scheme S is given by the following data.

1. For each $i = 1, \dots, n+1$, a functor from the category of the multichain $\mathcal{L}$ to the category of $\mathcal{O}_B \otimes \mathcal{O}_S$-modules on S

$$\Lambda \mapsto F^i_\Lambda \ , \ \Lambda \in \mathcal{L}.$$

2. For $i = 1, \dots, n+1$, a morphism of functors

$$j^i_\Lambda : F^i_\Lambda \to \Lambda \otimes \mathcal{O}_S.$$

We are requiring that the following conditions are satisfied:

a) For each $\Lambda \in \mathcal{L}$, $i = 1, \dots, n+1$, the homomorphism j^i_Λ is injective (and so it identifies F^i_Λ with a $\mathcal{O}_B \otimes \mathcal{O}_S$-submodule of $\Lambda \otimes \mathcal{O}_S$). Both F^i_Λ and the quotient $(\Lambda \otimes \mathcal{O}_S)/F^i_\Lambda$ are finite locally free $\mathcal{O}_S$-modules.

b) If b is a unit of B which normalizes $\mathcal{O}_B$ there are "periodicity" $\mathcal{O}_B \otimes \mathcal{O}_S$-linear isomorphisms

$$\theta_{b,\Lambda} : (F^i_\Lambda)^b \xrightarrow{\sim} F^i_{b\Lambda}$$

which make the diagrams

$$\begin{array}{ccc} (F^i_\Lambda)^b & \stackrel{j^i_\Lambda}{\to} & (\Lambda \otimes \mathcal{O}_S)^b \\ \theta_{b,\Lambda} \downarrow & & \downarrow b \\ F^i_{b\Lambda} & \stackrel{j^i_{b\Lambda}}{\to} & b\Lambda \otimes \mathcal{O}_S \end{array}$$

commutative.

c) For the action of $\mathcal{O}_B$ on F^1_Λ, we have the following identity of polynomial functions

$$\det_{\mathcal{O}_S}(a \mid F^1_\Lambda) = \det_{\mathcal{O}_S}(a \mid V_1), \quad a \in \mathcal{O}_B \ .$$

d) We have $F^{n+1}_\Lambda = (0)$. For $i = 1, \dots, n$, $F^{i+1}_\Lambda \subset F^i_\Lambda$, the quotient $F^i_\Lambda / F^{i+1}_\Lambda$ is $\mathcal{O}_S$-locally free of rank $r_i := r_{\phi_i}$ and is annihilated by

$$a \otimes 1 - 1 \otimes \phi_i(a) \in \mathcal{O}_B \otimes \mathcal{O}_S, \quad \text{for all } a \in \mathcal{O}_F.$$

e) Note that (a) implies that F^i_Λ is a locally direct $\mathcal{O}_S$-summand of $\Lambda \otimes \mathcal{O}_S$. We will denote by $(F^i_\Lambda)^\perp$ its orthogonal complement in $\hat{\Lambda} \otimes \mathcal{O}_S$ under the perfect pairing

$$(\ , \ ) : (\Lambda \otimes \mathcal{O}_S) \times (\hat{\Lambda} \otimes \mathcal{O}_S) \to \mathcal{O}_S \ .$$

For every $\Lambda \in \mathcal{L}$ and $i = 1, \dots, n+1$, we require that $F^i_{\hat{\Lambda}} \subset (F^i_\Lambda)^\perp$.

f) In addition to the above, we require that:

f1) If $F = F_1$, for every $i = 1, \dots, n$ and $\Lambda \in \mathcal{L}$

$$\prod_{1 \le k \le i} (a \otimes 1 - 1 \otimes \phi_k(a))((F^{i+1}_\Lambda)^\perp) \subset F^{i+1}_{\hat{\Lambda}}$$

for all $a \in \mathcal{O}_F$.

f2) If $F \neq F_1$, for every $h = 1, \dots, m = [F_1 : \mathbf{Q}_p]$ and $\Lambda \in \mathcal{L}$

$$\prod_{1 \leq k \leq 2h} (a \otimes 1 - 1 \otimes \phi_k(a))((F_\Lambda^{2h+1})^\perp) \subset F_{\hat{\Lambda}}^{2h+1}$$

for all $a \in \mathcal{O}_F$.

