SHIMURA VARIETIES AND GERBES

R. P. LANGLANDS IN PRINCETON AND *M. RAPOPORT* IN BONN

Contents

Translator's note

This document presents an English translation of the paper *Shimuravarietäten und Gerben* by Langlands and Rapoport, originally published in *Journal für die reine und angewandte Mathematik,* 378 (1987), pages 113–220.

The translation was completed by Yihang Zhu. While the translator is not fluent in German, he has prior familiarity with certain parts of the paper. The translation process was assisted by AI tools, which also helped generate the LaTeX code for the formulas. The re-typed version [LRo] served as the input for these tools.

One of the main challenges in this project was the need to manually correct, and in some cases completely re-type, the AI-generated formulas by comparing them with the original text. This was partly due to the inherent imprecision of AI and partly because the re-typed version mentioned above contained additional typos. Further complicating the process, the notations in the original text were occasionally so intricate—and not entirely free of errors—that accurately discerning the intended formulas required a non-trivial mathematical understanding.

The translator corrected obvious typos and misprints in the original text without explicit mention. However, in cases where corrections were less straightforward, he chose to remain faithful to the original text rather than making speculative modifications.

All footnotes have been added by the translator with the aim of providing clarifications, though it was not his intention to offer thorough annotations. The original page numbers are displayed in the margins. In Sections 3 and 5, the places where proofs begin and end are marked in the margins.

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A recent trend in the mathematical literature surrounding this paper favors spelling "gerbe" as "gerb" in English. However, this translation adheres to the more established spelling "gerbe".

A few words about later developments following this paper are in order. The main construction in §3, that of the quasimotivic Galois group $\mathscr{Q}$, contains a well-known mistake, and corrections have been provided by Pfau [Pfa93, Pfa96] and Reimann [Rei97]. See the footnote above (3.e) for details. The "future works of Kottwitz" mentioned several times in the paper have appeared as [Kot90] and [Kot92]. The conjecture in §5 is now known as the *Langlands–Rapoport Conjecture*. The paper [Mil92] contains some useful discussions of this conjecture; see also [Mil03]. Some special cases of the conjecture are proved in [Rei97]. Further refinements and generalizations of the conjecture are given in [Rap05], whose emphasis is on parahoric level, and in [Kis17], where the assumption that G_{der} is simply connected is removed, and a refined version involving connected components is formulated, among other things. Moreover, in [Kis17], a weak version of the Langlands–Rapoport Conjecture for Shimura varieties of abelian type of hyperspecial level is proved. This work has the sequel [KSZ], where the weak version is strengthened, from which a stable trace formula for the Frobenius–Hecke action on the cohomology is derived. (In the case of Hodge type, the paper [Lee21] derives a Lefschetz number formula from results in [Kis17] in the style of Kottwitz [Kot90], circumventing the Langlands–Rapoport Conjecture.) The paper [vH24] extends, in some cases, the results of [Kis17] to parahoric level.

The translator thanks Michael Rapoport and Sug Woo Shin for some useful suggestions.

Corrections and suggestions are welcome—please send them to Yihang Zhu

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1. Introduction

113 The problem of the extension of Hasse-Weil zeta functions and, more generally, of motivic L-functions remains a central issue in number theory. It is often divided into two problems that need to be solved separately. First, it is necessary to show that every motivic L-function is equal to an automorphic L-function, and second, to prove that every automorphic L-function can be extended. Both problems have been solved in very few cases, and then only thanks to the efforts of many mathematicians over a long period of time.

After abelian varieties, Shimura varieties are probably the most accessible in arithmetic terms, and this work aims to contribute to the first problem for the motivic L-functions associated with them. Shimura varieties arise from the theory of automorphic functions and are defined by a reductive group G with additional structure. Experience suggests that the solution to the problem in a given case consists of two components: a group-theoretic description of the points of the reduced variety, and combinatorial arguments that include the fundamental lemma from [L3], in order to transform the resulting expression into a form that can be compared with the Arthur–Selberg trace formula.

Here, we focus solely on the first problem and refer to future works by R. Kottwitz for the transformation.

The goal of this work is to formulate a conjecture about the reduction modulo p of Shimura varieties, make it plausible by relating it to the standard conjectures of Grothendieck, and prove the facts leading to a purely group-theoretic formula for the number of points of the reduction modulo p. We have to make some restrictive assumptions about the prime number p, which, however, allow certain cases of bad reduction in addition to the case of good reduction. Let $S = \mathrm{Sh}(G,h) \quad (K = K^pK_p \subset G(\mathbb{A}_f))$ be a Shimura variety, defined over the reflex field $E = E(G,h)$, and let $\mathfrak{p}$ be a prime of E above p, which is good. Let κ be the residue field of $E_{\mathfrak{p}}$, and κ_m the degree m extension. The formula takes the following form: The number $|S(\kappa_m)|$ is equal to

$$\sum_{\{\varepsilon\}_{\mathrm{st}}} \iota(\varepsilon)\cdot \mathrm{vol}\left(G_\varepsilon(\mathbb{Q})Z_K\backslash G_\varepsilon(\mathbb{A}_f)\right)\cdot \sum_{\substack{\{\gamma\},\{\delta\}\\ \kappa(\varepsilon;\gamma,\delta)=1}} O_\gamma(f^p)TO_\delta(\phi_p). \tag{1.1}$$

114 Here Z_K is equal to $Z(\mathbb{A}_f)\cap K$; the outer sum runs over the stable conjugacy classes of $Z(\mathbb{Q})\cap Z_K\backslash G(\mathbb{Q})$. The number $\iota(\varepsilon)$ is a cohomological invariant:

$$\iota(\varepsilon) = \left|\ker\left(H^1(\mathbb{Q},G_\varepsilon)\to H^1(\mathbb{A},G_\varepsilon)\times H^1(\mathbb{Q},G_{\mathrm{ab}})\right)\right|.$$

The inner sum runs over conjugacy classes of elements $\gamma\in G(\mathbb{A}_f^p)$ and twisted conjugacy classes of elements $\delta\in G(L)$, where L is the unramified extension of $\mathbb{Q}_p$ with residue field κ_m. The symbols O_γ and TO_δ refer to orbit integrals and twisted orbit integrals, and the function on $G(\mathbb{A}_f^p)$ is given by

$$f^p = \frac{1}{\mathrm{vol}(K^p)}\cdot \mathrm{char}_{K^p},$$

while ϕ_p is an element defined by the Shimura variety in the Hecke algebra of $G(L)$ with respect to K_p. The pairs γ,δ are required to satisfy that γ is stably conjugate to ε, the norm of δ is stably conjugate to ε, and most importantly, that the Kottwitz invariant $\kappa(\varepsilon;\gamma,\delta)$ is defined and equal to 1. Although this formula does not appear in our work, and is reserved for a future work by Kottwitz, it can be easily derived from our results. This follows from the results of Kottwitz [K3].

In the case where the Shimura variety S_K is compact, the numbers in (1.1) appear in the power series expansion of the logarithm of the Hasse-Weil Zeta function of S_K at the good prime p. To compare the right hand side of (1.1) with the corresponding power series expansion of automorphic L-functions, associated with automorphic representations of G, one would first need to rewrite the twisted orbital integral of ϕ_p into a standard orbital integral of a function f_p on $G(\mathbb{Q}_p)$. If the fundamental lemma were generally available, this could be achieved after stabilization, and with Kottwitz's theorem [K3], the function f_p could then be written explicitly. However, it is the appearance of the condition on the Kottwitz invariant that complicates this simple approach. This condition can be seen as the reason for introducing the endoscopic groups in this context. The way out of this dilemma is to compare the Hasse-Weil Zeta function not with an automorphic L-function associated with G, but with a product of automorphic L-functions associated with automorphic representations of endoscopic groups of G such as a product of the form

$$\prod_H \prod_\Pi L(s, \Pi, r_H)^{m(H)}. \tag{1.2}$$

Here, Π is an L-packet, r_H is a virtual representation of the L-group ${}^L H$, and the exponent m is possibly a rational number. One might expect that the power series expansion of the logarithm of (1.2) can be written as a sum of stabilized traces in a certain sense:

$$\sum_H ST(f_H). \tag{1.3}$$

The functions f_H are dictated by the form of (1.2). The further strategy is now to 115
express the individual terms of (1.3) in terms of the stable trace formulas for the various H, and on the other hand, to carry out the stabilization of the right hand side of (1.1), which also leads to a sum of stable trace formulas for the endoscopic groups corresponding to appropriate functions f'_H. Finally, the functions f_H and f'_H would need to be compared. Although this program is still future work, Kottwitz has succeeded in writing the elliptic part of the sum on the right hand side of (1.1) in the appropriate form.

As a side note, it follows easily from the representation of the Hasse-Weil Zeta function in the form (1.2) that the eigenvalues of the Frobenius element are roots of polynomials whose coefficients are formed from the eigenvalues of Hecke operators. This statement is also a consequence of congruence relations, which in many cases are relatively simple to prove. However, while in the case of the group GL(2), the representation of the Hasse-Weil Zeta function as a product of automorphic L-functions is a consequence of congruence relations, this is hardly to be expected for other groups.

For the computation of the Zeta function, it suffices to have the formula (1.1) available, instead of our conjecture, and the attempt to prove this formula in the case of the group G of symplectic similitudes leads to a relatively elementary but subtle question about principally polarized abelian varieties, which we will explicitly state at this point. Let (A, Λ) be an abelian variety of CM type over $\mathbb{C}$ with a $\mathbb{Q}$-class of polarizations. Then (A, Λ) is already defined over a finite extension of $\mathbb{Q}$, and at a given prime $\mathfrak{p}$ it has good reduction $(\bar{A}, \bar{\Lambda})$ over the finite field

$\mathbb{F}_q, q = p^n$. The Frobenius endomorphism of $(\bar{A}, \bar{\Lambda})$ over $\mathbb{F}_q$ lifts to an endomorphism of (A, Λ), which, through its action on rational cohomology, defines an element $\varepsilon \in G(\mathbb{Q})$. On the other hand, the l-adic cohomology $H^1(\bar{A} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_l)$ for all $l \neq p$ defines a conjugacy class $\gamma \in G(\mathbb{A}_f^p)$, and crystalline cohomology (i.e., the theory of Dieudonné modules) defines a twisted conjugacy class $\delta \in G(L)$. Kottwitz's conjecture is that the invariant $\kappa(\varepsilon; \gamma, \delta)$ equals 1.

The focus of the present work, however, is not on proving our conjecture or its weakening, but on its formulation. The conjecture describes the points of reduction modulo p as a disjoint union of double cosets, parametrized by conjugacy classes of *admissible* homomorphisms of *gerbes* into the neutral gerbe associated with G,

$$\phi : \mathscr{P} \to \mathscr{G}_G.$$

Here, $\mathscr{P}$ is an explicitly constructed gerbe. In order to carry out this construction, we have chosen to define the concept of a gerbe in the simplest possible way, as an extension of a Galois group by an algebraic group (the "kernel" of the gerbe), since we felt unable to work with the abstract notion. The construction of $\mathscr{P}$ is a major concern of this work, as is the construction of the homomorphism

$$\psi_{T,\mu} : \mathscr{P} \to \mathscr{G}_T,$$

116 associated with an algebraic torus T defined over $\mathbb{Q}$ and a cocharacter μ of T. If $(T, h_T) \subset (G, h)$ defines a "special point" of the Shimura variety, we obtain from $\psi_{T,\mu}$, with $\mu = \mu_{h_T}$, by composition with the inclusion $\mathscr{G}_T \subset \mathscr{G}_G$, a homomorphism $\psi_{T,\mu} : \mathscr{P} \to \mathscr{G}_G$, which turns out to be admissible. In the case of good reduction, every admissible homomorphism is conjugate to a homomorphism of this form for an appropriate (T, h_T). Since there is an element δ in the kernel of $\mathscr{P}$ (more precisely, a sequence of elements δ_n, which are powers of one another), whose image in T under $\psi_{T,\mu}$ is rational, we obtain from $\psi_{T,\mu}$ an element $\gamma \in G(\mathbb{Q})$. It is easy to show that (γ, h_T) is a Frobenius pair in the sense of [L1], and that the mapping $\phi \mapsto (\gamma, h)$ gives a bijection between the *local* equivalence classes of admissible homomorphisms ϕ and the equivalence classes of Frobenius pairs. Here, two homomorphisms of gerbes over $\mathbb{Q}$ are said to be locally equivalent if their localizations at each place are conjugate. This shows that the conjecture formulated here follows from the one first stated in [L1]; however, the new conjecture is more precise, insofar as the embedding of the group $I(\mathbb{A}_p^f)$ into $G(\mathbb{A}_p^f)$ defined in [L1] is not defined there with sufficient precision to derive the formula (1.1) from it.

However, the new conjecture also has another advantage over the old one, which even historically is its origin. The old conjecture fails in the simplest cases of bad reduction. This is reflected, on the one hand, in the fact that not every admissible homomorphism is of the form $\psi_{T,\mu}$, and on the other hand, in the fact that the lifting theorem of Zink [Z1] no longer holds in these cases. In this work, we have largely neglected the case of bad reduction. The second author intends to treat some cases of bad reduction in more detail in a forthcoming work, thus shedding light on the conjecture from a second perspective.

The nature of the new conjecture has its philosophical background in Grothendieck's theory of motives. The existence of the category of motives over a finite field can so far only be proven using still unproven standard conjectures. On the other hand, the theory of Tannakian categories, designed by Grothendieck and developed by Saavedra Rivano in [Sa] and by Deligne and Milne in [DM], associates a gerbe (we have kept the French word because the closest translations seem to be already taken) to an abelian category with additional structures, particularly a tensor product. The gerbe $\mathscr{P}$ is the one that would arise from the category of motives over finite fields, if it existed.

We now explain the organization of the article. In §2, we define a gerbe $\mathscr{T}_{\nu_1,\nu_2}$ with additional structures over a global field, which arises from a torus T and two cocharacters ν_1, ν_2. If these cocharacters arise from "averaging" a single cocharacter μ, we construct an explicit neutralization $\mathscr{T}_{\nu_1,\nu_2} \cong \mathscr{G}_T$. The main results of this section are Theorems 2.1 and 2.2. The construction uses class field theory and the theory of Tate-Nakayama and proceeds similarly to the construction of the Taniyama group. We believe it is a useful endeavor to characterize the construction via a universal property. This would replace our inconsistent cocycle verifications and perhaps provide more insight into the prevailing relations here. In §3, we apply these constructions to the torus T, whose character group is generated by the set of Weil-q numbers. We show that this torus has such simple cohomology that the 117
gerbe $\mathscr{P}$ constructed in this way, as well as the homomorphisms $\psi_{T,\mu} : \mathscr{P} \to \mathscr{G}_T$ associated with (T, μ), can be uniquely characterized. Sections 2 and 3 provide the basis for the formulation of our conjecture. The conjecture about the reduction modulo p of a Shimura variety $S(G, h)$ is formulated at the beginning of §5. There, we also precisely explain what conditions we place on G and the prime p. The rest of §5 is dedicated to proving Theorems 5.21 and 5.25, which allow the transition from the conjecture to the formula (1.1). A significant portion of this section is devoted to the definition of the Kottwitz invariant. The remaining sections are of less central importance to our work. In §4, we show, assuming the Tate and Hodge conjectures, that the gerbe corresponding to the Tannakian category $M_{\mathbb{F}}$ of motives over finite fields, constructed using standard conjectures, is isomorphic to the gerbe $\mathscr{P}$ from Section 3, together with additional structures defined, for example, by various cohomology theories.

This section is essentially a somewhat more detailed, though still sketchy, exposition of Chapter VI in [Sa]. In §6, we show that the identification of gerbes made in §4 implies the conjecture of §5 for Shimura varieties that solve certain moduli problems, such as for the variety corresponding to the symplectic group. This section relies on the work of T. Zink [Z1], [Z2]. Since this is a conditional result, we have kept it brief. However, we explicitly point out the proof of Theorem 6.3, where Theorem 4.4 is used. It is at this point that the Kottwitz conjecture formulated earlier for abelian varieties should come into play, so that the formula (1.1) should follow from it, at least in the case of the symplectic group. We have indeed structured the proof intentionally so that it may possibly apply to the proof of formula (1.1). In §7, we provide two examples: one shows that in the conjecture of §5, we must assume the derived group to be simply connected, and the other shows that in the case of bad reduction, many admissible homomorphisms ϕ are not of the form $\psi_{T,\mu}$.

Two questions that are almost unavoidable when one tries to prove the conjecture in general are not addressed in this work. First, if $G \to G'$ is a central extension with $G_{\mathrm{der}} = G'_{\mathrm{der}} = G_{\mathrm{sc}}$, and if the conjecture holds for G', does it also hold for G? Secondly, it often happens that a Shimura variety Sh_1 is naturally embedded into a second Sh_2. This is especially the case for special points, for which Sh_1 is zero-dimensional. How does this embedding reflect in the description of the reduction of both varieties?

We would like to thank Kottwitz for sharing his results and ideas with us, and for making his notes available. The first author presented the current results during the Issai Schur Memorial Lecture in Tel Aviv in May 1984 and the Ritt Lecture at Columbia University in February 1985, and thanks the audiences at both venues for their patience and understanding when listening to still provisional material. We also both thank the Humboldt Foundation for supporting the first author's repeated visits to Heidelberg, during which this work was carried out.

In the following, let p be a fixed prime number, $\overline{\mathbb{Q}}$ the field of algebraic numbers, which we have embedded into an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We denote by $\mathbb{F}$ the corresponding algebraic closure of $\mathbb{F}_p$.

118 2. Cohomological constructions

This section is devoted to some constructions in Galois cohomology. First we explain our terminology.

Let k be a field of characteristic zero, which for us will either be a global or local field. Let G be an algebraic group over an algebraic closure $\overline{k}$. If $G = \operatorname{Spec} A$, where $A = \Gamma(G)$, and if $\sigma \in \operatorname{Gal}(\overline{k}/k)$, an automorphism κ of $G(\overline{k})$ is called σ-linear if there exists a σ-linear automorphism κ' of the algebra A such that

$$\kappa'(f)(\kappa(g)) = \sigma(f(g)), \quad f \in A,\, g \in G(\overline{k}).$$

The simplest example arises from the action of $\operatorname{Gal}(\overline{k}/k)$ on $G(\overline{k})$ if G is defined over k. A Galois gerbe over k is a group extension

$$1 \longrightarrow G(\overline{k}) \longrightarrow \mathscr{G} \longrightarrow \operatorname{Gal}(\overline{k}/k) \longrightarrow 1,$$

together with a section $\sigma \mapsto g_\sigma$ for $\sigma \in \operatorname{Gal}(\overline{k}/K)$, for a suitable finite extension K of k, such that the σ-linear automorphism

$$\kappa(\sigma) : g \longmapsto g_\sigma g g_\sigma^{-1}, \quad g \in G(\overline{k}),$$

arises from a K-structure on G. Additionally, for each $\sigma \in \operatorname{Gal}(\overline{k}/k)$ and each representative g_σ, the automorphism $\kappa(\sigma)$ must be σ-linear. It is understood that the finite extension K can be replaced by a larger finite extension K', so that in reality, we are dealing with a germ of sections $\{g_\sigma\}$ for the Krull topology on $\operatorname{Gal}(\overline{k}/k)$. In the notation, we often omit the germ of sections and simply denote a Galois gerbe, sometimes just called a gerbe, by $\mathscr{G}$. We call G the *kernel* of the gerbe. A homomorphism of gerbes is a homomorphism of the corresponding extensions that maps the germs of sections into each other and whose restriction to the kernel is algebraic. An element g in the kernel defines an automorphism of a gerbe by conjugation. Indeed, for a sufficiently large finite extension K/k, the element g is K-rational. We have

$$g_\sigma g g_\sigma^{-1} = \sigma(g) = g,$$

so conjugation by g preserves the germ of sections. Two homomorphisms ϕ_1 and ϕ_2 between two gerbes are called *equivalent*, if $\phi_2 = \operatorname{ad} g \circ \phi_1$ with g in the kernel of the second gerbe. We refer to §4 for a comparison of our terminology with that commonly used in the theory of Tannakian categories. The simplest example of a gerbe is given by an algebraic group G defined over k, the semi-direct product

$$\mathscr{G}_G = G(\overline{k}) \rtimes \operatorname{Gal}(\overline{k}/k).$$

Such a gerbe is called *neutral*.

We now introduce some gerbes over local fields, starting with the weight gerbe $\mathscr{W}$, which is defined over $\mathbb{R}$. It is an extension 119

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathscr{W} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1.$$

If $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \iota\}$, then $\mathscr{W}$ is generated by $\mathbb{C}^\times$ and $w = w(\iota)$, where

$$w(\iota)^2 = -1 \in \mathbb{C}^\times$$

and

$$wzw^{-1} = \iota(z) = \overline{z}, \quad z \in \mathbb{C}^\times.$$

Here, $\mathbb{C}^\times$ is to be understood as $\mathbb{G}_m(\mathbb{C})$.

Next, we introduce the Dieudonné gerbes. The actual Dieudonné gerbe will turn out to be a direct limit of such gerbes. Let K be a finite Galois extension of $\mathbb{Q}_p$. We define an extension as follows:

$$1 \longrightarrow K^\times \longrightarrow \mathscr{D}^K_K \longrightarrow \mathrm{Gal}(K/\mathbb{Q}_p) \longrightarrow 1.$$

It is generated by $K^\times$ and $d^K_K(\sigma)$, where $d^K_K(\sigma) \mapsto \sigma$, $d^K_K(\sigma)z = \sigma(z)d^K_K(\sigma)$ for $z \in K^\times$, and $d^K_K(\varrho)d^K_K(\sigma) = d^K_{\varrho,\sigma}d^K_K(\varrho\sigma)$. Here, $d^K_{\varrho,\sigma}$ is a 2-cocycle in the fundamental class of the extension $K/\mathbb{Q}_p$. The extension is uniquely determined up to isomorphism, and by Theorem 90, this isomorphism is unique up to conjugation by an element of $K^\times$.

We obtain the gerbe $\mathscr{D}^K$ by pulling back and pushing forward using the diagram

$$\begin{tikzcd}
 & & & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \arrow[d] & \\
1 \arrow[r] & K^\times \arrow[r] \arrow[d] & \mathscr{D}^K_K \arrow[r] & \mathrm{Gal}(K/\mathbb{Q}_p) \arrow[r] & 1 \\
 & \overline{\mathbb{Q}}_p^\times & & &
\end{tikzcd}$$

The group $\overline{\mathbb{Q}}_p^\times$ is again to be understood as $\mathbb{G}_m(\overline{\mathbb{Q}}_p)$.

We denote the representative of σ in $\mathscr{D}^K$ by $d^K_\sigma = d_\sigma$, and regard $d^K_{\varrho,\sigma}$ as a cocycle of the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $K \subset K'$. Then

$$(d^{K'}_{\varrho,\sigma})^k = d^K_{\varrho,\sigma} c_\varrho \varrho(c_\sigma) c^{-1}_{\varrho\sigma}, \quad k = [K' : K].$$

We can thus define a homomorphism $\mathscr{D}^{K'} \to \mathscr{D}^K$ by sending $z \in \overline{\mathbb{Q}}_p^\times$ to z^k and $d^{K'}_\varrho$ to $c^{-1}_\varrho d^K_\varrho$. By Theorem 90, this homomorphism is uniquely determined up to conjugation by an element of $\overline{\mathbb{Q}}_p^\times$.

Lemma 2.1. *Let $\phi : \mathscr{D}^K \to \mathscr{G}$ be a homomorphism of gerbes. Then there exists an unramified extension L of $\mathbb{Q}_p$, a homomorphism $\psi : \mathscr{D}^L \to \mathscr{G}$, and an extension K_1 containing both K and L, such that the following diagram commutes:* 120

$$\begin{tikzcd}
\mathscr{D}^{K_1} \arrow[r] \arrow[d] & \mathscr{D}^L \arrow[d, "\psi"] \\
\mathscr{D}^K \arrow[r, "\phi"'] & \mathscr{G}.
\end{tikzcd}$$

Let L be the unramified extension of $\mathbb{Q}_p$ whose degree over $\mathbb{Q}_p$ is equal to that of K, and let K_1 be the compositum of L and K. Since the two cocycles $d^K_{\varrho,\sigma}$ and $d^L_{\varrho,\sigma}$ are equivalent, $\mathscr{D}^K$ and $\mathscr{D}^L$ are isomorphic. The isomorphism is uniquely determined up to conjugation by an element of $\overline{\mathbb{Q}}_p^\times$. The existence of ψ is therefore clear.

It is clear that the procedure we used to define $\mathscr{D}^K$ can be applied over any local field. Over $\mathbb{R}$, it leads to the group $\mathscr{D}^{\mathbb{C}} = \mathscr{W}$, and we actually only need it over $\mathbb{R}$ and $\mathbb{Q}_p$. Nevertheless, for clarity, we will introduce the global gerbes of this section more generally than strictly necessary, and for this, we need the general $\mathscr{D}^K$'s. The trivial gerbe

$$1 \to \mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(\overline{k}/k) \to 1$$

is denoted by Gal_k, and when $k = \mathbb{Q}_l$, by $\mathscr{G}_l$.

Now let T be a torus over the finite number field F, which splits over the Galois extension L. As usual, let

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m), \quad X_*(T) = \mathrm{Hom}(X^*(T), \mathbb{Z}).$$

The element corresponding to $\mu \in X_*(T)$ in $\mathrm{Hom}(\mathbb{G}_m, T)$ is written as $x \mapsto x^\mu$. We fix two places v_1 and v_2 of F, which we extend to L. The extended places are denoted by v_1' and v_2'. Let ν_1 and ν_2 be two elements of $X_*(T)$ with the following properties:

(2.a) ν_i is invariant under $\mathrm{Gal}(L_{v_i'}/F_{v_i})$.

(2.b) We have

$$\sum_{\sigma \in \mathrm{Gal}(L/F)/\mathrm{Gal}(L_{v_1'}/F_{v_1})} \sigma\nu_1 + \sum_{\sigma \in \mathrm{Gal}(L/F)/\mathrm{Gal}(L_{v_2'}/F_{v_2})} \sigma\nu_2 = 0.$$

According to the Tate-Nakayama theory, ν_i corresponds to a class α_i in

$$H^2(\mathrm{Gal}(L_{v_i'}/F_{v_i}), T(L_{v_i})).$$

Due to (2.b), there also exists a global class in $H^2(\mathrm{Gal}(L/F), T(L))$ whose local components outside v_1 and v_2 are trivial and whose components at v_i are equal to α_i. This global class, though not uniquely defined, plays an important role in the construction of the gerbe $\mathscr{T}_{\nu_1,\nu_2}$ associated with T, ν_1, ν_2. It is now explicitly introduced using the Weil group.

121 The gerbe $\mathscr{T} = \mathscr{T}_{\nu_1,\nu_2}$ is a gerbe over F with kernel T, hence an extension

$$1 \to T(\overline{F}) \to \mathscr{T} \to \mathrm{Gal}(\overline{F}/F) \to 1.$$

To define it, we need a 2-cocycle $\{t_{\varrho,\sigma}\}$ of $\mathrm{Gal}(\overline{F}/F)$ with values in $T(\overline{F})$. Then $\mathscr{T}$ is generated by t_ϱ for $\varrho \in \mathrm{Gal}(\overline{F}/F)$ and $T(\overline{F})$, and

$$t_\varrho t_\sigma = t_{\varrho,\sigma} t_{\varrho\sigma}.$$

The localization $\mathscr{G}_v$ of $\mathscr{G}$ at the place v of F is a gerbe over F_v, which can be defined by the following diagram:

$$\begin{tikzcd}
 & & & \mathrm{Gal}(\overline{F}_v/F_v) \arrow[d] \\
1 \arrow[r] & G(\overline{F}) \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] & \mathrm{Gal}(\overline{F}/F) \\
 & G(\overline{F}_v) & &
\end{tikzcd}$$

A homomorphism ϕ from a gerbe $\mathscr{H}$ defined over F_v to $\mathscr{G}$ is a homomorphism from $\mathscr{H}$ to $\mathscr{G}_v$.

Usually, it is better to choose a finite extension K of F such that $\mathscr{G}$ can be defined over K, that is, by a diagram:

$$\begin{tikzcd}
 & & & \mathrm{Gal}(\overline{F}/F) \arrow[d] & \\
1 \arrow[r] & G(K) \arrow[r] \arrow[d] & \mathscr{G}_K \arrow[r] & \mathrm{Gal}(K/F) \arrow[r] & 1 \\
 & G(\overline{K}) & & &
\end{tikzcd}$$

If v_0' is an extension of v to K, then $\mathscr{G}_v$ can be defined over $K_{v_0'}$, and if K is sufficiently large, $\mathscr{H}$ is also defined over $K_{v_0'}$ and ϕ (is induced) by $\phi : \mathscr{H}_{K_{v_0'}} \to \mathscr{G}_{K_{v_0'}}$.

To avoid the choice of v_0', we can use the embedding

$$G(K) \to \prod_{v'|v} G(K_{v'})$$

to introduce an extension

$$1 \to \prod_{v'|v} G(K_{v'}) \to \mathscr{G}_v^* \to \mathrm{Gal}(K/F) \to 1.$$

In the case where G is defined over F and $g_\sigma g g_\sigma^{-1} = \sigma(g)$, and we are not interested 122
in any other, the extension $\mathscr{G}_v^*$ can be defined using $\mathscr{G}_v$ alone.

Indeed, to each v', we assign a $\mu = \mu_{v'} \in \mathrm{Gal}(K/F)$ such that $|\mu x|_{v'} = |x|_{v_0'}$ for $x \in K$. Then $\mu : x \mapsto \mu x$ extends to an isomorphism $\mu : K_{v_0'} \to K_{v'}$, which in turn defines $\mu : G(K_{v_0'}) \to G(K_{v'})$. Consequently,

$$\prod_{v'|v} G(K_{v'})$$

is an induced object for the group $\mathrm{Gal}(K/F)$. The cocycle $\{g_{\varrho,\sigma}\}$ takes values in this group and indeed in its center. We can introduce a second cocycle by writing

$$\sigma\mu = \mu'\sigma_{\mu'}, \quad \varrho\mu' = \mu''\varrho_{\mu''}, \quad \varrho_{\mu''}, \sigma_{\mu'} \in \mathrm{Gal}(K_{v_0'}/F_v)$$

and setting

$$g'_{\varrho,\sigma} = \prod_{\mu''} \mu'' g_{\varrho_{\mu''},\sigma_{\mu'}}.$$

Here, μ, μ', μ'' lie in $\{\mu_{v'}\}$, which is a system of representatives for $\mathrm{Gal}(K_{v'_0}/F_v)$ in $\mathrm{Gal}(K/F)$, and the product extends over this set. By the Shapiro lemma applied to the center, the two cocycles $\{g_{\varrho,\sigma}\}$ and $\{g'_{\varrho,\sigma}\}$ are equivalent.

We can apply the same procedure to $\mathscr{H}$ to obtain $\mathscr{H}^*$. Then $\phi : \mathscr{H} \to \mathscr{G}_v$ also defines $\phi^* : \mathscr{H}^* \to \mathscr{G}_v^*$. If we apply the Shapiro lemma again, this time to G, we immediately see that two homomorphisms ϕ, ϕ_1 are equivalent if and only if $\phi_1^* = \operatorname{ad} g \circ \phi^*$ with $g \in \prod_{v'|v} G(K_{v'})$.

These somewhat cumbersome considerations are presented because T is to be equipped with homomorphisms

$$\zeta_{v_i} : \mathscr{D}^{L_{v'_i}} \to \mathscr{T}, \quad i = 1, 2, \quad \zeta_v : \mathrm{Gal}_{F_v} \to \mathscr{T}, \quad v \neq v_i.$$

The restriction of ζ_{v_i} to the kernel is defined by $x \mapsto x^{\nu_i}$. Let $d^1_{\varrho,\sigma}$ and $d^2_{\varrho,\sigma}$ be the cocycles of $\mathrm{Gal}(\overline{F}/F)$ defined by $(d^{L_{v'_1}}_{\varrho,\sigma})^{\nu_1}$ and $(d^{L_{v'_2}}_{\varrho,\sigma})^{\nu_2}$, respectively, with values in $\prod_{v'|v'_i} T(L_{v'})$.

According to the preceding remarks, ζ_{v_i} and ζ_v will be defined if we can assign to each $\varrho \in \mathrm{Gal}(L/F)$ an element $e_\varrho \in T(\mathbb{A}_L)$ such that

$$e_\varrho \varrho(e_\sigma) e_{\varrho\sigma}^{-1} t_{\varrho,\sigma} = d^1_{\varrho,\sigma} d^2_{\varrho,\sigma}.$$

The mapping ζ_v^*, where v can be equal to v_i, is obtained by projecting e_ϱ onto $\prod_{v'|v} T(L_{v'})$ to get $e_\varrho(v)$, and then sending d_ϱ (if $v = v_i$) or ϱ (if $v \neq v_i$) to $e_\varrho(v) t_\varrho$.

123 To obtain the cocycle $\{t_{\varrho,\sigma}\}$ and the elements e_ϱ, we choose in the usual way [MS1] diagrams

$$\begin{array}{ccccccccc}
1 & \longrightarrow & L^\times_{v'_i} & \longrightarrow & W_{L_{v'_i}/F_{v_i}} & \longrightarrow & \mathrm{Gal}(L_{v'_i}/F_{v_i}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C_L & \longrightarrow & W_{L/F} & \longrightarrow & \mathrm{Gal}(L/F) & \longrightarrow & 1
\end{array}$$

We further choose, as in [MS1], systems of representatives $\mathfrak{S}_i$ of $\mathrm{Gal}(L/F)/\mathrm{Gal}(L_{v'_i}/F_{v_i})$ with $1 \in \mathfrak{S}_i$. Finally, we choose representatives $\omega_\sigma \in W_{L_{v'_i}/F_{v_i}}$ of $\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})$ and $\omega_\mu \in W_{L/F}$ of $\mu \in \mathfrak{S}_i$ with $\omega_1 = 1$ in both cases and set

$$\omega_{\mu\sigma} = \omega_\mu \omega_\sigma, \quad \mu \in \mathfrak{S}_i, \quad \sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i}).$$

These representatives, as well as the corresponding cocycle defined by the equations

$$\omega_\varrho \omega_\sigma = A_{\varrho,\sigma}(i) \omega_{\varrho\sigma}, \quad \varrho, \sigma \in \mathrm{Gal}(L/F),$$

depend on i. The following conditions hold:

(i) $A_{\varrho,\sigma}(i) = d^{L_{v'_i}}_{\varrho,\sigma}$, $\quad \varrho, \sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})$,
(ii) $A_{\mu,\sigma}(i) = 1$, $\quad \sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})$, $\quad \mu \in \mathfrak{S}_i$,
(iii) $A_{\mu\varrho,\sigma}(i) = \mu A_{\varrho,\sigma}(i) \in \mu(L_{v'_i})$, $\quad \varrho, \sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})$, $\quad \mu \in \mathfrak{S}_i$.

To define $d^i_{\varrho,\sigma}$, we need a system of representatives of $\mathrm{Gal}(L_{v'_i}/F_{v_i})$. We take the ω_σ.

Let

$$\eta = \sum_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{v_1'}/F_{v_1})} \sigma\nu_1 = -\sum_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{v_2'}/F_{v_2})} \sigma\nu_2.$$

The cocycles $\{A_{\varrho,\sigma}(1)\}$ and $\{A_{\varrho,\sigma}(2)\}$ are cohomologous. Let

$$A_{\varrho,\sigma}(1)A^{-1}_{\varrho,\sigma}(2) = B_\varrho \varrho(B_\sigma)B^{-1}_{\varrho\sigma},$$

with $B_\varrho \in C_L$. Then also

$$A^{\eta}_{\varrho,\sigma}(1)A^{-\eta}_{\varrho,\sigma}(2) = B^{\eta}_\varrho \varrho(B_\sigma)^{\eta}B^{-\eta}_{\varrho\sigma}. \tag{2.c}$$

We introduce the following elements of $C_L \otimes X_*(T)$:

$$C_\varrho(i) = \prod_{\mu\in\mathfrak{S}_i} A^{-\varrho\mu\nu_i}_{\varrho,\mu}(i); \quad E_\varrho = C_\varrho(1)C_\varrho(2)B^{\eta}_\varrho.$$

A simple calculation shows that

$$E_\varrho \varrho(E_\sigma)E^{-1}_{\varrho\sigma} = d^1_{\varrho,\sigma}d^2_{\varrho,\sigma}, \quad \varrho,\sigma \in \mathrm{Gal}(L/F). \tag{2.d}$$

This equation only makes sense when one notes that 124

$$\prod_{v'|v_i} T(L_{v'}) \hookrightarrow C_L \otimes X_*(T).$$

The coboundary of $C_\varrho(i)$ is

$$\left\{\prod_\mu A^{-\varrho\mu\nu_i}_{\varrho,\mu}\right\}\left\{\prod_\mu \varrho A^{-\varrho\sigma\mu\nu_i}_{\sigma,\mu} A^{\varrho\sigma\mu\nu_i}_{\varrho\sigma,\mu}\right\}$$

or

$$A^{\pm\eta}_{\varrho,\sigma}(i)\prod_\mu A^{-\varrho\mu\nu_i}_{\varrho,\mu}(i)A^{\varrho\mu\nu_i}_{\varrho,\mu\sigma_\mu}(i),$$

with the sign equal to $(-1)^i$. Furthermore,

$$\prod_\mu A^{-\varrho\mu\nu_i}_{\varrho,\mu}(i)A^{\varrho\mu\nu_i}_{\varrho,\mu\sigma_\mu}(i) = \prod_\mu A^{\varrho\mu\nu_i}_{\varrho\mu,\sigma_\mu}(i) = \prod_{\mu'} \mu' A^{\mu'\nu_i}_{\varrho_{\mu'},\sigma_\mu}(i) = d^i_{\varrho,\sigma}.$$

Thus, (2.d) follows from (2.c).

The elements $d^i_{\varrho,\sigma}$ already belong to $\prod_{v'|v_i} T(L_{v'}) \subset T(\mathbb{A}_L)$. We lift E_ϱ to $e_\varrho \in T(\mathbb{A}_L)$ and thus obtain from (2.d) the equations

$$e_\varrho \varrho(e_\sigma)e^{-1}_{\varrho\sigma}t_{\varrho,\sigma} = d^1_{\varrho,\sigma}d^2_{\varrho,\sigma}$$

with $t_{\varrho,\sigma} \in T(L)$.

It must now be checked that $\mathscr{T}$ is independent of all the choices made during its construction, with the additional local homomorphisms, noting that a change of e_ϱ by $f\varrho(f^{-1})e_\varrho$, $f \in T(\mathbb{A}_L)$, is allowed, as the local homomorphisms are then replaced by equivalent ones.

We first enumerate these choices:

(i) The liftings e_ϱ of E_ϱ;
(ii) The bounding cochain $\{B_\varrho\}$;
(iii) The system of representatives $\mathfrak{S}_i$ and the representatives ω_μ;
(iv) The embedding $W_{L_{v_i'}/F_{v_i}} \to W_{L/F}$ and the representatives ω_ϱ, $\varrho \in \mathrm{Gal}(L_{v_i'}/F_{v_i})$;
(v) The place v_i' dividing v_i.

(i) If we replace e_ϱ by $e'_\varrho = s_\varrho e_\varrho$, we obtain the cocycle

$$t'_{\varrho,\sigma} = s_\varrho^{-1}\varrho(s_\sigma)^{-1}s_{\varrho\sigma}t_{\varrho,\sigma}.$$

The homomorphism defined by $t'_\varrho \mapsto s_\varrho^{-1}t_\varrho$ transfers the primed local homomorphisms into the unprimed ones.

(ii) If we replace B_ϱ by $F\varrho(F^{-1})B_\varrho$, then E_ϱ is replaced by $E_\varrho F^\eta\varrho(F^{-\eta})$. We can thus replace e_ϱ by $e_\varrho f\varrho(f^{-1})$, where f is a lifting of F^η. Consequently, $\{t_{\varrho,\sigma}\}$ remains unchanged.

125 (iii) The representatives ω_μ, $\mu \in \mathfrak{S}_1$, can be replaced by $b_\mu\omega_\mu$. Then, in general, ω_σ is replaced by $b_\sigma\omega_\sigma$, where $b_{\mu\varrho} = b_\mu$, $\mu \in \mathfrak{S}_1$, $\varrho \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. We would then have

$$B'_\varrho = b_\varrho B_\varrho$$

instead of B_ϱ and

$$C'_\varrho(1) = C_\varrho(1)\cdot\prod_\mu b_\varrho^{-\varrho\mu\nu_1}\varrho(b_\mu)^{-\varrho\mu\nu_1}b_{\varrho\mu}^{\varrho\mu\nu_1} = C_\varrho(1)b_\varrho^{-\eta}F\varrho(F)^{-1}$$

with

$$F = \prod_\mu b_\mu^{\mu\nu_i},$$

since

$$b_{\varrho\mu}^{\varrho\mu\nu_i} = b_{\mu'}^{\mu'\nu_i},$$

when $\varrho\mu = \mu'\varrho_{\mu'}$, $\varrho_{\mu'} \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. Therefore, we can replace e_ϱ by $e_\varrho f\varrho(f^{-1})$ with f a lifting of F, and $t_{\varrho,\sigma}$ remains unchanged.

The system of representatives $\{\mu\}$ can be replaced by $\{\mu' = \mu\sigma(\mu)\}$, where $\sigma(\mu) \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. We choose $\omega'_{\mu'} = \omega_{\mu'} = \omega_\mu\omega_{\sigma(\mu)}$. Then, in general,

$$\omega'_{\mu'\sigma} = \omega_\mu\omega_{\sigma(\mu)}\omega_\sigma = \mu(A_{\sigma(\mu),\sigma}(1))\omega_{\mu'\sigma}, \quad \sigma \in \mathrm{Gal}(L_{v'_1}/F_{v_1}).$$

We set

$$b_{\mu'\sigma} = \mu A_{\sigma(\mu),\sigma}(1).$$

This lies in $\mu(L_{v'_i})$. The new cocycle is

$$A'_{\varrho,\sigma}(1) = \varrho(b_\sigma)b_\varrho b_{\varrho\sigma}^{-1}A_{\varrho,\sigma}(1).$$

Furthermore,

$$C'_\varrho(1) = \prod_{\mu'} A'_{\varrho,\mu'}(1)^{-\varrho\mu'\nu_1} = \left\{\prod_\mu A_{\varrho,\mu\sigma(\mu)}(1)^{-\varrho\mu\nu_1}\right\}\left\{\prod_\mu \left(b_\varrho b_{\varrho\mu'}^{-1}\varrho\mu(A_{\sigma(\mu),1})\right)^{-\varrho\mu\nu_1}\right\}.$$

The second factor equals

$$\prod(b_\varrho b_{\varrho\mu'}^{-1})^{-\varrho\mu\nu_1},$$

since $A_{\sigma(\mu),1} = 1$. The first factor is

$$C_\varrho(1)\cdot\prod_\mu A_{\varrho\mu,\sigma(\mu)}^{-\varrho\mu\nu_1}.$$

Consequently,

$$E'_\varrho = E_\varrho\left\{\prod_\mu A_{\varrho\mu,\sigma(\mu)}^{-\varrho\mu\nu_1}b_{\varrho\mu'}^{\varrho\mu\nu_1}\right\},$$

126 if $B'_\varrho = b_\varrho B_\varrho$. The second factor lies in $\prod_{v'|v_1} T(L_{v'})$ and is therefore automatically lifted.

Thus, $t'_{\varrho,\sigma}$ and $t_{\varrho,\sigma}$ differ only by an element of $\prod_{v'|v_1} T(L_{v'})$. Since $t_{\varrho,\sigma}$ and $t'_{\varrho,\sigma}$ lie in $T(L)$, we have $t_{\varrho,\sigma} = t'_{\varrho,\sigma}$. Similarly, e'_ϱ and e_ϱ differ only by an element of $\prod_{v'|v_1} T(L_{v'})$. Consequently, the local homomorphisms outside the place v_1 are the same. To show that they are also the same at v_1, it suffices to show that the projection of

$$\prod_\mu A_{\varrho\mu,\sigma(\mu)}^{-\varrho\mu\nu_1} b_{\varrho\mu}^{\varrho\mu\nu_1}$$

onto $T(L_{v'_1})$ is equal to 1 when $\varrho \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. The only term in the product contributing to this projection is the one corresponding to $\mu = 1$, and it is 1, since $\sigma(1) = 1$ and

$$A_{\varrho\mu,1} = 1, \quad b_\varrho = 1.$$

(iv) The embedding $\phi : W_{L_{v'_1}/F_{v'_1}} \to W_{L/F}$ can only be altered by replacing ϕ with $\mathrm{ad}\, x \circ \phi$, $x \in C_L$. We can thus replace ω_σ by $x\sigma(x^{-1})\omega_\sigma$ for each $\sigma \in \mathrm{Gal}(L/F)$. All cocycles remain unchanged.

If we replace ω_ϱ by $b_\varrho\omega_\varrho$, $b_\varrho \in L_{v'_1}^\times$, we can keep ω_μ unchanged, so that $A_{\varrho,\sigma}(1)$ is replaced by

$$A'_{\varrho,\sigma}(1) = A_{\varrho,\sigma}(1) b_\varrho \varrho(b_\sigma) b_{\varrho\sigma}^{-1},$$

where

$$b_{\mu\varrho} = \mu(b_\varrho), \quad \varrho \in \mathrm{Gal}(L_{v'_1}/F_{v_1}).$$

Since $\mu(b_\varrho) \in \mu(L_{v'_1})$, this change has no effect on $t_{\varrho,\sigma}$ or the local homomorphisms outside v_1. That this change also does not affect the equivalence class of ζ_{v_1} is easily checked by noting that the altered cocycle on $\mathrm{Gal}(L_{v'_1}/F_{v_1})$ also appears in the definition of $\mathscr{D}^{L_{v'_1}}$. The representatives ω_σ, $\sigma \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$, are replaced by $\omega'_\sigma = b_\sigma\omega_\sigma$, and the projection of e_σ onto $T(L_{v'_1})$ is multiplied by b_σ^η.

(v) The last arbitrariness to address is the choice of v'_1. Any other place v''_1 of L dividing v_1 is obtained by choosing $\lambda \in \mathfrak{S}_1$ and defining v''_1 by

$$|\lambda x|_{v''_1} = |x|_{v'_1}.$$

Then,

$$\mathrm{Gal}(L_{v''_1}/F_{v_1}) = \lambda \mathrm{Gal}(L_{v'_1}/F_{v_1})\lambda^{-1}.$$

Accordingly, we obtain the cocycles and objects corresponding to the place v''_1 by conjugating with λ. For example,

$$A'_{\varrho,\sigma}(1) = \lambda\left(A_{\lambda^{-1}\varrho\lambda,\lambda^{-1}\sigma\lambda}(1)\right).$$

This cocycle is cohomologous to the old one, and a simple calculation yields 127

$$A'_{\varrho,\sigma}(1) = A_{\varrho,\sigma}(1) D_\varrho \varrho(D_\sigma) D_{\varrho\sigma}^{-1}$$

with

$$D_\varrho = \lambda\left(A^{-1}_{\lambda^{-1},\varrho\lambda}(1)\right) A^{-1}_{\varrho,\lambda}(1) A_{\lambda,\lambda^{-1}}(1).$$

Furthermore, if we abbreviate $A'_{\varrho,\sigma}(1)$ as $A'_{\varrho,\sigma}$, then

$$C'_\varrho(1) = \prod_\mu (A'_{\varrho,\lambda\mu\lambda^{-1}})^{-\varrho\lambda\mu\nu_1},$$

since $\nu'_1 = \lambda\nu_1$. Let $\lambda\mu = \mu'\lambda_{\mu'}$ with $\mu' \in \mathfrak{S}_1$ and $\lambda_{\mu'} \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. Then,

$$A'_{\varrho,\lambda\mu\lambda^{-1}} = A'_{\varrho,\mu'\lambda_{\mu'}\lambda^{-1}} = \varrho(A'_{\mu',\lambda_{\mu'}\lambda^{-1}})^{-1} A'_{\varrho,\mu'} A'_{\varrho\mu',\lambda_{\mu'}\lambda^{-1}},$$

and

$$A'_{\varrho,\mu'} = A_{\varrho,\mu'}\,\varrho(D_{\mu'})D_\varrho D^{-1}_{\varrho\mu'}.$$

From this, we immediately conclude that E_ϱ and E'_ϱ differ by the factor

$$\text{(2.e)}\qquad \left\{\prod_\mu \varrho(A'_{\mu,\lambda_\mu\lambda^{-1}})^{\varrho\mu\nu_1}\right\}\left\{\prod_\mu A'_{\varrho\mu,\lambda_\mu\lambda^{-1}}\right\}^{-\varrho\mu\nu_1}\left\{\prod_\mu \varrho(D_\mu)^{-\varrho\mu\nu_1} D^{\varrho\mu\nu_1}_{\varrho\mu}\right\}$$

where we write μ instead of μ'.

We consider

$$\text{(2.f)}\qquad \prod_\mu \left(A'_{\varrho\mu,\lambda_\mu\lambda^{-1}} D^{-1}_{\varrho\mu}\right)^{\varrho\mu\nu_1},$$

which depends on ϱ and show that it can be expressed as a product

$$\varrho(X)YZ_\varrho,$$

where X and Y are independent of ϱ, and Z_ϱ lies in $\prod_{v'|v_1} T(L_{v'})$. Then, (2.e) equals

$$\varrho(Y)Y^{-1}\varrho(Z_1)Z_\varrho^{-1},$$

from which it immediately follows that $\{t_{\varrho,\sigma}\}$ is preserved, as well as the local homomorphisms outside v_1.

Instead of (2.f), we consider

$$\text{(2.g)}\qquad A'_{\varrho\mu,\lambda_\mu\lambda^{-1}} D^{-1}_{\varrho\mu},$$

and write it as

$$\varrho(X_\mu)Y_{\mu'}Z_{\mu,\varrho}.$$

128 Here, $\varrho\mu = \mu'\varrho_{\mu'}$, $\varrho_{\mu'} \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$. The expression (2.g) resembles

$$A_{\varrho\mu,\lambda_\mu\lambda^{-1}}\,\varrho\mu(D_{\lambda_\mu\lambda^{-1}})D^{-1}_{\varrho\mu\lambda_\mu\lambda^{-1}},$$

which we expand to obtain a product with seven terms:

$$A_{\varrho\mu,\lambda_\mu\lambda^{-1}}\,\varrho\mu\lambda(A^{-1}_{\lambda^{-1},\lambda_\mu})\varrho\mu(A^{-1}_{\lambda_\mu\lambda^{-1},\lambda})\varrho\mu(A_{\lambda,\lambda^{-1}})\lambda(A_{\lambda^{-1},\varrho\mu\lambda_\mu})A_{\varrho\mu\lambda_\mu\lambda^{-1},\lambda}A^{-1}_{\lambda,\lambda^{-1}}.$$

We set

$$X_\mu = \mu\lambda(A^{-1}_{\lambda^{-1},\lambda_\mu})\mu(A_{\lambda,\lambda^{-1}}).$$

From the entire product, we extract

$$A_{\varrho\mu,\lambda_\mu\lambda^{-1}}\,\varrho\mu(A^{-1}_{\lambda_\mu\lambda^{-1},\lambda})A_{\varrho\mu\lambda_\mu\lambda^{-1},\lambda} = A_{\varrho\mu,\lambda_\mu},$$

and add it to $Z_{\mu',\varrho}$. The term $A^{-1}_{\lambda,\lambda^{-1}}$ is added to $Y_{\mu'}$. What remains is

$$\lambda(A_{\lambda^{-1},\varrho\mu\lambda_\mu}) = \lambda(A_{\lambda^{-1},\mu'\varrho_{\mu'}\lambda_\mu}) = A^{-1}_{\mu',\varrho_{\mu'}\lambda_\mu}\lambda(A_{\lambda^{-1}\mu',\varrho_{\mu'}\lambda_\mu})\lambda(A_{\lambda^{-1},\mu'}).$$

From this, $Z_{\mu,\varrho}$ gets the product

$$A^{-1}_{\mu',\varrho_{\mu'}\lambda_\mu}\lambda(A_{\lambda^{-1}\mu',\varrho_{\mu'}\lambda_\mu}) = \lambda(A_{\lambda^{-1}\mu',\varrho_{\mu'}\lambda_\mu}),$$

and $Y_{\mu'}$ gets the term $\lambda(A_{\lambda^{-1},\mu'})$.

As already emphasized, to show the equivalence of the homomorphisms at v_1, we only need to consider the homomorphisms from $\mathscr{D}^{L_{v''_1}}$ to $\mathscr{T}$. The homomorphism defined by v''_1 corresponds to the cocycle $A'_{\varrho,\sigma}(1)$, $\varrho, \sigma \in \mathrm{Gal}(L_{v''_1}/F_{v_1})$. The one defined by v'_1 corresponds to the projection of $d^1_{\varrho,\sigma}$ onto $T(L_{v''_1})$, $\varrho, \sigma \in \mathrm{Gal}(L_{v''_1}/F_{v_1})$. Since $\varrho\lambda = \lambda\lambda^{-1}\varrho\lambda = \lambda\varrho_\lambda$, $\varrho \in \mathrm{Gal}(L_{v''_1}/F_{v_1})$, this projection is nothing other than

$A'_{\varrho,\sigma}(1)$. Up to a coboundary, for $\varrho \in \mathrm{Gal}(L_{v''_1}/F_{v_1})$, the projection of (2.e) onto $T(L_{v''_1})$ equals that of $\varrho(Z_1)Z_\varrho^{-1}$, which in turn equals

$$\varrho(A_{\lambda,1})\varrho\lambda(A_{1,1})A^{-1}_{\varrho\mu,1}\lambda(A^{-1}_{1,\varrho_\lambda}) = 1,$$

since $\lambda_\lambda = 1$ and $\varrho\lambda = \lambda\varrho_\lambda$.

There is one more compatibility to verify, which is even more troublesome than the previous ones. Namely, let $L' \supset L$ be a second Galois extension of F, and let w'_i be primes of L' dividing v'_i. Let

$$\nu'_i = [L'_{w'_i} : L_{v'_i}]\nu_i = l_i\nu_i.$$

We can introduce $\mathscr{T}_{\nu'_1,\nu'_2}$ along with additional local homomorphisms, and we must show that the thus equipped $\mathscr{T}_{\nu_1,\nu_2}$ and $\mathscr{T}_{\nu'_1,\nu'_2}$ are canonically isomorphic.

We simplify the notations as follows: 129

$$G = \mathrm{Gal}(L/F), \quad G' = \mathrm{Gal}(L'/F), \quad H = \mathrm{Gal}(L'/L),$$

$$G_i = \mathrm{Gal}(L_{v'_i}/F_{v_i}), \quad G'_i = \mathrm{Gal}(L'_{w'_i}/F_{v_i}), \quad H_i = \mathrm{Gal}(L'_{w'_i}/L_{v'_i}).$$

The group H_i is $H \cap G_i$, and we choose representatives λ of H_i in H. Then we choose representatives μ' of HG'_i in G'. As representatives of G'_i in G', we take the $\mu'\lambda$. Finally, we take the projections μ of μ' onto G as representatives of G_i in G. In the Weil group $W_{L'/F}$, we choose $\omega_{\mu'\lambda} = \omega_{\mu'} \cdot \omega_\lambda$ with $\omega_\lambda \in W_{L'/L}$. We also choose right representatives ν of H_i in G'_i and set

$$\omega_{\gamma\nu} = \omega_\gamma\omega_\nu, \quad \gamma \in H_i.$$

Then we have

$$\omega_{\lambda\cdot\gamma}\omega_\nu := \omega_\lambda \cdot \omega_\gamma \cdot \omega_\nu = \omega_{\lambda\gamma\nu},$$

so that

$$A_{\gamma,\nu} = 1,$$

when γ is in H and ν is contained in the right representative system.

Let

$$D_{\varrho'}(i) = \prod_{\gamma \in H} A_{\varrho',\gamma}(i).$$

We consider the cocycle

$$A'_{\varrho',\sigma'}(i) = A^l_{\varrho',\sigma'}(i)D^{-1}_{\varrho'}(i)\varrho'(D^{-1}_{\sigma'}(i))D_{\varrho'\sigma'}(i)$$

with $l = [H : 1]$. It is easy to verify that it equals

$$\prod_\gamma A^{-1}_{\varrho',\gamma}A_{\varrho',\sigma'\gamma}.$$

Consequently, it depends only on the projection σ of σ' onto G and is 1 if $\sigma' \in H$.

We first show that

$$A'_{\varrho',\sigma'}(i) = 1, \tag{2.h}$$

when $\varrho' \in H$ and $\sigma' \in HG'_i$. The element $\sigma'\gamma$ can be written as $\delta\nu$ with $\delta \in H$ and ν from the right representative system. Then we have

$$A'_{\varrho',\sigma'\gamma}(i) = \varrho'(A'_{\delta,\nu}(i))^{-1}A'_{\varrho'\delta,\nu}(i)A'_{\varrho',\delta}(i) = A'_{\varrho',\delta}(i).$$

Consequently,

$$\prod_\gamma A'_{\varrho',\sigma'\gamma}(i) = \prod_\delta A'_{\varrho',\delta}(i).$$

Since δ also runs through H, (2.h) holds.

130 From the cocycle condition, we conclude that $A'_{\varrho',\sigma'}(i)$ depends only on the projections ϱ, σ and is contained in C_L if ϱ', σ' lie in HG'_i. Indeed, it lies in $L_{v'_i} = C_L \cap \prod_{w'|v'_i} L'_{w'}$, since

$$A'_{\varrho',\sigma'}(i) = \prod_{\gamma} A^{-1}_{\varrho',\gamma} A_{\varrho',\sigma'\gamma} = \prod_{\gamma} A^{-1}_{\varrho',\gamma} A_{\varrho',\gamma\nu}$$

with $\nu = \nu(\sigma)$. The right hand side is itself equal to

$$\prod_{\gamma} \varrho'(A^{-1}_{\gamma,\nu}) A_{\varrho'\gamma,\nu} = \prod_{\gamma} A_{\varrho'\gamma,\nu},$$

and $\varrho'\gamma \in HG'_i$.

Since the cocycles $A^l_{\varrho',\sigma'}$ and $A_{\varrho,\sigma}$ are cohomologous anyway, we can choose $A_{\varrho,\sigma}$ such that

$$A_{\varrho,\sigma} = A'_{\varrho',\sigma'}, \quad \varrho', \sigma' \in HG'_i.$$

In general, there exists a cochain $B_{\varrho'}(i)$ such that

$$A^l_{\varrho',\sigma'}(i) = A_{\varrho,\sigma}(i) B_{\varrho'}(i) \varrho'(B_{\sigma'}(i)) B^{-1}_{\varrho'\sigma'}(i).$$

The product $B_{\varrho'}(i) D^{-1}_{\varrho'}(i)$ is a cocycle on HG'_i. Consequently,

$$B_{\varrho'}(i) = D_{\varrho'}(i) F' \varrho'(F')^{-1}, \quad F' \in C_{L'}.$$

Since we can replace $B_{\varrho'}(i)$, $\varrho' \in G'$, by

$$B_{\varrho'}(i) F'^{-1} \varrho'(F'),$$

we may assume that

$$B_{\varrho'}(i) = D_{\varrho'}(i), \quad \varrho' \in HG'_i.$$

After carefully and appropriately choosing the occurring cocycles and cochains, we will now be able to verify the desired compatibility without too much effort.

We have

$$\sum_{G'/G'_1} \sigma\nu'_1 = \eta' = l\eta.$$

If

$$A_{\varrho',\sigma'}(1) A^{-1}_{\varrho',\sigma'}(2) = B_{\varrho'} \varrho'(B_{\sigma'}) B^{-1}_{\varrho'\sigma'},$$

then the coboundary of $B^l_{\varrho'}$ is equal to that of $B_\varrho B_{\varrho'}(1) B^{-1}_{\varrho'}(2)$. Consequently, it follows that

$$B^l_{\varrho'} = B_\varrho B_{\varrho'}(1) B^{-1}_{\varrho'}(2) F' \varrho(F')^{-1}, F' \in C_{L'}.$$

131 We first compute $C_{\varrho'}(i)$, where we sometimes omit the i:

$$\begin{aligned} C_{\varrho'}(i) &= \prod_{\mu'} \prod_{\lambda} A^{-\varrho\mu'\lambda\nu'_i}_{\varrho',\mu'\lambda} \\ &= \prod_{\mu'} \prod_{\lambda} \left\{ \varrho'(A^{\varrho\mu'\lambda\nu'_i}_{\mu',\lambda}) A^{-\varrho\mu'\lambda\nu'_i}_{\varrho'\mu',\lambda} A^{-\varrho\mu'\nu'_i}_{\varrho',\mu'} \right\} \\ &= \prod_{\mu',\lambda} \left\{ A^{-\varrho\mu'\lambda\nu'_i}_{\varrho'\mu',\lambda} A^{-\varrho\mu'\nu'_i}_{\varrho',\mu'} \right\}. \end{aligned}$$

We expand

$$\prod_{\mu',\lambda} A^{-\varrho\mu'\nu'_i}_{\varrho',\mu'}$$

as

$$\left\{\prod_{\mu} A_{\varrho,\mu}^{-\varrho\mu\nu_i}\right\}\left\{\prod_{\mu'} B_{\varrho'}(i)^{-\varrho\mu'\nu_i}\varrho'(B_{\mu'}(i))^{-\varrho\mu'\nu_i}B_{\varrho'\mu'}(i)^{\varrho\mu'\nu_i}\right\}$$

and note that

$$\prod_{\mu'} B_{\varrho'}(i)^{-\varrho\mu'\nu_i} = B_{\varrho'}(i)^{(-1)^i\eta}.$$

From these equations, we conclude that $E_{\varrho'}$ is the product of E_ϱ, a coboundary, and the expressions

$$\text{(2.i)} \qquad \left\{\prod_{\mu',\lambda} A_{\varrho'\mu',\lambda}^{-\varrho\mu'\nu_i'}\right\}\left\{\prod_{\mu'} \varrho(B_{\mu'}(i))^{-\varrho\mu'\nu_i}B_{\varrho'\mu'}(i)^{\varrho\mu'\nu)i}\right\}, \quad i=1,2.$$

We multiply this expression by the coboundary

$$\varrho\Bigg(\prod_{\mu'} B_{\mu'}(i)^{\mu'\nu_i}\Bigg)\prod_{\mu'} B_{\mu'}(i)^{-\mu'\nu_i}$$

and obtain

$$\left\{\prod_{\mu',\lambda} A_{\varrho'\mu',\lambda}^{-\varrho\mu'\nu_i'}\right\}\left\{\prod_{\mu'} B_{\varrho'\mu'}(i)^{\varrho\mu'\nu_i}B_{\mu'}(i)^{-\mu'\nu_i}\right\}.$$

Let $\varrho'\mu' = \mu_1'\varrho'_{\mu_1'}$, $\varrho'_{\mu_1'} \in HG_i'$. We have

$$B_{\varrho'\mu'}(i)B_{\mu_1'}(i)^{-1} = A_{\mu_1',\varrho'_{\mu_1'}}A^{-1}_{\mu_1',\varrho'_{\mu_1'}}\mu_1'(B_{\varrho'_{\mu_1'}}(i)) = \mu_1'(B_{\varrho'_{\mu_1'}}(i))$$

and

$$A_{\mu_1'\varrho'_{\mu_1'},\lambda}(i) = \mu_1'\big(A_{\varrho'_{\mu_1'},\lambda}(i)\big)A_{\mu_1',\varrho_{\mu_1'}\lambda}(i)A^{-1}_{\mu_1',\varrho_{\mu_1'}}(i) = \mu_1'\big(A_{\varrho'_{\mu_1'},\lambda}(i)\big).$$

Consequently, (2.i) is replaced by

$$\text{(2.j)} \qquad \left\{\prod_{\mu',\lambda} \mu'\big(A_{\varrho'_{\mu'},\lambda}(i)\big)^{-\mu'\nu_1'}\right\}\left\{\prod_{\mu'} \mu'\big(B_{\varrho'_{\mu'}}(i)\big)^{\mu'\nu_1}\right\}.$$

We write μ' instead of μ_1' to simplify the notation. The element $\varrho'_{\mu'}$ lies in HG_i', so 132

$$\begin{aligned} B_{\varrho'_{\mu'}}(i) &= \prod_{\gamma\in H} A_{\varrho'_{\mu'},\gamma}(i) \\ &= \prod_{\lambda}\prod_{\gamma\in H_i} A_{\varrho'_{\mu'},\lambda\gamma}(i) \\ &= \prod_{\lambda}\prod_{\gamma} A_{\varrho'_{\mu'}\lambda,\gamma}(i)A_{\varrho'_{\mu'},\lambda}(i). \end{aligned}$$

Since $[H_i : 1] = l_i$ and $\nu_i' = l_i\nu_i$, we conclude that (2.j) is equal to

$$\text{(2.k)} \qquad \prod_{\mu',\lambda,\gamma} \mu'(A_{\varrho'_{\mu'}\lambda,\gamma})^{\mu'\nu_i},$$

which lies in $\prod_{w'|v_i} T(L'_{w'})$. This immediately yields the equation

$$\text{(2.l)} \qquad t_{\varrho',\sigma'} = t_{\varrho,\sigma}, \quad \varrho',\sigma' \in \mathrm{Gal}(L'/F),$$

since $t_{\varrho,\sigma}^{-1}t_{\varrho',\sigma'}$ on one hand lies in $T(L')$ and on the other hand lies in

$$\prod_{w'|v_1} T(L'_{w'}) \prod_{w'|v_2} T(L'_{w'}).$$

It is also clear that the local homomorphisms outside v_1 and v_2 are equivalent. We also want to show that, for example,

$$\begin{tikzcd} \mathscr{D}^{L'_{w'_1}} \arrow[r] \arrow[d] & \mathscr{T}'_{\nu'_1,\nu'_2} \arrow[d] \\ \mathscr{D}^{L_{v'_1}} \arrow[r] & \mathscr{T}_{\nu_1,\nu_2} \end{tikzcd}$$

is commutative. On the right is the isomorphism resulting from (2.e). The commutativity on the kernel of $\mathscr{D}^{L'_{w'_1}}$ is clear. What else needs to be considered is the projection of (2.k) onto $T(L'_{w'_1})$, namely

$$\prod_{\gamma\in H_1} A^{\nu_1}_{\varrho',\gamma}(1) = G^{\nu_1}_{\varrho'}.$$

Therefore, it suffices to show the following equation:

$$A'_{\varrho',\sigma'}(1) = A^{l_1}_{\varrho',\sigma'}(1)G^{-1}_{\varrho'}\varrho'(G_{\sigma'})^{-1}G_{\varrho'\sigma'}, \quad \varrho',\sigma' \in G'_1,$$

where both expressions are contained in the group $L'^{\times}_{w'_1}$.

133 We have already seen that

$$\begin{aligned} A'_{\varrho',\sigma'}(1) &= \prod_{\gamma\in H} A^{-1}_{\varrho',\gamma}(1)A_{\varrho',\sigma'\gamma}(1) \\ &= \prod_{\gamma} A^{-1}_{\varrho',\gamma}(1)A_{\varrho',\gamma\nu}(1), \end{aligned}$$

where ν is the representative of σ' in the right representative system. We have also seen that the product on the right hand side is equal to $\prod_\gamma A_{\varrho'\gamma,\nu}(1)$. We are interested in this product as an element of $L'_{w'_1}$, so we can project it onto the corresponding coordinate and obtain

$$\prod_{\gamma\in H_1} A_{\varrho'\gamma,\nu}(1).$$

This product, in turn, is equal to

$$\prod_{\gamma\in H_1} A^{-1}_{\varrho',\sigma'}(1)A_{\varrho',\sigma'\gamma}(1) = A^{l_1}_{\varrho',\sigma'}(1)G^{-1}_{\varrho'}\varrho'(G^{-1}_{\sigma'})G_{\varrho'\sigma'}.$$

The construction of $\mathscr{T}_{\nu_1,\nu_2}$ is obviously functorial in T, ν_1, and ν_2. Namely, if we have a homomorphism $\phi : T \to T'$ defined over F and $\nu'_i = \phi(\nu_i)$ for $i = 1, 2$, then we immediately obtain a homomorphism ϕ from $\mathscr{T}_{\nu_1,\nu_2}$ to $\mathscr{T}_{\nu'_1,\nu'_2}$.

We formulate some of the results obtained above in a theorem.

Theorem 2.2. *Let T be a torus defined over F that splits over the Galois extension L of F. Let v_1 and v_2 be two places of F, and let v'_1 and v'_2 be chosen extensions to L. Let two cocharacters ν_1 and ν_2 in $X_*(T)$ be given, satisfying conditions* (2.a) *and* (2.b). *Then the above construction defines a gerbe* $\mathscr{T} = \mathscr{T}_{\nu_1,\nu_2}$ *over* F *with kernel* T*, equipped with homomorphisms*

$$\zeta_v : \mathrm{Gal}_{F_v} \longrightarrow \mathscr{T}, \quad v \neq v_1, v_2,$$
$$\zeta_{v_i} : \mathscr{D}^{L_{v'_i}} \longrightarrow \mathscr{T}, \quad i = 1, 2,$$

such that ζ_{v_i} *on the kernel is given by* $x \mapsto x^{\nu_i}$. *This gerbe is uniquely determined up to an isomorphism that transforms the local homomorphisms into equivalent ones. If* $L' \supset L$ *is a larger Galois extension to which the places* v'_i *are extended to* w'_i*, then the above construction yields an isomorphism*

$$\mathscr{T}'_{\nu'_1,\nu'_2} \xrightarrow{\sim} \mathscr{T}_{\nu_1,\nu_2},$$

where 134

$$\nu'_i = [L'_{w'_i} : L_{v'_i}] \cdot \nu_i \quad \textit{for} \quad i = 1, 2,$$

such that the following two diagrams commute up to equivalence:

$$\begin{tikzcd} & \mathscr{T}'_{\nu'_1,\nu'_2} \arrow[dd, "\sim"] \\ \mathrm{Gal}_{F_v} \arrow[ur, "\zeta'_v"] \arrow[dr, "\zeta_v"'] & \\ & \mathscr{T}_{\nu_1,\nu_2} \end{tikzcd} \qquad \textit{for } v \neq v_1, v_2,$$

$$\begin{tikzcd} \mathscr{D}^{L'_{w'_i}} \arrow[r] \arrow[d] & \mathscr{T}'_{\nu'_1,\nu'_2} \arrow[d, "\sim"] \\ \mathscr{D}^{L_{v'_i}} \arrow[r] & \mathscr{T}_{\nu_1,\nu_2}. \end{tikzcd}$$

The construction is functorial in T, ν_1, *and* ν_2.

One way to achieve the above situation is for ν_1 and ν_2 to arise from averaging the same cocharacter. Let T be a torus over F, and let μ be an element of $X_*(T)$. We set

$$\nu_i = (-1)^{i+1} \sum_{\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})} \sigma\mu.$$

We will introduce a canonical homomorphism τ_μ from the corresponding gerbe $\mathscr{T}_{\nu_1,\nu_2}$ to the neutral gerbe $\mathscr{G}_T$.

Before doing so, we define in a very simple way a homomorphism ξ_μ from $\mathscr{D}^{L_{v'}}$ to $\mathscr{G}_T$. The field L splits T, and v' is a prime of L dividing a prime v of F. Let

$$\nu = \sum_{\sigma \in \mathrm{Gal}(L_{v'}/F_v)} \sigma\mu.$$

The homomorphism ξ_μ is defined as follows:

$$\xi_\mu(z) = z^\nu, \quad z \in L_{v'}^\times,$$
$$\xi_\mu(d_\varrho) = \prod_{\sigma \in \mathrm{Gal}(L_{v'}/F_v)} d_{\varrho,\sigma}^{\varrho\sigma\mu} \rtimes \varrho,$$

where $d_{\varrho,\sigma}$ is the cocycle defining $\mathscr{D}^{L_{v'}}$, and $d_\varrho = d_\varrho^{L_{v'}}$. It is easy to verify that ξ_μ is indeed a homomorphism.

The homomorphism τ_μ to be introduced has the following properties:

(i) $\tau_\mu \circ \zeta_{v_1}$ is equivalent to ξ_μ;
(ii) $\tau_\mu \circ \zeta_{v_2}$ is equivalent to $\xi_{-\mu}$;
(iii) $\tau_\mu \circ \zeta_v$ is the canonical neutralization of $\mathscr{G}_T$.

The homomorphism τ_μ is defined by

$$\tau_\mu : t_\varrho \longmapsto s_\varrho \rtimes \varrho,$$

135 where $s_\varrho \in T(L)$ and

$$s_\varrho \varrho(s_\sigma) s_{\varrho\sigma}^{-1} = t_{\varrho,\sigma}.$$

Furthermore, there exists $f \in T(\mathbb{A}_L)$ such that

$$f e_\varrho s_\varrho \varrho(f^{-1}) = e'_\varrho$$

satisfies the following properties:

(i) the projection of e'_ϱ onto $L_{v'}$ is 1 if v' divides neither v_1 nor v_2;
(ii) the projection of e'_ϱ onto $T(L_{v'_i})$ is

$$\prod_{\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})} A_{\varrho,\sigma}^{(-1)^{i+1}\varrho\sigma\mu}(i).$$

Let $\{B_\varrho\}$ be the cocycle appearing in (2.c). We set

$$F = \prod_{\varrho \in \mathrm{Gal}(L/F)} B_\varrho^{-\varrho\mu},$$

$$E_\varrho(i) = \prod_{\tau \in \mathfrak{S}_i} \prod_{\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})} A_{\varrho\tau,\sigma}^{(-1)^{i+1}\varrho\tau\sigma\mu}(i),$$

so

$$E_\varrho(i) \in \prod_{v'|v_i} T(L_{v'}).$$

A straightforward calculation yields the equation

$$E_\varrho = E_\varrho(1) E_\varrho(2) F \varrho(F^{-1}). \tag{2.m}$$

Indeed, for $\tau \in \mathfrak{S}_i$ and $\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})$, we have

$$A_{\varrho,\tau\sigma}(i) = A_{\varrho\tau,\sigma}(i) A_{\varrho,\tau}(i),$$

so $C_\varrho(i)$ is the product of

$$\prod_{\sigma \in \mathrm{Gal}(L/F)} A_{\varrho,\sigma}^{(-1)^{i}\varrho\sigma\mu}(i)$$

and

$$\prod_{\tau \in \mathfrak{S}_i} \prod_{\sigma \in \mathrm{Gal}(L_{v'_i}/F_{v_i})} A_{\varrho\tau,\sigma}^{(-1)^{i+1}\varrho\tau\sigma\mu} = E_\varrho(i).$$

Consequently,

$$\begin{aligned} E_\varrho &= \left\{ \prod_\sigma A_{\varrho,\sigma}^{-\varrho\sigma\mu}(1) A_{\varrho,\sigma}^{\varrho\sigma\mu}(2) \right\} \left\{ B_\varrho^\eta E_\varrho(1) E_\varrho(2) \right\} \\ &= \left\{ \prod_\sigma \left(B_\varrho \varrho(B_\sigma) B_{\varrho\sigma}^{-1} \right)^{-\varrho\sigma\mu} \right\} \left\{ B_\varrho^\eta E_\varrho(1) E_\varrho(2) \right\} \\ &= F^{-1} \varrho(F) E_\varrho(1) E_\varrho(2). \end{aligned}$$

The elements $E_\varrho(i)$ have already been lifted to $T(\mathbb{A}_L)$. We lift F to an element 136
f and obtain

$$s_\varrho e_\varrho f \varrho(f^{-1}) = E_\varrho(1) E_\varrho(2)$$

with $s_\varrho \in T(L)$. Thus,

$$e'_\varrho = E_\varrho(1) E_\varrho(2),$$

so e'_ϱ clearly possesses the desired properties.

It must be checked once again whether s_ϱ is independent of the choices made in its construction, up to a coboundary. These choices are enumerated as (i) to (v) below. The possibility mentioned under (v) requires special attention.

(i) If we take a lift $e''_\varrho = r_\varrho e_\varrho$ with $r_\varrho \in T(L)$, then $t_{\varrho,\sigma}$ is modified to $t''_{\varrho,\sigma} = r_\varrho^{-1} \varrho(r_\sigma^{-1}) r_{\varrho\sigma} t_{\varrho,\sigma}$, and the isomorphism between the two resulting gerbes is defined by

$$t''_\varrho \longmapsto r_\varrho^{-1} t_\varrho.$$

Since s_ϱ is replaced by $r_\varrho^{-1} s_\varrho$, t''_ϱ maps to $r_\varrho^{-1} s_\varrho \rtimes \varrho$ in the neutral gerbe $\mathscr{G}_T$, which is also the image of $r_\varrho^{-1} t_\varrho$.

(ii) If we replace B_ϱ by $B_\varrho G \varrho(G^{-1})$, then F is multiplied by

$$\prod_\varrho G^{-\varrho\mu} \varrho(G)^{\varrho\mu}.$$

If we lift G to g, we can choose

$$f' = f \prod_\varrho g^{-\varrho\mu} \varrho(g)^{\varrho\mu}$$

instead of f, and

$$f' \varrho(f')^{-1} = f \varrho(f^{-1}) g^{-\eta} \varrho(g^\eta).$$

We have already seen that e_ϱ is replaced by $e_\varrho g^\eta \varrho(g^{-\eta})$. Thus, $\{s_\varrho\}$ remains unchanged.

(iii) A similar calculation shows compatibility under changes of the representatives ω_τ for $\tau \in \mathfrak{S}_i$. If we change the system of representatives $\mathfrak{S}_i$ itself, then B_ϱ is multiplied by an element of $\prod_{v'|v_i} L_{v'}^\times$. Consequently, neither $t_{\varrho,\sigma}$ nor s_ϱ is changed.

(iv) A change in the embedding $W_{L_{v'_1}/F_{v_1}} \to W_{L/F}$ does not alter the cocycles and thus does not change $\{s_\varrho\}$. A change in ω_ϱ for $\varrho \in \mathrm{Gal}(L_{v'_1}/F_{v_1})$ multiplies B_ϱ by an element of $\prod_{v'|v_1} T(L_{v'})$ and causes no change in $\{s_\varrho\}$.

(v) If we change v'_1 and replace it with v''_1, 137

$$|\lambda x|_{v''_1} = |x|_{v'_1},$$

then

$$\sum_{\sigma \in \mathrm{Gal}(L_{v_1''}/F_{v_1})} \sigma\mu = \lambda \sum_{\sigma \in \mathrm{Gal}(L_{v_1'}/F_{v_1})} \sigma\lambda^{-1}\mu,$$

and in general, this will only equal $\lambda\nu_1$ if λ can be chosen such that $\lambda^{-1}\mu = \mu$. Therefore, we assume this condition. However, even in this case, τ_μ is not independent of v_1''. Indeed, let $\widetilde{A}_{\varrho,\sigma}$ be liftings of $A_{\varrho,\sigma}$, and let $C_{\varrho,\sigma,\tau}$ be the Teichmüller cocycle defined by

$$\varrho(\widetilde{A}_{\sigma,\tau})\widetilde{A}^{-1}_{\varrho\sigma,\tau}\widetilde{A}_{\varrho,\sigma\tau}\widetilde{A}^{-1}_{\varrho,\sigma} = C_{\varrho,\sigma,\tau}.$$

Then

$$r_\varrho = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} C^{-\varrho\sigma\mu}_{\varrho,\sigma,\lambda}$$

is a 1-cocycle of $\mathrm{Gal}(L/\mathbb{Q})$ with values in $T(L)$, and τ_μ is replaced by $\tau'_\mu : t_\varrho \mapsto r_\varrho\tau_\mu(t_\varrho)$ when v_1' is replaced by v_1''.