There is a morphism $\pi : \mathcal{M} \to M^{\text{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_K$, where M^{naive} is the functor of the "naive" local model of [RZ] (denoted by M^{loc} in loc. cit.) given by sending the S-point of $\mathcal{M}$ given by $\Lambda \mapsto (F_\Lambda^i \subset \Lambda \otimes \mathcal{O}_S)_{1 \leq i \leq n+1}$ to $\Lambda \mapsto t_\Lambda := (\Lambda \otimes \mathcal{O}_S)/F_\Lambda^1$. Indeed, the functor $\Lambda \mapsto t_\Lambda$ satisfies the conditions of loc. cit., Definition 3.27. For example, (c) and (e) together with the fact that $F_{\hat{\Lambda}}^1$, $(F_\Lambda^1)^\perp$ are locally direct $\mathcal{O}_S$-summands of $\hat{\Lambda} \otimes \mathcal{O}_S$ imply that $F_{\hat{\Lambda}}^1 = (F_\Lambda^1)^\perp$ and so t_Λ satisfies condition (iii) of loc. cit.

It is clear that $\mathcal{M}$ is representable by a projective scheme over $\operatorname{Spec} \mathcal{O}_K$ and that the morphism π is projective. We can also see that, on the generic fibers, π induces an isomorphism

$$\pi \otimes_{\mathcal{O}_K} K : \mathcal{M} \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} M^{\text{naive}} \otimes_{\mathcal{O}_E} K.$$

Let us use the same symbol π for the composed morphism $\pi : \mathcal{M} \to M^{\text{naive}} \otimes_{\mathcal{O}_E} O_K \to M^{\text{naive}}$. The scheme theoretic image $\pi(\mathcal{M}) \subset M^{\text{naive}}$ is a closed subscheme of M^{naive} which has the same generic fiber as M^{naive}. One can now set

$$M^{\text{loc}} = \pi(\mathcal{M}).$$

We believe that, if we exclude the orthogonal and certain unitary cases, then M^{loc} is a good integral model of its generic fiber. [1] More precisely, assume that we are either in case (I), or in case (II) with $*$ an orthogonal involution (then G is a form of a symplectic group), or in case (III) with F/F_1 unramified. Recall here that an involution of the first kind on a central simple algebra is called orthogonal resp. symplectic, if after a base change that splits the algebra it becomes the adjoint involution with respect to a symmetric resp. alternating form. Then it seems that the methods of the present paper prove that M^{loc} is flat over $\operatorname{Spec} \mathcal{O}_E$, with reduced special fiber, and such that all irreducible components of the special fiber are normal with rational singularities. Furthermore, let L denote the completion of the maximal unramified extension of $\mathbf{Q}_p$ and let $\tilde{K} = \tilde{K}_{\mathcal{L}}$ be the parahoric subgroup of $G(L)$ which fixes the lattice chain $\mathcal{L} \otimes \mathcal{O}_L$ in $V \otimes_{\mathbf{Q}_p} L$. Then $\tilde{K}$ acts on $M^{\text{loc}}(\overline{\mathbf{F}}_p)$ and the orbits are in bijective correspondence with the *μ-admissible subset* $\operatorname{Adm}_{\tilde{K}}(\mu)$ of $\tilde{K} \backslash G(L) / \tilde{K}$. We refer to [R], section 3, for the definition of the μ-admissible subset in the general case, cf. also [KR]. Our work in the previous sections shows that all these statements hold true in the following situations (and we believe that the general case, as limited above, may be reduced to these cases):

a) Let F_1 be a finite field extension of $\mathbf{Q}_p$ and consider $B = F_1 \times F_1$ with the involution $(a_1, a_2)^* = (a_2, a_1)$. Let $\mathcal{O}_B = \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_1}$ and take $V = B^d = W_1 \oplus W_2$, $W_i = F_1 \cdot e_1^i \oplus \cdots \oplus F_1 \cdot e_d^i$, for $i = 1$ or 2, with the alternating form $(\ ,\ )$ defined by

$$(e_k^i, e_l^i) = 0, \quad (e_k^1, e_l^2) = \delta_{kl} \ , i = 1, 2; \ k, l = 1, \dots, d \ .$$

[1] Genestier has pointed out to us that the orthogonal case is problematic in this respect.