Indeed, E_ϱ is multiplied by the expression (2.e), which is equal to

$$\varrho(Y)Y^{-1}\varrho(Z_1)Z_\varrho^{-1}.$$

On the other hand, F is multiplied by

$$U = \prod_{\sigma \in \mathrm{Gal}(L/F)} D_\sigma^{-\sigma\mu}.$$

By definition,

$$U = \prod_\sigma \lambda(A_{\lambda^{-1},\sigma\lambda})^{\sigma\mu} A^{\sigma\mu}_{\sigma,\lambda} A^{-\sigma\mu}_{\lambda,\lambda^{-1}}$$

and

$$Y = \prod_{\tau \in \mathfrak{S}_1} A^{-\tau\nu_1}_{\lambda,\lambda^{-1}} \prod_{\tau \in \mathfrak{S}_1} \lambda(A_{\lambda^{-1},\tau})^{\tau\nu_1}.$$

Consequently,

$$UY^{-1} = \prod_\sigma \Big(\lambda(A_{\lambda^{-1},\sigma\lambda})^{\sigma\mu} A^{\sigma\mu}_{\sigma,\lambda}\Big) \prod_\tau \lambda(A_{\lambda^{-1},\tau})^{\tau\nu_1}.$$

Due to the cocycle property, the first product on the right hand side is equal to

$$\prod_\sigma \lambda(A_{\lambda^{-1}\sigma,\lambda}A_{\lambda^{-1},\sigma})^{\sigma\mu} = \prod_\sigma A^{\sigma\mu}_{\sigma,\lambda} \prod_\sigma \lambda(A_{\lambda^{-1},\sigma})^{\sigma\mu}.$$

138 If σ lies in $\tau\mathrm{Gal}(L_{v_1'}/F_{v_1})$, then

$$\lambda(A_{\lambda^{-1},\sigma}) \equiv \lambda(A_{\lambda^{-1},\tau}) \mod \tau(L_{v_1'}).$$

We can therefore lift UY^{-1} to the product of $\prod_\sigma \widetilde{A}^{\sigma\mu}_{\sigma,\lambda}$ and an element of $\prod_{v'|v_1} T(L_{v'})$. Furthermore,

$$\prod_\sigma \widetilde{A}^{\sigma\mu}_{\sigma,\lambda} \prod_\sigma \varrho(\widetilde{A}_{\sigma,\lambda})^{-\varrho\sigma\mu} = \prod_\sigma C^{-\varrho\sigma\mu}_{\varrho,\sigma,\lambda} = r_\varrho,$$

since

$$\prod_\sigma \widetilde{A}^{\varrho\sigma\mu}_{\varrho,\sigma\lambda}\widetilde{A}^{-\varrho\sigma\mu}_{\varrho,\sigma} = 1.$$

Thus, $s'_\varrho = r_\varrho s_\varrho$.

Finally, we want to show that τ_μ is independent of L. Let $L' \supset L$ be a Galois extension of F. We reuse the earlier formulas. To avoid using the same symbol with two meanings, we write

$$B^l_{\varrho'} = B_\varrho B_{\varrho'}(1) B^{-1}_{\varrho'}(2) G\varrho(G)^{-1}.$$

From the earlier calculations, it follows that $E_{\varrho'}$ is the product of E_ϱ and two other factors. One of these factors is irrelevant for $s_{\varrho'}$ because it lies in

$$\prod_{v'|v_1} T(L_{v'}) \prod_{v'|v_2} T(L_{v'}).$$

The other factor is

$$G^\eta \varrho(G)^{-\eta} \prod_{i=1}^{2} \left\{ \prod_{\mu'} B_{\mu'}(i)^{\mu'\nu_i} \prod_{\mu'} \varrho(B_{\mu'}(i))^{-\varrho\mu'\nu_i} \right\}.$$

On the other hand,

$$F' = \prod_{\varrho' \in \mathrm{Gal}(L'/F)} B^{-\varrho'\mu}_{\varrho'}.$$

We show that

$$F'F^{-1}G^\eta \prod_{i=1}^{2} \prod_{\mu'} B_{\mu'}(i)^{\mu'\nu_i} \equiv V \quad \text{modulo} \quad \prod_{v'|v_1,v_2} T(L_{v'}),$$

where V is not only invariant but actually the image of an element in $T(\mathbb{A}_F)$. Thus, $s_{\varrho'} = s_\varrho$. Indeed, we show that

$$W = \left(\prod_{\gamma \in H} B^{-1}_\gamma \right) G$$

lies in C_L, and

$$V = \prod_{\varrho \in \mathrm{Gal}(L/\mathbb{Q})} \varrho(W)^{\varrho\mu},$$

which suffices, since $\mathbb{A}_L \to C_L$ is surjective.

Let $\varrho' \in \mathrm{Gal}(L'/F)$, and let μ'_i be its representative modulo HG'_i. It must be shown that 139

$$\text{(2.n)} \qquad \varrho'^{-1} \left(\prod_{\gamma \in H} B^{-1}_{\varrho'\gamma} B_\varrho G B_{\mu'_1}(1) B^{-1}_{\mu'_2}(2) \right)$$

is congruent to

$$\text{(2.o)} \qquad \prod_{\gamma \in H} B^{-1}_\gamma G,$$

with equality if $\varrho' \in H$.

First, we have

$$\varrho'^{-1} \left(\prod_{\gamma \in H} B^{-1}_{\varrho'\gamma} \right) = \prod_\gamma B_{\sigma'} \cdot \prod_\gamma B^{-1}_\gamma \cdot A^{-1}_{\sigma',\varrho'\gamma}(1) A_{\sigma',\varrho'\gamma}(2)$$
$$= B^l_{\sigma'} \prod_\gamma \left(A^{-1}_{\sigma',\varrho'\gamma}(1) A_{\sigma',\varrho'\gamma}(2) \right) \prod_\gamma B^{-1}_\gamma,$$

where $\sigma' = \varrho'^{-1}$. Thus, (2.n) is equal to

$$B_{\sigma'}(1)B_{\sigma'}^{-1}(2)\prod_{\gamma}\left(A^{-1}_{\sigma',\varrho'\gamma}(1)A_{\sigma',\varrho'\gamma}(2)\right)\sigma'\left(B_{\mu_1'}(1)B^{-1}_{\mu_2'}(2)\right)\sigma(B_\varrho)B_\sigma \tag{2.p}$$

times (2.o).

Second, we have

$$\sigma'\left(B_{\mu_i'}(i)\right)B_{\sigma'}(i) = B_{\sigma'\mu_i'}(i)A^l_{\sigma',\mu_i'}(i)A^{-1}_{\sigma,\mu_i}(i),$$

and

$$B_{\sigma'\mu_i'}(i) = \prod_{\gamma\in H} A_{\sigma'\mu_i',\gamma}(i),$$

since $\sigma'\mu_i' \in HG_i'$.

If $\varrho' \in H$, then $B_\varrho = B_\sigma = 1$, $\mu_i' = 1$, and

$$\prod_\gamma A_{\sigma',\varrho'\gamma}(i) = \prod_\gamma A_{\sigma',\gamma}(i),$$

which yields the desired equality.

In general, if $\varrho' \in \mu_i' H\nu_i$ with ν_i from the right representative system of H_i in G_i', then

$$\prod_\gamma A_{\sigma',\varrho'\gamma}(i) = \prod_\gamma A_{\sigma',\mu_i'\gamma\nu_i}(i)$$

and

$$A_{\sigma',\mu_i'\gamma\nu_i}(i)A^{-1}_{\sigma',\mu_i'\gamma}(i) = \sigma'\left(A_{\mu_i'\gamma,\nu_i}(i)^{-1}\right)A_{\sigma'\mu_i'\gamma,\nu_i}(i) \in \sigma'\mu_i'\gamma(L'_{w_i'}).$$

Furthermore,

$$A^{-1}_{\sigma',\mu_i'\gamma}(i)A_{\sigma'\mu_i',\gamma}(i)A_{\sigma',\mu_i'}(i) = \sigma'\left(A_{\mu_i',\gamma}(i)\right) = 1.$$

140 Thus, (2.p) is equal to

$$\sigma(B_\varrho)B_\sigma A^{-1}_{\sigma,\mu_1}(1)A_{\sigma,\mu_2}(2).$$

But

$$A_{\sigma,\mu_i}(i) \equiv A_{\sigma,\varrho}(i) \pmod{L'_{w_i'}},$$

and

$$\sigma(B_\varrho)B_\sigma A^{-1}_{\sigma,\varrho}(1)A_{\sigma,\varrho}(2) = 1,$$

since $\varrho\sigma = 1$.

We now formulate the main result in a theorem.

Theorem 2.3. *In the situation of Theorem 2.2, let μ be an element of $X_*(T)$ that yields the cocharacters ν_i by averaging:*

$$\nu_i = (-1)^{i+1}\sum_{\sigma\in \mathrm{Gal}(L_{v_i'}/F_{v_i})}\sigma\mu.$$

Then the above construction provides a homomorphism τ_μ from the gerbe $\mathscr{T}_{\nu_1,\nu_2}$ to the neutral gerbe $\mathscr{G}_T$, such that $\tau_\mu\circ\zeta_v$ for $v \neq v_1, v_2$ is equivalent to the canonical neutralization, and such that $\tau_\mu\circ\zeta_{v_1}$ is equivalent to ξ_μ and $\tau_\mu\circ\zeta_{v_2}$ is equivalent to $\xi_{-\mu}$ (the definition of ξ_μ appears after Theorem 2.2). The homomorphism τ_μ is uniquely determined up to composition with an automorphism of $\mathscr{G}_T$ that is equivalent to the identity automorphism at all places. The construction is functorial in T and μ.

3. The pseudomotivic Galois group

In this section, we introduce two gerbes: not only the truly important one, which we call the pseudomotivic Galois group, but also a subordinate one, which exists solely to be able to formulate the conjecture in the fifth section for the most general Shimura variety.

The pseudomotivic Galois group $\mathscr{P}$ is defined as a direct limit. Let L be a finite Galois extension of $\mathbb{Q}$ in the field $\overline{\mathbb{Q}}$ of algebraic numbers in $\mathbb{C}$, and let m be a natural number. We first define a gerbe $\mathscr{P}(L,m)$, whose kernel is a torus $P(L,m)$, which we define by prescribing the Galois module of its characters. Let $q = p^m$, where p is the fixed rational prime.

Definition 3.1. The group $X(L,m)$ of Weil numbers associated with L and m consists of those $\pi \in L$ with the following properties:

(a) There exists an integer $\nu_1 = \nu_1(\pi)$ such that for all Archimedean places v of L, we have
$$\Big| \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{R})} \sigma\pi \Big| = q^{\nu_1}.$$

(b) For each prime v of L above p, there exists an integer $\nu_2(v) = \nu_2(\pi, v) \in \mathbb{Z}$ such that
$$|\pi|_v = \Big| \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_p)} \sigma\pi \Big|_p = q^{\nu_2(v)}.$$

(c) At all finite places outside p, π is a unit. 141

From Dirichlet's Unit Theorem, it follows that $X(L,m)$ is finitely generated. We divide out the finite group of the roots of unity contained in $X(L,m)$ to obtain a torsion-free module $X^*(L,m)$. The corresponding torus is denoted by $P(L,m)$. The module $X^*(L,m)$ is clearly a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Consequently, $P(L,m)$ is defined over $\mathbb{Q}$.

If necessary, to avoid misunderstandings, we denote by χ_π the character of $P(L,m)$ corresponding to the Weil number $\pi \in X(L,m)$. We fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ once and for all, and consequently a prime place v_2 of every finite-degree number field L contained in $\overline{\mathbb{Q}}$. Let v_1 be the Archimedean place of L given by $L \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We define the cocharacters ν_1, ν_2 in $X_*(L,m) = X_*(P(L,m))$ as follows:

$$\langle \nu_1, \chi_\pi \rangle = \nu_1(\pi), \tag{3.a}$$

$$\langle \nu_2, \chi_\pi \rangle = \nu_2(\pi, v_2) \tag{3.b}$$

The relation (2.a) is obvious, while (2.b) follows from the product formula. It should be noted that this ν_1 is even invariant under the full group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The following functorialities are present:

(3.c) For $L \subset L'$, we have $\phi^* = \phi^*_{L,L'} : X^*(L,m) \to X^*(L',m)$ sending π to itself.

(3.d) For $m|m'$, we have $\phi^* = \phi^*_{m,m'} : X^*(L,m) \to X^*(L,m')$ sending π to $\pi^{m'/m}$.

The contragredient maps are homomorphisms from the $\mathbb{Q}$-tori $P(L',m)$ and $P(L,m')$ into $P(L,m)$.

Lemma 3.2. *(a) We have*

$$\phi_{m,m'}(\nu_i') = \nu_i.$$

(b) We have

$$\phi_{L,L'}(\nu_i') = [L'_{v_i'} : L_{v_i}]\nu_i,$$

where v_i' are the places of L' defined by $L' \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $L' \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Proof of Lem. 3.2.

If m is replaced by m', then q is replaced by $q^{m'/m}$. From this, the first claim follows immediately. The second claim follows from the equation

$$\prod_{\sigma\in \mathrm{Gal}(L'_{v_i'}/\mathbb{Q}_p)} \sigma\pi = \left(\prod_{\sigma\in\mathrm{Gal}(L_{v_i}/\mathbb{Q}_p)} \sigma\pi\right)^{[L'_{v_i'}:L_{v_i}]},$$

QED Lem. 3.2. which holds for $\pi \in L$.

142 The procedure from the previous section associates the gerbe $\mathscr{P}(L,m)$ with the triple $P(L,m),\nu_1,\nu_2$ and the field L. From Lemma 3.2(a), it follows that there is a canonical homomorphism

$$\phi = \phi_{m,m'} : \mathscr{P}(L,m') \to \mathscr{P}(L,m).$$

In the previous section, it was shown that the gerbe defined by $P(L,m),\nu_1,\nu_2$, and L is isomorphic to the gerbe defined by $P(L,m),[L'_{v_1'} : L_{v_1}]\nu_1,[L'_{v_2'} : L_{v_2}]\nu_2$, and L'. Consequently, there is also a canonical homomorphism

$$\phi = \phi_{L,L'} : \mathscr{P}(L',m) \to \mathscr{P}(L,m).$$

As preparation for the next section, where we derive a canonical isomorphism between the pseudomotivic Galois group

$$\mathscr{P} = \varprojlim_{L,m} \mathscr{P}(L,m)$$

and the actual motivic Galois group based on the standard conjectures and the Tate conjecture, we need to demonstrate certain cohomological properties of the inverse system $P(L,m)$. Before doing so, we introduce a second, somewhat artificial gerb, which, for lack of a better name, we call the quasimotivic Galois group. It is defined using quasi-Weil numbers.

Definition 3.3. The group $Y(L,m)$ of quasi-Weil numbers associated with L and m consists of those $\pi \in L$ satisfying the following properties:

(a) There exists an integer $\nu_1 = \nu_1(\pi)$ such that for all Archimedean places v of L,

$$\Big|\prod_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})} \sigma\pi\Big|^{[L_v:\mathbb{R}]} = q^{\nu_1[L:\mathbb{Q}]}.$$

(b) For each place v of L above p, there exists $\nu_2(v) = \nu_2(\pi,v) \in \mathbb{Z}$ such that

$$|\pi|_v = \Big|\prod_{\sigma\in\mathrm{Gal}(L_v/\mathbb{Q}_p)} \sigma\pi\Big|_p = q^{\nu_2(v)}.$$

(c) At all finite prime places v outside p, π is a unit.

Just as $X(L,m)$, the groups $Y(L,m)$ also form an inverse system. To obtain $Y^*(L,m)$, one divides $Y(L,m)$ by the (typically infinite) group of all units, not just the roots of unity. This gives rise to an inverse system of gerbes $\mathscr{Q}(L,m)$, whose limit $\mathscr{Q}$ we call the quasimotivic Galois group.

The group $P(L,m)$ contains a sequence $\{\delta_n\}$, where $m|n$ and n is sufficiently large, of distinguished rational points defined by the equations

$$\chi_\pi(\delta_n) = \pi^{\frac{n}{m}},$$

where $\frac{n}{m}$ must be divisible by the order of the torsion of $X(L,m)$. The rationality of δ_n follows from the relation

$$\sigma(\chi_\pi(\delta_n)) = (\sigma\pi)^{\frac{n}{m}} = \chi_{\sigma\pi}(\delta_n).$$

The equations 143

$$\phi_{m,m'}(\delta_n) = \delta_n, \quad \phi_{L,L'}(\delta_n) = \delta_n$$

are easily verified. In $Q(L,m)$, the kernel of $\mathscr{Q}(L,m)$, there is no uniquely determined δ. We can define several, and this lack of uniqueness will mostly be ignored. If k is sufficiently large, then there exists a commutative diagram of $\mathrm{Gal}(L/\mathbb{Q})$-modules:

$$\begin{tikzcd}
& Y(L,m) \arrow[dd] \\
k \cdot Y^*(L,m) \arrow[ur, "\eta"] \arrow[dr, hook] & \\
& Y^*(L,m)
\end{tikzcd}$$

We define δ_n by the equations

$$\chi_\pi(\delta_n) = \eta(\pi^k)^{\frac{n}{mk}}.$$

Thus, for every π, the number $\pi^{-n}\chi_\pi(\delta_n)$ is a unit.

Let T be a torus defined over $\mathbb{Q}$, and let $\mu \in X_*(T)$. If T splits over L, and m is sufficiently large, we associate to the pair (T,μ) a homomorphism

$$\psi_\mu : Q(L,m) \to T$$

defined over $\mathbb{Q}$.[1] If

$$(\sigma - 1)(\iota + 1)\mu = (\iota + 1)(\sigma - 1)\mu = 0, \tag{3.e}$$

then ψ_μ can be factored through $P(L,m)$ and can be regarded as a homomorphism from $P(L,m)$ to T. It should be noted that the embedding

$$X(L,m) \hookrightarrow Y(L,m)$$

[1] This part of the article contains a well-known mistake. For the current definition of $Q(L,m)$, the map ψ_μ cannot be constructed as claimed. The element $\pi_\lambda = \lambda(\gamma)$ defined below does not satisfy Definition 3.3(a) in general and hence is not an element of $Y(L,m)$. (If (3.e) holds, then π_λ is indeed an element of $X(L,m) \subset Y(L,m)$ as claimed here, so we have $\psi_\mu : Q(L,m) \to T$ factoring through $P(L,m)$.) For correction, one needs to appropriately modify the definition of $Q(L,m)$, and thereby modify the definition of the gerbe $\mathscr{Q}$. Such modifications have been proposed by Pfau [Pfa93, Pfa96], and later by Reimann [Rei97] (see §B.2). (References are at the end of Translator's Note.) The relationship between their versions is explained in Remark B2.13 of *loc. cit.*.

defines a homomorphism

$$Q(L,m) \to P(L,m).$$

The homomorphism ψ_μ is defined via a homomorphism of Galois modules $X^*(T) \to Y^*(L,m)$, specifically via $X^*(T) \to Y(L,m), \lambda \mapsto \pi_\lambda$, where it must hold that

$$\pi_{\sigma\lambda} = \sigma(\pi_\lambda).$$

If the equations (3.e) hold, then the following will also be true:

$$\sigma\big(\iota(\pi_\lambda)\cdot\pi_\lambda\big) = \pi_{\sigma\iota\lambda+\sigma\lambda} = \pi_{(\iota+1)\sigma\lambda} = \pi_{(\iota+1)\lambda} = \iota(\pi_\lambda)\cdot\pi_\lambda,$$

and

$$\iota(\pi_\lambda)\pi_\lambda = \pi_{\iota\lambda+\lambda} = \pi_{\iota'\lambda+\lambda} = \iota'(\pi_\lambda)\pi_\lambda, \quad \iota' = \sigma\iota\sigma^{-1}.$$

Thus, $\iota(\pi_\lambda)\pi_\lambda$ is a rational number, and $\pi_\lambda \in X(L,m)$.

144 To define $\lambda \mapsto \pi_\lambda$, we choose a suitable $\gamma \in T(\mathbb{Q})$ and set

$$\pi_\lambda = \lambda(\gamma).$$

For the group $\mathscr{P}$, we then have $\psi_\mu(\delta_m) = \gamma$. For the group $\mathscr{Q}$, $\lambda\big(\gamma^{-1}\psi_\mu(\delta_m)\big)$ is a unit for every λ. Thus, we can replace γ with $\psi_\mu(\delta_m)$ and assume that $\psi_\mu(\delta_m) = \gamma$.

The homomorphism ψ_μ must satisfy the conditions[2]

$$\psi_\mu(\nu_i) = -(-1)^i \sum_{\sigma\in\mathrm{Gal}(L_{v_i}/\mathbb{Q}_{v_i'})} \sigma\mu,$$

which, when translated to γ, means that

$$\Big|\prod_{\sigma\in\mathrm{Gal}(L_{v_i}/\mathbb{Q}_{v_i'})} \sigma\lambda(\gamma)\Big| = q^{-(-1)^i\left\langle\lambda,\ \sum_{\sigma\in\mathrm{Gal}(L_{v_i}/\mathbb{Q}_{v_i'})}\sigma\mu\right\rangle}.$$

Finally, $\lambda(\gamma)$ is a unit outside the infinite places and p.

We choose m large enough so that there exists an element a in L such that the ideal (a) is a power $\mathfrak{p}^r$ of the prime ideal $\mathfrak{p}$ corresponding to the place v_2, and such that

$$\big|\mathrm{Nm}_{L_{v_2}/\mathbb{Q}_{v_2}} a\big|_p = q.$$

The element γ is then defined by the equations

$$\lambda(\gamma) = \prod_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})} \sigma(a)^{\langle\lambda,\sigma\mu\rangle}.$$

Then γ is rational, and

$$\begin{aligned}\Big|\prod_{\sigma\in\mathrm{Gal}(L_{v_2}/\mathbb{Q}_{v_2})}\sigma\lambda(\gamma)\Big|_p &= \Big|\prod_{\varrho\in\mathrm{Gal}(L_{v_2}/\mathbb{Q}_{v_2})}\ \prod_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})}\varrho\sigma(a)^{\langle\lambda,\sigma\mu\rangle}\Big|_p\\ &= \Big|\prod_{\varrho,\sigma\in\mathrm{Gal}(L_{v_2}/\mathbb{Q}_{v_2})}\varrho\sigma(a)^{\langle\lambda,\sigma\mu\rangle}\Big|_p\\ &= q^{\left\langle\lambda,\sum_\sigma\sigma\mu\right\rangle}.\end{aligned}$$

According to the previous section, ψ_μ can canonically be extended to a homomorphism of gerbes: $\psi_\mu : \mathscr{Q}(L,m) \to \mathscr{G}_T$ or $\psi_\mu : \mathscr{P}(L,m) \to \mathscr{G}_T$.

From now on in this section, we will focus exclusively on the gerbes $\mathscr{P}(L,m)$ and begin with a simple remark.

[2] Here $v_i' = \infty, p$ for $i = 1, 2$.

Lemma 3.4. *Let K be the maximal CM subfield of L. Then $X(L,m) = X(K,m)$.*

Proof of Lem. 3.4. 145

We recall that the Galois extension K is a CM field if the complex conjugation ι lies in the center of $\mathrm{Gal}(K/F)$. We therefore need to show that

$$\iota(\pi) = \iota'(\pi)$$

when $\pi \in X(L,m)$, and $\iota' = \sigma\iota\sigma^{-1}, \sigma \in \mathrm{Gal}(L/F)$. If L is a real field, this is clear. Otherwise, we have

$$\iota(\pi)\pi = q^{\nu_1(\pi)} = \iota'(\pi)\pi,$$

and consequently $\iota'(\pi) = \iota(\pi)$. *QED Lem. 3.4.*

We now assume without loss of generality that L is a CM field. Let L_0 be its totally real subfield.

Lemma 3.5. *$H^2(\mathbb{Q}, P(L,m))$ satisfies the Hasse principle.*

Proof of Lem. 3.5.

Clearly, it suffices to show that the homomorphism

$$H^1\left(\mathrm{Gal}(L'/\mathbb{Q}), P(\mathbb{A}_{L'})\right) \to H^1\left(\mathrm{Gal}(L'/\mathbb{Q}), X_* \otimes C_{L'}\right)$$

is surjective when $P = P(L,m)$ and $L_1 \subset L_2$. So, with Tate-Nakayama, we need to establish the surjectivity of

$$\bigoplus_{v \in \mathbb{Q}} H^{-1}\left(\mathrm{Gal}(L'_v/\mathbb{Q}_v), X_*\right) \to H^{-1}\left(\mathrm{Gal}(L'/\mathbb{Q}), X_*\right).$$

There is an exact sequence of Galois modules

$$1 \to X'_* \to X_* \to \mathbb{Z} \to 1,$$

where ι acts as -1 on X'_*. An element from $H^{-1}(\mathrm{Gal}(L'/Q), X_*)$ is represented by $\lambda \in X^*$ with

$$\sum_{\sigma \in \mathrm{Gal}(L'/Q)} \sigma\lambda = 0.$$

Consequently, λ lies in X'_*, and

$$\sum_{\sigma \in \mathrm{Gal}(L'_{v_1}/\mathbb{R})} \sigma\lambda = 0,$$

so that λ represents an element from $H^{-1}(\mathrm{Gal}(L'_{v_1}/\mathbb{R}), X_*)$. Thus, the lemma is proven. *QED Lem. 3.5.*

There are still additional cohomological properties of the groups $P(L,m)$ to address. Before that, we make some remarks about the modules $X_*(L,m)$. The Serre group ${}^L S$ is a torus over $\mathbb{Q}$ defined by its character module $X^*({}^L S)$:

$$X^*({}^L S) = \left\{ \lambda = \sum_{\varrho : L \to \overline{\mathbb{Q}}} n_\varrho[\varrho] \mid n_\varrho + n_{\iota\varrho} \equiv k(\lambda), k(\lambda) \in \mathbb{Z} \right\}.$$

The group ${}^L S$ is equipped with a distinguished cocharacter μ: 146

$$\mu : \sum_\varrho n_\varrho[\varrho] \mapsto n_1,$$

where 1 denotes the given embedding $L \to \overline{\mathbb{Q}}$. The cocharacter μ defines $\psi_\mu : P(L,m) \to {}^L S$ and $\psi_\mu^* : X^*({}^L S) \to X^*(L,m)$.

Lemma 3.6. *For sufficiently large m, the homomorphism ψ_μ^* is surjective, and thus $P(L,m) \to {}^LS$ is injective.*

To each $\varrho \in \mathrm{Gal}(L/\mathbb{Q})$, a prime v_ϱ dividing p is assigned, with the condition:

$$|x|_{v_\varrho} = |\varrho^{-1}x|_{v_2}.$$

If $\sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_{v_2})$, then $v_{\varrho\sigma} = v_\varrho$. The module $X_*(L,m)$ contains the distinguished elements $\nu_\infty = \nu_1$ and $\nu_\varrho = -\varrho\nu_2$. The following relations hold:

$$\nu_{\varrho\sigma} = \nu_\varrho, \quad \sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_{v_2}),$$

and

$$\nu_\varrho + \nu_{\iota\varrho} = \nu_\infty k, \quad k = 2[L_{v_2} : \mathbb{Q}_p][L : L_0]^{-1}.$$

The first relation is clear. The second follows from

$$q^{\nu_\varrho(\pi)+\nu_{\iota\varrho}(\pi)} = \Big| \prod_{\sigma \in \mathrm{Gal}(L_{v_\varrho}/\mathbb{Q}_p)} \varrho^{-1}\sigma(\pi\iota(\pi)) \Big|_p = |\pi\iota(\pi)|_p^{[L_{v_2}:\mathbb{Q}_p]}.$$

Lemma 3.7. *(a) If the p-dividing places of L_0 do not split in L (note that either none of them splits, or all of them split), then*

$$\nu_\varrho = \frac{k}{2}\nu_\infty,$$

and for sufficiently large m, $\{\nu_\infty\}$ is a $\mathbb{Z}$-basis of $X_*(L,m)$.

(b) If the (p-dividing) places of L_0 split in L, let $\varrho_1, \dots, \varrho_r$ be a system of representatives for the double cosets $\mathrm{Gal}(L_{v_1}/\mathbb{R})\backslash\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)$. Then, for sufficiently large m, the set

$$\{\nu_\infty, \nu_{\varrho_1}, \dots, \nu_{\varrho_r}\}$$

is a $\mathbb{Z}$-basis of $X_*(L,m)$.

Proof of Lem. 3.6, 3.7.

In case (a), $\nu_\varrho = \nu_{\iota\varrho}$, and the given relation is clear. In both cases it is clear that the elements that are to form a basis generate a submodule of finite index, since from

$$\nu_\infty(\pi) = \nu_\varrho(\pi) = 0$$

for each ϱ, we conclude that π is a root of unity. We only need to show that in case (a), ν_∞, and in case (b), ν_∞ and $\nu_{\varrho_1}, \dots, \nu_{\varrho_r}$, can be arbitrarily prescribed.[3] We will prove this statement together with Lemma 3.6.

147 The map $X^*({}^LS) \to X^*(L,m)$ is dual to the map $\psi_\mu : X_*(L,m) \to X_*({}^LS)$, which is defined by

$$\psi_\mu(\nu_\infty) = \mu + \iota\mu, \quad L_0 \neq L, \quad \psi_\mu(\nu_\infty) = \mu, \quad L_0 = L,$$

and

$$\psi_\mu(\nu_{\varrho_i}) = \sum_{\sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)} \varrho_i\sigma\mu.$$

Thus,

$$\psi_\mu(\nu_{\varrho_i})(\sum a_\varrho[\varrho]) = \sum_\sigma a_{\varrho_i\sigma},$$

$$\psi_\mu(\nu_\infty)(\sum a_\varrho[\varrho]) = \begin{cases} a_1 + a_\iota, & L_0 \neq L, \\ a_1, & L_0 = L. \end{cases}$$

[3] The authors probably mean that one can always find $\chi \in X^*(L,m)$ such that the pairings $\langle \nu_\bullet, \chi \rangle$ are equal to arbitrarily prescribed integers.

Since a_1 and a_ι can be chosen arbitrarily, it is clear that we can find a χ in the image of $X^*({}^L S)$ with the given $\langle \nu_\infty, \chi \rangle$. In case (b), we can choose $a_{\varrho_1}, \ldots, a_{\varrho_r}$ and $a_{\iota\varrho_1}$ arbitrarily, and at the same time require that $a_{\varrho_i \sigma} = a_{\iota \varrho_1 \sigma} = 0$ if $\sigma \neq 1$, $\sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)$.[4] Thus, both lemmas are proven. *QED Lem. 3.6, 3.7.*

Corollary 3.8. *For sufficiently large m and $m|m'$, the map $\phi_{m,m'} : X_*(L, m') \to X_*(L, m)$ is an isomorphism.*

We can simplify the notation a bit by using the symbols $P(L)$, sometimes P_L, or $X^*(L)$ instead of $P(L, m)$ or $X^*(L, m)$, with the understanding that m is sufficiently large.

The next statements concern the first cohomology group of P_L. Since they are trivial in case (a) of Lemma 3.7, we can restrict ourselves to case (b) in the proofs. We consider the exact sequence of $\mathrm{Gal}(L/\mathbb{Q})$-modules:

$$0 \to \mathbb{Z}\nu_\infty \to X_*(L) \to X_*(N) \to 0. \tag{3.f}$$

Here, N is a torus, and $X_*(N)$ is defined by this sequence. Let $\bar{\nu}_\varrho$ denote the image of ν_ϱ in $X_*(N)$.

The structure of the Galois module $X_*(N)$ can be immediately inferred from Lemma 3.7. Let $H = \mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)$ and $H_0 = H \cup H\iota$. We obtain an exact sequence of $\mathrm{Gal}(L/\mathbb{Q})$-modules:

$$0 \to \mathrm{Ind}_{H_0}^G 1 \to \mathrm{Ind}_H^G 1 \to X_*(N) \to 1. \tag{3.g}$$

Lemma 3.9. *$H^1(\mathbb{Q}, P_L)$ satisfies the Hasse principle.*

Proof of Lem. 3.9.

From (3.f) and Theorem 90, it suffices to show that the Hasse principle holds for $H^1(\mathbb{Q}, N)$. This follows from (3.g), Theorem 90, the Shapiro Lemma, and the Hasse principle for $H^2(H_0, K_0^\times)$, where K_0 denotes the fixed field of H_0. *QED Lem. 3.9.*

Lemma 3.10. *Let L be a CM-field. Then there exists a CM-extension L' of L,* 148
such that for every prime v of $\mathbb{Q}$, the transition homomorphism

$$H^1(\mathbb{Q}_v, P_{L'}) \to H^1(\mathbb{Q}_v, P_L)$$

is zero.

Proof of Lem. 3.10.

We use the Tate-Nakayama theory to prove that

$$H^{-1}(\mathrm{Gal}(L'_{v'}/\mathbb{Q}_v), X_*(L')) \to H^{-1}(\mathrm{Gal}(L_v/\mathbb{Q}_v), X_*(L))$$

is zero, where v and v' denote extensions of v in L and L', respectively. Clearly, it suffices to show that

$$H^{-1}(\mathrm{Gal}(L'_{v'}/\mathbb{Q}_v), X_*(N')) \to H^{-1}(\mathrm{Gal}(L_v/\mathbb{Q}_v), X_*(N))$$

is zero, where $X_*(N')$ is defined by the sequence

$$0 \to \mathbb{Z}\nu'_\infty \to X_*(L') \to X_*(N') \to 0.$$

An element of $H^{-1}(\mathrm{Gal}(L_v/\mathbb{Q}_v), X_*(N))$ is represented by

$$\nu = \sum_{\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)} a_\varrho \bar{\nu}_\varrho, \quad a_\varrho \in \mathbb{Z}.$$

Let $s = [L : \mathbb{Q}]$; then ν represents the trivial class if $s | a_\varrho$ for every ϱ.

[4] Here one takes $\varrho_1 = 1$. One should delete the requirement that $a_{\iota\varrho_1\sigma} = 0$.

An element of $H^{-1}(\mathrm{Gal}(L'_{v'}/\mathbb{Q}_v), X_*(N'))$ is similarly represented by

$$\nu' = \sum_{\mathrm{Gal}(L'/\mathbb{Q})/\mathrm{Gal}(L'_{v'_2}/\mathbb{Q}_p)} a_{\varrho'}\bar{\nu}_{\varrho'},$$

and the image $\phi_{L,L'}(\nu_{\varrho'})$ is $r\nu_{\varrho}$, where ϱ is the image of ϱ' in $\mathrm{Gal}(L/\mathbb{Q})$, and

$$r = [L'_{v'_2} : L_{v_2}].$$

The task is therefore to choose L' such that $s|r$. This can be achieved by setting
QED Lem. 3.10. $L' = KL$, where K is a suitably chosen abelian extension of $\mathbb{Q}$.

Applying global Tate-Nakayama theory in the same way yields the following statement.

Lemma 3.11. *Let L be a CM-field with idele class group C_L. Then there exists a CM-extension L' of L such that the transition homomorphism*

$$H^1(\mathrm{Gal}(L'/\mathbb{Q}), X_*(L') \otimes C_{L'}) \to H^1(\mathrm{Gal}(L/\mathbb{Q}), X_*(L) \otimes C_L)$$

is zero.

149 The following corollary immediately follows from Lemmas 3.9 and 3.10:

Corollary 3.12. *Let L be a CM-field. Then there exists a CM-extension L' of L such that the transition homomorphism*

$$H^1(\mathbb{Q}, P_{L'}) \to H^1(\mathbb{Q}, P_L)$$

is zero.

By Lemma 3.6, there exists an embedding

$$\psi_L = \psi_\mu : \mathscr{P}_L \hookrightarrow \mathscr{G}_{{}^LS}.$$

From the definition of ψ_μ and Lemma 3.7, the commutativity of the following diagram (up to equivalence) follows:

$$\text{(3.h)} \qquad \begin{tikzcd} \mathscr{P}_{L'} \arrow[r, "\psi_{L'}"] \arrow[d] & \mathscr{G}_{L'} \arrow[d] \\ \mathscr{P}_L \arrow[r, "\psi_L"] & \mathscr{G}_L, \end{tikzcd}$$

where $\mathscr{G}_L = \mathscr{G}_{{}^LS}$ and $\mathscr{G}_{L'} = \mathscr{G}_{{}^{L'}S}$. Furthermore, from the definitions in the previous section, the following hold:

$$\text{(3.i)} \qquad \psi_\mu \circ \zeta_{v_1} \cong \xi_\mu, \quad \psi_\mu \circ \zeta_{v_2} \cong \xi_{-\mu}.$$

(3.j) If v neither divides ∞ nor p, then $\psi_\mu \circ \zeta_v$ is equivalent to the canonical neutralization of $\mathscr{G}_L$.

That the local homomorphisms are compatible with modifications of L, up to equivalence, follows from the results of the previous section.

Two local homomorphisms differ by an element in

$$H^1\left(\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v), {}^LS(\overline{\mathbb{Q}}_v)\right).$$

The following lemma thus implies that, for a *non-archimedean* place v of $\mathbb{Q}$, any two compatible families of local homomorphisms are automatically equivalent. In particular, the condition (3.j) as well as the second part of (3.i) are automatically satisfied.

Lemma 3.13. *Let v be a non-archimedean place of $\mathbb{Q}$. For every CM-field L, there exists a CM-field L' that contains L such that*

$$H^1\left(\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v), {}^{L'}S(\overline{\mathbb{Q}}_v)\right) = 0.$$

Proof. We consider the exact sequence of algebraic tori over $\mathbb{Q}_v$ ([De2]):

$$1 \to R_{L_0'/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m \times R_{L'/\mathbb{Q}}\mathbb{G}_m \to {}^{L'}S \to 1.$$

The associated long exact sequence of cohomology gives: 150

$$0 \to H^1(\mathbb{Q}_v, {}^{L'}S) \to \bigoplus_{v_0'} \mathrm{Br}(L'_{0v_0'}) \to \Big(\bigoplus_{v'} \mathrm{Br}(L'_{v'})\Big) \oplus \mathrm{Br}(\mathbb{Q}_v).$$

The sums are taken over the places lying above v in the respective fields. The claim follows because we can find a Galois extension of CM type $L' \supset L$ such that all places in L_0' above v split in L'. Specifically, we choose a totally real Galois extension K of L_0 that splits L at every place dividing v, and set $L_0' = KL_0$, $L' = KL$. □

In the following section, we will encounter an inverse system of gerbes $\mathscr{M}_L$ with properties similar to those of $\mathscr{P}_L$. Firstly, there exist embeddings

$$\phi_\mu : \mathscr{M}_L \hookrightarrow \mathscr{G}_L,$$

and secondly, there exist local homomorphisms ζ_v for which the analogues of (3.h), (3.i), and (3.j) hold. Furthermore, the following will hold:

$$\phi_\mu(M_L) = \psi_\mu(P_L). \tag{3.k}$$

We therefore identify M_L and P_L.

Lemma 3.14. *Assume that for every place v of $\mathbb{Q}$ and every L, there exists an isomorphism*

$$\eta_L(v) : \mathscr{M}_L(v) \to \mathscr{P}_L(v),$$

which is the identity on the kernel M_L. Then, there exists an isomorphism $\eta_L : \mathscr{M}_L \to \mathscr{P}_L$, such that $\eta_L(v)$ is equivalent to the localization of η_L for every v. The family $\{\eta_L\}$ is uniquely determined up to equivalence, and the diagrams

(3.l)
$$\begin{tikzcd} \mathscr{M}_{L'} \arrow[r] \arrow[d, "\eta_{L'}"'] & \mathscr{M}_L \arrow[d, "\eta_L"] \\ \mathscr{P}_{L'} \arrow[r] & \mathscr{P}_L \end{tikzcd}$$

commute up to equivalence.

Proof of Lem. 3.14.