This form identifies W_2 with the dual of W_1. A selfdual multichain of lattices in V now is given by a pair $\mathcal{L} = \{\Lambda_k\}_k$, $\hat{\mathcal{L}} = \{\hat{\Lambda}_l\}_l$ where $\mathcal{L}$ is a chain of $\mathcal{O}_{F_1}$-lattices in W_1 and $\hat{\mathcal{L}}$ is the dual chain. In this case,

$$G = \{(g, c \cdot (g^t)^{-1}) \mid g \in \mathrm{GL}(W_1), c \in \mathbf{G}_m\} \subset \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \ .$$

Therefore, $G \simeq \mathrm{Res}_{F_1/\mathbf{Q}_p}(\mathrm{GL}_d) \times \mathbf{G}_m$. Let us assume that F_1 is totally ramified over $\mathbf{Q}_p$. The schemes $\mathcal{M}$ (for various choices of the cocharacter μ and the multichain $\mathcal{L}$) can be identified with the splitting models for $\mathrm{Res}_{F_1/\mathbf{Q}_p}\mathrm{GL}_d$ of §5. To see this we observe that by using conditions (e) and (f2) and an argument as in Lemma 9.1 we can show that there is a 1-1 correspondence between submodules

$$F^i_{\Lambda_k \oplus \hat{\Lambda}_l} = G^{n+1-i}_{\Lambda_k} \oplus G'^{n+1-i}_{\hat{\Lambda}_l} \subset (\Lambda_k \otimes \mathcal{O}_S) \oplus (\hat{\Lambda}_l \otimes \mathcal{O}_S), \ (i = 1, \ldots, n+1)$$

which correspond to S-points of $\mathcal{M}$ and submodules $G^j_{\Lambda_k} \subset (\Lambda_k \otimes \mathcal{O}_S)$ which correspond to S-points in the splitting model of §5. Theorem 5.3 now implies that the schemes $\mathcal{M}$ are flat over $\mathrm{Spec}\,\mathcal{O}_K$.

b) Let $B = F = F_1$ a finite field extension of $\mathbf{Q}_p$ and let $(V, \{\ ,\ \})$ be the standard symplectic vector space over F of dimension $2g$ with basis $e_1, \ldots, e_g, f_1, \ldots, f_g$, i.e

$$\{e_i, e_j\} = \{f_i, f_j\} = 0, \quad \{e_i, f_j\} = \delta_{ij}. \tag{14.3}$$

We set $(v, w) = \mathrm{Tr}_{F/\mathbf{Q}_p}(\{v, w\})$. In this case, $G = \mathrm{Res}_{F/\mathbf{Q}_p}\mathrm{GSp}_{2g}$ and, in case F_1 is totally ramified over $\mathbf{Q}_p$, the scheme $\mathcal{M}$ can be identified with the splitting model for $\mathrm{Res}_{F/\mathbf{Q}_p}\mathrm{GSp}_{2g}$ of §9; here Theorem 9.4 implies the truth of the above conjecture.

Remark 14.2 An example where the methods of the previous sections do not directly apply is provided by the case of a group of unitary similitudes for a ramified quadratic extension of $\mathbf{Q}_p$. However, even in this case, there are instances in which we can show that M^{loc} as defined above, is flat over $\mathrm{Spec}\,\mathcal{O}_E$. We review some results from [P]. Let $B = F$ a ramified quadratic extension of $\mathbf{Q}_p$, p odd, with the involution given by the non-trivial Galois automorphism. Let $V = F^n$ and denote by e_i, $1 \leq i \leq n$, the canonical $\mathcal{O}_F$-generators of the standard lattice $\Lambda_0 := \mathcal{O}_F^n \subset V$. Let π be a uniformizer of $\mathcal{O}_F$ which satisfies $\pi^* = -\pi$. We define a non-degenerate alternating $\mathbf{Q}_p$-bilinear form $(\ ,\ ) : V \times V \to \mathbf{Q}_p$ which satisfies $(ax, y) = (x, a^* y)$ for $a \in F$ by setting

$$(e_i, e_j) = 0, \quad (e_i, \pi e_j) = \delta_{ij} \ , \ i, j = 1, \ldots, n \ \ .$$