Let $\mathscr{M}_L$ be defined by the cocycle $\{m_{\varrho,\sigma} \mid \varrho, \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$, and $\mathscr{P}_L$ by $\{p_{\varrho,\sigma}\}$. Then, $\{m_{\varrho,\sigma}\}$ and $\{p_{\varrho,\sigma}\}$ are locally equivalent. By the Hasse principle from Lemma 3.5, we may assume $m_{\varrho,\sigma} \equiv p_{\varrho,\sigma}$.

Any two locally equivalent isomorphisms η_L and $\eta_L' : \mathscr{M}_L \to \mathscr{P}_L$ differ by a class in $H^1(\mathbb{Q}, P_L)$. Consequently, Lemma 3.9 implies that two locally equivalent isomorphisms are also globally equivalent. Thus, the uniqueness of η_L is established.

The commutativity of diagram (3.l) is proved similarly. Let $p_\varrho' \in \mathscr{P}_{L'}$ be repre- 151 sentatives of elements in the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with

$$p'_\varrho p'_\sigma = p'_{\varrho,\sigma} p'_{\varrho\sigma},$$

and let $b_\varrho p_\varrho$, $b_\varrho \in P_L(\overline{\mathbb{Q}})$, be the image of p'_ϱ. Similarly, let m'_ϱ be representatives with

$$m'_\varrho m'_\sigma = m'_{\varrho,\sigma} m'_{\varrho\sigma},$$

where $m'_{\varrho,\sigma} = p'_{\varrho,\sigma}$, and let $a_\varrho m_\varrho$ be the image of m'_ϱ. Finally, let

$$\eta_{L'}(m'_\varrho) = c'_\varrho p'_\varrho, \quad c'_\varrho \in \mathscr{P}_{L'},$$

and let c_ϱ be the image of c'_ϱ in $\mathscr{P}_L$. The isomorphism η_L could have been defined through the commutativity of (3.l), namely:

$$\eta_L(m_\varrho) = a_\varrho^{-1} b_\varrho c_\varrho p_\varrho. \tag{3.m}$$

Thus, the commutativity follows from uniqueness.

To define η_L, we use Lemma 3.10 and choose L' such that the transition homomorphisms $H^1(\mathbb{Q}_v, P_{L'}) \to H^1(\mathbb{Q}_v, P_L)$ are zero for all v. A possible isomorphism $\xi : \mathscr{M}_{L'} \to \mathscr{P}_{L'}$ is defined by $m'_\varrho \to p'_\varrho$, since $m'_{\varrho,\sigma} = p'_{\varrho,\sigma}$. The localization $\xi(v)$ differs from $\eta_{L'}(v)$ by an element α_v of $H^1(\mathbb{Q}_v, P_{L'})$. We define η_L by equation (3.m), via c'_ϱ and the resulting c_ϱ not using $\eta_{L'}$ (which has not been defined yet), but using ξ instead. Then $\eta_L(v)$ and the localization of η_L differ by the image of α_v in $H^1(\mathbb{Q}_v, P_L)$, which by assumption is zero.

QED Lem. 3.14.

In the next section, we will need a statement that guarantees we can identify the gerbes $\mathscr{M}_L$ and $\mathscr{P}_L$ as sub-gerbes of $\mathscr{G}_L$.

Lemma 3.15. *With the notation from the previous lemma, assume further that the local homomorphisms at the infinite place, $\psi_\mu \circ \zeta_{v_1}$ and $\phi_\mu \circ \zeta'_{v_1}$, from $\mathscr{W}$ to $\mathscr{G}_L$, are equivalent to each other for all L. Then the following diagram is commutative up to equivalence:*

(3.n)

$$\begin{tikzcd}
\mathscr{M}_L \arrow[dr, "\phi_\mu"] \arrow[dd, "\eta_L"'] & \\
& \mathscr{G}_L \\
\mathscr{P}_L \arrow[ur, "\psi_\mu"'] &
\end{tikzcd}$$

Proof. By assumption, for each place the local homomorphisms ζ_v and ζ'_v form a compatible family for varying L. Therefore, the same holds for the local homomorphisms $\psi_\mu \circ \zeta_v$ and $\phi_\mu \circ \zeta'_v$ for $\mathscr{G}_L$. For a non-archimedean place, these local homomorphisms are, by Lemma 3.13, equivalent, and this is true for the infinite place by assumption. After identifying $\mathscr{M}_L$ with $\mathscr{P}_L$, the embeddings ϕ_μ and ψ_μ differ by a class from $H^1(\mathbb{Q}, {}^L S)$, which is thus locally trivial. The statement follows from the Hasse principle for $H^1(\mathbb{Q}, {}^L S)$ ([De2], C.1). □

152 We note that in §2, we explicitly constructed the gerbe $\mathscr{T}_{\nu_1,\nu_2}$ and the homomorphism τ_μ, without providing a unique characterization. For the gerbe $\mathscr{P}$ and the homomorphism τ_μ, associated with a cocharacter of a torus that satisfies the Serre condition, such a characterization follows directly from Lemmas 3.14 and 3.15. We

leave the formulation to the reader (one can use the universal property of the Serre group).

4. The motivic Galois group

In this section, we introduce the motivic Galois group by defining and describing the corresponding Tannakian category. First, we recall the relationship between gerbes and Tannakian categories in a way we find understandable and show how the terminology of §2 compares with that in [DM] and [Sa].

The concept of a Galois gerbe over a field k of characteristic zero, with the associated kernel G, was already defined at the beginning of §2. In that section, the notion of a homomorphism between Galois gerbes was also explained. Additionally, we encountered examples, including the neutral Galois gerbe $\mathscr{G}_G$ associated with a linear algebraic group G defined over k. If V is a finite-dimensional vector space over the algebraic closure $\bar{k}$ of k, we can introduce a Galois gerbe $\mathscr{G}_V$. This gerbe consists of all isomorphisms $g : V \to V$ that are additive and σ-linear with respect to some $\sigma = \sigma(g) \in \mathrm{Gal}(\bar{k}/k)$. The homomorphism $g \mapsto \sigma(g)$ defines an exact sequence:

$$1 \to \mathrm{GL}(V) \to \mathscr{G}_V \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

The section germ is determined by the choice of a K-basis of V for a finite extension K of k. A representation of a Galois gerbe $\mathscr{G}$ is a homomorphism $\mathscr{G} \to \mathscr{G}_V$. The category of all representations of a given Galois gerbe $\mathscr{G}$ is equipped with an obvious k-linear structure, a tensor product, and a unit object $\mathbf{1}$, which is the representation defined by the natural projection onto $\mathrm{Gal}(\bar{k}/k)$.

Although these concepts of a gerbe and its associated Tannakian category are not the usual ones, they are closely related to those in [DM] and [Sa] for the cases of interest to us. From [DM], §3.10, it follows that any gerbe in the sense of Giraud that corresponds to a Tannakian category over k is an inverse limit of algebraic gerbes. Let $\underline{\mathscr{G}}$ be an algebraic gerbe in the sense of Giraud (we underline the symbol for clarity, distinguishing it from Galois gerbes), and let $Q \in \mathrm{ob}\, \underline{\mathscr{G}}_{\mathrm{Spec}\, \bar{k}}$ be an object. We will associate a Galois gerbe $\mathscr{G}$ with the pair $(\underline{\mathscr{G}}, Q)$ and show that this association defines an equivalence of categories. A homomorphism $(\underline{\mathscr{G}}, Q) \to (\underline{\mathscr{H}}, R)$ in the first category consists of a pair: a cartesian functor $\Phi : \underline{\mathscr{G}} \to \underline{\mathscr{H}}$ and an isomorphism

$$\tau : \Phi(Q) \cong R$$

in $\underline{\mathscr{H}}_{\mathrm{Spec}\, \bar{k}}$. In the correspondence between algebraic Tannakian categories and 153
algebraic gerbes, when replacing Giraud gerbes with Galois gerbes, the Tannakian categories must be replaced by algebraic Tannakian categories, *together* with a fiber functor over $\bar{k}$.

According to the last proposition in the appendix to [DM], there exists a finite extension k' of k and an object $Q_{k'}$ in $\underline{\mathscr{G}}_{\mathrm{Spec}\, \bar{k}'}$, whose inverse image in $\underline{\mathscr{G}}_{\mathrm{Spec}\, \bar{k}}$ is equipped with an isomorphism with Q. In other words (see the appendix to [DM]), Q is equipped with descent data over k'. Let ${}^{\sigma}Q \in \mathrm{ob}\, \underline{\mathscr{G}}_{\mathrm{Spec}\, \bar{k}}$ be the object obtained by pulling back via $\sigma : \bar{k} \to \bar{k}$. The descent data is nothing other than a system of isomorphisms $\phi_\sigma : Q \to {}^{\sigma}Q$ for $\sigma \in \mathrm{Gal}(\bar{k}/k)$ that satisfies the usual cocycle condition. We define

$$\mathscr{G} = \{\phi : Q \to {}^{\sigma}Q \mid \sigma \in \mathrm{Gal}(\bar{k}/k)\}.$$

Because there is an obvious isomorphism

$$\sigma : \mathrm{Isom}(Q, {}^{\varrho}Q) \to \mathrm{Isom}({}^{\sigma}Q, {}^{\sigma\varrho}Q),$$

as well as a pairing

$$\mathrm{Isom}(Q, {}^{\varrho}Q) \times \mathrm{Isom}({}^{\varrho}Q, {}^{\varrho\sigma}Q) \to \mathrm{Isom}(Q, {}^{\varrho\sigma}Q),$$

the set $\mathscr{G}$ forms a group, specifically an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$, where G is by definition $\mathrm{Aut}(Q)$. By assumption, G is an algebraic group over $\bar{k}$. The descent data defines a splitting of the extension over $\mathrm{Gal}(\bar{k}/k')$. In this way, we obtain a Galois gerbe $\mathscr{G}$.

A quasi-inverse is easy to define. Let $\mathscr{G}$ be a Galois gerbe. Then the Tannakian category $\mathrm{Rep}\,\mathscr{G}$ is equipped with a fiber functor over $\bar{k}$, which assigns to a representation its underlying vector space over $\bar{k}$. Thus, we obtain the Giraud gerbe $\underline{\mathscr{G}}$ associated with the Tannakian category $\mathrm{Rep}\,\mathscr{G}$, along with an element Q in $\mathrm{ob}\,\underline{\mathscr{G}}_{\mathrm{Spec}\,\bar{k}}$, the fiber functor. Since for any given $\underline{\mathscr{G}}$, two objects in $\mathrm{ob}\,\underline{\mathscr{G}}_{\mathrm{Spec}\,\bar{k}}$ are isomorphic, $\mathscr{G}$ is uniquely determined up to isomorphism. The isomorphism between two possible $\mathscr{G}$ is uniquely determined up to conjugation by an element of $G(\bar{k})$. Consequently, we do not distinguish too much between $\underline{\mathscr{G}}$ and $\mathscr{G}$.

The category of motives over an arbitrary field can be constructed assuming the standard conjectures, and we will sketch the construction below. This category is a rigid abelian tensor category over $\mathbb{Q}$ with $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$. If the field has positive characteristic, the category does not possess a fiber functor over $\mathbb{Q}$, although étale cohomology provides a fiber functor over $\mathbb{Q}_\ell$. This introduces unexpected difficulties because the proof of Theorem III, 3.2.2 in [Sa] is incomplete, as noted in [DM]. As a result, the category of motives is not immediately a Tannakian category in the sense of [DM]. According to the note at the end of [DM], this gap may not be serious. But it turns out that the problem is irrelevant for us here, since it is solved by the (very strong) standard conjectures anyway.

In order not to interrupt the later discussion, we will carry out the necessary considerations at this point.

154 Let C be a rigid abelian tensor category over k with $\mathrm{End}(\mathbf{1}) = k$, and let k' be an extension of k without further specific properties. As in [DM], p. 156, we introduce the subcategory of essentially constant ind-objects C^e of $\mathrm{Ind}(C)$ equivalent to C (see [Sa], II.2.3.4). The functor $i : C^e \to \mathrm{Ind}(C)_{(k')}$ is formally defined by the same formulas as in [DM]. We have

$$\mathrm{Hom}(i(X), i(Y)) \cong k' \otimes_k \mathrm{Hom}(X, Y).$$

Let $C_{k'}$ be the abelian tensor subcategory of $\mathrm{Ind}(C)_{(k')}$ generated by $i(X)$ for X in C^e ([DM], 1.14). We consider i as a functor from C to $C_{k'}$. If ω is a k'-valued fiber functor on C, we can define ω' as in [DM] and get the same commutative diagram.[5]

(4.d)
$$\begin{tikzcd} C \arrow[r] \arrow[dr, "\omega"'] & C_{k'} \arrow[d, "\omega'"] \\ & \mathrm{Vec}_{k'} \end{tikzcd}$$

We assume that there exists an ω for which ω' is exact and faithful, and we show that C is then a Tannakian category.

[5]The equation numbers (4.a)-(4.c) are missing in the original paper.

There is little left to prove. From (4.d), it follows that:

$$\mathrm{Aut}^{\otimes}\omega = \mathrm{Aut}^{\otimes}\omega'.$$

Then, by [DM], Th. 2.11, we conclude that $\mathrm{FIB}(C)$ ([DM], §3.6) is an affine Giraud gerbe $\underline{\mathscr{G}}$. It remains to show that

$$C \xrightarrow{\lambda} \mathrm{Rep}_k\underline{\mathscr{G}}$$

is an equivalence of categories. If $C_0 \subset C$ is a finitely generated subcategory, there exists a functor $\mathrm{FIB}(C) \to \mathrm{FIB}(C_0)$. Since any $\Phi \in \mathrm{Rep}_k\underline{\mathscr{G}}$ factors through $\mathrm{FIB}(C_0)$ with C_0 finitely generated, we may, without loss of generality, assume that C is finitely generated ([DM], §1.14), as C is a filtered union of its finitely generated subcategories. In this case, due to the proposition at the end of [DM], k' can be replaced by a finite Galois extension without losing the exactness and faithfulness of ω'. So let k' be finite and Galois.

We can then identify $C_{k'}$ with $C_{(k')}$, identifying $i(x)$ in the above sense with the $i(x)$ of [DM], p. 156. Via the morphism $\mathrm{id} \to ij$, X in $C_{k'}$ becomes a subobject of $ij(X)$. When passing to dual objects, every X in $C_{k'}$ becomes a quotient of an object in $i(C)$.

Let $\underline{\mathscr{G}}' = \mathrm{FIB}(C_{k'})$, i^* the induced functor $i^* : \mathrm{FIB}(C_{k'}) \to \mathrm{FIB}(C)$, and i^{**} : 155
$\mathrm{Rep}_k\underline{\mathscr{G}} \to \mathrm{Rep}_{k'}\underline{\mathscr{G}}'$. The diagram

$$\begin{tikzcd}
C \arrow[r, "\lambda"] \arrow[d, "i"] & \mathrm{Rep}_k\underline{\mathscr{G}} \arrow[d, "i^{**}"] \\
C_{k'} \arrow[r, "\lambda'"] & \mathrm{Rep}_{k'}\underline{\mathscr{G}}'
\end{tikzcd}$$

commutes.

The functor i is exact and faithful, and

$$\dim_k \mathrm{Hom}(X, Y) = \dim_{k'} \mathrm{Hom}(i(X), i(Y)).$$

If Φ_1 and Φ_2 lie in $\mathrm{Rep}_k\underline{\mathscr{G}}$, then the map

$$\mathrm{Hom}(\Phi_1, \Phi_2) \otimes_k k' \to \mathrm{Hom}(\Phi_1(\omega), \Phi_2(\omega)) \tag{4.e}$$

is an embedding. We expose our naivety below by giving the reason for this. It follows that

$$\dim_k \mathrm{Hom}(\Phi_1, \Phi_2) \leq \dim_{k'} \mathrm{Hom}(i^{**}\Phi_1, i^{**}\Phi_2). \tag{4.f}$$

Since λ' is an equivalence of categories, λ is consequently fully faithful, and the inequality (4.f) becomes an equality.

Because $i^{**}\Phi_1$ is a subobject of some $\lambda' i(X) = i^{**}\lambda(X)$, Φ_1 is a subobject of $\lambda(X)$. The quotient Φ_2 is a subobject of $\lambda(Y)$, and Φ_1 is the kernel of a morphism $\lambda(X) \to \lambda(Y)$. Since λ is fully faithful, Φ_1 lies in the essential image of λ.

By definition ([DM], p. 153), for every automorphism σ of k' over k, there exist linear isomorphisms

$$\alpha_i : \Phi_i(\omega) \longrightarrow \Phi_i(\omega) \otimes_\sigma k', \quad i = 1, 2,$$

which are compatible with $\mathrm{Hom}(\Phi_1, \Phi_2)$. Let $\phi \in \mathrm{Hom}(\Phi_1, \Phi_2)$. Then

$$\begin{aligned} &\alpha_2\phi = (\phi \otimes 1)\alpha_1, \\ &\alpha_2 a = (1 \otimes a)\alpha_1 = (\sigma^{-1}(a) \otimes 1)\alpha_1, \quad a \in k. \end{aligned} \tag{4.g}$$

If (4.e) were not an embedding, there would exist a minimal r and

$$\phi_1, \dots, \phi_r \in \operatorname{Hom}(\Phi_1, \Phi_2), \quad a_1, \dots, a_r \in k',$$

such that $\{\phi_1, \dots, \phi_r\}$ is linearly independent over k, while

$$\sum_{i=1}^{r} a_i \phi_i = 0$$

156 in $\operatorname{Hom}(\Phi_1(\omega), \Phi_2(\omega))$. Necessarily, $r \geq 2$. From (4.g), we deduce for every σ,

$$\sum_{i=1}^{r} \sigma^{-1}(a_i) \phi_i = 0,$$

from which we immediately obtain a contradiction.

We now proceed to the construction of the category of motives. Let $V_{\mathbb{F}}$ be the category of smooth projective varieties (not necessarily connected) over $\mathbb{F}$. If $X \in \mathrm{ob} V_{\mathbb{F}}$, let $C^*(X)$ denote the graded $\mathbb{Q}$-vector space of algebraic cycles (graded by codimension) modulo *numerical equivalence*. We construct a new category $CV_{\mathbb{F}}$ from $V_{\mathbb{F}}$ with the same objects, which we denote by $h(X)$ for clarity, and with morphisms given by algebraic correspondences of degree zero from X to Y, ([Sa], A.0.3.3)

$$\operatorname{Hom}_{CV_{\mathbb{F}}}\big(h(X), h(Y)\big) = CV^0(X, Y).$$

If X is irreducible of dimension n, then $CV^0(X, Y) = C^n(X \times Y)$. The category $CV_{\mathbb{F}}$ is a $\mathbb{Q}$-linear category equipped with a direct sum $\oplus$ (corresponding to disjoint union) and a tensor product $\otimes$ (corresponding to the Cartesian product), with the unit object $\mathbf{1} = h(\operatorname{Spec} \mathbb{F})$. These operations are endowed with natural associativity and commutativity laws (see [DM], §1.1). The assignment $X \mapsto h(X)$ defines a *contravariant* functor (the graph of a morphism):

$$h : V_{\mathbb{F}}^{\mathrm{opp}} \to CV_{\mathbb{F}}.$$

The "false" category of *effective* motives $\dot{M}_{\mathbb{F}}^{+}$ is the pseudo-abelian (Karoubi) envelope of $CV_{\mathbb{F}}$, obtained by formally adding images and kernels of projectors (see [DM], p. 201). In $\dot{M}_{\mathbb{F}}^{+}$, the motive $h(\mathbb{P}^1)$ decomposes as:

$$h(\mathbb{P}^1) = \mathbf{1} \oplus L.$$

Here, $\operatorname{Hom}(M, N) \xrightarrow{\sim} \operatorname{Hom}(M \otimes L, N \otimes L)$ for two effective motives M and N (see [Sa], VI, 4.1.2.5). The "false" category of motives $\dot{M}_{\mathbb{F}}$ is obtained from $\dot{M}_{\mathbb{F}}^{+}$ by formally inverting the object L. Let $T = L^{-1}$ be the "Tate motive", and define $M(n) = M \otimes T^n$. Then:

$$\operatorname{Hom}_{\dot{M}_{\mathbb{F}}}\big(M(m), N(n)\big) = \operatorname{Hom}(M \otimes L^{N-m}, N \otimes L^{N-n}), \quad N \geq m, n.$$

We aim to turn this constructed tensor category into a ($\mathbb{Z}$)-graded polarized Tannakian category. To achieve this, we must assume the standard conjectures about algebraic cycles. Fix a prime $l \neq p$ and work with the graded l-adic cohomology: $H_l^*(X) = \bigoplus_i H^i(X, \mathbb{Q}_l)$. This cohomology is equipped with the cycle map

$$\gamma^i : C^i(X) \to H_l^{2i}(X)(i),$$

and the trace map

$$\operatorname{Tr}_X : H_l^{2n}(X)(n) \to \mathbb{Q}_l, \quad n = \dim X.$$

157 The Künneth formula, Poincaré duality, and the hard Lefschetz theorem hold. The

standard conjectures state:

1. The cycle maps γ^i are injective.
2. The map
$$l^{n-2p}: C^p(X) \to C^{n-p}(X)$$
defined by an ample divisor is bijective for $0 \leq 2p \leq n = \dim X$.
3. On the primitive algebraic cycles
$$C^p_{\mathrm{pr}}(X) \underset{\mathrm{Def}}{=} \mathrm{Ker}\ l^{n-2p+1} = C^p(X) \cap H^{2p}_{\mathrm{pr}}(X),$$
the symmetric bilinear form:
$$C^p_{\mathrm{pr}}(X) \times C^p_{\mathrm{pr}}(X) \to \mathbb{Q}, \quad (x,y) \mapsto (-1)^p \cdot \mathrm{Tr}_X(l^{n-2p}xy)$$
is positive definite for $0 \leq 2p \leq n$. By decomposing into primitive components and appropriate sign conventions (see [Sa], VI. A.2.2.2.3), one obtains a positive-definite symmetric bilinear form on all of $C^p(X)$:
$$(x,y) \mapsto \mathrm{Tr}_X(x \cdot *y).$$

(These conjectures are not independent of each other; see [K].)

From these conjectures, it follows that there exist elements
$$\pi^i \in C^{2n}(X \times X), \quad n = \dim X,$$
which induce the decomposition into Künneth components of the diagonal in l-adic cohomology:
$$\Delta_X = \sum \pi^i \in \bigoplus H^{2n-i}_l(X) \otimes H^i_l(X).$$
(This last fact has actually been proven over $\mathbb{F}$; see [K-M].)

Thus, we obtain a decomposition of $\mathrm{id}_{h(X)}$ into pairwise orthogonal idempotents
$$\mathrm{id}_{h(X)} = \pi^0 \oplus \cdots \oplus \pi^n,$$
and correspondingly, a decomposition in $\dot{M}^+_{\mathbb{F}}$:
$$h(X) = h^0(X) \oplus \cdots \oplus h^n(X).$$
This grading of objects in $CV_{\mathbb{F}}$ extends in an obvious way to a grading of objects in $\dot{M}_{\mathbb{F}}$, ensuring that the Künneth formula holds.

To obtain the category of *genuine* motives $M_{\mathbb{F}}$ from the category of *false* motives 158
$\dot{M}_{\mathbb{F}}$, we modify the commutativity law ψ for *homogeneous* objects by setting
$$\psi = (-1)^{r\cdot s}\dot{\psi}: M \otimes N \xrightarrow{\sim} N \otimes M, \quad \deg M = r, \deg N = s,$$
and extending it linearly. We aim to show that this construction yields a semisimple Tannakian category over $\mathbb{Q}$. To this end, we first verify that the standard conjectures imply that the l-adic cohomology defines a faithful, exact functor: $H_l : (M_{\mathbb{F}})_{\mathbb{Q}_l} \to \mathrm{Vec}_{\mathbb{Q}_l}$. This is an immediate consequence of the injectivity of the canonical map
$$\mathrm{Hom}(M,N) \otimes \mathbb{Q}_l \to \mathrm{Hom}\big(H^*_l(M), H^*_l(N)\big). \tag{4.h}$$
Let X be a variety. The bilinear form $\mathrm{Tr}_X(x \cdot y)$ on $C^*(X)$ takes values in $\mathbb{Q}$ and, due to the standard conjectures, is non-degenerate. It is the restriction of the corresponding bilinear form with values in $\mathbb{Q}_l$ on l-adic cohomology. Consequently, the kernel of the natural map
$$C^*(X) \otimes \mathbb{Q}_l \to H^*_l(X)$$

is contained in the kernel of the bilinear form on $C^*(X) \otimes \mathbb{Q}_l$, which arises from the $\mathbb{Q}_l$-linear extension of the bilinear form on $C^*(X)$. It is therefore zero. The above claim is thus proven if M and N are motives associated with varieties. The claim then follows easily for effective motives and, finally, for all motives. Next, we note that the proof of Proposition 6.5 in [DM] (see [Sa], VI.4.2.2) shows that every indecomposable motive is simple. (In *loc. cit.*, it is not used that the faithful functor employed there takes values in $\mathrm{Vec}_{\mathbb{Q}}$; a faithful $\mathbb{Q}$-linear functor into $\mathrm{Vec}_{k'}$ for any extension k' of $\mathbb{Q}$ serves the same purpose.) We can therefore (with the same remark) apply Lemma 6.6 in [DM] and conclude that $M_{\mathbb{F}}$ is a semisimple $\mathbb{Q}$-linear abelian tensor category. Clearly, $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$, so it remains only to prove the rigidity of the category. This follows as in [DM], p. 204, bottom. One obtains

$$h(X)^\vee = h(X)(n),$$

if X is an irreducible variety of dimension n. That $M_{\mathbb{F}}$ is a Tannakian category now follows from the injectivity of (4.h), which yields the faithful exact functor ω' in (4.d).

The category of motives has additional structures. More precisely, $M_{\mathbb{F}}$, with its $\mathbb{Z}$-grading w and the Tate motive T, is a Tate triple over $\mathbb{Q}$ (see [DM], §5). There exists a polarization π_M of this Tate triple, characterized by the property that for a variety X, the polarization set $\pi_M(h^r(X))$ contains the form $\mathrm{Tr}_X(x \cdot *y)$.

If we take the field $\overline{\mathbb{Q}}$ of algebraic numbers as the base field instead of $\mathbb{F}$, a category of motives $M_{\overline{\mathbb{Q}}}$ was constructed in [DM]. In this construction, the algebraic correspondences (modulo numerical equivalence) are replaced by cohomological correspondences induced by *absolute Hodge cycles*. An advantage of this approach is
159 that no unproven conjectures need to be used. A disadvantage is that the reduction modulo p of an absolute Hodge cycle is not defined. In the following, we are only interested in the sub-Tannakian category $CM_{\overline{\mathbb{Q}}}$ of $M_{\overline{\mathbb{Q}}}$ generated by abelian varieties of CM type and $\mathbb{P}^1$ (see [DM], 1.14). For the constructions of $CM_{\overline{\mathbb{Q}}}$ and $M_{\mathbb{F}}$ to be compatible, we must in the following *assume the Hodge conjecture for abelian varieties of CM type*. The standard conjectures about algebraic cycles for these varieties are then, incidentally, a consequence of Hodge theory. An abelian variety of CM type over $\overline{\mathbb{Q}}$ has good reduction modulo p, as does the projective line. More precisely, such an abelian variety is already defined over a finite extension K of $\mathbb{Q}$. After possibly enlarging K, it has good reduction at the place over p distinguished by the fixed embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ (see [Se-Ta], Th. 6).

Here, the specialization map on the Chow rings is a homomorphism ([Fu], 20.3.1.)

$$A(X) \to A(X_{\mathbb{F}}).$$

From the standard conjectures, it follows that this map factors through the *numerical* equivalence classes and is compatible with the cycle maps in l-adic cohomology, i.e., the following diagram commutes:

$$\begin{tikzcd}
C^*(X) \arrow[r] \arrow[d] & C^*(X_{\mathbb{F}}) \arrow[d] \\
H_l^{2*}(X) \arrow[r, "\sim"] & H_l^{2*}(X_{\mathbb{F}})(*).
\end{tikzcd}$$

Indeed, if $x \in A(X)$ is numerically equivalent to zero, then $\mathrm{Tr}_X(x \cdot *x) = 0$, and thus the image of x in $A(X_{\mathbb{F}})$ is also numerically equivalent to zero. It is easy to see that this yields a tensor functor describing the reduction modulo p of these motives

$$\mathrm{red} : CM_{\overline{\mathbb{Q}}} \to M_{\mathbb{F}}.$$

More precisely, let $L \subset \overline{\mathbb{Q}}$ be a CM field, and let ${}^{L}CM_{\overline{\mathbb{Q}}}$ be the sub-Tannakian category of $CM_{\overline{\mathbb{Q}}}$ generated by abelian varieties of CM type by L and by $\mathbb{P}^1$. For $L \subset L'$, we obtain

$${}^{L}CM_{\overline{\mathbb{Q}}} \to {}^{L'}CM'_{\overline{\mathbb{Q}}}.$$

Here, we regard an abelian variety X with complex multiplication by L as a direct summand of the abelian variety (up to isogeny) with complex multiplication by L', $X \otimes_L L'$ (with the obvious definition). The rational cohomology defines a natural fiber functor of ${}^{L}CM_{\overline{\mathbb{Q}}}$ over $\mathbb{Q}$:

$${}^{L}CM_{\overline{\mathbb{Q}}} \to \mathrm{Vec}_{\mathbb{Q}}, \quad h(X) \mapsto \bigoplus H^i(X \times_{\overline{\mathbb{Q}}} \mathbb{C}, \mathbb{Q}).$$

Deligne ([D2], (A)) has shown that the corresponding neutralized gerbe over $\mathbb{Q}$ is the 160
Giraud gerbe $\underline{\mathscr{G}}_L = \underline{\mathscr{G}}_{{}^{L}S}$ associated with the Serre group (where $\underline{\mathscr{G}}_{{}^{L}S}$ corresponds to the Galois gerbe $\mathscr{G}_L = \mathscr{G}_{{}^{L}S}$), and that the homomorphism corresponding to the above functors is the natural projection:

$$\mathscr{G}_{L'} \to \mathscr{G}_L.$$

Let ${}^{L}M_{\mathbb{F}}$ be the sub-Tannakian category of $M_{\mathbb{F}}$ generated by the image of ${}^{L}CM_{\overline{\mathbb{Q}}}$ under the reduction functor ([DM], 1.14). Since ${}^{L}CM_{\overline{\mathbb{Q}}}$ is an algebraic Tannakian category, ${}^{L}M_{\mathbb{F}}$ is also an algebraic Tannakian category ([DM], 2.20). Its gerbe will be denoted by $\mathscr{M}_L$. If $L \subset L'$, then the homomorphism of gerbes $\mathscr{M}_{L'} \to \mathscr{M}_L$ is surjective because the functor ${}^{L}M_{\mathbb{F}} \to {}^{L'}M_{\mathbb{F}}$ is fully faithful, and every subobject of an image is the image of a subobject ([DM], 2.21, see Note 1 at the end of the section). We obtain commutative diagrams:

$$\begin{array}{ccc} \underline{\mathscr{M}}_{L'} & \xrightarrow{\phi_{L'}} & \underline{\mathscr{G}}_{L'} \\ \downarrow & & \downarrow \\ \underline{\mathscr{M}}_{L} & \xrightarrow{\phi_{L}} & \underline{\mathscr{G}}_{L}. \end{array}$$

To verify the properties of $\mathscr{M}_L$ that we desire, we must *assume the Tate conjecture* for the l-adic cohomology (for the fixed l) of algebraic varieties over finite fields [Sa], A.4. Let $\mathrm{Tate}_{\mathbb{F}_{p^m}}$ be the Tannakian category over $\mathbb{Q}_l$ of continuous semi-simple l-adic representations of the Galois group $T_m = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$, equipped with a natural fiber functor:

$$\omega_m : \mathrm{Tate}_{\mathbb{F}_{p^m}} \to \mathrm{Vec}_{\mathbb{Q}_l}.$$

We form the direct limit (restriction):

$$\mathrm{Tate}_{\mathbb{F}} = \varinjlim \mathrm{Tate}_{\mathbb{F}_{p^m}},$$

and thus obtain a neutralized, semi-simple Tannakian category over $\mathbb{Q}_l$. The Frobenius element in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$ defines an isomorphism $T_m \cong \widehat{\mathbb{Z}}$. The pro-algebraic group $\mathbb{T}_m = \mathrm{Aut}^{\otimes}(\omega_m)$ is the algebraic hull of T_m over $\mathbb{Q}_l$, and we have:

$$\mathrm{Rep.cont.}_{\mathbb{Q}_l}(T_m) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Q}_l}(\mathbb{T}_m)$$

(cf. [Sa], V.0.3.1). In particular, $\mathbb{T}_m$ is abelian, and consequently ([DM], 2.23), since $\mathrm{Rep}_{\mathbb{Q}_l}(\mathbb{T}_m)$ is a semi-simple Tannakian category, $\mathbb{T}_m$ is a projective limit of algebraic groups of multiplicative type. We want to determine the character group of $\mathbb{T}_m$. To do this, we extend the scalars from $\mathbb{Q}_l$ to $\overline{\mathbb{Q}}_l$, as explained at the beginning of this section.

$$(\text{Rep.cont.}_{\mathbb{Q}_l} T_m)_{\overline{\mathbb{Q}}_l} \xrightarrow{\sim} (\text{Rep}_{\mathbb{Q}_l} \mathbb{T}_m)_{\overline{\mathbb{Q}}_l} = \text{Rep}_{\overline{\mathbb{Q}}_l} \mathbb{T}_m.$$

161 (For the last identification, see [DM], 3.12.) It is easy to see that the category on the left is simply the category of diagonalizable representations of T_m in a $\overline{\mathbb{Q}}_l$-vector space, where the eigenvalues of the Frobenius element are l-adic units. These form a semi-simple Tannakian category over $\overline{\mathbb{Q}}_l$, in which the simple objects have rank 1 ([DM], 1.7.3). Furthermore, the simple objects are parametrized by the group $\mathcal{O}^*_{\overline{\mathbb{Q}}_l}$ of l-adic units. We conclude ([Sa], VI.3.5.1) that

$$X^*(\mathbb{T}_m) \cong \mathcal{O}^*_{\overline{\mathbb{Q}}_l}.$$

The transition map $X^*(\mathbb{T}_m) \to X^*(\mathbb{T}_{m'})$ for $m|m'$ sends π to $\pi^{m'/m}$.

In the following lemma, we transition to Galois gerbes. Consequently, ϕ_L is only determined up to conjugation by an element of ${}^L S(\overline{k})$.

Lemma 4.1. *Let L be Galois. The canonical homomorphism*

$$\mathscr{M}_L \xrightarrow{\phi_L} \mathscr{G}_L$$

is injective. The kernels M_L of $\mathscr{M}_L$ and P_L of $\mathscr{P}_L$ are identical as subtori of ${}^L S$.

Proof. From the definitions, it follows that every object in ${}^L M_{\mathbb{F}}$ is isomorphic to a subquotient of $\text{red}(X)$ for some $X \in \text{ob}\, {}^L CM_{\overline{\mathbb{Q}}}$. Therefore ([DM], 2.21), the first statement is clear (see Note 1). We have previously shown that the l-adic cohomology defines a faithful functor of Tannakian categories over $\mathbb{Q}_l$

$$({}^L M_{\mathbb{F}})_{\mathbb{Q}_l} \xrightarrow{H_l} \text{GradTate}_{\mathbb{F}}.$$

The Tate conjecture over $\mathbb{F}$ is *equivalent* to the statement that this functor is fully faithful ([Sa], A.4). (This conjecture is a consequence of the Tate conjecture over finite fields.) Because the Tannakian category $\text{GradTate}_{\mathbb{F}}$ is semisimple, every subobject of an image is the image of a subobject. Consequently, the homomorphism on the character modules of the kernels is injective ([DM], 2.21, see Note 1)

$$X^*(M_L) \to X^*(\mathbb{T}) = \varinjlim X^*(\mathbb{T}_m).$$

Since M_L is an algebraic group, it follows from the definition of the direct limit on the right hand side that $X^*(M_L)$ is torsion-free, and thus M_L is a torus.

Consider the diagram with the obvious arrows:

$$\begin{tikzcd}
& X^*(M_L) \arrow[dl, hook] & \\
X^*(\mathbb{T}) & & X^*({}^L S) \arrow[ul, two heads] \arrow[dl, two heads, "X^*(\psi_L)"] \\
& X^*(P_L) \arrow[ul, hook] &
\end{tikzcd}$$

162 Here, the surjectivity of the upper diagonal arrow is already clear, while for the

lower diagonal arrow, it follows from Lemma 3.6. Thus, it suffices to prove the commutativity of the diagram. Consider an element $\chi \in X^*({}^L S)$ of the form

$$\chi = \sum n_\sigma \cdot [\sigma], \quad n_\sigma + n_{\iota\sigma} = 1, \quad n_\sigma = 0 \text{ or } 1.$$

Its image under $X^*(\psi_L)$ is (see calculation after 3.7) the Weil number $\pi_\chi = \pi \in L$ with

(1) $\pi \cdot \overline{\pi} = q$,

(2) $\left| \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_p)} \sigma\pi \right|_p = q^{-\sum_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_p)} n_\sigma}$ for $v|p$.

Here, $q = p^m$ for sufficiently large m. On the other hand, χ corresponds to an abelian variety of CM type (L, Φ) up to isogeny, where $\Phi = \{\sigma \mid n_\sigma = 1\}$ ([Mu2], p. 212). This abelian variety is defined over a finite extension of $\mathbb{Q}$ and has good reduction modulo p over a finite field $\mathbb{F}_q$. Let π be the Frobenius endomorphism of this abelian variety over $\mathbb{F}_q$, regarded as an element of L. This is a unit outside ∞ and p (existence of l-adic cohomology) and satisfies (1) (Weil conjectures) and (2) (Shimura-Taniyama formula). Consequently, the images of χ under the two homomorphisms from $X^*({}^L S)$ to $X^*(\mathbb{T})$ are identical. The claim follows because such elements generate $X^*({}^L S)$ ([D2], A.2). $\square$

Corollary 4.2. *Let L be Galois. ${}^L M_{\mathbb{F}}$ is the sub-Tannakian category of motives whose Frobenius endomorphisms, with respect to a sufficiently large finite field, have all eigenvalues in L.*

The Tannakian category ${}^L M_{\mathbb{F}}$ is equipped with the fiber functor over $\mathbb{Q}_l$ defined by the l-adic cohomology for each $l \neq p$:

$${}^L M_{\mathbb{F}} \xrightarrow{H_l} \mathrm{Vec}_{\mathbb{Q}_l}.$$

This defines a trivialization ζ_l' of $\mathscr{M}_L$ over $\mathbb{Q}_l$. The crystalline cohomology (tensored with $\mathbb{Q}$) assigns to each motive an (F)-isocrystal over the fraction field $k = K(\mathbb{F})$ of the Witt vectors over $\mathbb{F}$ (see Note 2 at the end of this section). The gerbe associated with the Tannakian category of isocrystals over k is the Dieudonné gerbe $\mathscr{D}$. More precisely, this Tannakian category belongs to the filtered inverse system of gerbes $\mathscr{D}^{L_n}$, where L_n denotes the unramified extension of $\mathbb{Q}_p$ of degree n. This follows from [Sa], VI.3.3.2 (see also the note in the next section). The crystalline cohomology thus defines a homomorphism of gerbes

$$\zeta_p' : \mathscr{D} \to \mathscr{M}_L.$$

The locally constructed homomorphisms are obviously compatible with respect to 163
changes in L. For $v = l \neq p$, the localized gerbes $\mathscr{P}_L(v)$ and $\mathscr{M}_L(v)$ are isomorphic as neutral gerbes with the same kernel. For $v = p$, the isomorphism classes of $\mathscr{P}_L(v)$ and $\mathscr{M}_L(v)$ as gerbes over $\mathbb{Q}_p$ with $M_L = P_L$ are uniquely determined by the restrictions of the local homomorphisms ζ_p and ζ_p' to the kernel. To show that these restrictions are identical, it suffices to compare the restrictions ν^{L_n} and ν'^{L_n} of $\psi_L \circ \zeta_p : \mathscr{D}^{L_n} \to \mathscr{G}_L$ and $\phi_L \circ \zeta_p' : \mathscr{D}^{L_n} \to \mathscr{G}_L$ respectively. The first is given by the formula

$$\left| \prod_{\sigma \in \mathrm{Gal}(L_n/\mathbb{Q}_p)} \sigma\pi_\chi \right|_p = q^{-\langle \nu^{L_n}, \chi \rangle}, \quad \chi \in X^*({}^L S).$$

The homomorphism $\phi_L \circ \zeta'_p : \mathscr{D}^{L_n} \to \mathscr{G}_L$ defines an element b in Kottwitz's set $B({}^L S)$ (cf. the note to §5). The homomorphism ν'^{L_n} is given by the sequence of slopes of the Newton polygon of the isocrystal structure associated with $\phi_L \circ \zeta'_p$ on the representations of ${}^L S$ (cf. [K4], §4.2). Kottwitz has shown ([K4], 4.3) that

$$\langle \nu'^{L_n}, \chi \rangle = n \cdot \mathrm{val}(\chi(b)),$$

where val denotes the normalized valuation. The desired equality follows because, in the case where $\chi = \chi_\Phi$ belongs to an abelian variety of CM type (L, Φ) over $\mathbb{F}_q$, its Frobenius endomorphism π_χ is related to the F of the Dieudonné module as follows ([Dem], p. 63):

$$\pi_\chi = b \cdot \sigma(b) \cdots \sigma^{m-1}(b).$$

To define the local additional structure of $\mathscr{M}_L$ at the infinite place, we use the graded polarization π_M of the Tate triple $\left({}^L M_{\mathbb{F}}, w, \mathbf{1}(1)\right)$. Since there are motives of odd degree in ${}^L M_{\mathbb{F}}$, we may apply [DM, 5.20]. The Tate triple $(\mathbf{V}, w, T)$ described there is precisely the graded Tannakian category associated with the $\mathbb{R}$-gerbe $\mathscr{W}$. Thus, we obtain a homomorphism of gerbes, unique up to equivalence,

$$\zeta'_\infty : \mathscr{W} \to \mathscr{M}_L,$$

which defines a functor on the associated Tate triples

$$\left({}^L M_{\mathbb{F}}, w, \mathbf{1}(1)\right) \to (\mathbf{V}, w, T),$$

characterized up to isomorphism by the property that it transforms the polarization π_M into the canonical polarization π_{can} of $(\mathbf{V}, w, T)$. We want to show that the composition $\phi_L \circ \zeta'_\infty$ is equivalent to $\xi_\mu : \mathscr{W} \to \mathscr{G}_L$. However, just like ${}^L M_{\mathbb{F}}$, ${}^L CM_{\overline{\mathbb{Q}}}$ is naturally equipped with a graded polarization π_{CM}, and under the reduction functor, π_{CM} transforms into π_M. Thus, the composition $\phi_L \circ \zeta'_\infty$ is the homomorphism from $\mathscr{W}$ to $\mathscr{G}_L$ associated with $({}^L CM_{\overline{\mathbb{Q}}}, \pi_{CM})$ by Prop. 5.20 in [DM]. This homomorphism is easy to determine. On the kernel, it is given by the grading, and thus equals the weight homomorphism

$$\nu = \mu + \iota\mu : \mathbb{G}_m(\mathbb{C}) \to {}^L S(\mathbb{C}).$$

164 The graded polarization π_{CM} on the neutral Tannakian category ${}^L CM_{\overline{\mathbb{Q}}}$ is of the form π_C for a well-defined Hodge element $C \in {}^L S(\mathbb{R})$ ([DM], 4.22 and 4.25 (b)). From Hodge theory, it follows that $C = \nu(i)$. Indeed, C induces on an $\mathbb{R}$-rational representation V of ${}^L S$ precisely the Weil operator [W] associated with the Hodge structure induced on V, and the bilinear form defined by Weil [W, IV, Cor. to Th. 7] lies in $\pi_{CM}(V)$. By the definition of the homomorphism $\phi_L \circ \zeta'_\infty$ ([DM], 5.20), the chosen generator $w_\iota \in \mathscr{W}$ with $w_\iota^2 = -1$ is mapped to

$$\nu(i) \rtimes \iota \in {}^L S(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

It is clear that this homomorphism is equivalent to $\xi_\mu : \mathscr{W} \to \mathscr{G}_L$:

$$\mu(i)(\nu(i) \rtimes \iota)\mu(i)^{-1} = \mu(-1) \rtimes \iota.$$

In summary, we have shown that at every place v, the gerbes $\mathscr{P}_L(v)$ and $\mathscr{M}_L(v)$ are isomorphic. Moreover, due to the preceding remarks, the conditions of Lemma 3.15 are satisfied. We thus obtain the following theorem:[6]

[6]The numbering 4.3 is missing in the original text.

Theorem 4.4. *There exists a family of isomorphisms, $\eta_L : \mathscr{M}_L \to \mathscr{P}_L$, such that $\eta_L \circ \zeta'_v \cong \zeta_v$, and the following diagrams commute up to equivalence:*

$$\begin{tikzcd} \mathscr{M}_{L'} \arrow[r] \arrow[d, "\eta_{L'}"'] & \mathscr{M}_L \arrow[d, "\eta_L"] \\ \mathscr{P}_{L'} \arrow[r] & \mathscr{P}_L \end{tikzcd} \qquad \begin{tikzcd} \mathscr{M}_L \arrow[dd] \arrow[dr, "\phi_L"] & \\ & \mathscr{G}_L \\ \mathscr{P}_L \arrow[ur, "\psi_L"'] & \end{tikzcd}$$

Note 1. The cited reference, however, only deals with tensor functors $\omega : C' \to C$ between *neutral* Tannakian categories over k. To apply it here, it suffices to verify that if ω satisfies one of the two conditions below, then the functor $\omega_{(k_1)} : C'_{(k_1)} \to C_{(k_1)}$, arising after a finite base field extension k_1/k, also satisfies the same condition. The conditions are:

1. ω is fully faithful, and every subobject of $\omega(X')$, $X' \in \operatorname{ob} C'$, is isomorphic to the image of a subobject of X'.
2. Every object in C is a subquotient of an object of the form $\omega(X')$, $X' \in \operatorname{ob} C'$.

The Tannakian category $C_{(k_1)}$ can be identified with the category of k_1-modules $(X, \alpha_X : k_1 \to \operatorname{End}(X))$ in C ([DM], p. 156). Here, the functor $i = i_{k_1/k} : C \to C_{(k_1)}$ (external tensor product with k_1) is left adjoint to $j = j_{k_1/k} : C_{(k_1)} \to C$, which sends (X, α_X) to X. We have $k_1 \otimes_k \operatorname{Hom}(X, Y) = \operatorname{Hom}(i(X), i(Y))$. Now, suppose ω satisfies condition 1. For $(X', \alpha_{X'}), (Y', \alpha_{Y'}) \in \operatorname{ob} C'_{(k_1)}$,

$$\operatorname{Hom}\big((X', \alpha_{X'}), (Y', \alpha_{Y'})\big) = \operatorname{Hom}\big(\,(\omega(X'), \omega(\alpha_{X'}))\,, (\omega(Y'), \omega(\alpha_{Y'}))\,\big)$$

as subsets of $\operatorname{Hom}(\omega(X'), \omega(Y'))$, since ω is fully faithful by assumption. Thus, $\omega_{(k_1)}$ is fully faithful. Similarly, let (Y, α_Y) be a subobject of $(\omega(X'), \omega(\alpha_{X'}))$. By assumption, Y is isomorphic to an object of the form $\omega(Y')$ for some subobject Y' of X'. Due to the full faithfulness of ω, α_Y then defines a k_1-module structure $\alpha_{Y'} : k_1 \to \operatorname{End}(Y')$ on Y', and (Y, α_Y) is isomorphic to the image under $\omega_{(k_1)}$ of the subobject $(Y', \alpha_{Y'})$ of $(X', \alpha_{X'})$. Now, suppose ω satisfies condition 2. Let X_1 165
be an object of $C_{(k_1)}$, and let $j(X_1)$ be a subquotient of $\omega(X')$. The adjunction homomorphism identifies X_1 with a subobject of $i \circ j(X_1)$, which in turn is a subquotient of $i\left(\omega(X')\right) = \omega_{(k_1)}\left(i(X')\right)$.

Note 2. According to a communication from L. Illusie, the problem of defining a cycle class in crystalline cohomology with the usual formal properties has now been solved. For *rational* crystalline cohomology, such a solution was provided by H. Gillet and W. Messing (unpublished); for "integral" crystalline cohomology, a solution was achieved by M. Gros (Thèse de 3ème cycle, Orsay 1983).

5. A conjecture on Shimura varieties and its consequences

For the definition of a Shimura variety, the reader is referred to [De1]. A Shimura variety is associated with an algebraic group G defined over $\mathbb{Q}$, a homomorphism h defined over $\mathbb{R}$ from $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ to G, and an open compact subgroup K of $G(\mathbb{A}_f)$. Let $\mathscr{G} = \mathscr{G}_G$ be the neutral gerbe

$$G(\overline{\mathbb{Q}}) \rtimes \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

What is given is not actually h, but the conjugacy class X_∞ of h under $G(\mathbb{R})$. To this class, we assign an equivalence class of homomorphisms of gerbes over $\mathbb{R}$,

$$\xi_\infty : \mathscr{W} \to \mathscr{G}.$$

The group $\mathbb{S}$ is equipped with a canonical cocharacter μ, and the map $(z, w) \mapsto z^\mu w^{\bar\mu}$, where $\bar\mu = \iota(\mu)$, is an isomorphism of $\mathbb{S}$ with $\mathbb{G}_m \times \mathbb{G}_m$, which is defined over $\mathbb{C}$. We define $\xi_\infty(z) = h(z^{\mu+\bar\mu})$ for $z \in \mathbb{C}^\times$, and $\xi_\infty(w) = h((-1)^\mu) \rtimes \iota$ when $w = w(\iota)$.

Since $\mu + \overline{\mu}$ is central, ξ_∞ defines a homomorphism of the gerbe $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ to $\mathscr{G}_{G_{\mathrm{ad}}} = \mathscr{G}_{\mathrm{ad}}$, and therefore a class a in $H^1(\mathbb{R}, G_{\mathrm{ad}})$.

Lemma 5.1. *Let G' be the $\mathbb{R}$-form of G corresponding to the class a. Then $G'_{\mathrm{ad}}(\mathbb{R})$ is compact.*

Proof of Lem. 5.1.

Let T be a Cartan subgroup defined over $\mathbb{R}$ through which h factors. Then $h(\mu)$ is a cocharacter of T, which we also denote by μ. If α is a root of T, we can choose the root vectors $X_\alpha, X_{-\alpha}$ such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha, \quad \alpha(H_\alpha) = 2, \quad \iota(X_\alpha) = \pm X_{-\alpha}.$$

If the sign is positive (resp. negative), the root α is non-compact (resp. compact). The sign is given by $-(-1)^{\langle \mu, \alpha \rangle}$ (see [Sh], §4). When transitioning to G', the action of ι is replaced by that of $(-1)^\mu \iota = \iota'$. Therefore, for ι', we have

$$\iota'(X_\alpha) = -X_{-\alpha}$$

QED Lem. 5.1. for each α, and $G'_{\mathrm{ad}}(\mathbb{R})$ is compact because each root of G' is compact.

166 We have fixed a prime p once and for all. It is assumed that G splits over an unramified extension of $\mathbb{Q}_p$. It is further assumed that the compact subgroup that defines the Shimura variety is a product:

$$K = K^p \cdot K_p$$

with $K^p \subset G(\mathbb{A}_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$. The time is certainly not ripe to formulate a conjecture about the reduction of Shimura varieties without imposing strong restrictions on K_p, although special cases are certainly accessible ([C], [DR]). We assume one of the following two conditions. Either (a) K_p is hyperspecial in the sense of [T], §3.1, or (b) G_{ad} is anisotropic over $\mathbb{Q}_p$, and K_p is the maximal compact subgroup of $G(\mathbb{Q}_p)$. The reader can likely generalize the considerations of this section a bit. However, since this work mainly considers case (a), and in a further work to be written by the second author, case (b), and since the natural framework for the conjectures below is not at all clear to us, we have preferred to remain in these two cases.

We also assume that the derived group G_{der} of G is simply connected, an assumption that does not restrict generality, and the reason for which we will explain later. Let $\mathfrak{k}$ be the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{un}}$ of $\mathbb{Q}_p$. The group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ acts on $\mathfrak{k}$. The building $\mathscr{B}(G, \mathfrak{k})$ is defined in [T]. In the cases (a) and (b), there are facets $w_1, \ldots, w_r$ of minimal dimension and points $x_i \in w_i$, such that the set $\{x_1, \ldots, x_r\}$ is preserved by $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$, and such that K_p is the stabilizer in $G(\mathbb{Q}_p)$ of the center of the simplex spanned by $\{x_1, \ldots, x_r\}$.

We now introduce the concept of an admissible homomorphism $\phi : \mathscr{Q} \to \mathscr{G}$. This homomorphism can be factored through a $\mathscr{Q}_L$, and we often work with $\mathscr{Q}_L$ instead of $\mathscr{Q}$. Four conditions must be satisfied. First, let G_{ab} be the quotient G/G_{der} and h_{ab} the composition,

$$\mathbb{S} \to G \to G_{\mathrm{ab}}.$$

We write $h_{\mathrm{ab}}(z,w) = z^{\mu_{\mathrm{ab}}} \cdot w^{\bar{\mu}_{\mathrm{ab}}}$. Since the group G_{ab} is a torus, we can introduce the homomorphism $\psi_{\mu_{\mathrm{ab}}} : \mathscr{Q} \to \mathscr{G}_{\mathrm{ab}} = \mathscr{G}_{G_{\mathrm{ab}}}$ associated with μ_{ab}.

(5.a) *The composition* $\phi_{\mathrm{ab}} : \mathscr{Q} \xrightarrow{\phi} \mathscr{G} \to \mathscr{G}_{\mathrm{ab}}$ *is equivalent to* $\psi_{\mu_{\mathrm{ab}}}$.

There are several homomorphisms to $\mathscr{G}$:

(i) $\zeta_\infty = \zeta_{v_1} : \mathscr{W} \to \mathscr{G}$,
(ii) $\zeta_p = \zeta_{v_2} : \mathscr{D} \to \mathscr{G}$,
(iii) $\zeta_l : \mathscr{G}_l \to \mathscr{G}, \quad l \neq p$.

(5.b) *The composition* $\phi \circ \zeta_\infty$ *is equivalent to* ξ_∞.

(5.c) *The composition* $\phi \circ \zeta_l$ *is equivalent to the canonical neutralization* ξ_l *of* $\mathscr{G}$ *over* $\mathbb{Q}_l$.

The condition at the place p is more complicated to clarify. Let $w_1^0, \dots, w_r^0$ and 167
$x_i^0 \in w_i^0$ be the special facets and points that define K_p, and let

$$\mathscr{X} = \{(x_1, \dots, x_r) | \exists g \in G(\mathfrak{k}) \text{ such that } x_i = g \cdot x_i^0 \text{ for all } i\}.$$

We define $\sigma(i)$, $1 \le i \le r$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, by the equation

$$\sigma(x_i^0) = x_{\sigma(i)}^0.$$

By Lemma 2.1, there exists an unramified extension L of $\mathbb{Q}_p$, such that $\zeta_p = \phi \circ \zeta_p$ can be viewed as a homomorphism from $\mathscr{D}^L$ to $\mathscr{G}$, which itself is defined by a homomorphism ξ_p from the extension

$$1 \to L_1^\times \to \mathscr{D}_{L_1}^L \to \mathrm{Gal}(L_1/\mathbb{Q}_p) \to 1$$

to $G(L_1) \rtimes \mathrm{Gal}(L_1/\mathbb{Q}_p)$, where L_1 is a finite Galois extension of $\mathbb{Q}_p$. Let L_2 be the maximal unramified subfield of L_1 over L. The cocycle defining $\mathscr{D}_{L_1}^L$ is actually defined over $\mathrm{Gal}(L/\mathbb{Q}_p)$. Therefore, $\mathrm{Gal}(L_1/L)$ and, in particular, $\mathrm{Gal}(L_1/L_2)$, are embedded in $\mathscr{D}_{L_1}^L$, and the restriction of ξ_p to $\mathrm{Gal}(L_1/L_2)$ is defined by a 1-cocycle of this group with values in $G(L_1)$. According to a theorem of Steinberg (see [Se], p. 3), there exists an unramified extension L' of L such that this cocycle, when viewed as a cocycle of $\mathrm{Gal}(L_1L'/L_2L')$ with values in $G(L_1L')$, becomes trivial. Consequently, we can replace L by L_2L' and ξ_p by an equivalent homomorphism ξ_p', and assume that $L_1 = L$.

The homomorphism ξ_p' then maps $\mathscr{D}_L^L$ to $G(L) \rtimes \mathrm{Gal}(L/\mathbb{Q}_p)$. The group $\mathscr{D}_L^L$ is isomorphic to the Weil group $W_{L/\mathbb{Q}_p}$, and $W_{L/\mathbb{Q}_p}$ is naturally mapped to $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Let ξ be the homomorphism from $W_{L/\mathbb{Q}_p}$ to $G(\mathfrak{k}) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ defined by the commutativity of the following diagram:

$$\begin{tikzcd}
 & W_{L/\mathbb{Q}_p} \arrow[dl] \arrow[d] \arrow[dr] & \\
G(L) \rtimes \mathrm{Gal}(L/\mathbb{Q}_p) & G(L) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \\
 & G(\mathfrak{k}) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p). &
\end{tikzcd}$$

Let w be a preimage of the Frobenius element σ under the canonical homomorphism $W_{L/\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. The element w is uniquely determined up to a unit in $L^\times$. Therefore, a second possible w' can be written as $x \cdot \sigma(x)^{-1} \cdot w$ with $x \in \mathfrak{k}^\times$. The restriction of ξ_p' to $\mathscr{D}_L^L = \mathbb{G}_m$ is algebraic. Let $c = \xi_p'(x)$. Then $\xi(w') = c\xi(w)c^{-1}$. Hence, $F = \xi(w)$ is uniquely determined by ξ_p' alone, up to conjugation by an element of $\xi_p'(\mathfrak{k}^\times)$.

When we factor h by a torus T, it defines a cocharacter μ of T. Although μ itself depends on h and not only on its conjugacy class, the orbit of μ under the Weyl group is well-defined and independent of h and, in an obvious sense, of T.
168 Therefore, for any Cartan subgroup T of G, we can associate an orbit $\{\mu\}$ in $X^*(T)$ to the Shimura variety. On the other hand, any two points y, z in $\mathscr{B}(G,\mathfrak{k})$ lie in an apartment, and $z = y + a$ with $a \in X_*(T) \otimes \mathbb{R}$. Again, the orbit of a under the Weyl group (rather than a) is uniquely determined, which we denote by $\mathrm{inv}(z,y)$.

The group $G(\mathfrak{k}) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ acts on $\mathscr{X}$

$$g \rtimes \sigma : (x_1, \ldots, x_r) \mapsto (g\sigma(x_1), \ldots, g\sigma(x_r))$$

and also acts in the same way on $\mathscr{B}(G,\mathfrak{k})$. Let X_p be the set of $x = (x_1, \ldots, x_r)$ in $\mathscr{X}$ such that for every i,

$$\mathrm{inv}(x_{\sigma(i)}, Fx_i) = \{\mu\}.$$

(5.d) *The set X_p is non-empty.*

We define

$$X_l = \{x \in G(\overline{\mathbb{Q}}_l) \mid \phi \circ \zeta_l = \mathrm{ad}\, x \circ \xi_l\}, \quad l \neq p.$$

Since

$$G(\mathbb{Q}_l) = \{g \in G(\overline{\mathbb{Q}}_l) \mid \mathrm{ad}\, g \circ \xi_l = \xi_l\},$$

the group $G(\mathbb{Q}_l)$ acts on the right on X_l simply transitively. We want to introduce a restricted product $X^p = \prod_l' X_l$. According to [T], §3.9, we can find a finite set S and for each $l \notin S$, a hyperspecial point x_l in $\mathscr{B}(G,\mathbb{Q}_l)$. Neither S nor the x_l's are uniquely determined, but two possible families of x_l's are almost everywhere equal. Therefore, for each finite extension L' of $\mathbb{Q}$ and almost every place v of L', there is a distinguished hyperspecial subgroup K_v. Let ϕ be defined by $q_\varrho \mapsto g_\varrho \rtimes \varrho$. Then, with the notation from the second section, $\phi \circ \zeta_l$ is defined by $\varrho \mapsto \phi\,(e_\varrho(l)) \cdot g_\varrho \rtimes \varrho$, where $\varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$. However, we can choose a finite extension L' of $\mathbb{Q}$ such that $\phi(e_\varrho) \cdot g_\varrho$ lies in $G(\mathbb{A}_{L'})$ for each ϱ. Therefore, $\phi\,(e_\varrho(l)) \cdot g_\varrho$ lies in $\prod_{v/l} K_v$ for almost all l and each ϱ. The cocycle $\{\phi\,(e_\varrho(l)) \cdot g_\varrho\}$ can then be bounded for almost every l in $\prod_{v/l} K_v$, which provides a distinguished subset of X_l for almost every l, and thus a restricted product.

Let
$$I_\phi = \{g \in G(\overline{\mathbb{Q}}) \mid \operatorname{ad} g \circ \phi = \phi\}.$$
The group I_ϕ acts on X_l, $l \neq p$, and on X^p. Let
$$J_\phi = \{g \in G(\overline{\mathbb{Q}}) \mid \operatorname{ad} g \circ \xi_p = \xi_p\}$$
and
$$J'_\phi = \{g \in G(\mathfrak{k}) \mid \operatorname{ad} g \circ \xi'_p = \xi'_p\}.$$
The group J'_ϕ acts on X_p. If $\xi'_p = \operatorname{ad} h \circ \xi_p$, then $g \mapsto hgh^{-1}$ maps the groups I_ϕ 169
and J_ϕ into J'_ϕ. Thus, I_ϕ acts on X_p. Let
$$X_\phi(K) = I_\phi \backslash X_p \times X^p / K^p.$$
Let $\mathfrak{p}$ now be a prime of E, the reflex field of the Shimura variety
$$\mathrm{Sh} = \mathrm{Sh}_K = \mathrm{Sh}_K(G, h).$$
We assume that $\mathfrak{p}$ is defined by the embeddings $E \subset \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. The embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ can always be chosen in this way. E is unramified at the place p. Let $r = [E_p : \mathbb{Q}_p]$ and $\Phi(x_1, \ldots, x_r) = F^r(x_1, \ldots, x_r)$. Then, Φ acts on X_p and on X_ϕ.

Let $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $E_\mathfrak{p}$. We are looking for a model of Sh over $\mathcal{O}_\mathfrak{p}$ with at least the following properties, and we conjecture that it exists if K^p is sufficiently small.

(5.e) *Let κ be the residue field of $\mathcal{O}_\mathfrak{p}$ and $\bar\kappa$ its algebraic closure. Then,* $\mathrm{Sh}(\bar\kappa)$ *as a set with the action of the Frobenius element is isomorphic to the union over equivalence classes of admissible ϕ of the sets $X_\phi(K)$ with the action of Φ.*

(5.f) *If* Sh *is proper over E, then the model is proper over $\mathcal{O}_\mathfrak{p}$.*

(5.g) *If K_p is hyperspecial* [T], *the model over $\mathcal{O}_\mathfrak{p}$ is smooth.*

In the next section we will use the results of the previous section to justify this conjecture. In this section we will try to make it more clear and prepare the transition to further results of Kottwitz. We will freely use terms and results from him.

We start with a simple example that allows us to check the signs. Let $G = \mathbb{G}_m$, so that $X_*(G) = \mathbb{Z}$, and let h be the homomorphism $(z, w) \to zw$, so that $\mu = 1 \in \mathbb{Z}$. The corresponding variety Sh is of dimension zero, and its set of points is

$$\mathrm{Sh}(\overline{\mathbb{Q}}) \cong \mathbb{Q}^\times \backslash I_f / K^p \cdot K_p, \quad I_f = \mathbb{A}_f^\times. \tag{5.h}$$

We only consider the case where $K_p = \mathbb{Z}_p^\times$. In Deligne's definition, which we use, two inverses appear (see [De1], §2.2.3), namely the inverse of the reciprocity isomorphism in class field theory, which itself is the inverse of the conventional one. Therefore, the Frobenius element acts on $\mathrm{Sh}(\overline{\mathbb{Q}})$ as $I_f \ni a \mapsto p^a$, where p is considered as an element of $\mathbb{Q}_p^\times$.

On the other hand, according to (5.a), $\phi = \psi_\mu$ represents the only admissible class. If we choose $x \in X^p$, we can use the map $t \mapsto xt$ to identify X^p with I_f^p. Furthermore, $I_\phi = \mathbb{Q}^\times$. Let $\mathcal{O}$ be the ring of integers of $\mathfrak{k}$. Then $\mathscr{X} = \mathfrak{k}^\times / \mathcal{O}^\times = \mathbb{Q}_p^\times / \mathbb{Z}_p^\times$.

The homomorphism $\psi_\mu \circ \zeta_p$ is $\xi_{-\mu}$-equivalent. Since G splits over $\mathbb{Q}$, $\xi_{-\mu}$ is defined on $\mathscr{D}_{\mathbb{Q}_p}^{\mathbb{Q}_p}$ as
$$z \mapsto z^{-1}.$$

170 Because we take the inverse of the conventional reciprocity isomorphism, p^{-1} corresponds to the Frobenius element, and p^{-1} is mapped to $p \rtimes 1 \in G(\mathbb{Q}_p) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Consequently, F sends the element $a \in \mathfrak{k}^\times \pmod{\mathcal{O}^\times}$ to pa. According to the sign that appears in [T], Eq. 1.2(1), we have:

$$\mathrm{inv}(a, pa) = -\omega(p^{-1}) = 1 = \mu.$$

Thus, $X_p = \mathscr{X} = \mathbb{Q}_p^\times/K_p$, and

$$I_\phi \backslash X_p \times X^p/K^p \cong \mathbb{Q}^\times \backslash I_f/K^p \cdot K_p \tag{5.i}$$

The mappings in (5.h) and (5.i) establish an identification between $I_\phi \backslash X_p \times X^p/K^p$ and $\mathrm{Sh}(\overline{\mathbb{Q}})$, under which $F = \Phi$ corresponds to the action of the Frobenius element. We emphasize that this identification is by no means canonical.

It is evident that Sh has a model over $\mathbb{Z}_p$ with good reduction, and in this case, the conjecture is trivially valid. In general, if G is a torus, the conjecture can be easily verified.

Then $\mathscr{X}$ is $T(\mathfrak{k})/T(\mathcal{O})$. If T splits over the unramified extension L of $\mathbb{Q}_p$, let $k = [L : \mathbb{Q}_p]$ and let σ be a Frobenius element in $\mathrm{Gal}(L/\mathbb{Q}_p)$. The fundamental class is given as follows:

$$a_{i,j} = 1, \text{ for } 0 \leq i, j \leq k, i + j < k,$$
$$a_{i,j} = p^{-1}, \text{ for } 0 \leq i, j \leq k, i + j \geq k.$$

The element $1 \times \sigma \in \mathscr{D}_L^L$ corresponds to the preimage in $W_{L/\mathbb{Q}_p}$ of a Frobenius element. Thus, $F = p^\mu \times \sigma$.

If ν is defined as in [T], §1.2, then:

$$\mathrm{inv}(x, Fx) - \mu = \mathrm{inv}(x, \sigma(x) + \nu(p^\mu)) - \mu = \mathrm{inv}(x, \sigma(x)),$$

from which it easily follows that:

$$X_p \cong T(\mathbb{Q}_p)/T(\mathbb{Z}_p).$$

Let r be the smallest power of σ that fixes μ. The field E is unramified over $\mathbb{Q}_p$, and $r = [E : \mathbb{Q}_p]$. Consequently,

$$\Phi = p^{-\nu_2} \times \sigma^r,$$

where σ now denotes the Frobenius automorphism of $\mathfrak{k}/\mathbb{Q}_p$, and

$$-\nu_2 = \mu + \sigma\mu + \cdots + \sigma^{r-1}\mu.$$

Thus, Φ acts on X_p as translation by $p^{-\nu_2} \in T(\mathbb{Q}_p)$, and $p^{-\nu_2}$ is precisely the image of $p \in E^\times$ under the homomorphism $\mathrm{NR}(\mu) : E^\times \to T$, as defined in [De1], §2.2.3.

These remarks allow us to identify $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K^p \cdot K_p$ with $I_\phi \backslash X_p \times X^p/K^p$, so that Φ corresponds to the Frobenius substitution at $\mathfrak{p}$, and thus to test the conjecture.

171 Let T be a Cartan subgroup of G, defined over $\mathbb{Q}$. Assume T is fundamental, meaning the group $T_{\mathrm{ad}}(\mathbb{R})$ is compact, where T_{ad} is the image of T in the adjoint group G_{ad}. Then h can be chosen to factor through T, thereby defining a cocharacter μ of T:

$$h(z, w) = z^\mu w^{\bar{\mu}}.$$

Let $\psi_{T,\mu}$ denote the composition

$$\mathscr{Q} \xrightarrow{\psi_\mu} \mathscr{G}_T \subset \mathscr{G} = \mathscr{G}_G.$$

Lemma 5.2. *$\psi_{T,\mu}$ is admissible.*

Proof of Lem. 5.2.

The conditions (5.a) and (5.c) are clear, as is (5.b), since the fundamental class at infinity is defined by $a_{\iota,\iota} = -1$, and consequently, ξ_∞ is, by definition, equivalent to $\psi_\mu \circ \xi_\infty$.

Let

$$\nu_p = \nu_2 = -\sum_{\sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)} \sigma\mu.$$

Let P be the parabolic subgroup defined over $\mathbb{Q}_p$ that contains T, with roots defined by $\langle \nu_p, \alpha \rangle \leq 0$. The group P contains the Levi factor J, whose roots are defined by $\langle \nu_p, \alpha \rangle = 0$. The field $\mathbb{Q}_p^{\mathrm{un}}$ splits J.

We may assume that the homomorphism defining X_p is given by $\xi_p' = \mathrm{ad}\, h \circ \xi_p$ with $h \in T(\overline{\mathbb{Q}}_p)$. Then $J_\phi \subset J(\overline{\mathbb{Q}}_p)$ resp. $J_\phi' \subset J(\mathfrak{k})$. Indeed, J is the centralizer of the image of the kernel of $\mathscr{D}$, so that ξ_p resp. ξ_p' define a twisted form $\mathscr{J}_\phi$ resp. $\mathscr{J}_\phi'$ of J. The groups J_ϕ and J_ϕ' are the groups of $\mathbb{Q}_p$-rational points on these twisted forms.

The Bruhat-Tits buildings $\mathscr{B}(G,\mathfrak{k})$, $\mathscr{B}(J,\mathfrak{k})$, as well as $\mathscr{B}(G,\mathbb{Q}_p)$, $\mathscr{B}(J,\mathbb{Q}_p)$, are described in [T], §2.1. Since J contains a maximal split torus of G, this characterization yields a commutative diagram of embeddings (see also [T], §2.6):

$$\begin{tikzcd} \mathscr{B}(J,\mathbb{Q}_p) \arrow[r, hook] \arrow[d, hook] & \mathscr{B}(G,\mathbb{Q}_p) \arrow[d, hook] \\ \mathscr{B}(J,\mathfrak{k}) \arrow[r, hook] & \mathscr{B}(G,\mathfrak{k}). \end{tikzcd}$$

We also need an elliptic Cartan subgroup T' of J, defined over $\mathbb{Q}_p$, that splits over $\mathfrak{k}$, and such that the maximal compact subgroup of $T'(\mathbb{Q}_p)$ is contained in K_p. In case (b), the second condition is automatically satisfied, and for the existence of T', we can refer to [Kn], p. 271. In case (a), we also refer to [T], §3.4, and [SGAD]. Then $x_1^0, \ldots, x_r^0$ lie in the apartment of T' (see [T], §3.6).

Let μ' be a cocharacter of T', conjugate to μ under J. We will first show that if 172
F' is defined using $\xi_{-\mu'}$ instead of ξ_p or the equivalent $\xi_{-\mu}$, but otherwise defined as F, then the following holds:

$$\mathrm{inv}(x_{\sigma(i)}^0, F'x_i^0) = \{\mu'\} = \{\mu\}. \tag{5.j}$$

The group T' splits over an unramified extension. Consequently, as we have already calculated in the abelian case, we have $F' = p^{\mu'} \rtimes \sigma$, and

$$F'x_i^0 = \sigma(x_i^0) - \mu'.$$

Thus (5.j) would be proven.

Our goal, however, was to show that X_p is non-empty, and for this, it suffices to show that $\xi_{-\mu}$ and $\xi_{-\mu'}$, as homomorphisms from $\mathscr{D}$ to $\mathscr{G}_J$, are equivalent when the image of T' in the adjoint group of J is anisotropic, as we may assume. This follows immediately from the results of Kottwitz [K4] (see the Note at the end of this section). We denote the set of equivalence classes of these homomorphisms by $C(J)$. Since the image T' in J_{ad} is anisotropic, we know that

$$\nu_p' = -\sum_{\sigma \in \mathrm{Gal}(L_{v_2}/\mathbb{Q}_p)} \sigma\mu' = \nu_p$$

is central. Here, it is assumed that L is sufficiently large such that L_{v_2} splits the torus T'. Consequently, the classes of $\xi_{-\mu}$ and $\xi_{-\mu'}$ lie in $C(J)_b \cong B(J)_b$ (see the Note) and are labeled, according to Kottwitz, by elements of $X^*(Z(\widehat{J})^{\Gamma(p)})$, where $\Gamma(p) = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ and $Z(\widehat{J})$ denotes the center of the connected component of the L-group. By definition, $\xi_{-\mu}$ corresponds to the character $z \mapsto z^{\mu}$, and $\xi_{-\mu'}$ corresponds to the character $z \mapsto z^{\mu'}$, and these characters are equal on $Z(\widehat{J})$. More precisely, in the bijection defined by Kottwitz, a sign is involved. We choose it so
QED Lem. 5.2. that $\xi_{-\mu}$ corresponds to the character $z \mapsto z^{\mu}$.

We can now explain why we assumed G_{der} to be simply connected. Let G be arbitrary. Then we can find a z-extension G' in the sense of Kottwitz [K1] and a factorization

$$\begin{tikzcd} \mathbb{S} \arrow[r, "h'"] \arrow[dr, "h"'] & G' \arrow[d, "\alpha"] \\ & G \end{tikzcd}$$

We can also assume ([MS2], §3.4) that the reflex field $E(G', h')$ is the field $E = E(G, h)$. Let A be the kernel of α.

173 Let K' be an open compact subgroup of $G'(\mathbb{A}_f)$ such that $\alpha(K') \subset K$. The map

$$G'(\mathbb{Q})\backslash G'(\mathbb{A})/K' \to G(\mathbb{Q})\backslash G(\mathbb{A})/K$$

is the restriction on the set of complex-valued points of a morphism from $\mathrm{Sh}_{K'}(G', h')$ to $\mathrm{Sh}_K(G, h)$ defined over E. This map is surjective because $G'(\mathbb{Q}_v) \to G(\mathbb{Q}_v)$ is surjective for every place v of $\mathbb{Q}$. It follows from [T], §3.9.1, that $\alpha(K')$ is a subgroup of finite index in K, and we may even assume that $\alpha(K')$ is a normal subgroup of K.

Then, the finite group

$$B = A(\mathbb{Q})\backslash\alpha^{-1}(K)/K'$$

acts on the right on $\mathrm{Sh}_{K'}(G', h')$, and the quotient is $\mathrm{Sh}_K(G, h)$, since $G'(\mathbb{Q}) \to G(\mathbb{Q})$ is surjective.

Ultimately, we are interested in the cohomology groups of certain sheaves on $\mathrm{Sh}_K(G, h)$ (see [L1]), which are associated with a representation ξ of G. These can be obtained as the fixed points of B in the cohomology of the pullback of the sheaves, defined via $\xi \circ \alpha = \xi'$, on $\mathrm{Sh}_{K'}(G', h')$. Consequently, we may, without loss of generality, work with G' instead of G.

Nevertheless, we can introduce the concept of an admissible homomorphism $\phi : \mathscr{Q} \to \mathscr{G} = \mathscr{G}_G$. Later, we will demonstrate with an example that the conjecture, as stated above, cannot hold simultaneously for both G and G' in certain cases. For this we need a theorem, which we will prove first and which is important anyway for the transition to the trace formula.

Theorem 5.3. *Assume the Hasse principle holds for G_{sc}. If G is quasi-split over $\mathbb{Q}_p$ and K_p is hyperspecial, then every admissible ϕ is equivalent to some $\psi_{T,\mu}$.*

We start the proof with a lemma.

Lemma 5.4. *Assume the Hasse principle holds for G_{sc}. Let G be quasi-split over $\mathbb{Q}_p$, and let ϕ be admissible. Then ϕ is equivalent to some ϕ' such that $\phi'(\delta_n)$ lies in $G(\mathbb{Q})$ for sufficiently large n.*

Proof of Lem. 5.4.

We treat ϕ as a map $\phi : \mathscr{Q}(L,m) \to \mathscr{G}$. Let $\gamma_n = \phi(\delta_n)$, and let $g_\varrho \rtimes \varrho$ denote the image of q_ϱ. Then

$$\gamma_n = \phi(\delta_n) = \phi(q_\varrho \delta_n q_\varrho^{-1}) = g_\varrho \varrho(\gamma_n) g_\varrho^{-1}.$$

Hence the conjugacy class of γ_n is rational.

Let G^* be the quasi-split form of G, and let $\psi : G \to G^*$ be an isomorphism over $\overline{\mathbb{Q}}$ such that $\psi^{-1}\sigma(\psi)$ is inner for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Define $\gamma_n^* = \psi(\gamma_n)$. Its conjugacy class is also rational. Replacing n by a sufficiently large multiple if necessary, we can ensure that the group $C_{\gamma_n^*}$, which appears in Theorem 4.7 of [K1], is trivial. This theorem allows us to replace ϕ with $\operatorname{ad} g \circ \phi$, making γ_n^* rational.

Lemma 5.5. *The Zariski closure of $\{\delta_n^k \mid k \in \mathbb{Z}\}$ in $Q(L,m)$ is $Q(L,m)$.* 174

Proof of Lem. 5.5.

This closure is an algebraic subgroup of $Q(L,m)$. If it were not $Q(L,m)$ itself, there would exist a nontrivial character χ_π of $Q(L,m)$ such that

$$1 = \chi_\pi(\delta_n) = \pi^{\frac{n}{m}}.$$

This implies that π is a root of unity, and $\chi_\pi = 1$, a contradiction. *QED Lem. 5.5.*

For simplicity, we use ψ to identify G and G^*. Then the action of the Galois group on G^* is defined by

$$\sigma^*(g) = b_\sigma \sigma(g) b_\sigma^{-1},$$

where $b_\sigma \in G(\overline{\mathbb{Q}})$, and $b_\varrho \varrho(b_\sigma) b_{\varrho\sigma}^{-1} \in Z(\overline{\mathbb{Q}})$, with Z denoting the center of G. Let I^* be the centralizer of γ_n^* in G^*, which is also the centralizer of $\phi(Q(L,m))$.

We set

$$g_\varrho = h_\varrho b_\varrho.$$

Then

$$\gamma_n^* = \gamma_n = g_\varrho \varrho(\gamma_n) g_\varrho^{-1} = h_\varrho b_\varrho \varrho(\gamma_n) b_\varrho^{-1} h_\varrho^{-1} = h_\varrho \varrho^*(\gamma_n^*) h_\varrho^{-1} = h_\varrho \gamma_n^* h_\varrho^{-1},$$

and $h_\varrho \in I^*$.