The restriction $(\ ,\ ) : \mathcal{O}_F^n \times \mathcal{O}_F^n \to \mathbf{Z}_p$ is a perfect $\mathbf{Z}_p$-bilinear form. Therefore, we have $\hat{\Lambda}_0 = \Lambda_0$ and more generally $\widehat{\pi^n \Lambda_0} = \pi^{-n}\Lambda_0$. Let $\mathcal{L}$ be the selfdual lattice chain $\{\pi^n \Lambda_0\}_{n \in \mathbf{Z}}$. Using the duality isomorphism $\mathrm{Hom}_F(V, F) \simeq \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p)$ given by composing with the trace $\mathrm{Tr}_{F/\mathbf{Q}_p} : F \to \mathbf{Q}_p$ we see that there exists a unique non-degenerate hermitian form $\phi : V \times V \to F$ such that

$$(x, y) = \mathrm{Tr}_{F/\mathbf{Q}_p}(\pi^{-1}\phi(x, y)), \quad x, \ y \in V.$$

Hence, in this case the group G can be identified with the group of unitary similitudes of the form ϕ. Now let r, s be two non-negative integers such that $n = r + s$. Fix a cocharacter $\mu_F : \mathbf{G}_{mF} \to G_F$ such that the corresponding subspace V_1 of $V_F = F^n \otimes F$, when considered as an F-module via the first factor, is isomorphic to $F^r \oplus F_\tau^s$ where F_τ is the module obtained by F by restriction of scalars via $\tau : F \to F$. The "naive" local models that correspond to these choices have been studied in [P]. As was shown there, when $|r - s| > 1$, they are not flat over $\operatorname{Spec} \mathcal{O}_E$.

Given these choices of PEL data, we can see that $K = F$ and that for any $\operatorname{Spec} \mathcal{O}_F$-scheme S the points $\mathcal{M}(S)$ are now pairs (F^2, F^1) of $\mathcal{O}_F \otimes \mathcal{O}_S$-submodules of $\Lambda_0 \otimes \mathcal{O}_S$ which are locally direct summands as $\mathcal{O}_S$-modules and satisfy

i) F^1 is isotropic for the form $(\ ,\ )$ on $\Lambda_0 \otimes \mathcal{O}_S$;

ii) $F^2 \subset F^1$; and F^1, F^2 have ranks n and r respectively;

iii) $\det_{\mathcal{O}_S}(T \cdot I - a \otimes 1 \mid F^1) = (T - a)^r (T - \tau(a))^s \in \mathcal{O}_S[T]$, for every $a \in \mathcal{O}_F$;

iv) $(a \otimes 1 - 1 \otimes a)(F^2) = (0)$, $(a \otimes 1 - 1 \otimes \tau(a))(F^1) \subset F^2$, for every $a \in \mathcal{O}_F$

(the tensor products are in $\mathcal{O}_F \otimes \mathcal{O}_F$ which maps to $\mathcal{O}_F \otimes \mathcal{O}_S$).

For simplicity, let us assume that $r \neq s$; then $K = E = F$. The naive local model M^{naive} classifies isotropic $\mathcal{O}_F \otimes \mathcal{O}_S$-submodules F^1 of $\Lambda_0 \otimes \mathcal{O}_S$ which are locally direct summands of rank n as $\mathcal{O}_S$-modules and satisfy condition (iii) above; the morphism $\pi : \mathcal{M} \to M^{\text{naive}}$ corresponds to forgetting F^2. We can see that the scheme theoretic image $M^{\text{loc}} := \pi(\mathcal{M}) \subset M^{\text{naive}}$ is contained in the closed subscheme $M'_{r,s}$ of M^{naive} described by

$$\wedge^{r+1}(a \otimes 1 - 1 \otimes \tau(a) \mid F^1) = (0),\ \wedge^{s+1}(a \otimes 1 - 1 \otimes a \mid F^1) = (0).$$

By [P] Theorem 4.5 and its proof, $M'_{r,s}$ is flat over $\operatorname{Spec} \mathcal{O}_E$ when $r = n - 1$, $s = 1$. Note that the scheme $M'_{r,s}$ has the same generic fiber as M^{naive}. Hence, by the above, if $r = n - 1$, $s = 1$, $M^{\text{loc}} = M'_{r,s}$ is flat over $\operatorname{Spec} \mathcal{O}_E$.