We now show that $h_\varrho \varrho^*(h_\sigma) h_{\varrho\sigma}^{-1}$ lies in the center of I^*, so that $\{h_\varrho\}$ defines a form I' of I^* on which the Galois group acts according to the equations

$$\sigma'(h) = h_\sigma \sigma^*(h) h_\sigma^{-1}.$$

Specifically,

$$h_\varrho \varrho^*(h_\sigma) h_{\varrho\sigma}^{-1} = h_\varrho b_\varrho \varrho(h_\sigma) b_\varrho^{-1} h_{\varrho\sigma}^{-1} = g_\varrho \varrho(g_\sigma) \varrho(b_\sigma^{-1}) b_\varrho^{-1} h_{\varrho\sigma}^{-1}.$$

Since

$$z = b_{\varrho\sigma} \varrho(b_\sigma^{-1}) b_\varrho^{-1} \in Z(\overline{\mathbb{Q}}),$$

the last term in this equation becomes

$$g_\varrho \varrho(g_\sigma) b_{\varrho\sigma}^{-1} h_{\varrho\sigma}^{-1} z = g_\varrho \varrho(g_\sigma) g_{\varrho\sigma}^{-1} z = \phi(g_{\varrho,\sigma}) z,$$

and hence it lies in the center of I^*.

According to [K1], Lemma 3.3, we can assume that I^* is quasi-split. Since I' contains a Cartan subgroup defined over $\mathbb{Q}$, there exists a Cartan subgroup T^* of I^* defined over $\mathbb{Q}$ and an element $h \in I^*$ such that $h h_\varrho \varrho^*(h^{-1}) \in T^*$ for every ϱ. We replace ϕ by $\operatorname{ad} h \circ \phi$ and assume that $h_\varrho \in T^*$.

We first show that T^*, which is also a Cartan subgroup of G^*, transfers to the 175 group G (in the sense of [L3]). This means that there exists $g \in G(\overline{\mathbb{Q}})$ such that

$$\sigma^*(t) = g^{-1}\big(\sigma(gtg^{-1})\big)g, \quad t \in T^*.$$

Then gT^*g^{-1} is a Cartan subgroup T of G defined over $\mathbb{Q}$. Replacing ϕ by $\operatorname{ad} g \circ \phi$, γ_n becomes $g\gamma_n g^{-1}$, which lies in $T(\mathbb{Q})$ because γ_n lies in $T^*(\mathbb{Q})$. Thus Lemma 5.4 would be proven.

To show that T^* transfers to G, we need a sequence of considerations from [L3].

Lemma 5.6. *Assume the Hasse principle holds for G_{sc}. If the Cartan subgroup T^* transfers locally everywhere to G, and if T^*_{ad} is anisotropic at at least one place of $\mathbb{Q}$, then T^* transfers globally to G.*

Proof of Lem. 5.6.

In [L3] (see Lemma 7.15), an obstruction is introduced, whose vanishing is necessary and sufficient for T^* to transfer to G. This obstruction lies in

$$H^{-1}(\mathrm{Gal}(L/\mathbb{Q}), X_*(T^*_{\mathrm{sc}}))$$

modulo

$$\bigoplus_v \xi_v \left(H^{-1}(\mathrm{Gal}(L_v/\mathbb{Q}_v), X_*(T^*_{\mathrm{sc}}))\right),$$

where ξ_v denotes the natural map

$$H^{-1}(\mathrm{Gal}(L_v/\mathbb{Q}_v), X_*(T^*_{\mathrm{sc}})) \to H^{-1}(\mathrm{Gal}(L/\mathbb{Q}), X_*(T^*_{\mathrm{sc}})).$$

But if T^*_{ad} is anisotropic at a place v, then ξ_v is clearly surjective, and the obstruction is automatically zero.

QED Lem. 5.6.

We apply Lemma 5.6 to our T^*. It is clear that T^* transfers over $\mathbb{Q}_p$, because G is quasi-split over $\mathbb{Q}_p$ by assumption. At infinity, one obtains the twist I' by restricting ξ_∞ to I^*. Then I' is contained in the group G' from Lemma 5.1. Since T^* is a Cartan subgroup of I', it follows from that lemma that T^*_{ad} is anisotropic over $\mathbb{R}$. Thus T^* transfers to G (at infinity).

By assumption, $\sigma \mapsto e_\sigma(l)g_\sigma \rtimes \sigma = a_\sigma \rtimes \sigma,\ \sigma \in \mathrm{Gal}(L_l/\mathbb{Q}_L)$ is equivalent to the canonical neutralization of $\mathscr{G}$. Hence, $\sigma_1 : g \mapsto a_\sigma \sigma(g) a_\sigma^{-1}$ defines a form of G that is isomorphic to G. However, $e_\sigma(l)$ lies in $\phi(Q(L,m))$, which is contained in the center of I^*. For $t \in T^*$, it therefore holds that

$$\sigma_1(t) = \sigma^*(t),$$

and T^* transfers to G over $\mathbb{Q}_l$.

QED Lem. 5.4.

Proof of Thm. 5.3.

We now return to Theorem 5.3, continuing to assume that $\gamma_n = \phi(\delta_n)$ is rational for sufficiently large n. The centralizer of γ_n is $\mathscr{I}$. Replacing n by an appropriate multiple, we may assume that $\mathscr{I}$ is connected. The homomorphism ϕ defines a twist $\mathscr{I}_\phi$ of $\mathscr{I}$, with $\mathscr{I}_\phi(\mathbb{Q}) = I_\phi$.

176 We have $\phi(q_\varrho) = g_\varrho \rtimes \varrho$, where g_τ now lies in $\mathscr{I}$ because γ_n is rational. We need a torus T in $\mathscr{I}$ defined over $\mathbb{Q}$, and an $a \in I(\overline{\mathbb{Q}})$ such that $a g_\varrho \varrho(a)^{-1} \in T$ for every ϱ. Then we can replace ϕ with $\operatorname{ad} a \circ \phi$ and assume that

$$g_\varrho \varrho(t) g_\varrho^{-1} = \varrho(t), \tag{5.k}$$

for every ϱ and $t \in T(\overline{\mathbb{Q}})$. Thus, T becomes also a $\mathbb{Q}$-torus in $\mathscr{I}_\phi$.

The group $\mathscr{I}$ satisfies the Hasse principle with G.[7] Hence, we can apply the following lemma to $\mathscr{I}$ to obtain T. Although the lemma must be well-known, we have not found a reference in the literature.

Lemma 5.7. *Assume the Hasse principle holds for G,[8] and let $\psi_1 : G \to G_1$ be an isomorphism defined over $\overline{\mathbb{Q}}$ such that $\psi_1^{-1}\sigma(\psi_1)$ is inner. Then there exist Cartan subgroups T and T_1 of G and G_1, both defined over $\mathbb{Q}$, and an element $g \in G_1(\overline{\mathbb{Q}})$ such that $\operatorname{ad} g \circ \psi_1$ maps T to T_1 and is an isomorphism between T and T_1 defined over $\mathbb{Q}$.*

Proof of Lem. 5.7.

The group G_1 is an inner form of G^*, and the lemma asserts the existence of a Cartan subgroup T^* that transfers to both G and G_1. This is a consequence of Lemma 5.6 and the following three lemmas, which we apply to G and G_1.

Lemma 5.8 ([Sh], Corollary 2.9). *If T^* is fundamental in G^* over $\mathbb{R}$, then it transfers locally at infinity to G.*

Lemma 5.9. *Let v be a finite place of $\mathbb{Q}$, and let T^* be a Cartan subgroup of G^* over $\mathbb{Q}_v$ with T^*_{ad} anisotropic. Then T^* transfers to G.*

Proof of Lem. 5.9.

Let N be the kernel of $G^*_{\mathrm{sc}} \to G^*_{\mathrm{ad}}$. The lemma follows as a well-known consequence of the existence of a bijection $H^1(\mathbb{Q}_p, G^*_{\mathrm{ad}}) \to H^2(\mathbb{Q}_p, N)$, (see [Kn], Theorem 2) and the exact sequence:

$$H^1(\mathbb{Q}_p, T^*_{\mathrm{ad}}) \to H^2(\mathbb{Q}_p, N) \to H^2(\mathbb{Q}_p, T^*_{\mathrm{sc}}) = 0.$$

For the existence of such T^*, refer to [Kn], p. 271. *QED Lem. 5.9.*

Lemma 5.10. *Let S be a finite set of places of $\mathbb{Q}$, and for each $v \in S$ let T^*_v be a Cartan subgroup of G^* over $\mathbb{Q}_v$. Then there exists a Cartan subgroup T^* of G^* defined over $\mathbb{Q}$ such that for each $v \in S$, T^* is conjugate to T^*_v over $\mathbb{Q}_v$.*

Proof of Lem. 5.10.

It follows easily from the Bruhat decomposition that the variety G^*_{sc} is rational. Thus, the weak approximation theorem holds for G^*_{sc}, and we immediately conclude the lemma by approximating a product of regular elements $\prod_v t_v$ from $T_v(\mathbb{Q}_v)$. *QED Lem. 5.10, 5.7.*

The torus T, for which (5.k) holds, is also a Cartan subgroup of G defined over $\mathbb{Q}$. From Lemma 5.1 and assumption (5.a), it follows that T_{ad} is anisotropic over $\mathbb{R}$. Therefore, we can choose h such that it can be factored through T:

$$h(z, w) = z^{\mu_h} w^{\bar{\mu}_h}.$$

Lemma 5.11. *There exists μ in the orbit of μ_h under the Weyl group such that on* 177
the kernel ϕ and $\psi_{T,\mu}$ are equal, and such that

$$\phi(q_\sigma) = b_\sigma \psi_{T,\mu}(q_\sigma),$$

where $\{b_\sigma\}$ is a 1-cocycle with values in T.

Proof of Lem. 5.11.

We emphasize that $\psi_{T,\mu}$ may not be admissible, as μ may not be μ_h. The restriction of ϕ to $Q(L, m)$ maps $Q(L, m)$ into T and is defined over $\mathbb{Q}$. We can assume, by replacing $Q(L, m)$ with $Q(L, n)$, that $n = m$. Let $\nu_p = \phi(\nu_2)$, which is

[7] The authors probably mean that if one assumes that Hasse principle holds for G_{sc}, then it also holds for $\mathscr{I}_{\mathrm{sc}}$. The Hasse principle for all semi-simple simply connected groups have been proved unconditionally.

[8] Here G should be replaced by G_{sc}.

a cocharacter of T. It suffices to show that there exists a μ in the orbit of μ_h such that

$$\mathrm{Nm}_{L_p/\mathbb{Q}_p}(\mu) = -\nu_p, \tag{5.l}$$

since

$$\begin{aligned}\Big|\prod_{\sigma\in\mathrm{Gal}(L_p/\mathbb{Q}_p)}\sigma\lambda(\gamma_m)\Big|_p &= \Big|\prod_{\sigma\in\mathrm{Gal}(L_p/\mathbb{Q}_p)}\sigma\phi^*(\lambda)(\delta_m)\Big|_p\\ &= \Big|\prod_{\sigma\in\mathrm{Gal}(L_p/\mathbb{Q}_p)}\sigma\pi_\lambda\Big|_p\\ &= q^{\langle\nu_2,\chi_{\pi_\lambda}\rangle}.\end{aligned}$$

Therefore, by the definition of $\psi_{T,\mu}$, it is defined on $Q(L,m)$ by $\delta_m\mapsto\gamma_m$.

We will conclude the existence of μ from the fact that X_p is non-empty. We need to replace the homomorphism $\xi_p=\phi\circ\zeta_p$ with $\xi'_p=\mathrm{ad}\,u\circ\xi_p$ to define F. We can assume that $u\in T$.

Let L be a finite unramified extension of $\mathbb{Q}_p$ such that ξ'_p is defined on $\mathscr{D}_L^L$, and let

$$\xi'_p(d_\sigma)=g_\sigma\rtimes\sigma,$$

where σ is the Frobenius element in $\mathrm{Gal}(L/\mathbb{Q}_p)$. To the homomorphism ξ'_p, we associate a parabolic subgroup P and a Levi factor J. Let M be the centralizer of a maximal split subtorus of T. The group M is a Levi factor of a parabolic subgroup Q of G defined over $\mathbb{Q}_p$ and contained in P. The group M is quasi-split.

We have already seen that we can embed the building $\mathscr{B}(M,\mathfrak{k})$ into $\mathscr{B}(G,\mathfrak{k})$ in such a way that the point x^0 used to define K_p is contained in $\mathscr{B}(M,\mathfrak{k})$. We can even assume that x^0 lies in the apartment of a Cartan subgroup T' of M which is defined over $\mathbb{Q}_p$ and split over $\mathfrak{k}$. A point $x\in X_p$ can be written as $x=hx^0$ with $h=nm\in Q(\mathfrak{k})$, where $m\in M(\mathfrak{k})$ and n lies in the unipotent radical of Q. Then

$$\mathrm{inv}(x,F_x)=\mathrm{inv}(hx^0,g_\sigma\sigma(h)x^0)=\mathrm{inv}(x^0,h^{-1}g_\sigma\sigma(h)x^0), \tag{5.m}$$

since $\sigma(x^0)=x^0$.

178 Let $\{\mu'\}$ be the orbit in $X_*(T')$ corresponding to μ'. Define $t_{\mu'}$ by

$$\chi(t_{\mu'})=p^{\langle\chi,\mu'\rangle},\quad \chi\in X^*(T'),$$

and let $f_{\mu'}$ be the characteristic function of the double class $K_0t_{\mu'}K_0$, where K_0 is the stabilizer of x^0 in $G(\mathfrak{k})$. From (5.m), it follows that

$$f_{\mu'}(m^{-1}g_\sigma\sigma(m)n')=1,$$

where

$$n'=\sigma(m)^{-1}g_\sigma^{-1}n^{-1}g_\sigma\sigma(n)\sigma(m)$$

is in the unipotent radical of Q.

There exists $\mu''\in X_*(T')$ such that

$$m^{-1}g_\sigma\sigma(m)\in(K_0\cap M)t_{\mu''}(K_0\cap M).$$

From Lemmas 2.3.3 and 2.3.9 of [K3], it follows easily that μ'' lies in the orbit of μ', because μ' is a “minuscule weight”.

If λ is the homomorphism $M(\mathfrak{k})\to X^*(Z(\widehat{M})^\Gamma)$ introduced in Lemma 3.3 of [K3], then $\lambda(m^{-1}g_\sigma\sigma(m))$ is the restriction of μ'' to $Z(\widehat{M}^\Gamma)$.

The definition of [K3] is actually introduced only for finite extensions of $\mathbb{Q}_p$, but it can clearly be extended to $M(\mathfrak{k})$.

Equation (5.l) holds if and only if

$$[L_p : \mathbb{Q}_p]\langle \mu, \chi\rangle = -\langle \nu_p, \chi\rangle \tag{5.n}$$

for every character χ of T defined over $\mathbb{Q}_p$. Such a character can be viewed as a character of M, and if μ is a cocharacter of T conjugate to μ'' in M, then

$$\langle \mu, \chi\rangle = \langle \mu'', \chi\rangle.$$

Furthermore, μ lies in the orbit of μ_h under the Weyl group of G.

By the definition of λ, we have

$$|\chi(m^{-1}g_\sigma\sigma(m))| = p^{-\langle \mu'', \chi\rangle}.$$

Since χ is defined over $\mathbb{Q}_p$, we have

$$\chi(m^{-1}g_\sigma\sigma(m))^l = \chi(m^{-1}g_\sigma\sigma(g_\sigma)\dots\sigma^{l-1}(g_\sigma)\sigma^l(m)),$$

where $l = [L_p : \mathbb{Q}_p]$, and

$$p^{-l\langle \mu'', \chi\rangle} = |\chi(m^{-1}g_\sigma\sigma(g_\sigma)\dots\sigma^{l-1}(g_\sigma)\sigma^l(m))|.$$

By the definition, we also know that 179

$$g_\sigma\sigma(g_\sigma)\dots\sigma^{l-1}(g_\sigma) = \xi'_p(p^{-1}) = p^{-\nu_p},$$

and

$$|\chi(p^{-\nu_p})| = p^{\langle \nu_p, \chi\rangle}.$$

Since

$$|\chi(m)| = |\chi(\sigma^l(m))|,$$

we have proven equation (5.n). *QED Lem. 5.11.*

Theorem 5.3 is not yet proven. As the next approximation, we prove the following lemma.

Lemma 5.12. *There exist a torus T' over $\mathbb{Q}$ (inside G) with T'_{ad} anisotropic over $\mathbb{R}$, and an h' factoring through T', such that ϕ is equivalent to ϕ', where $\phi' = \psi_{T', \mu_{h'}}$ on the kernel of $\mathcal{Q}$, and*

$$\phi'(q_\sigma) = b_\sigma \psi_{T', \mu_{h'}}(q_\sigma),$$

where $\{b_\sigma\}$ is a 1-cocycle with values in T'.

Proof of Lem. 5.12.

Let $\mu = \omega \cdot \mu_h$ with ω in the Weyl group. Then we have a 1-cocycle α^∞ : $\mathrm{Gal}(L_\infty/\mathbb{Q}_\infty) \to T_{\mathrm{sc}}(\mathbb{C})$ defined by $1 \mapsto 1, \iota \mapsto (-1)^{\omega\mu - \mu}$, assuming that $L_\infty = \mathbb{C}$. There exists a w in the normalizer of T_{sc} in $G_{\mathrm{sc}}(\mathbb{C})$ that represents ω, and for this, we have:

$$\alpha^\infty_\sigma = w\sigma(w^{-1}), \quad \sigma \in \mathrm{Gal}(L_\infty/\mathbb{Q}_\infty),$$

(see [Sh], Prop. 4.2).

According to Lemma 7.16 of [L3], there exists a global cocycle α with values in T_{sc} that is equivalent to α^∞ at infinity. We can even assume that by appropriately modifying α^∞ and w, α^∞ is the restriction of α to $\mathrm{Gal}(L_\infty/\mathbb{Q}_\infty)$. The image of α^∞ in $H^1(\mathbb{Q}_\infty, G_{\mathrm{sc}})$ is trivial. By the Hasse principle, we conclude that α defines the trivial class in $H^1(\mathbb{Q}, G_{\mathrm{sc}})$. Let

$$\alpha_\sigma = u^{-1}\sigma(u), \quad u \in G_{\mathrm{sc}}(\overline{\mathbb{Q}}).$$

The torus $T' = uTu^{-1}$ and the homomorphism $t \mapsto t' = utu^{-1}$ are both defined over $\mathbb{Q}$. We replace ϕ by $\phi' = \operatorname{ad} u \circ \phi$, and thus γ_n by $\gamma_n' = u\gamma_n u^{-1}$. Let μ' be the cocharacter of T' corresponding to μ. The homomorphism $\psi_{T',\mu'}$ is defined, and $\psi_{T',\mu'} = \phi'$ on the kernel. It follows that

$$\phi'(g_\sigma) = b_\sigma' \psi_{T',\mu'}(q_\sigma), \quad b_\sigma' \in T'.$$

The goal is to show that $\mu' = \mu_{h'}$, where h' is still to be defined.

180 We have

$$u^{-1}\sigma(u) = w\sigma(w^{-1}), \quad \sigma \in \operatorname{Gal}(L_\infty/\mathbb{Q}_\infty).$$

Hence $uw \in G_{\text{sc}}(\mathbb{R})$. Let $h' = \operatorname{ad} uw \circ h$. Then

$$\mu_{h'} = \operatorname{ad} u \circ \mu = \mu'.$$

QED Lem. 5.12.

Up to this point, we have not used the assumption $G_{\text{der}} = G_{\text{sc}}$. All lemmas hold without this assumption. It is only needed in the final step. From the previous lemma, we replace ϕ, T, μ by $\phi', T', \mu' = \mu_{h'}$.

The cocycle $\{b_\sigma'\}$ takes values in T and therefore in G. Since ϕ is admissible, the image of $\{b_\sigma\}$ in G_{ab} represents the trivial class. We can therefore assume that $b_\sigma \in T_{\text{sc}}(\overline{\mathbb{Q}})$.

The next lemma is a remark from [De1].

Lemma 5.13. *If $G_{\text{sc}} = G_{\text{der}}$ and G_{sc} satisfies the Hasse principle, then the Hasse principle holds for the image of $H^1(\mathbb{Q}, G_{\text{sc}})$ in $H^1(\mathbb{Q}, G)$.*

Proof of Lem. 5.13.

We have a diagram with exact sequences:

$$\begin{array}{ccccc}
H^0(\mathbb{Q}, G_{\text{ab}}) & \longrightarrow & H^1(\mathbb{Q}, G_{\text{sc}}) & \longrightarrow & H^1(\mathbb{Q}, G) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\mathbb{R}, G_{\text{ab}}) & \longrightarrow & H^1(\mathbb{R}, G_{\text{sc}}) & \longrightarrow & H^1(\mathbb{R}, G).
\end{array}$$

Moreover, the image of $H^0(\mathbb{Q}, G_{\text{ab}})$ in $H^0(\mathbb{R}, G_{\text{ab}})$ is dense, and a 1-cocycle with values in a small neighborhood of the identity in $G_{\text{sc}}(\mathbb{C})$ is a coboundary. Thus the lemma follows.

QED Lem. 5.13.

Lemma 5.14. *Let G' be a connected reductive group over $\mathbb{R}$ with $G'_{\text{ad}}(\mathbb{R})$ compact, and let T' be a Cartan subgroup of G'. Then $H^1(\mathbb{R}, T') \to H^1(\mathbb{R}, G')$ is injective.*

Proof of Lem. 5.14.

Let $\{t_\sigma\}$ be a cocycle with values in T', which is a coboundary in G':

$$t_\sigma = g^{-1}\sigma(g).$$

Then $gT'g^{-1}$ is also a Cartan subgroup over $\mathbb{R}$. Since all Cartan subgroups over $\mathbb{R}$ are conjugate in $G(\mathbb{R})$, we can assume that $T' = gT'g^{-1}$. Then g defines an element of the Weyl group, which can also be defined by an element in $G(\mathbb{R})$. Therefore $\{t_\sigma\}$ is a coboundary in $T'(\mathbb{C})$.

QED Lem. 5.14.

181 Theorem 5.3 has not been proven yet, but we are at the stage where ϕ can be

written as a product of the cocycle $\{b_\sigma\}$ and $\psi_{T,\mu}$. In particular,

$$\begin{aligned}\phi(q_\sigma) &= g_\sigma \rtimes \sigma,\\ \psi_{T,\mu}(q_\sigma) &= h_\sigma \rtimes \sigma,\\ g_\sigma &= h_\sigma b_\sigma,\end{aligned}$$

with h_σ, b_σ in T.

By assumption (5.b), for $\sigma \in \mathrm{Gal}(L_\infty/\mathbb{Q}_\infty)$, we have

$$g_\sigma \rtimes \sigma = u(h_\sigma \rtimes \sigma)u^{-1},$$

with $u \in G(L_\infty)$. Hence the restriction of $\{b_\sigma\}$ to $\mathrm{Gal}(L_\infty/\mathbb{Q}_\infty)$ is trivial in the twisted group G', and therefore already trivial in T. From Lemma 5.13, we conclude that there exists a $v \in G(L)$ such that

$$b_\sigma = v\sigma(v^{-1}), \quad \sigma \in \mathrm{Gal}(L/\mathbb{Q}).$$

Then,

$$v^{-1}(g_\sigma \rtimes \sigma)v = v^{-1}(h_\sigma b_\sigma \rtimes \sigma)v = v^{-1}h_\sigma v \rtimes \sigma. \tag{5.o}$$

But the isomorphism $t \mapsto v^{-1}tv$ is a $\mathbb{Q}$-isomorphism between the two $\mathbb{Q}$-tori T and $T' = v^{-1}Tv$, and if μ' is the image of μ, it follows from (5.o) that

$$\mathrm{ad}\ v^{-1} \circ \phi = \psi_{T',\mu'}.$$

Thus, Theorem 5.3 is proven. *QED Thm. 5.3.*

Kottwitz's unpublished results show, at least in the case of an anisotropic group G, how to calculate the local zeta function based on a good enumeration of points on the reduced variety using partially proven and partially unproven results from harmonic analysis. The desired enumeration, to be given in Theorem 5.21 and Theorem 5.25, arises from the description conjectured above, but not without further effort. The purpose of the remaining part of this section is to present the necessary considerations for this. We begin with the definition of an invariant introduced by Kottwitz, which we will need and which we will call the Kottwitz invariant (or shorter, the K-invariant).

The Shimura variety corresponds to a group G and a class X_∞. It is still assumed that G_{der} is simply connected. The purpose of [K3] and other works is to determine the alternating sum of the traces on suitable cohomology groups of a product of a power of the Frobenius at a place $\mathfrak{p}$ of E with a Hecke operator. We do not want to repeat the considerations from [K3] and we do not want to anticipate further considerations from Kottwitz. We only note, for clarity, that when it involves the m-th power of the Frobenius, the natural number n that we define now is equal to mr, where $r = [E_\mathfrak{p} : \mathbb{Q}_p]$. Let $L_n \subset \overline{\mathbb{Q}}_p$ be the unramified extension of $\mathbb{Q}_p$ of degree n.

Consider a semi-simple element ε in $G(\mathbb{Q})$, an element $\gamma = (\gamma(l))_{l\neq p}$ with $\gamma(l) \in$ 182
$G(\mathbb{Q}_l)$, and an element $\delta \in G(L_n)$. The K-invariant is associated with the triple $(\gamma, \delta; \varepsilon)$. Let $G(\varepsilon)$ denote the centralizer of ε in G. It is connected because G_{der} is simply connected. Kottwitz denotes the connected components of the L-groups of G and $G(\varepsilon)$ by $\widehat{G}$ and $\widehat{G}(\varepsilon)$, and their centers by $Z(\widehat{G})$ and $Z(\widehat{G}(\varepsilon))$, respectively. The global Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is denoted by Γ, and the local Galois group at a place v by $\Gamma(v)$. We adopt these notations and simplify $Z(\widehat{G})$ and $Z(\widehat{G}(\varepsilon))$

to Z and $Z(\varepsilon)$, respectively. Kottwitz also considers the group $(Z(\varepsilon)/Z)^\Gamma$ of Γ-invariant elements in the quotient $Z(\varepsilon)/Z$, as well as the group $\pi_0((Z(\varepsilon)/Z)^\Gamma)$ of its connected components and its dual group X. We note that $Z(\varepsilon)/Z = Z(\widehat{H})$ and $\widehat{G}(\varepsilon)/Z = \widehat{H}$, where $H = G(\varepsilon) \cap G_{\text{der}}$. For each place v, let Y_v be the group of those characters of $\pi_0((Z(\varepsilon)/Z)^{\Gamma(v)})$ that are trivial on $\pi_0(Z(\varepsilon)^{\Gamma(v)})$, and X_v its image in X. The invariant $\kappa(\gamma, \delta; \varepsilon) = \kappa(\gamma, \delta)$ associated with the triple is in the group $X/\prod_v X_v$. This is the group $\mathfrak{K}(I_0/F)^D$, which appears in formula (6.5.1) of [K5].

To define $\kappa(\gamma, \delta)$, we need to impose some conditions on $(\gamma, \delta; \varepsilon)$. First, we require:

(i) ε is elliptic in $G(\mathbb{R})$.

(ii) For every $l \neq p$, $\gamma(l)$ and ε are stably conjugate. Since G_{der} is simply connected, this means that ε and $\gamma(l)$ are conjugate in $G(\overline{\mathbb{Q}}_l)$. For almost all l, ε and $\gamma(l)$ should be conjugate in $G(\mathbb{Q}_l)$.

(iii) If σ denotes the Frobenius element in $\mathrm{Gal}(\mathbb{Q}_p^{\text{un}}/\mathbb{Q}_p)$ and

$$N_{L_n/\mathbb{Q}_p}\delta = \delta\sigma(\delta)\cdots\sigma^{n-1}(\delta),$$

then $N_{L_n/\mathbb{Q}_p}\delta$ should be conjugate to ε in $G(\overline{\mathbb{Q}}_p)$.

Every $h \in X_\infty$ can be written as

$$h(z, w) = z^{\mu_h} w^{\bar{\mu}_h} = z^{\mu} w^{\bar{\mu}}.$$

Let $\mathscr{M}_{\mathbb{C}}$ be the conjugacy class of homomorphisms from $\mathbb{G}_m$ to G containing μ. It depends only on X_∞ and not on h, and it also defines $\mathscr{M}_{\overline{\mathbb{Q}}}$ and $\mathscr{M}_{\overline{\mathbb{Q}}_p}$, as $\overline{\mathbb{Q}}$ is embedded in both $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$.

All elements of $\mathscr{M}_{\overline{\mathbb{Q}}_p}$ (or $\mathscr{M}_{\mathbb{C}}$) define the same homomorphism μ_2 in

$$\mathrm{Hom}(\mathbb{G}_m, G_{\text{ab}}) = X^*(Z).$$

On the other hand, δ clearly defines an element of the set $B(G_{\text{ab}})$ introduced in [K4] and, consequently, by the theorem proven there, an element of $X^*(Z^{\Gamma(p)})$. It is important to emphasize that $Z = \widehat{G}_{\text{ab}}$. We impose the following condition on δ.

183 $(*(\delta))$ The element in $X^*(Z^{\Gamma(p)})$ defined by δ is the restriction of μ_2.

The second condition is a requirement on ε. Let $M = M(\varepsilon)$ denote the centralizer in G of the maximally split torus over $\mathbb{Q}_p$ in the center of $G(\varepsilon)$. If $\mu : \mathbb{G}_m \to M$, then μ can be factored through at least one Cartan subgroup T, and therefore defines an element of $X_*(T)$, whose conjugacy class is well-defined. For any Cartan subgroup $\widehat{T}$ of $\widehat{M}$, μ thus defines an orbit in $X^*(\widehat{T})$ under the Weyl group of M, and finally a well-defined element $\mu_1 \in X^*(Z(\widehat{M}))$ that does not depend on T or $\widehat{T}$. Since G splits over an unramified extension of $\mathbb{Q}_p$, so does M, and the homomorphism $\lambda = \lambda_M : M(\mathbb{Q}_p) \to X^*(Z(\widehat{M}))^{\Gamma(p)}$ defined in [K3], §3, is available.

The second condition is as follows:

$(*(\varepsilon))$ There exists a $\mu : \mathbb{G}_m \to M$ defined over L_n, which, through the embedding $M \subset G$, defines an element in $\mathscr{M}_{\overline{\mathbb{Q}}_p}$, and such that the equation

$$\lambda(\varepsilon) = N_{L_n/\mathbb{Q}_p}\mu_1$$

holds.

The invariant $\kappa(\gamma,\delta;\varepsilon)$ is defined as a character on the preimage U of $(Z(\varepsilon)/Z)^\Gamma$ in $Z(\varepsilon)$, specifically as a product:

$$\kappa(\gamma,\delta;\varepsilon)=\prod_v \beta(v). \tag{5.p}$$

Here, $\beta(v)$ is the restriction to U of a character $\beta'(v)$ on the preimage U_v of $(Z(\varepsilon)/Z)^{\Gamma(v)}$. Almost all $\beta'(v)$ are trivial, and the product is trivial on Z.

For $v=l\neq p$, ε and γ_l are stably conjugate. Consequently, there exists a $c\in G(\overline{\mathbb{Q}}_l)$ such that

$$\gamma_l=c\varepsilon c^{-1},$$

and $\{b_\sigma\}=\{c^{-1}\sigma(c)\}$ is a 1-cocycle of $\Gamma(v)$ with values in $G(\varepsilon)$. According to [K2], Proposition 6.4, this cocycle defines an element of $\pi_0(Z(\varepsilon)^{\Gamma(v)})^D$. Since the image of the cocycle in G_{ad} is trivial, this element is trivial on $\pi_0(\widehat{G}_{\mathrm{ab}}^{\Gamma(v)})=\pi_0(Z^{\Gamma(v)})$. Therefore, it defines a character of $\pi_0(Z(\varepsilon)^{\Gamma(v)})/\pi_0(Z^{\Gamma(v)})$, which we can extend to $\pi_0((Z(\varepsilon)/Z)^{\Gamma(v)})$ by noting that the connected component of $(Z(\varepsilon)/Z)^{\Gamma(v)}$ is the image of the connected component of $Z(\varepsilon)^{\Gamma(v)}$. By lifting the character thus obtained from U_v, we get $\beta'(v)$. It is trivial on Z.

At the place p, we have:

$$N_{L_n/\mathbb{Q}_p}\delta=c\varepsilon c^{-1},$$

where c is an element of G with coefficients in $\overline{\mathbb{Q}}_p$. By Steinberg's theorem, we can assume that $c\in G(\mathbb{Q}_p^{\mathrm{un}})$. We have

$$\delta^{-1}c\varepsilon c^{-1}\delta=\sigma(\delta)\cdots\sigma^n(\delta)=\sigma(c\varepsilon c^{-1})=\sigma(c)\varepsilon\sigma(c^{-1}).$$

Thus, $b=c^{-1}\delta\sigma(c)$ commutes with ε and defines (using the notation of [K4]) an element of $B(G(\varepsilon))$, which depends only on ε and δ.

Lemma 5.15. *b is basic in the sense of [K4].* 184

Proof of Lem. 5.15.

Let n' be divisible by n, so that $n'=kn$. Then

$$b\sigma(b)\cdots\sigma^{n-1}(b)=\varepsilon c^{-1}\sigma^n(c),$$

and so, first, $c^{-1}\sigma^n(c)\in G(\varepsilon)$, and second,

$$b\sigma(b)\cdots\sigma^{n'-1}(b)=\varepsilon^k c^{-1}\sigma^{n'}(c).$$

From the definitions ([K4], §4), it must be shown that for sufficiently large n', there exists a homomorphism ν' from $\mathbb{G}_m$ to the center of $G(\varepsilon)$ such that

$$\varepsilon^k c^{-1}\sigma^{n'}(c)=\nu'(p)d^{-1}\sigma^{n'}(d),$$

with $d\in G(\mathfrak{k},\varepsilon)$.[9] Let T be a Cartan subgroup of $G(\varepsilon)$ over $\mathbb{Q}_p$. Let

$$|\lambda(\varepsilon)|_p^k=p^{-\nu'(\lambda)},\quad \lambda\in X^*(T).$$

If k is sufficiently large, then $\nu'(\lambda)\in\mathbb{Z}$ for every $\lambda\in X^*(T)$, and consequently $\nu'\in X_*(T)$. It is clear that ν' is invariant under the Galois group. Therefore, $\nu':x\mapsto x^{\nu'}$ is a homomorphism from $\mathbb{G}_m$ to *the center* of $G(\varepsilon)$ defined over $\mathbb{Q}_p$. For sufficiently large n', $c^{-1}\sigma^{n'}(c)$ is contained in a given neighborhood of 1. Since the eigenvalues of $\nu'(p)^{-1}\varepsilon^k$ are units, we can replace k with a large multiple ak, and ν' with $a\nu'$, and assume that

$$\varepsilon^k c^{-1}\sigma^{n'}(c)=\nu'(p)\cdot e,$$

[9] Here $G(\mathfrak{k},\varepsilon)=G(\varepsilon)(\mathfrak{k})$.

where e lies in a given neighborhood V of 1 in $G(\mathfrak{k}, \varepsilon)$. But as implicitly noted in the proof of [K3], Lemma 1.4.9, there exists a V such that for any natural number
QED Lem. 5.15. n' we have $V \subset \{x\sigma^{n'}(x)^{-1} \mid x \in G(\mathfrak{k}, \varepsilon)\}$.

Let $\alpha(p)$ denote the element of $X^*(Z(\varepsilon)^{\Gamma(p)})$ corresponding to b in the sense of [K4]. Due to condition $*(\delta)$, the restriction of $\alpha(p)$ to $Z^{\Gamma(p)}$ must equal the restriction of μ_2. Consequently, there exists a character $\beta'(p)$ of U_p, which equals $\alpha(p)$ on $Z(\varepsilon)^{\Gamma(p)}$ and μ_2 on Z. Moreover, $\beta'(p)$ is uniquely determined up to a character of $\pi_0((Z(\varepsilon)/Z)^{\Gamma(p)})$, which is trivial on $\pi_0(Z(\varepsilon)^{\Gamma(p)})$.

To define $\beta'(\infty)$, we choose an $h \in X_\infty$ that factors through $G(\varepsilon)$, which is possible because ε is elliptic. Let $h(z,w) = z^\mu w^{\overline{\mu}}$.

Lemma 5.16. *(a) The restriction $\alpha(\infty)$ of $-\mu$ to $Z(\varepsilon)^{\Gamma(\infty)}$ does not depend on the choice of h.*
(b) On $Z^{\Gamma(\infty)}$, $\alpha(\infty)$ equals $-\mu_2$.

Proof of Lem. 5.16.

The lemma, whose second statement is already clear, allows us to define $\beta'(\infty)$ in the same way we defined $\beta'(p)$, except that it should equal $-\mu_2$ on Z. The first statement is to be understood as follows: if $\widehat{T}$ is a Cartan subgroup of $\widehat{G}(\varepsilon)$, then μ defines an orbit under the Weyl group $\Omega(\widehat{T}, \widehat{G}(\varepsilon))$ in $X^*(\widehat{T})$. All elements in the orbit have the same restriction on $Z(\varepsilon)$ and hence on $Z(\varepsilon)^{\Gamma(\infty)}$. This handles the
QED Lem. 5.16. restriction.

185 Let h_1 and h_2 be two elements of X_∞ that factor through $G(\varepsilon)$. We want to prove that the corresponding μ_1 and μ_2 have the same restriction to $Z(\varepsilon)^{\Gamma(\infty)}$. This is clear if h_1 and h_2 are conjugate under $G(\varepsilon, \mathbb{R})$. We can therefore assume that they also factor through a common elliptic Cartan subgroup $T \subset G(\varepsilon)$. Let $h_2 = h_1^g = \operatorname{ad} g^{-1} \circ h_1, g \in G(\mathbb{R})$. Then $K_2 = K_1^g$, where K_i is the centralizer of h_i in $G(\mathbb{R})$. Consequently, T and T^g are Cartan subgroups of K_2. We can thus replace g with gk, $k \in K_2$, and assume $T = T^g$. The element g represents an element of the Weyl group, which we associate with a class in $H^1(\mathbb{R}, T)$ in the usual manner ([Sh]). This class is defined as

$$\iota \mapsto (-1)^{g^{-1}\mu_1 - \mu_1} = (-1)^{\mu_2 - \mu_1}$$

([MS2], Prop. 4.2). On the other hand, it is trivial. Hence $\mu_2 - \mu_1$ defines the trivial class in $H^1(\mathbb{R}, X_*(T))$, from which it immediately follows that $\mu_2 - \mu_1$ is trivial on $\widehat{T}^{\Gamma(\infty)}$. In particular, it is trivial on $Z(\varepsilon)^{\Gamma(\infty)}$, because $Z(\varepsilon) \subset \widehat{T}$ and the actions of $\Gamma(\infty)$ on both groups are compatible.

It remains to show that the product $\prod \beta(v)$ on the identity component $[(Z(\varepsilon)/Z)^\Gamma]^0$ is trivial, i.e., that $\beta(\infty) \cdot \beta(p)$ is trivial on $[Z(\varepsilon)^\Gamma]^0$. We consider the image of $\alpha(p)$ under the composition of maps

$$X^*(Z(\varepsilon)^{\Gamma(p)}) \to X^*(Z(\widehat{M})^{\Gamma(p)}) \to X^*(Z(\widehat{M})^{\Gamma(p)}) \otimes \mathbb{Q}$$

$$X^*(Z(\widehat{M})^{\Gamma(p)}) \otimes \mathbb{Q} = X^*(Z(\varepsilon))^{\Gamma(p)} \otimes \mathbb{Q} \to X^*(Z(\varepsilon))^{\Gamma} \otimes \mathbb{Q}.$$

The image of $\alpha(p)$ in $X^*(Z(\widehat{M})^{\Gamma(p)}) \otimes \mathbb{Q}$ is equal to

$$\frac{1}{n}\lambda_M(\mathrm{Nm}_{L_n/\mathbb{Q}_p} b)|_{Z(\widehat{M})^{\Gamma(p)}} = \frac{1}{n}\lambda_M(\varepsilon)|_{Z(\widehat{M})^{\Gamma(p)}}.$$

On the other hand, the image of $\mu \in X^*(Z(\varepsilon)^{\Gamma(\infty)})$ in $X^*(Z(\varepsilon))^{\Gamma} \otimes \mathbb{Q}$ equals the image of $\mu_1 \in X^*(Z(\widehat{M}))$, that is, $\frac{1}{n}\mathrm{Nm}_{L_n/\mathbb{Q}_p}\mu_1$. The claim follows from condition $*(\varepsilon)$.

The invariant $\kappa(\gamma, \delta)$ is now defined via the product (5.p). Let ε and ε' be stably conjugate elements of $G(\mathbb{Q})$. Conditions (i), (ii), (iii) also hold for $(\gamma, \delta, \varepsilon')$, as well as condition $*(\delta)$.

Lemma 5.17. *Let $\varepsilon \in G(\mathbb{Q}_p)$ satisfy condition $*(\varepsilon)$, and let $\varepsilon' \in G(\mathbb{Q}_p)$ be stably conjugate to ε. Then ε' satisfies condition $*(\varepsilon')$.*

Proof of Lem. 5.17.

Let $\varepsilon' = \varepsilon^g$, $g \in G(\overline{\mathbb{Q}}_p)$. Then $M' = M^g$, and the induced map $M_{\mathrm{ab}} \to M'_{\mathrm{ab}}$ is defined over $\mathbb{Q}_p$. This equation also allows us to identify $Z(\widehat{M})$ with $Z(\widehat{M'})$. Then μ and $\operatorname{ad} g^{-1} \circ \mu$ correspond to the same orbit in $X_*(T)$, so $\mu_1 = \mu_1'$. Since μ is defined over L_n, the conjugacy class of $\operatorname{ad} g^{-1} \circ \mu$ is defined over L_n. From [K3], Lemma 1.1.3, it follows that there exists $\mu' : \mathbb{G}_m \to M'$ defined over L_n, which is conjugate to $\operatorname{ad} g^{-1} \circ \mu$ in $M'(\overline{\mathbb{Q}}_p)$. Since G, and hence M', is quasi-split over $\mathbb{Q}_p$, if we can show that M_{der} and M'_{der} are simply connected then we immediately get $\lambda_M(\varepsilon) = \lambda_{M'}(\varepsilon')$. The lemma will then be proven.

We consider M alone and show, for a later purpose, more than is currently 186
necessary. M is a Levi factor of a quasi-split group G defined over $\mathbb{Q}_p$, with G_{der} simply connected. Since $M_{\mathrm{der}} \subset M \cap G_{\mathrm{der}}$, we temporarily assume that $G = G_{\mathrm{der}}$. Let $T \subset M$ be a Cartan subgroup of G defined over $\mathbb{Q}_p$ which is contained in a Borel subgroup B defined over $\mathbb{Q}_p$. The simple roots $\alpha_1, \ldots, \alpha_s$ of T in G with respect to B can be divided into two sets: the roots $\alpha_1, \ldots, \alpha_r$ of T in M, and the others $\alpha_{r+1}, \ldots, \alpha_s$. As a basis for $X^*(T)$, we choose $\lambda_1, \ldots, \lambda_s$, where $\lambda_i(\widehat{\alpha}_j) = \delta_{ij}$, and $\widehat{\alpha}_j$ is the coroot corresponding to α_j. Since the coroots with respect to G are also coroots with respect to M, M_{der} is simply connected. Furthermore, M is the semidirect product of M_{der} and the kernel $R \subset T$ of $\{\lambda_1, \ldots, \lambda_r\}$. The group R is defined over $\mathbb{Q}_p$, and $\{\lambda_{r+1}, \ldots, \lambda_s\}$ forms a basis for $X^*(R)$. Since $\lambda_{r+1}, \ldots, \lambda_s$ are permuted by the Galois group, it follows that $H^1(\mathbb{Q}_p, R) = \{1\}$ and $H^1(\mathbb{Q}_p, M) = \{1\}$. In the general case, only $H^1(\mathbb{Q}_p, M \cap G_{\mathrm{der}}) = \{1\}$ holds. *QED Lem. 5.17.*

Lemma 5.18. $\kappa(\gamma, \delta; \varepsilon') = \kappa(\gamma, \delta; \varepsilon)$.

To make sense of this lemma, we must identify the groups in which the invariants lie. Let $\varepsilon' = g^{-1}\varepsilon g$, with $g \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$. Then $u \mapsto u' = g^{-1}ug$ is an isomorphism from $G(\varepsilon)$ to $G(\varepsilon')$, which allows us to treat $G(\varepsilon')$ as an inner twist of $G(\varepsilon)$, since for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the element $a_\sigma = g\sigma(g^{-1})$ lies in $G(\varepsilon)$. We can therefore identify $\widehat{G}(\varepsilon')$ with $\widehat{G}(\varepsilon)$ as well as $Z(\varepsilon')$ with $Z(\varepsilon)$, and consequently $X'/\prod_v X'_v$ with $X/\prod_v X_v$. We identify $G(\varepsilon')$ with $G(\varepsilon)$ via the map $u \mapsto u'$. This results in a new action of the Galois group on $G(\varepsilon)$, defined by the equations:

$$g^{-1}\sigma'(u)g = \sigma(g^{-1}ug).$$

Thus we have $\sigma'(u) = a_\sigma \sigma(u) a_\sigma^{-1}$.

To prove Lemma 5.18, we require two additional lemmas, which we prove not only for $G(\varepsilon)$ but also for any (reductive) group G. Let $\{a_\sigma\}$ be a cocycle of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in G, and let G' be the corresponding inner twist. It is easy to verify that the map $\{v_\sigma\} \mapsto \{v'_\sigma = v_\sigma a_\sigma^{-1}\}$ defines a bijection $\xi : H^1(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G')$, as well as local bijections

$$\xi(v) : H^1(\mathbb{Q}_v, G) \to H^1(\mathbb{Q}_v, G').$$

As above, we can identify $Z' = Z(\widehat{G}')$ with Z, and consequently $\pi_0(Z'^{\Gamma(v)})^D$ with $\pi_0(Z^{\Gamma(v)})^D$. The cocycle $\{a_\sigma\}$ defines a class $\alpha \in H^1(\mathbb{Q}, G)$ with localizations $\alpha(v)$. Let $\delta(v)$ be the element of $\pi_0(Z^{\Gamma(v)})^D$ corresponding to $\alpha(v)$ (see [K2], §6.4, for a finite place, and [K5], §1.2, for the general case), and $\eta(v)$ the isomorphism arising from multiplication by $\delta(v)^{-1}$:

$$\pi_0(Z^{\Gamma(v)})^D \to \pi_0(Z^{\Gamma(v)})^D = \pi_0(Z'^{\Gamma(v)})^D.$$

187 **Lemma 5.19.** *The diagram*

$$\begin{tikzcd} H^1(\mathbb{Q}_v, G) \arrow[r, "\xi(v)"] \arrow[d] & H^1(\mathbb{Q}_v, G') \arrow[d] \\ \pi_0(Z^{\Gamma(v)})^D \arrow[r, "\eta(v)"'] & \pi_0(Z'^{\Gamma(v)})^D \end{tikzcd}$$

is commutative.

Proof of Lem. 5.19.

We require a central extension

$$1 \to Z \to H \to G \to 1,$$

with H_{der} simply connected, for which $H^1(\mathbb{Q}_v, H) \to H^1(\mathbb{Q}_v, G)$ is surjective. For a non-archimedean place, such an extension is constructed in [K2], §6.4. For an archimedean place, any z-extension (see [K1], §1), for which Z is a torus induced from $\mathbb{C}$, has the desired property. It suffices to prove the lemma for H instead of G. Then, according to the definitions, we can replace H by H_{ab}, and for H_{ab} the claim is clear.

QED Lem. 5.19.

The second lemma concerns an element $a \in G(\mathfrak{k})$ such that for a sufficiently large n the product $a\sigma(a)\cdots\sigma^{n-1}(a)$ is central in G, where σ is the Frobenius element. Then a defines a twist G' of G, for which

$$G'(\mathbb{Q}_p) = \{g \in G(\mathfrak{k}) \mid a\sigma(g)a^{-1} = g\}.$$

We have a bijection ξ_p from $B(G)$ to $B(G')$ defined by $b \mapsto b' = ba^{-1}$.

We have

$$b'\sigma'(b')\cdots\sigma'^{n-1}(b') = b\sigma(b)\cdots\sigma^{n-1}(b)\sigma^{n-1}(a)^{-1}\cdots\sigma(a)^{-1}a^{-1}.$$

It then follows easily from [K4], Prop. 4.3.3, that ξ_p is also a bijection between $B(G)_b$ and $B(G')_b$. The element $\delta(p)$ associated with a lies in $X^*(Z^{\Gamma(p)})$. By multiplying with $\delta(p)^{-1}$, we obtain an isomorphism

$$\eta(p) : X^*(Z^{\Gamma(p)}) \to X^*(Z^{\Gamma(p)}) = X^*(Z'^{\Gamma(p)}).$$

Lemma 5.20. *The diagram*

$$\begin{tikzcd} B(G)_b \arrow[r, "\xi(p)"] \arrow[d] & B(G')_b \arrow[d] \\ X^*(Z^{\Gamma(p)}) \arrow[r, "\eta(p)"'] & X^*(Z'^{\Gamma(p)}) \end{tikzcd}$$

is commutative.

Proof of Lem. 5.20.

188 This lemma is proved in exactly the same way as the previous one. One chooses, as in [K2], §6.4, a central extension

$$1 \to Z \to H \to G \to 1,$$

with $H_{\mathrm{der}} = H_{\mathrm{sc}}$, and uses the definition from [K4]. *QED Lem. 5.20.*

Proof of Lem. 5.18.

To prove Lemma 5.18, we apply the two lemmas to $G(\varepsilon) \cap G_{\mathrm{der}}$. According to Prop. 2.6 from [K5], the lemma would be proven if we could show that when we replace ε with ε', $\beta'(v)$ can be replaced by $\delta^{-1}(v)\beta'(v)$. Firstly, it should be noted that we had chosen $g \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$ so that the cocycle $\{a_\varrho\}$ lies in $G(\varepsilon) \cap G_{\mathrm{der}}$, and secondly, it should be remembered that $Z(\varepsilon)/Z = Z(\widehat{H})$, where $H = G(\varepsilon) \cap G_{\mathrm{der}}$.

When $v = l \neq p$, we have

$$\gamma = c\varepsilon c^{-1} = cg\varepsilon' g^{-1} c^{-1}$$

and

$$g^{-1}c^{-1}\varrho(c)\varrho(g) = g^{-1}(b_\varrho a_\varrho^{-1})g.$$

Therefore, the class $\alpha(v)$ of $\{b_\varrho\}$ is replaced by $\xi(v)(\alpha(v))$. We can thus replace $\beta'(v)$ by $\delta^{-1}(v)\beta'(v)$.

At the place p, we have

$$\mathrm{Nm}_{L_n/\mathbb{Q}_p}(\delta) = cg\varepsilon' g^{-1}c^{-1}.$$

Unfortunately, g may not lie in $G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{un}})$. However, it can be written as a product $g = kg_1$ with $g_1 \in G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{un}})$ and $k \in G_{\mathrm{der}}(\overline{\mathbb{Q}}_p) \cap G(\varepsilon)$.

Thus we have

$$g\varrho(g)^{-1} = kg_1\varrho(g_1)^{-1}\varrho(k)^{-1},$$

so g and g_1 give the same class in $H^1(\mathbb{Q}_p, G(\varepsilon) \cap G_{\mathrm{der}})$. The identification of $\widehat{G}(\varepsilon')$ and $\widehat{G}(\varepsilon)$ defined by g is also the one defined by g_1. Therefore, we can replace g with g_1 if we want to show that $\beta'(v)$ can be replaced by $\delta^{-1}(v)\beta'(v)$.

We have

$$\mathrm{Nm}_{L_n/\mathbb{Q}_p}(\delta) = cg_1\varepsilon' g_1^{-1}c^{-1}.$$

Thus, we replace b with

$$b' = g_1^{-1}\left(b\sigma(g_1)g_1^{-1}\right)g_1,$$

so that the desired replacement follows from Lemma 5.19 and the compatibility ([K4]) of the two bijections $B(G)_b \to X^*(Z^{\Gamma(p)})$ and $H^1(\mathbb{Q}_p, G) \to \pi_0(Z^{\Gamma(p)})^D$.

For the infinite case, the claim that $\beta'(\infty)$ can be replaced by $\delta^{-1}(\infty)\beta'(\infty)$ is 189
local and transitive. It holds when ε and ε' are conjugated within $G_{\mathrm{der}}(\mathbb{R})$. Hence, we can assume that there is a Cartan subgroup T of $G(\varepsilon)$ defined over $\mathbb{R}$ such that $T^g = T$. We can further assume that $h : \mathbb{S} \to T$. When we identify the two groups $Z(\varepsilon)$ and $Z(\varepsilon')$ via g, then $\alpha'(\infty)$ is the restriction of $-g\mu$ to $Z(\varepsilon)^{\Gamma(\infty)}$, i.e., the product of $\alpha(\infty)$ with the inverse of the restriction of $g\mu - \mu$. According to [K5], Th. 1.2, and [MS], Prop. 4.2, this restriction is precisely $\delta(\infty)$. *QED Lem. 5.18.*

We now return to our presumed description of points on a reduced Shimura variety. The conditions of Theorem 5.3 are assumed to hold. Let m be a given natural number. We consider pairs (ϕ, ε) with the following properties:

(i) $\phi : \mathscr{Q} \to \mathscr{G}_G$ is admissible.
(ii) $\varepsilon \in I_\phi$.

(iii) There exist $\gamma = (\gamma(l)) \in G(\mathbb{A}_f^p)$, $y \in X^p$, and x in the $G(\mathfrak{k})$-orbit of the point x^0, such that

$$\varepsilon y = y\gamma \quad \text{and} \quad \varepsilon x = \Phi^m x.$$

Such pairs are called *admissible*. It is not required that x lies in X_p. If y is replaced by yh, where $h \in G(\mathbb{A}_f^p)$, then γ changes to $h^{-1}\gamma h$. Therefore, the conjugacy class of γ is uniquely determined by the pair (ϕ, ε). Since $\mathscr{I}_\phi$ is a twisted form of a subgroup of G, it is meaningful both globally and locally to state that ε is stably conjugate to an element of G. In particular, it is stably conjugate to each $\gamma(l)$. It also follows from Lemma 5.23 below that ε is stably conjugate to an element $\varepsilon_0 \in G(\mathbb{Q})$. Then, ε_0 is stably conjugate to each $\gamma(l)$. It follows easily from the definition of X_p and Greenberg's Lemma [G] that ε_0 and $\gamma(l)$ are conjugate almost everywhere.

To define X_p, we have replaced ξ_p with ξ_p' and introduced $F = \xi(w) = b \times \sigma$, where $b \in G(\mathfrak{k})$. If $\xi_p' = \operatorname{ad} u \circ \xi_p$, we must replace ε with $\varepsilon' = u\varepsilon u^{-1}$ in order to obtain its action on X_p and also for the condition $\varepsilon x = \Phi^m x$ on $\mathscr{B}(G, \mathfrak{k})$. Therefore, the condition should rather be $\varepsilon' x = \Phi^m x$. According to Lemma 1.4.9 from [K3], condition (iii) implies the existence of $c \in G(\mathfrak{k})$ and $\delta \in G(L_n)$ such that $cb\sigma c^{-1} = \delta\sigma$ and $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta = c\varepsilon' c^{-1} = cu\varepsilon u^{-1}c^{-1} = cug\varepsilon_0 g^{-1}u^{-1}c^{-1}$, if $\varepsilon = g\varepsilon_0 g^{-1}$. Thus, the triple $\gamma, \delta, \varepsilon_0$ satisfies conditions (ii) and (iii) (where ε is replaced by ε_0), which were used to define the K-invariant. That condition (i) is also satisfied follows immediately from (5.b). The condition $*(\delta)$ follows from (5.a), because b and δ define the same class in $B(G_{\mathrm{ab}})$. The condition $*(\varepsilon)$ is also satisfied, as follows from Theorem 5.21. We can therefore introduce the invariant $\kappa(\gamma, \delta; \varepsilon_0)$. We immediately emphasize that the b that appears in the definition of the invariants is the same as the one that appears in $F = b \times \sigma$.

Theorem 5.21 below describes those ε_0 for which there exists an admissible pair (ϕ, ε) with ε stably conjugate to ε_0. Two admissible pairs (ϕ, ε) and (ϕ', ε') are called *equivalent*, if there exists $g \in G(\overline{\mathbb{Q}})$ such that $\phi' = \operatorname{ad} g \circ \phi, \varepsilon' = \operatorname{ad} g(\varepsilon)$. Theorem 5.25 counts the classes of (ϕ, ε) with ε stably conjugate to a given ε_0. For the significance of this enumeration, we refer to [K3] and other unpublished works by Kottwitz.

190 **Theorem 5.21.** *There exists an admissible pair (ϕ, ε) with ε stably conjugate to ε_0*
Proof of Thm. 5.21. *if and only if ε_0 is elliptic at infinity and satisfies the condition $*(\varepsilon)$ with $n = rm$.*

The fact that ε_0 must be elliptic at infinity follows from conditions (i) and (ii) for admissibility. To prove the necessity of the second condition, we need some preparation that will also be useful in the proof of Theorem 5.25.

The map ϕ is called *well-positioned*[10] if $\gamma_n = \phi(\delta_n)$ lies in $G(\mathbb{Q})$ for large n. According to Lemma 5.4, we can assume that ϕ is favorable. Then $I_\phi = \mathscr{I}_\phi(\mathbb{Q})$ where $\mathscr{I}_\phi$ is a twisted form of the centralizer $\mathscr{I}$ of γ_n (for sufficiently large n). If $\phi(q_\varrho) = b_\varrho \times \varrho$, then b_ϱ lies in $\mathscr{I}$, and $\{b_\varrho\}$ defines the twisting. The pair (ϕ, ε) is called *well-positioned* if ϕ is favorable and $\varepsilon \in I_\phi \subset \mathscr{I}_\phi(\overline{\mathbb{Q}}) = \mathscr{I}(\overline{\mathbb{Q}}) \subset G(\overline{\mathbb{Q}})$ also lies in $G(\mathbb{Q})$. Then b_ϱ lies in $G(\varepsilon)$, which is necessarily connected because G_{der} is simply connected.

Lemma 5.22. *Every pair (ϕ, ε), with $\varepsilon \in I_\phi$, is equivalent to a well-positioned pair (ϕ', ε').*

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[10] Translated from the German term "*günstig gelegen*".

It is useful to prove a stronger lemma. Let T be an elliptic Cartan subgroup defined over $\mathbb{Q}$ and let $h \in X_\infty$ factor through T. Let $\mu = \mu_h$. We say that (ϕ, ε) is *nested* in (T, h) if $\phi = \psi_{T,\mu}$ and $\varepsilon \in T(\mathbb{Q})$.

Lemma 5.23. *For every pair (ϕ, ε), with $\varepsilon \in I_\phi$, we can find an equivalent pair (ϕ', ε') and T', h' such that (ϕ', ε') is nested in (T', h').*

Proof of Lem. 5.23.

It is clear that Lemma 5.22 follows from Lemma 5.23. The proof of Lemma 5.23 partially repeats the proof of Theorem 5.3. We can assume that ϕ is well-positioned. Let ε_1 be an element of I_ϕ whose centralizers in $\mathscr{I}_\phi, \mathscr{I}$, or G are Cartan subgroups that contain ε. These centralizers are then all the same and are denoted by T. As a subgroup of $\mathscr{I}_\phi$, T is defined over $\mathbb{Q}$, but it may not be so as a subgroup of G. The conjugacy class of ε_1 in $G(\overline{\mathbb{Q}})$ is rational because $\mathscr{I}$ is an inner twisting of $\mathscr{I}_\phi$. We had fixed $\psi : G \to G^*$ up to inner automorphism. According to Steinberg's theorem (see [K1]), we can assume that $\varepsilon_1^* = \psi(\varepsilon_1)$ is rational. Let $T^* = \psi(T)$ be the centralizer of ε_1^*. We use Lemma 5.6 to prove that T^* transfers to G. We show T^* transfers locally and that it is elliptic at infinity.

That T^* transfers to G at p is clear because G is quasi-split over $\mathbb{Q}_p$. At a place $l \neq p$, we have $b_\varrho = c_\varrho d_\varrho, \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$, where c_ϱ is in the image of Q and thus in the center of $\mathscr{I}$, and $d_\varrho = u\varrho(u^{-1}), u \in G(\overline{\mathbb{Q}}_l)$. Then $d_\varrho \varrho(\varepsilon_1) d_\varrho^{-1} = \varepsilon_1$ and $u^{-1}\varepsilon_1 u = \varrho(u^{-1}\varepsilon_1 u)$. Thus, the centralizer T_l of $u^{-1}\varepsilon_1 u$ is defined over $\mathbb{Q}_l$, and $\psi \circ \mathrm{ad}\, u(T_l) = T^*$. At infinity, $b_\varrho = c_\varrho d_\varrho$ with c_ϱ in the image of Q and

$$d_\iota \times \iota = u\xi_\infty(w(\iota))u^{-1},$$

where ξ_∞ is defined at the beginning of the section. Therefore, $g \mapsto u^{-1}gu$ is an embedding over $\mathbb{R}$ of $\mathscr{I}_\phi$ into the group G' defined by ξ_∞. In particular, $u^{-1}\varepsilon_1 u$ lies in $G'(\mathbb{R})$. On the other hand, if ξ is defined by $h \in X_\infty$ and h factors through T_1, then $T_1(\mathbb{R})$ lies in $G'(\mathbb{R})$. It follows that $\varepsilon_1 = v^{-1}\varepsilon_2 v$ with $\varepsilon_2 \in T_1(\mathbb{R})$. Since $\varepsilon_1^* = \psi \circ \mathrm{ad}\, v(\varepsilon_2)$, T^* transfers over $\mathbb{R}$ and in particular it is elliptic as T_1.

Thus, there exists a $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{ad}\, g(\psi^{-1}(T^*))$ is defined over $\mathbb{Q}$, and 191
such that the homomorphism $t^* \mapsto t = \mathrm{ad}\, g(\psi^{-1}(t^*)) \in \mathrm{ad}\, g(T)$ is also defined over $\mathbb{Q}$. Consequently, $\mathrm{ad}\, g(\varepsilon_1) \in G(\mathbb{Q})$. Since ε and γ_n lie in $T(\mathbb{Q})$, $\mathrm{ad}\, g(\varepsilon)$ and $\mathrm{ad}\, g(\gamma_n)$ are contained in $\mathrm{ad}\, g(T)$. We now replace ε, ϕ, and T with $\mathrm{ad}\, g(\varepsilon)$, $\mathrm{ad}\, g \circ \phi$, and $\mathrm{ad}\, g(T)$. We are now in the same situation as we were during the proof of Theorem 5.3, before we proved Lemma 5.11, except that we are now given an additional datum, namely $\varepsilon \in T(\mathbb{Q})$. If we apply Lemmas 5.11 and 5.12, as well as the final consideration in the proof of Theorem 5.3, to adjust ϕ and T step by step, we can carry ε along, ultimately obtaining the desired $\phi', \varepsilon', T', h'$. *QED Lem. 5.23.*

We now come to the necessity of condition $*(\varepsilon)$. We can assume that there exist T and h such that (ϕ, ε) is nested in (T, h). Furthermore, we can assume that $\varepsilon_0 = \varepsilon$. We want to show that under these circumstances, $M \subset \mathscr{J}$. This also holds more generally when (ϕ, ε) is well-positioned and $\varepsilon_0 = \varepsilon$. The general claim follows immediately from the special case.

As in the proof of Lemma 5.2, we can assume that $\xi'_p = \xi_{-\mu'}$, where μ' is a cocharacter of a $\mathfrak{k}$-split Cartan subgroup T' of $\mathscr{J}$. Then $F = p^{\mu'} \times \sigma$, as in the example preceding Lemma 5.2. Let $\xi_{-\mu'} = \mathrm{ad}\, v(\xi_p)$, $\varepsilon' = \mathrm{ad}\, v(\varepsilon)$, where $v \in \mathscr{J}(\overline{\mathbb{Q}}_p)$. Then we have

$$\Phi^{tm} = F^{tn} = p^{-t'\nu_2} \times \sigma^{tn},$$

where for sufficiently large s,

$$\nu_2 = -\mu' \cdot \sigma\mu' \cdots \sigma^{s-1}\mu', \quad tn = t's.$$

The element ε' commutes with Φ^{tm}. However, when t is large enough such that T' splits over L_{tn}, it follows easily from the Bruhat decomposition that ε' can only commute with Φ^{tm} if it also commutes with σ^{tn} and $p^{-t\nu_2}$.

From the equation $\varepsilon' x = \Phi^m x$, it generally follows that

$$(\varepsilon')^t x = \Phi^{tm} x, \quad t \in \mathbb{Z}.$$

Let t be large enough such that $\sigma^t(x) = x$. Then $k' = p^{t'\nu_2} \cdot (\varepsilon')^t$ has a fixed point, and its eigenvalues all have absolute value 1.

The decomposition

$$(\varepsilon')^t = p^{-t'\nu_2} k'$$

is a kind of polar decomposition of $(\varepsilon')^t$ and is uniquely determined. From the uniqueness, it follows that $p^{-t'\nu_2}$ lies in the center of $G(\varepsilon)$. Since $\mathscr{J}$ is the centralizer of $p^{-t'\nu_2}$ and $p^{-t'\nu_2}$ obviously generates a one-dimensional splitting torus, it follows that M is contained in $\mathscr{J}$.

192 From [K3], §3, it follows easily that

$$\lambda_M(\varepsilon) = \frac{1}{t}\lambda_M(\varepsilon^t) = \frac{1}{t}\lambda_M(p^{-t'\nu_2}) = -\frac{t'}{t}\nu_2,$$

and the restriction of $-\frac{t'}{t}\nu_2$ to $Z(\widehat{M}) \subset \widehat{T}$ is equal to $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\mu'_1$. Although μ' itself may not be defined over L_n, by the definition of E and Lemma 1.1.3 from [K3], there exists a $\mu : \mathbb{G}_m \to M$ conjugate to μ' and defined over L_n. It is clear that $\mu_1 = \mu'_1$.

We now assume that ε_0 satisfies the two conditions of Theorem 5.21 and prove the existence of an admissible pair (ϕ, ε) with ε stably conjugate to ε_0. We can find a Cartan subgroup T of $G(\varepsilon_0)$ defined over $\mathbb{Q}$ and a cocharacter μ of T such that T is elliptic at infinity and at p, and such that μ, though not necessarily the homomorphism from $\mathbb{G}_m$ to M required by assumption (b), is at least conjugate to it in M. It might be the case that $\psi_{T,\mu}$ is not admissible. In that case, we use the proof of Lemma 5.12 to find a $u \in G(\overline{\mathbb{Q}})$ such that $T' = uTu^{-1}$ and $t \mapsto t' = utu^{-1}$ are both defined over $\mathbb{Q}$, while $\mu' = \operatorname{ad} u \circ \mu = \mu_h, h \in X_\infty$. If necessary, we replace ε_0 by $\varepsilon'_0 = \operatorname{ad} u(\varepsilon_0)$ and T, μ by T', μ', so that $\psi_{T,\mu}$ becomes admissible. Then X^p contains a point y from $\prod_{l \neq p} T(\overline{\mathbb{Q}}_l)$, which necessarily commutes with ε_0, and so $\varepsilon_0 y = y\varepsilon_0$. Consequently, the pair $(\phi, \varepsilon) = (\psi_{T,\mu}, \varepsilon_0)$ satisfies the conditions (i), (ii), and the part of (iii) about the existence of the point y.

To verify the second part of (iii), we can assume that ξ'_p, which defines X_p and Φ, is equal to $\operatorname{ad} u \circ \xi_p = \xi_{-\mu'}$, where $u \in M(\overline{\mathbb{Q}}_p)$, and where $\xi_{-\mu'}$ corresponds to an elliptic Cartan subgroup T' of M split over $\mathfrak{k}$ and a cocharacter μ' conjugate to μ. Let $\varepsilon' = \operatorname{ad} u(\varepsilon)$. Then

$$\Phi^m = p^\nu \times \sigma^n$$

with

$$\nu = \sum_{j=0}^{n-1} \sigma^j(\mu').$$

By assumption $*(\varepsilon)$, we have

$$\lambda_M(\varepsilon') = \lambda_M(\varepsilon) = \lambda_M(p^\nu).$$

Since the cocharacter of M_{ab} defined by μ' is defined over $\mathbb{Q}_p$, we conclude that the image of $(\varepsilon')^{-1}p^{\nu}$ lies in the maximal compact subgroup of $M_{\mathrm{ab}}(\mathfrak{k})$. Therefore, there exists $c \in T'(\mathfrak{k})$ such that $c^{-1}(\varepsilon')^{-1}\Phi^m c \in M_{\mathrm{sc}}(\mathfrak{k}) \times \sigma^n$.

To prove the existence of x, we must (see [K3], Lemma 1.4.9) show the existence of a $u \in G(\mathfrak{k})$ such that $u^{-1}(\varepsilon')^{-1}\Phi^m u = \sigma^n$. Instead, we seek something stronger: a $d \in M_{\mathrm{sc}}(\mathfrak{k})$ such that $d^{-1}c^{-1}(\varepsilon')^{-1}\Phi^m cd = \sigma^n$.

It is sufficient to show that $c^{-1}\varepsilon'^{-1}\Phi^m c\sigma^{-n} \in M_{\mathrm{sc}}(\mathfrak{k})$ is basic, since Prop. 5.4 of 193
[K4] states that $B(M_{\mathrm{sc}})_b$ is trivial over any finite extension of $\mathbb{Q}_p$, in particular over L_n. According to condition (4.3.3) of [K4], $c^{-1}\varepsilon'^{-1}\Phi^m c\sigma^{-n}$ is basic if for sufficiently large t,

$$c^{-1}(\varepsilon'^{-1}\Phi^m)^t c = e\sigma^{tn}e^{-1}. \tag{5.q}$$

For a large r, we have $\Phi^{mr} = p^{\nu'} \times \sigma^{rn}$ with

$$\nu' = \sum_{j=0}^{rn-1} \sigma^j(\mu),$$

and ε' commutes with $p^{\nu'}$ and σ^{rn}. Furthermore, ν' is central, and $(\varepsilon')^r$ has a polar decomposition $(\varepsilon')^r = p^{\nu'}k$. Therefore,

$$(\varepsilon'^{-1}\Phi^m)^{sr} = k^s \times \sigma^{srn},$$

and

$$c^{-1}(\varepsilon'^{-1}\Phi^m)^{sr}c = c^{-1}k^{-s}\sigma^{srn}(c) \times \sigma^{srn} = (c^{-1}k^{-s}c)(c^{-1}\sigma^{srn}(c)) \times \sigma^{srn}.$$

For large s and r, k^{-s} and $c^{-1}\sigma^{srn}(c)$ are inside an arbitrarily given neighborhood of 1. Thus, the existence of a solution to (5.q) is proven and thus Theorem 5.21. *QED Thm. 5.21.*

In Theorem 5.25 below, an invariant $i(\varepsilon_0)$ appears, namely the order of the kernel of

$$H^1(\mathbb{Q}, G(\varepsilon_0)) \to H^1(\mathbb{Q}, G_{\mathrm{ab}}) \times \prod_v H^1(\mathbb{Q}_v, G(\varepsilon_0)).$$

If ε_0 is replaced by a stably conjugate ε_0', $G(\varepsilon_0)$ is replaced by $G(\varepsilon_0')$, a twisted form of $G(\varepsilon_0)$. To ensure that $i(\varepsilon_0)$ is well-defined, we prove the following simple lemma.

Lemma 5.24. *Let G be a reductive group over $\mathbb{Q}$ without an E_8-factor, A a torus over $\mathbb{Q}$, and $G \to A$ a surjective homomorphism defined over $\mathbb{Q}$. Let G' be an inner twist of G. Then the orders of the kernels of the two homomorphisms*

$$H^1(\mathbb{Q}, G) \to H^1(\mathbb{Q}, A) \times \prod_v H^1(\mathbb{Q}_v, G)$$

and

$$H^1(\mathbb{Q}, G') \to H^1(\mathbb{Q}, A) \times \prod_v H^1(\mathbb{Q}_v, G')$$

are equal.

Proof of Lem. 5.24.

As in the proof of Lemma 4.3.2 from [K2], we can replace G by a z-extension and assume that G_{der} is simply connected. By Lemma 4.3.1 of the same work, we can then replace G by G_{ab}, and the statement is clear for G_{ab}. *QED Lem. 5.24.*

We have already noted that the conjugacy class of γ_l, which corresponds to 194 an admissible pair, is well-defined. The twisted conjugacy class of $\delta \in G(L_n)$ is also well-defined. For example, if $cb\sigma c^{-1} = \delta\sigma$ and $ucb\sigma c^{-1}u^{-1} = \delta'\sigma$ while $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta = c\varepsilon' c^{-1}$ and $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta' = uc\varepsilon' c^{-1}u^{-1}$, then $\delta' = u\delta\sigma(u)^{-1}$ and $uc\varepsilon' c^{-1}\sigma^n(u)^{-1} = u\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta\sigma^n(u)^{-1} = uc\varepsilon' c^{-1}u^{-1}$. Therefore, u lies in $G(L_n)$.

Theorem 5.25. *Let $\varepsilon_0 \in G(\mathbb{Q})$ be an element satisfying the conditions of Theorem 5.21, and suppose for each l a conjugacy class $\{\gamma_l\}$ in $G(\mathbb{Q}_l)$ is given, along with a twisted conjugacy class $\{\delta\}$ in $G(L_n)$. Assume that γ_l and ε_0 are stably conjugate for every l, and conjugate for almost every l. Also, assume that $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta$ and ε_0 are stably conjugate. Finally, assume that δ satisfies the condition $*(\delta)$. Then, $\gamma = (\gamma_l)$ and δ correspond to an admissible pair (ϕ, ε) with ε stably conjugate to ε_0 if and only if $\kappa(\gamma, \delta; \varepsilon_0) = 1$. In this case, (γ_l) and δ correspond to exactly $i(\varepsilon_0)$ non-equivalent pairs.*

Proof of Thm. 5.25.

We first prove: If (ϕ, ε) exists, then the K-invariant $\kappa(\gamma, \delta; \varepsilon_0) = 1$. We can assume that there is a pair (T, μ) such that (ϕ, ε) is nested in (T, h). We can also assume that $\varepsilon_0 = \varepsilon$. Then γ_l and ε are conjugate for every l. Therefore, we can choose $\beta(v)$, which appears in (5.p), to be equal to 1.

At the place p, the element b defined by δ and ε is the same as the one that appears in $F = b \times \sigma$. Since $\phi = \psi_{T,\mu}$, the corresponding element in $X^*(Z(\varepsilon)^{\Gamma(p)})$ is nothing other than the restriction of μ to $Z(\varepsilon)^{\Gamma(p)}$, as already mentioned in the proof of Theorem 5.21. Accordingly, we choose $\beta'(p)$ as the restriction of μ to U_p. On the other hand, $\mu = \mu h$ with some $h \in X_\infty$, which factors through $T \subset G(\varepsilon)$. Therefore, we can choose $\beta'(\infty)$ as the restriction of $-\mu$ to U_∞. It immediately follows that

$$\kappa(\gamma, \delta; \varepsilon) = \beta(\infty)\beta(p) = 1.$$

Each class of admissible pairs (ϕ, ε) with ε stably conjugate to ε_0 contains a pair (ϕ, ε_0). We now show that the restriction of ϕ to the kernel of $\mathscr{Q}$ is determined by ε_0 along, and that its image lies in the center of $G(\varepsilon_0)$. Let ϕ and ϕ' be two admissible homomorphisms from $\mathscr{Q}$ to $\mathscr{G}_G$ with $\varepsilon_0 \in I_\phi \cap G(\mathbb{Q})$, $\varepsilon_0 \in I_{\phi'} \cap G(\mathbb{Q})$. Let $\gamma_k = \phi(\delta_k)$, $\gamma'_k = \phi'(\delta_k)$. We must show that for sufficiently large k, γ_k and γ'_k are central in $G(\varepsilon_0)$ and equal. In any case, γ_k and γ'_k lie in $G(\varepsilon_0)$. It is also clear that $\gamma_k \equiv \gamma'_k \pmod{G_{\mathrm{der}}}$. Consequently, we can replace G by G_{ad} and ϕ and ϕ' by the corresponding compositions. Thus, for the moment, let G be an adjoint group.

Let T be a Cartan subgroup of $G(\varepsilon_0)$ defined over $\mathbb{Q}$, which is elliptic at infinity and contains γ_k. Since ε_0 lies in the center of $G(\varepsilon_0)$, the element ν' defined by the equations

$$|\lambda(\varepsilon_0)|_p = p^{\nu'(\lambda)}, \quad \lambda \in X^*(T)$$

is central in $X_*(T) \otimes \mathbb{Q}$, i.e., invariant under the Weyl group of $G(\varepsilon_0)$. There exists a natural number t and an element $\eta = p^{-t\nu'} \in T(\mathbb{Q})$, which is a unit outside p, such that

$$|\lambda(\eta)|_p = p^{t\nu'(\lambda)}, \quad \lambda \in X^*(T).$$

195 The element η can be chosen to be central in $G(\varepsilon_0)$. Let $s = \frac{k}{tn}$. We show that for sufficiently large k, $\gamma_k = \eta^s$. It remains to prove that $\eta^{-s}\gamma_k$ is a unit at every finite place, since γ_k is necessarily elliptic at infinity, and we can replace k by a multiple. The claim is clear outside p.

At the place p, consider the homomorphism $\xi_p = \phi \circ \zeta_p$ and the corresponding ν. On the one hand, ν is defined by

$$|\lambda(\gamma_k)|_p = q^{-\langle\lambda,\nu\rangle}, \quad q = p^k, \lambda \in X^*(T),$$

(see the Note). On the other hand, since (ϕ, ε_0) is admissible, we have

$$|\lambda(\varepsilon_0)|_p = p^{-n\langle\lambda,\nu\rangle}$$

and

$$|\lambda(\eta)|_p = p^{-tn\langle\lambda,\nu\rangle}.$$

Thus, the claim is also proved at p, and the equation $\gamma_k = \gamma_k'$ follows.

We fix an admissible and well-positioned pair (ϕ_0, ε_0). If (ϕ, ε_0) is also admissible, then $\phi = \phi_0$ on the kernel. Let $G'(\varepsilon_0)$ be the twisting of $G(\varepsilon_0)$ defined by ϕ_0. We consider a homomorphism ϕ that agrees with ϕ_0 on the kernel. Then we have $\phi(q_\varrho) = a_\varrho \phi_0(q_\varrho)$, where $a = \{a_\varrho\}$ is a cocycle of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in $G'(\varepsilon_0)$. We write $\phi = a \cdot \phi_0$.

Lemma 5.26. *(ϕ, ε_0) is admissible if and only if the images of a in $H^1(\mathbb{Q}, G_{\mathrm{ab}})$ and in $H^1(\mathbb{R}, G'(\varepsilon_0))$ are both trivial.*

We assume this lemma for the moment and conclude the proof of Theorem 5.25. Let (γ_0, δ_0) and (γ, δ) be the pairs corresponding to admissible pairs (ϕ_0, ε_0) and (ϕ, ε), respectively. To define $\kappa(\gamma_0, \delta_0; \varepsilon_0)$ and $\kappa(\gamma, \delta; \varepsilon)$, we have introduced characters $\beta_0'(v)$ and $\beta'(v)$ for each place v. The difference $\beta'(v)(\beta_0'(v))^{-1}$ is a character on $(Z(\varepsilon_0)/Z)^{\Gamma(v)}$, although only its restriction to $Z(\varepsilon_0)^{\Gamma(v)}$ is uniquely determined. On the other hand, as shown in [K2], Prop. 6.4, for each v a character $\alpha(v)$ of

$$Z(\widehat{G}'(\varepsilon_0))^{\Gamma(v)} = Z(\widehat{G}(\varepsilon_0))^{\Gamma(v)} = Z(\varepsilon_0)^{\Gamma(v)}$$

is assigned to the class a.