In fact, the calculations described in loc. cit., 4.16 suggest that $M'_{r,s}$, and therefore also M^{loc}, should be flat over $\operatorname{Spec} \mathcal{O}_E$ for all values of r, s. The discussion in loc. cit., 4.16 shows that this flatness statement follows if one knows that the subscheme of $n \times n$-matrices over $\mathbf{F}_p$ defined by

$$\{A \in \operatorname{Mat}_{n \times n} |\ A^2 = 0,\ A = A^t,\ \wedge^{s+1} A = 0,\ \wedge^{r+1} A = 0,\ \det(T \cdot I - A) \equiv T^n\}$$

is reduced. This can be viewed as the symmetric matrix version of a result of Strickland [St] (compare to [PR] Cor. 5.10) and it can be verified (for various primes p) using Macaulay when $r, s \leq 5$.

In the case considered in this remark, the parahoric subgroup fixing the lattice chain $\mathcal{L}$ is a special maximal parahoric. For more general lattice chains one encounters additional problems.

15 Moduli spaces of abelian varieties

In this section we briefly indicate the construction of moduli spaces of abelian varieties corresponding, in a sense made precise by the diagram (15.4) below, to the splitting and local models of the previous section. The use of the language of algebraic stacks in (15.4) replaces the method of *linear modifications* of [P]; its mathematical content is the same.

In this section we will use the following notation, taken from [RZ], ch. 6. Let B be a semi-simple algebra over $\mathbf{Q}$ and let $*$ be a positive involution on B. Let V be a finite-dimensional $\mathbf{Q}$-vector space with a nondegenerate alternating bilinear form $(\ ,\ )$ with values in $\mathbf{Q}$. We assume that V is equipped with a B-module structure such that

$$(bv, w) = (v, b^* w), \quad v, w \in V, \quad b \in B \quad .$$

Let $G \subset GL_B(V)$ be the closed algebraic subgroup over $\mathbf{Q}$ such that

$$G(\mathbf{Q}) = \{g \in GL_B(V) \mid (gv, gw) = c(g)(v, w),\ c(g) \in \mathbf{Q}\} \quad .$$

Let $\mathcal{S} = R_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m$ and let $h : \mathcal{S} \to G_{\mathbf{R}}$ be a homomorphism satisfying the usual Riemann bilinear relations (cf. loc.cit.). We have a corresponding Hodge decomposition

$$V \otimes \mathbf{C} = V_0 \oplus V_1$$

and a corresponding cocharacter μ of G defined over $\mathbf{C}$. We let $E \subset \overline{\mathbf{Q}}$ be the corresponding Shimura field. We now fix a prime number p and choose an embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. The corresponding ν-adic completion of E will be denoted E_ν. Let $C^p \subset G(\mathbf{A}_f^p)$ be an open compact subgroup.

We consider an order O_B of B such that $O_B \otimes \mathbf{Z}_p$ is a maximal order of $B \otimes \mathbf{Q}_p$. We assume that $O_B \otimes \mathbf{Z}_p$ is invariant under the involution. We also fix a selfdual periodic multichain $\mathcal{L}$ of $O_B \otimes \mathbf{Z}_p$-lattices in $V \otimes \mathbf{Q}_p$ with respect to the alternating form $(\ ,\ )$.

We recall from loc.cit. the definition of a moduli problem $\mathcal{A}_{C^p}$ over $(Sch/\mathrm{Spec}\ \mathcal{O}_{E_\nu})$. It associates to a $\mathcal{O}_{E_\nu}$-scheme S the following set of data up to isomorphism:

1. An $\mathcal{L}$-set of abelian varieties $A = \{A_\Lambda\}$
2. A $\mathbf{Q}$-homogeneous principal polarization $\overline{\lambda}$ of the $\mathcal{L}$-set A.
3. A C^p-level structure

$$\overline{\eta} : H_1(A, \mathbf{A}_f^p) \simeq V \otimes \mathbf{A}_f^p \bmod C^p \quad ,$$

which respects the bilinear forms on both sides up to a constant in $(\mathbf{A}_f^p)^\times$.

We require an identity of characteristic polynomials,

$$\det(T \cdot I - b \mid \mathrm{Lie}\ A_\Lambda) = \det(T \cdot I - b \mid V_0), \quad b \in O_B,\ \Lambda \in \mathcal{L} \quad .$$

For the definitions of the terms employed here we refer to loc.cit., 6.3–6.8. We only mention that A is a functor from the category $\mathcal{L}$ to the category of abelian schemes over S up to isogeny of order prime to p, with O_B-action, and that a polarization λ is a $\mathcal{O}_B$-linear homomorphism from A to the dual $\mathcal{L}$-set $\tilde{A}$ (for which $\tilde{A}_\Lambda = (A_{\Lambda^*})^\wedge$).

The functor $\mathcal{A}_{C^p}$ is representable by a quasi-projective scheme over $\mathcal{O}_{E_\nu}$, provided that C^p is sufficiently small.

We denote by M_Λ the Lie algebra of the universal extension of A_Λ. Then $\{M_\Lambda\}$ is a polarized multichain of $(\mathcal{O}_B \otimes \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathcal{O}_S$-modules on S of type $(\mathcal{L})$ in the sense of [RZ], Def. 3.14. Let $\tilde{\mathcal{A}}_{C^p}$ be the functor which to $S \in (Sch/\mathcal{O}_{E_\nu})$ associates the isomorphism classes of objects $(A, \overline{\lambda}, \overline{\eta})$ of $\mathcal{A}_{C^p}(S)$ and an isomorphism of polarized multichains between $\{M_\Lambda\}$ and $\mathcal{L} \otimes_{\mathbf{Z}_p} \mathcal{O}_S$. By [P], Thm. 2.2 (a slight extension of [RZ] Thm. 3.16), the forgetful morphism

$$\pi : \tilde{\mathcal{A}}_{C^p} \longrightarrow \mathcal{A}_{C^p} \tag{15.1}$$

is a principal homogeneous space, locally trivial for the étale topology, under the smooth group scheme $\mathcal{G} \times_{\mathrm{Spec}\ \mathbf{Z}_p} \mathrm{Spec}\ \mathcal{O}_{E_\nu}$. Here $\mathcal{G} = \underline{\mathrm{Aut}}(\mathcal{L})$ is the group scheme over Spec $\mathbf{Z}_p$ with $C_p = \mathcal{G}(\mathbf{Z}_p)$ the parahoric subgroup of $G(\mathbf{Q}_p)$ fixing the lattice chain $\mathcal{L}$.

The Lie algebra Lie A_Λ is a factor module t_Λ of M_Λ. Using the identification of M_Λ with $\Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}_S$ over $\tilde{\mathcal{A}}_{C^p}$ we therefore obtain a point of the naive local model M^{naive} defined in terms of the $\mathbf{Z}_p$-data $(B \otimes \mathbf{Q}_p, O_B \otimes \mathbf{Z}_p, V \otimes \mathbf{Q}_p, \mathcal{L})$ induced from our global data,

$$\tilde{\varphi} : \tilde{\mathcal{A}}_{C^p} \longrightarrow M^{\mathrm{naive}} \quad . \tag{15.2}$$

Since φ is obviously equivariant for the action of $\mathcal{G} \otimes_{\mathbf{Z}_p} \mathcal{O}_{E_\nu}$, $\tilde{\varphi}$ corresponds to a relatively representable morphism of algebraic stacks

$$\varphi : \mathcal{A}_{C^p} \longrightarrow \left[M^{\mathrm{naive}}/\mathcal{G} \otimes_{\mathbf{Z}_p} \mathcal{O}_{E_\nu}\right] \quad . \tag{15.3}$$

By [P], Thm. 2.2, (a slight extension of [RZ], Prop. 3.3), the morphism φ is smooth of relative dimension dim G. Let us form the cartesian product of φ with the morphisms $\mathcal{M} \to M^{\mathrm{loc}} \hookrightarrow M^{\mathrm{naive}}$, where $\mathcal{M}$ denotes the splitting model over $\mathcal{O}_K$, with K the Galois closure of E_ν,

$$\begin{array}{ccc} \mathcal{A}^{\mathrm{spl}}_{C^p} & \longrightarrow & [\mathcal{M}/\mathcal{G}_{\mathcal{O}_K}] \\ \downarrow & & \downarrow \\ \mathcal{A}^{\mathrm{loc}}_{C^p} & \longrightarrow & [M^{\mathrm{loc}}/\mathcal{G}_{\mathcal{O}_{E_\nu}}] \\ \downarrow & & \downarrow \\ \mathcal{A}_{C^p} & \longrightarrow & [M^{\mathrm{naive}}/\mathcal{G}_{\mathcal{O}_{E_\nu}}]. \end{array} \tag{15.4}$$

The scheme $\mathcal{A}^{\mathrm{loc}}_{C^p}$ is a closed subscheme of $\mathcal{A}_{C^p}$ and is the image of $\mathcal{A}^{\mathrm{spl}}_{C^p}$ in $\mathcal{A}_{C^p}$. The scheme $\mathcal{A}^{\mathrm{loc}}_{C^p}$ is a linear modification of $\mathcal{A}_{C^p}$ in the sense of [P]; likewise, $\mathcal{A}^{\mathrm{spl}}_{C^p}$ is a linear modification of $\mathcal{A}^{\mathrm{spl}}_{C^p} \otimes_{\mathcal{O}_{E_\nu}} \mathcal{O}_K$.

The $\mathcal{O}_K$-scheme $\mathcal{A}^{\mathrm{spl}}_{C^p}$ represents the following moduli problem on $(Sch/\mathcal{O}_K)$. It associates to S the set of isomorphism classes of objects $(A, \overline{\lambda}, \overline{\eta}, \mathcal{F})$. Here $(A = \{A_\Lambda\}, \overline{\lambda}, \overline{\eta})$ is an object of $\mathcal{A}_{C^p}(S)$.

Let F_1 be the invariants under $*$ in the center F of $B \otimes \mathbf{Q}_p$. Then F_1 is a direct sum of fields,

$$F_1 = F_{1,1} \oplus \ldots \oplus F_{1,r} \quad . \tag{15.5}$$

For $k = 1, \ldots, r$, let n_k be the degree of the direct summand of F corresponding to the direct summand $F_{1,k}$ of F_1. Let M_Λ be the Lie algebra of the universal extension of A_Λ and let F_Λ be the kernel of the factor map from M_Λ to Lie A_Λ. Then the action of $\mathcal{O}_{F_1}$ on M_Λ and F_Λ induces decompositions

$$M_\Lambda = \bigoplus_{k=1}^{r} M_{\Lambda,k} \quad , \quad F_\Lambda = \bigoplus_{k=1}^{r} F_{\Lambda,k} \quad .$$

The final ingredient $\mathcal{F}$ of an object of $\mathcal{A}^{\mathrm{spl}}_{C^p}(S)$ is a collection of functors $\Lambda \mapsto F^i_{\Lambda,k}$ for $k = 1, \ldots, r$ and $i = 1, \ldots, n_k$, with functor morphisms $j^i_{\Lambda,k} : F^i_{\Lambda,k} \to M_{\Lambda,k}$, satisfying for each $k = 1, \ldots, r$ the conditions in Definition 14.1 when $\Lambda \otimes \mathcal{O}_S$ is replaced by $M_{\Lambda,k}$ and $(F^i_\Lambda, j^i_\Lambda)$ by $(F^i_{\Lambda,k}, j^i_{\Lambda,k})$, and such that $F^1_{\Lambda,k} = F_{\Lambda,k}$.

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