Lemma 5.27. *$\alpha(v)$ is the restriction of $\beta'(v)(\beta_0'(v))^{-1}$ to $Z(\varepsilon_0)^{\Gamma(v)}$.*

The proof is deferred. From this, it follows that γ and γ_0 are conjugate everywhere (except at infinity and p) if and only if a is locally trivial outside of ∞ and p. If $\delta' = u\delta\sigma(u)^{-1}, u \in G(L_n)$ and $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta = c\varepsilon_0 c^{-1}$, then $\mathrm{Nm}_{L_n/\mathbb{Q}_p}\delta' = uc\varepsilon_0 c^{-1}u^{-1}$ and $b' = c^{-1}u^{-1}\delta' uc = b$, or at least $b' = vb\sigma(v)^{-1}, v \in G(\varepsilon_0)$, since c is only determined modulo $G(\varepsilon_0)$. Conversely, if $b' = vb\sigma(v)^{-1}$, we can choose $\delta' = \delta$. Thus, δ and δ_0 are twisted conjugates in $G(L_n)$ if and only if a is trivial at p. The last statement of Theorem 5.25 follows from the two previous lemmas. Let (γ, δ) be such that $\kappa(\gamma, \delta; \varepsilon_0) = 1$, and let $a(v)$ be the corresponding adelic cohomology class with a trivial component at infinity. From Lemma 5.27, it easily follows that we 196
can lift $(a(v))$ into a cohomology class $(\widetilde{a}(v))$ with values in $G'(\varepsilon_0) \cap G_{\mathrm{der}}$, which lies in the kernel of the natural map

$$\bigoplus_v H^1(\mathbb{Q}_v, G'(\varepsilon_0) \cap G_{\mathrm{der}}) \to \left(\pi_0(Z(\varepsilon_0)/Z)^\Gamma\right)^D.$$

From [K5], Prop. 2.6, we conclude that $(\widetilde{a}(v))$ is the localization of a global cohomology class. Its image $a \in H^1(\mathbb{Q}, G'(\varepsilon_0))$, defined according to Lemma 5.26, defines an admissible pair (ϕ, ε_0). This completes the proof of Theorem 5.25. *QED Thm. 5.25.*

Proof of Lem. 5.26, 5.27.

The proof of Lemmas 5.26 and 5.27 remains. The necessity of the first condition of Lemma 5.26 is clear. The necessity of the second condition follows from the injectivity of $H^1(\mathbb{R}, G'(\varepsilon_0)) \to H^1(\mathbb{R}, G')$, which itself is a consequence of Lemma 5.14 and the next lemma.

Lemma 5.28. *Assume the conditions of Lemma 5.14. Then $H^1(\mathbb{R}, T') \to H^1(\mathbb{R}, G')$ is surjective.*

Proof of Lem. 5.28.

The lemma can be immediately reduced to the case of an adjoint group, for which it asserts that every inner twisting of G' contains an elliptic Cartan subgroup. This is Lemma 5.8.

QED Lem. 5.28.

Now suppose a satisfies the conditions of Lemma 5.26. We will simultaneously show that (ϕ, ε_0) is admissible and that Lemma 5.27 holds. Conditions (5.a) and (5.b) are certainly satisfied. To check the other conditions, we can assume that $a_\varrho \in G(\varepsilon_0) \cap G_{\mathrm{der}}$ for each ϱ.

Let
$$\phi_0 \circ \zeta_l = \operatorname{ad} y \circ \xi_l, \quad \varepsilon_0 y = y\gamma_l^0, \quad \gamma_l^0 \in G(\mathbb{Q}_l).$$
Then $\{y^{-1}a_\varrho y \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)\}$ is a cocycle with values in G_{der}, and hence trivial: $y^{-1}a_\varrho y = c^{-1}\varrho(c)$. Let $y' = yc^{-1}$. Then we have
$$\phi \circ \zeta_l = \operatorname{ad} y' \circ \xi_l,$$
so condition (5.c) is satisfied. We also have
$$\varepsilon_0 y' = \varepsilon_0 y c^{-1} = y\gamma_l^0 c^{-1} = y' c\gamma_l^0 c^{-1}$$
and
$$\varrho(c\gamma_l^0 c^{-1}) = \varrho(cy^{-1}\varepsilon_0 y c^{-1}) = cy^{-1}\varepsilon_0 y c = c\gamma_l^0 c^{-1},$$
because $a_\varrho y \varrho(y^{-1})$ commutes with ε_0. Thus, $\gamma_l = c\gamma_l^0 c^{-1}$, and $\beta_0'(l)$ (resp. $\beta'(l)$) is defined by the cocycle $\{y\varrho(y)^{-1}\}$ (resp. $\{y'\varrho(y')^{-1}\} = \{a_\varrho y \varrho(y)^{-1}\}$). The group $G'(\varepsilon_0)$ is defined by the cocycle $\{y\varrho(y)^{-1}\}$, and we obtain $G(\varepsilon_0)$ from $G'(\varepsilon_0)$ via the cocycle $\{\varrho(y)y^{-1}\}$. We apply Lemma 5.18 to $G'(\varepsilon_0), G(\varepsilon_0)$ instead of G, G' to conclude Lemma 5.27 at the place l.

197 The remarks preceding Theorem 5.21 can be applied to $\mathscr{J}$. Consequently, $H^1(\mathbb{Q}_p, \mathscr{J} \cap G_{\mathrm{der}}) = \{1\}$ and $H^1(\mathbb{Q}_p, \mathscr{J}_{\phi_0} \cap G_{\mathrm{der}}) = \{1\}$. It is clear that ϕ_0 and ϕ define the same $\mathscr{J}$. Since a defines the trivial class in $\mathscr{J}_{\phi_0}$, ϕ_0 and ϕ define the same element in $B(\mathscr{J})_b$. Therefore, ϕ is an admissible homomorphism as ϕ_0 is. Except for the existence of x in condition (iii), it is also clear that the pair (ϕ, ε_0) is admissible. The existence of x is verified as in the proof of Theorem 5.21. Thus, Lemma 5.26 is proven.

ϕ_0 and ϕ are mappings from $\mathscr{Q}$ to $\mathscr{G}_{G(\varepsilon_0)} \subset \mathscr{G}_G$. To verify the claim of Lemma 5.27 at the place p, we can replace $\xi_p = \phi \circ \zeta_p$ by $\xi_p' = \operatorname{ad} u \circ \xi_p$ with $u \in G(\varepsilon_0)$. We can further assume that the localization of a is given by a cocycle from $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Let σ be the Frobenius element. Then, $b = a_\sigma \cdot b_\sigma$. Thus the claim at p follows immediately from Lemma 5.19, applied to $G(\varepsilon_0), G'(\varepsilon_0)$, and a_σ^{-1}.

QED Lem. 5.26, 5.27.

Note. Let $\mathscr{D}_n^n = \mathscr{D}_{L_n}^{L_n}$, and let $\mathscr{D}^n$ (resp. $\mathscr{D}_{\mathfrak{k}}^n$) be the group obtained by pushing out via $L_n \subset \mathbb{Q}_p^{\mathrm{un}}$ (resp. $L_n \subset \mathfrak{k}$) and pulling back via $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \to \mathrm{Gal}(L_n/\mathbb{Q}_p)$. We define $\mathscr{D}_n^n$ by the fundamental cocycle
$$a_{\sigma^i,\sigma^j} = 1, \quad 0 \le i, j < n, i + j < n$$
$$a_{\sigma^i,\sigma^j} = p^{-1}, \quad 0 \le i, j < n, i + j \ge n.$$

If σ is the Frobenius element in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$, the n-th power of d_σ is equal to p^{-1} in $\mathscr{D}^n$ and $\mathscr{D}^n_{\mathfrak{k}}$. Therefore, the homomorphisms $\mathscr{D}^{mn} \to \mathscr{D}^n$ (resp. $\mathscr{D}^{mn}_{\mathfrak{k}} \to \mathscr{D}^n_{\mathfrak{k}}$), which map $x \mapsto x^m$ for $x \in (\mathbb{Q}_p^{\mathrm{un}})^\times$ (resp. $\mathfrak{k}^\times$), and $d_\sigma \mapsto d_\sigma$, form a compatible family. In $D^n_{\mathfrak{k}}$, each element xd_σ with $x \in \mathfrak{k}^\times, |x| = 1$ is conjugate to d_σ.

Any homomorphism $\xi_p : \mathscr{D} \to \mathscr{G}_G$ is equivalent, by Steinberg's theorem (see [Se], p. 3), to a homomorphism ξ'_p that can be induced from

$$\xi'_p : \mathscr{D}_n \to G(\mathbb{Q}_p^{\mathrm{un}}) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p).$$

Let us assume that ξ_p itself can be defined in this way.

The element $\xi(w)$ that appears in the definition of X_p is nothing other than $\xi_p(d_\sigma)$. In general, $\xi_p(d_\sigma) = b \times \sigma$, and b defines a class in $B(G)$, which depends only on the class of ξ_p. If we choose n sufficiently large, the $n\nu$ appearing in [K4], equation (4.3.3), is the restriction of ζ_p^{-1} on $(\mathbb{Q}_p^{\mathrm{un}})^\times \subset \mathscr{D}^n$, which is a homomorphism of algebraic groups. We obtain in this way a map from $C(G)$, the set of classes in $\mathrm{Hom}(\mathscr{D}, \mathscr{G}_G)$, to $B(G)$. Let $C(G)_b$ be the preimage of $B(G)_b$. We do not know whether $C(G) \to B(G)$ is bijective. However, it follows easily from [K4] that $C(G)_b \to B(G)_b$ is bijective.

For tori, one has the functor[11] $C(T) \to B(T)$, as well as the functor $\mu \mapsto \xi_{-\mu}$ from $X_*(T)$ to $C(T)$. Since $\zeta_{-\mu}(d_\sigma) = p^\mu \times \sigma$, the composition maps $1 \in X_*(\mathbb{G}_m)$ to $b = p$. It therefore follows from Lemma 2.2 of [K4] that the composition is an isomorphism. Consequently, $C(T) \to B(T)$ is surjective. It is injective because the number of classes in $C(T)$ (resp. $B(T)$) for a given ν is equal to $|H^1(\mathbb{Q}_p, T)|$. For a general group, injectivity is also proved in this way. Surjectivity follows from [K4], Prop. 5.3.

6. The conjecture as a consequence of the Standard Conjectures in some cases 198

In this section, we want to show that, for certain Shimura varieties, the conjecture in §5 follows from the identification of the pseudo-motivic Galois group and the motivic Galois group, which was carried out in §4 under the assumption of the standard conjectures, the Tate conjecture, and the Hodge conjecture. In doing so, we closely follow Zink's work [Z2], so the reader is expected to have at least a superficial familiarity with this work. We begin with the definition of the relevant Shimura varieties and recall some group-theoretic facts, for whose proofs we refer to [De3] and [Z2].

Let D be a simple, finite-dimensional $\mathbb{Q}$-algebra of degree d^2 over its center L, equipped with a positive involution $x \mapsto x^*$. Let L_0 be the field of invariants of the involution in L. Let (V, ψ) be a symplectic D-module, i.e., a D-module equipped with a non-degenerate alternating $\mathbb{Q}$-bilinear form such that

$$\psi(xu, v) = \psi(u, x^* v), \quad x \in D, \quad u, v \in V.$$

Let $\dim_L V = d \cdot m$ and $[L_0 : \mathbb{Q}] = n$. We assume that $(D, *)$ is of one of the following two types:

(A) The involution is of the second kind, i.e., it induces a nontrivial automorphism of L. In this case, L_0 is a totally real number field.

(C) The involution is of the first kind, i.e., it induces the trivial automorphism on L. If $K \supset L$ is a splitting field of D, then we find an isomorphism $D \otimes_L K \cong$

[11] Here the two appearances of "functor" should be understood as "natural transformation".

$M_d(K)$ and a K-linear extension of the involution. In matrix notation, we assume $x^* = c\,{}^t x c^{-1}$ with ${}^t c = c$. In this case, L is a totally real number field.

Let G be the group of symplectic L_0-similarities, regarded as an algebraic group over $\mathbb{Q}$:

$$G = \{g \in \mathrm{GL}_D(V) \mid \psi(gu, gv) = \psi(c(g)u, v), \quad c(g) \in L_0^\times\}.$$

This is a connected reductive group over $\mathbb{Q}$, whose derived group is simply connected ([De3], 5.8). (In case (A), G_{der} is a product of special unitary groups, and in case (C), a product of symplectic groups). The groups G and G_{der} satisfy the Hasse principle ([De3], 5.11). Furthermore, there exists a homomorphism

$$h : \mathbb{S} \to G_{\mathbb{R}},$$

which, up to conjugation, is characterized by the fact that the Hodge structure induced on V is of type $(+1, 0) + (0, +1)$, and that $\psi(u, h(i)v)$ is symmetric and positive definite ([Z2], 3.1). The pair (G, h) defines a Shimura variety, where it should be noted that the weight homomorphism $\mu + \bar{\mu} : \mathbb{G}_m \to G$ is defined over $\mathbb{Q}$ and that $X_*(G_{\mathrm{ab}})$ satisfies the Serre condition (3.e). The D-module V is uniquely determined up to isomorphism by

$$t(x) = \mathrm{Tr}(x \mid V_h^{1,0}), \quad x \in D,$$

199 (see [De3], 5.10). Let

$$E = E(G, h) = \mathbb{Q}(t(x) \mid x \in D).$$

This is the Shimura field. (In case (C), $E = \mathbb{Q}$.) Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup. The Shimura variety $\mathrm{Sh}_K(G, h)$ has a canonical model over E and is a *coarse* moduli scheme for the moduli problem on (Sch/E), which associates to an E-scheme S the following data:

(6.a)

(i) *An abelian D-scheme up to isogeny $(A, \iota : D \to \mathrm{End}\, A)$ such that*

$$\mathrm{Tr}(x \mid \mathrm{Lie}^* A) = t(x), \quad x \in D.$$

Here, $\mathrm{Lie}^* A$ *denotes the cotangent space of A.*

(ii) *An L_0-homogeneous polarization Λ, which induces the involution $*$ on D.*

(iii) *A class of D-module isomorphisms of sheaves for the étale topology*

$$\bar{\eta} : H^1(A, \mathbb{A}_f) \xrightarrow{\sim} V \otimes \mathbb{A}_f \mod K,$$

which preserves the bilinear forms on both sides up to a constant in $L_0 \otimes \mathbb{A}_f$.

Here, $H^1(A, \mathbb{A}_f) = \prod H^1(A, \mathbb{Q}_l)$ denotes the l-adic cohomology with coefficients in $\mathbb{A}_f$.

We now fix our prime number p. Let O_D be an order in D and $V_{\mathbb{Z}}$ an O_D-invariant lattice in V, such that

(6.1)

a) p is unramified in D,

b) $D \otimes \mathbb{Q}_p$ is a product of matrix algebras, and $O_D \otimes \mathbb{Z}_p$ is a maximal order,

c) $\psi : V_{\mathbb{Z}_p} \times V_{\mathbb{Z}_p} \to \mathbb{Z}_p$ is a perfect pairing.

Then we have $*(O_D \otimes \mathbb{Z}_p) = O_D \otimes \mathbb{Z}_p$. Let

$$K_p = G(\mathbb{Q}_p) \cap \mathrm{End}_{O_D} V_{\mathbb{Z}_p}.$$

Then G is quasi-split over $\mathbb{Q}_p$ and split over an unramified extension, and $K_p \subset G(\mathbb{Q}_p)$ is the group of integer points, i.e.([T], 3.8) a hyperspecial subgroup. Let $K = K^p \cdot K_p$. To define a model of Sh_K over $\mathrm{Spec}(\mathcal{O}_E \otimes \mathbb{Z}_{(p)})$, we reformulate the moduli problem (6.a). A point of this moduli problem with values in an $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$-scheme S consists of:

(6.b)

(i) *An abelian O_D-scheme A, up to prime-to-p isogeny, such that*

$$\mathrm{Tr}(x \mid \mathrm{Lie}^* A) = t(x), \quad x \in O_D.$$

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(ii) *An L_0-homogeneous polarization Λ that induces the given involution $*$ on D, such that there exists $\lambda \in \Lambda$ whose degree is prime to p.*

(iii) *A class of $O_D \otimes \mathbb{A}_f^p$-module isomorphisms of sheaves for the étale topology*

$$\bar{\eta} : H^1(A, \mathbb{A}_f^p) \xrightarrow{\sim} V \otimes \mathbb{A}_f^p \mod K^p,$$

preserving the bilinear forms on both sides up to a constant in $L_0 \otimes \mathbb{A}_f^p$.

We refer to [Z2], 3.5, for a proof of the fact that the moduli problems (6.a) and (6.b) coincide over $\mathrm{Spec}\, E$. As in §5, let $\mathcal{O}_\mathfrak{p}$ denote the ring of integers in $E_\mathfrak{p}$.

Theorem 6.2. *The moduli problem* (6.b) *admits a coarse moduli scheme over* $\mathrm{Spec}\,(\mathcal{O}_\mathfrak{p})$. *If K^p is sufficiently small, then the moduli scheme is smooth, and if $L_0 = \mathbb{Q}$, it is also a fine moduli scheme.*

We outline a proof, which follows the proof of the corresponding Theorem 1.7 from [Z2]. The O_D-lattice $\eta^{-1}(V_\mathbb{Z} \otimes \widehat{\mathbb{Z}}^p) \subset H^1(A, \mathbb{A}_f^p)$ is independent of the choice of $\eta \in \bar{\eta}$ and defines an abelian O_D-scheme B in the isogeny class of A, equipped with an O_D-module isomorphism $\bar{\eta} : H^1(B, \widehat{\mathbb{Z}}^p) \cong V_\mathbb{Z} \otimes \widehat{\mathbb{Z}}^p \mod K^p$. This shows that the moduli problem (6.b) is equivalent to the following moduli problem.

(6.c)

(i) *An abelian O_D-scheme B, such that*

$$\mathrm{Tr}(x \mid \mathrm{Lie}^* B) = t(x), \quad x \in O_D.$$

(ii) *As in* (6.b), *(ii).*

(iii) *A class of O_D-module isomorphisms*

$$\bar{\eta} : H^1(B, \widehat{\mathbb{Z}}^p) \xrightarrow{\sim} V_\mathbb{Z} \otimes \widehat{\mathbb{Z}}^p \mod K^p,$$

preserving the bilinear forms on both sides up to a constant in $L_0 \otimes \mathbb{A}_f^p$.

By using the methods of Mumford [Mu2] (or those of Artin [A], if one is satisfied with the construction of an algebraic space), we obtain the existence of a fine moduli scheme if in (ii) we require the presence of an effective involution-preserving polarization and if K^p is taken to be sufficiently small. If $(B, \Lambda, \bar{\eta})$ is a T-valued point of the moduli problem (6.c) and $\lambda \in \Lambda$ is prime to p and $\eta \in \bar{\eta}$, then, if E^λ denotes the associated Riemann form,

$$\psi\big(\eta(x), \eta(y)\big) = E^\lambda(tx, y), \quad x, y \in H^1(B, \widehat{\mathbb{Z}}^p)$$

for some $t \in L_0^\times(\mathbb{A}_f^p)$. The residue class $\tau(B, c, \lambda, \bar{\eta})$ of t in the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\mathfrak{p})$-module $\big(L_0^\times(\mathbb{A}_f^p)/c(K^p)\big)(+1)$ is independent of the choice of $\eta \in \bar{\eta}$ and can be regarded as a section of this sheaf over T. Two polarizations λ and λ' differ by a totally positive el-
201 ement of L_0 that is a unit at p. Thus, we obtain a section of $L_0^+\backslash L_0^\times(\mathbb{A}_f)/c(K)\,(+1)$ that depends only on $(B, \Lambda, \bar{\eta})$. Let $\tau_1, \ldots, \tau_r$ be sections of $L_0^\times(\mathbb{A}_f^p)/c(K^p)\,(+1)$ over a finite unramified extension R of $\mathcal{O}_\mathfrak{p}$ that form a system of representatives. We consider the following moduli problem on (Sch/R):

(6.d)

(i) *As in* (6.c) *(i).*

(ii) *An involution-preserving polarization λ that is principal at p.*

(iii) *A class of O_D-module isomorphisms*

$$\bar{\eta} : H^1(B, \widehat{\mathbb{Z}}^p) \xrightarrow{\sim} V_\mathbb{Z} \otimes \widehat{\mathbb{Z}}^p \mod K^p,$$

such that $\tau(B, c, \lambda, \bar{\eta}) = \tau_i$ for some $i = 1, \ldots, r$.

If we choose the τ_i such that they are representatives of *integral* ideals, then λ is an effective polarization whose degree is determined by τ_i. Consequently, according to Mumford, the moduli problem (6.d) for sufficiently small K^p is representable by a quasi-projective R-scheme $\widetilde{\mathscr{M}}_K$. The group $L_0^+ \cap c(K)$ acts on $\widetilde{\mathscr{M}}_K$ (viewed as an $\mathcal{O}_\mathfrak{p}$-scheme) by changing the polarization. It is easy to verify that this action factors over the finite quotient $L_0^+ \cap c(K)/c(L^\times \cap K)$, and that the quotient is a coarse moduli scheme for the modular problem (6.c) over $\operatorname{Spec} \mathcal{O}_\mathfrak{p}$. If $L_0 = \mathbb{Q}$, then $L_0^+ \cap c(K)$ is trivial. We show that in general, for sufficiently small K^p, the action of $L_0^+ \cap c(K)/c(L^\times \cap K)$ on $\widetilde{\mathscr{M}}_K$ is free. Let $\zeta \in L_0^+ \cap c(K)$, and

$$\alpha : (B, \iota, \lambda, \bar{\eta}) \xrightarrow{\sim} (B, \iota, \zeta\lambda, \bar{\eta})$$

be an isomorphism. We must show that $\zeta \in c(L^\times \cap K)$. Let $K' = K'^p \cdot K_p$ be a sufficiently small open compact subgroup. We consider only $K \subset K'$, and denote the equivalence class of $\bar{\eta} \mod K'$ by $\bar{\eta}'$. Then, there is at most one isomorphism

$$\alpha' : (B, \iota, \lambda, \bar{\eta}') \xrightarrow{\sim} (B, \iota, \zeta\lambda, \bar{\eta}').$$

Let K be chosen small enough such that $L_0^+ \cap c(K) \subset (L_0^\times \cap K')^2$. Then, $\zeta = \zeta_1^2$, and ζ_1 defines an isomorphism between $(B, \iota, \lambda, \bar{\eta}')$ and $(B, \iota, \zeta\lambda, \bar{\eta}')$. But then ζ_1 coincides with α, and therefore $\zeta_1 \in K$, implying $\zeta = \zeta_1^2 \in c(L^\times \cap K)$.

To convince ourselves of the smoothness of $\widetilde{\mathscr{M}}_K$, we need to show that there are no deformation obstructions. We use the theory of Grothendieck and Messing [Me]. Let $T \subset T'$ be a nil-immersion with divided powers of affine R-schemes, on which p is nilpotent. Let $(B, \iota, \lambda, \bar{\eta})$ be a T-valued point of the moduli problem (6.d). We consider the value of the Dieudonné crystal at T',

$$\mathbb{D}(B)_{(T,T')},$$

which is equipped with an action of O_D and an alternating *perfect* bilinear form E^λ with values in $\mathcal{O}_{T'}$, satisfying the identity

$$E^\lambda(xu, v) = E^\lambda(u, x^* v).$$

202 Its restriction to T is equipped with the Hodge filtration, which is totally isotropic with respect to E^λ and for which

$$\mathrm{Tr}(x \mid \mathrm{Fil}) = t(x), \quad x \in O_D.$$

It must be shown that this filtration can be lifted to a totally isotropic filtration of $\mathbb{D}_{(T,T')}$. The decomposition $O_L \otimes \mathbb{Z}_p = \prod O_{L_v}$ provides an orthogonal decomposition $(\mathbb{D}_{(T,T')}, E^\lambda) = \bigoplus (D'_v, E'_v)$.

In case (C), $L_0 = L$. We choose an isomorphism $O_D \otimes_{O_L} O_{L_v} \cong M_d(O_{L_v})$. Then, the involution $*$ is induced by a perfect *symmetric* pairing β on $O^d_{L_v}$, cf. [Z2], 2.9. We introduce the bilinear form $\widetilde{E}'_v$ with values in $O_{L_v} \otimes \mathcal{O}_{T'}$:

$$\mathrm{Tr}_{L_v/\mathbb{Q}_p} f \cdot \widetilde{E}'_v(u,w) = E'_v(u, f \cdot w), \quad f \in O_{L_v}.$$

Then the $M_d(O_{L_v}) \otimes \mathcal{O}_{T'}$-module $(D'_v, \widetilde{E}'_v)$ decomposes as $D'_v = O^d_{L_v} \otimes V'$, $\widetilde{E}'_v = \beta \otimes \Phi'$, where Φ' is an *alternating* perfect bilinear form on the $\mathcal{O}_{T'}$-module V'. The corresponding result holds for $\mathbb{D}_{(T,T)}$. Here $\mathrm{Fil}_v \subset D_v$ is generated by a totally isotropic part (with respect to Φ) of a basis of V. Since this basis can be lifted to such a basis of V', the assertion follows.

The case (A) is handled analogously (cf. [Z2], 3.4).

With this, our sketch of the proof of Theorem 6.2 is complete. We have obtained a smooth model over $\mathcal{O}_\mathfrak{p}$, whose points over $\mathbb{F} = \bar{\kappa}$ we want to analyze, assuming the validity of the results of §4. A *special point* of the moduli problem (6.b) is defined by the following data ([Z2], 4.2): a point $(A, \iota, \Lambda, \bar{\eta})$ of the moduli problem, a CM-algebra R (= product of CM fields) with $\dim_\mathbb{Q} R = 2 \dim A/d$, and an involution-preserving embedding

$$\vartheta_A : R \to \mathrm{End}^0_D A.$$

Let $(A, \iota, \Lambda, \bar{\eta}, R, \vartheta)$ be a special point over $\mathbb{C}$. Then $D \otimes_L R$ is a matrix algebra over R ([Z2], 2.8), and we obtain a decomposition $A \cong B^d$, where B is an abelian variety with complex multiplication (R, Φ). Let

$$\mu : \mathbb{C}^\times \to (R \otimes \mathbb{C})^\times = \prod_{\varphi \in \Phi} \mathbb{C}^\times \times \prod_{\varphi \notin \Phi} \mathbb{C}^\times, \quad z \mapsto \prod \varphi(z) \times \prod 1.$$

Let $T = \{r \in R^\times \mid r \cdot \bar{r} \in L_0\}$, where $r \mapsto \bar{r}$ denotes the Rosati involution on R. Then T is a maximal torus defined over $\mathbb{Q}$ in G, and the above homomorphism factors through T and is of the form $\mu = \mu_h$ for some $h : \mathbb{S} \to G_\mathbb{R}$ in the conjugacy class we fixed. Thus, $(A, \iota, \Lambda, \bar{\eta}, R, \vartheta)$ defines a special point of $\mathrm{Sh}_K(\mathbb{C})$ in the usual sense ([De1], 2.2.4). The module $X_*({}^K S)$ contains a distinguished element $\mu = \mu_S$, which generates it. Since $X_*(G_{\mathrm{ab}})$ satisfies the Serre condition (3.e), so does $X_*(T)$, and for sufficiently large fields K, there exists a homomorphism $\psi : {}^K S \to T$ defined over $\mathbb{Q}$ that sends μ_S to μ. Let $G^0 = \{g \in G \mid c(g) \in \mathbb{Q}^\times\}$. Then μ is a cocharacter of $T^0 = T \cap G^0$ and $\psi : {}^K S \to T^0$. We denote also by ψ the corresponding homomorphism[12]

$$\mathscr{G}_S \longrightarrow \mathscr{G}_{T^0} \xrightarrow{\psi_{T^0,\mu}} \mathscr{G}_{G^0} \subset \mathscr{G}_G.$$

The group $\mathscr{G}_S$ is the motivic Galois group of the Tannakian category generated 203
by abelian varieties of CM type. Thus, ψ defines a motive with additional structure, namely the homogeneous component of degree 1 of $(A, \iota, \lambda, R, \vartheta)$ with $\lambda \in \Lambda$. We introduce the concept of *isogeny* between two points $(A_1, \iota_1, \Lambda_1, \bar{\eta}_1)$ and $(A_2, \iota_2, \Lambda_2, \bar{\eta}_2)$ of the moduli problem (6.b): an isogeny $\alpha : A_1 \to A_2$ that is compatible with ι_1 and Λ_1 and ι_2 and Λ_2. The trivializations of the Tate modules do

[12]The second map should simply be the one induced by the inclusion $T^0 \subset G^0$, and the symbol $\psi_{T^0,\mu}$ should be deleted.

not enter here. An isogeny prime to p is nothing other than an isomorphism

$$\alpha : (A_1, \iota_1, \Lambda_1) \xrightarrow{\sim} (A_2, \iota_2, \Lambda_2).$$

In terms of the moduli problem (6.c), an isogeny between abelian varieties $\beta : B_1 \to B_2$ is one that respects the actions of O_D and the polarization classes. One speaks of an isogeny between special points if there is an isomorphism $R_1 \cong R_2$ such that the isogeny additionally transfers the action ϑ_1 into the action ϑ_2.

Theorem 6.3. *The set of isogeny classes of points of* $\mathrm{Sh}(\bar{\kappa})$ *is in a one-to-one correspondence with the equivalence classes of admissible homomorphisms* $\phi : \mathscr{P} \to \mathscr{G}_G$.

We assume, however, that $\mathscr{P} \cong \mathscr{M}$ in $\mathscr{G}_S$. According to the definition of $\mathscr{M}$, the homogeneous motive $M^1(A)$ of degree 1 associated with an abelian variety corresponds to a homomorphism $\phi : \mathscr{M} \to \mathscr{G}_{V'}$. Since

$$\mathrm{Hom}(M^1(A), M^1(A')) \cong \mathrm{Hom}_{\mathbb{Q}}(A, A'),$$

A is defined as an abelian D-variety by an embedding $\iota' : D \to \operatorname{End} V'$ that is compatible with ϕ. Similarly, the dual variety A^* is defined by the contragredient homomorphism ϕ^* on V'^*. For T the Tate motive, we have $V'^* = \mathrm{Hom}(V', T)$. A polarization corresponds to a homomorphism $\psi : V' \to V'^*$ that is compatible with ϕ and ϕ^*, i.e., a bilinear form ψ' on V' for which the equation

$$\psi'(\phi(g)u, \phi(g)v) = \tau(g) \cdot \psi'(u, v)$$

holds, where τ denotes the action corresponding to T on $\overline{\mathbb{Q}}$. The isomorphism class of (A, ι, λ) is clearly determined by (V', ι', ψ'). Due to the existence of a normal form, we can assume that $(V', \iota', \psi') = (V, \iota, \psi)$. Then, the image of $\mathscr{M}$ lies in $\mathscr{G}_{G_0} \subset \mathscr{G}_V$. Two homomorphisms $\phi_1, \phi_2 : \mathscr{M} \to \mathscr{G}_{G_0}$ are equivalent under $G(\overline{\mathbb{Q}})$ if and only if $(A_1, \iota_1, \lambda_1) \cong (A_2, \iota_2, c\lambda_2), c \in L_0^\times$. We have $G_{\mathrm{ab}}^0 = \mathbb{Q}^\times \subset G_{\mathrm{ab}} = L_0^\times$. Since the cocharacter μ_{ab} factors through G_{ab}^0, every admissible $\phi : \mathscr{Q} \to \mathscr{G}_G$ factors through $\mathscr{G}_{G_0}$. Since $X_*(G_{\mathrm{ab}})$ satisfies the Serre condition (3.e), ϕ also factors through $\mathscr{P} = \mathscr{M}$.

To complete the proof, it suffices to show that if (A, ι, Λ) arises from $(A, \iota, \Lambda, \bar{\eta})$, then the corresponding ϕ is admissible, and vice versa. This is not difficult. However, we prefer a different approach that directly associates an admissible ϕ to each $(A, \iota, \Lambda, \bar{\eta})$ and vice versa, without assuming any unproven conjectures, though we do not at present prove the bijectivity of this correspondence. Furthermore, it is not straightforward to apply this if G is not quasi-split over $\mathbb{Q}_p$ (see the second example in §7).

204 Since A is already defined over a finite field, it is an abelian variety of CM type, and we can consider the point $(A, \iota, \Lambda, \bar{\eta})$ as a special point $(A, \iota, \Lambda, \bar{\eta}, R, \vartheta)$. According to Zink's lifting theorem [Z2], §4.4, there exists a mixed characteristic discrete valuation ring $\mathcal{O}$ with residue field $\bar{\kappa}$, and a special point $(\widetilde{A}, \widetilde{\iota}, \widetilde{\Lambda}, \widetilde{\bar{\eta}}, R, \widetilde{\vartheta})$, whose reduction is isogenous to the original point. Let (T, μ) be the corresponding torus with cocharacter in G. The homomorphism $\psi_{T,\mu} : \mathscr{P} \to \mathscr{G}_S \to \mathscr{G}_T \subset \mathscr{G}_G$ is admissible. Since $\phi \cong \psi_{T,\mu}$, it follows that ϕ is admissible. Conversely, if ϕ is admissible, it is equivalent to $\psi_{T,\mu}$. It defines $\psi_{T,\mu}$ and $(\widetilde{A}, \widetilde{\iota}, \widetilde{\Lambda}, \widetilde{\bar{\eta}}, R, \widetilde{\vartheta})$, which by reduction yields $(A, \iota, \Lambda, \bar{\eta})$.

In the following we fix an isogeny class $\mathscr{J} = (A, \iota, \Lambda)_{\mathbb{Q}}$ and the corresponding admissible homomorphism ϕ up to equivalence. We denote by $\mathscr{J}_K$ the set of points of $\mathrm{Sh}_K(\mathbb{F})$ in $\mathscr{J}$. Let $r = [E_{\mathfrak{p}} : \mathbb{Q}_p]$, and $q = p^r$ be the number of elements in κ. If $(A, \iota, \Lambda, \bar{\eta})$ is a point of $\mathrm{Sh}_K(\mathbb{F})$, then the inverse image $(A^{(q)}, \iota^{(q)}, \Lambda^{(q)}, \bar{\eta}^{(q)})$ under the r-th power of the Frobenius is again a point of $\mathrm{Sh}_K(\mathbb{F})$. The existence of the relative Frobenius

$$\mathrm{Fr}^r : (A, \iota, \Lambda, \bar{\eta}) \to (A^{(q)}, \iota^{(q)}, \Lambda^{(q)}, \bar{\eta}^{(q)})$$

shows that we obtain an action of $\mathrm{Gal}(\mathbb{F}/\kappa)$ on $\mathscr{J}_K$, which is also obviously compatible with the change of K^p.

Theorem 6.4. *There are bijections*

$$\mathscr{J}_K \cong X_\phi(K),$$

such that the action of Φ on the right hand set (see (5.e)*) corresponds to the action of the Frobenius in Gal($\mathbb{F}/\kappa$) on the left hand set, and they are compatible with the change of K^p.*

In the following we fix a point $(A_0, \iota_0, \Lambda_0, \bar{\eta}_0)$ in $\mathscr{J}_K$, which is the reduction modulo p of a special point $(\widetilde{A}_0, \widetilde{\iota}_0, \widetilde{\Lambda}_0, \widetilde{\bar{\eta}}_0)$ and defines, as in the proof of 6.2, the torus (T, μ).

An isogeny decomposes into a chain of a prime-to-p isogeny and a p-power isogeny. If an element $\eta_0 \in \bar{\eta}_0$ is fixed, an element $g \in G(\mathbb{A}_f^p)$ defines a prime-to-p isogeny:

$$(A_0, \iota_0, \Lambda_0, \bar{\eta}_0) \to (A_0, \iota_0, \Lambda_0, \overline{g^{-1} \circ \eta_0}).$$

Two elements differing by an element of K^p define the same isogeny. Conversely, any prime-to-p isogeny is induced by a well-defined element of $G(\mathbb{A}_f^p)/K^p$.

An isogeny $\alpha : (A_0, \iota_0, \Lambda_0) \to (A, \iota, \Lambda)$, whose degree is divisible by p, defines $(A, \iota, \Lambda, \bar{\eta})$, where $\bar{\eta}$ is the composition

$$H^1(A, \mathbb{A}_f^p) \xrightarrow[\sim]{\alpha} H^1(A_0, \mathbb{A}_f^p) \xrightarrow{\eta_0} V \otimes \mathbb{A}_f^p.$$

We now consider the p-divisible group X_0 of A_0 and its *contravariant* Dieudonné module M_0 ([Dem]). The cotangent space of X_0 canonically identifies with M_0/FM_0 ([Fo], III.4.3). Then M_0 is equipped with an action of $O_D \otimes \mathbb{Z}_p$ and a class of perfect 205
alternating bilinear forms ψ, which differ from one another by units in $O_{L_0} \otimes O_{\mathfrak{k}}$. Here $O_{\mathfrak{k}}$ denotes the ring of Witt vectors of $\mathbb{F}$. We extend the action of $D \otimes \mathbb{Q}_p$ and the class of alternating bilinear forms to the rational Dieudonné module $M_{0\mathbb{Q}}$. Let Y_p denote the set of sublattices $M \subset M_{0\mathbb{Q}}$ that satisfy the following conditions:

(i) M is invariant under the action of O_D,
(ii) There exists an element $c(M) \in (L_0 \otimes O_{\mathfrak{k}})^\times$ such that the dual space with respect to one of the forms ψ is $M^* = c(M) \cdot M$,
(iii) M is V- and F-invariant, and it satisfies

$$\mathrm{Tr}(x \mid M/FM) = t(x)$$

(equality of elements of $\mathbb{F}$).

Lemma 6.5. *The V-invariance required in (iii) follows from the condition (ii) and the F-invariance.*

Proof. We have

$$\psi(Fu, Fv) = p \cdot \psi(u, v)^\sigma,$$

where $\sigma : O_{\mathfrak{k}} \to O_{\mathfrak{k}}$ denotes the Frobenius on the Witt vectors. From $FV = p$, it follows that

$$\psi(u, Vv) = \psi(Fu, v)^{\sigma^{-1}}.$$

Thus, we have

$$(FM)^* = V^{-1}M^* = V^{-1} \cdot c(M)M = c(M)^{\sigma^{-1}} \cdot V^{-1}M.$$

From the inclusion $FM \subset M$, it follows that $VM \subset M$. □

Let $\alpha : (A_0, \iota_0, \Lambda_0, \bar{\eta}_0) \to (A, \iota, \Lambda, \bar{\eta})$ be an isogeny. The Dieudonné module of A becomes an element of Y_p via α. By decomposing α into its prime-to-p component and its p-primary component, we obtain an element of $(G(\mathbb{A}_f^p)/K^p) \times Y_p$. Its image in

$$Y(K) \underset{\text{Def}}{=} \mathrm{Aut}(A_0, \iota_0, \Lambda_0)\backslash\big(G(\mathbb{A}_f^p)/K^p \times Y_p\big)$$

depends only on $(A, \iota, \Lambda, \bar{\eta}) \in \mathscr{I}_K$ and not on the choice of the isogeny. Indeed, if α is a composition,

$$(A_0, \iota_0, \Lambda_0, \bar{\eta}_0) \xrightarrow{\alpha^p} (A', \iota', \Lambda', \bar{\eta}') \xrightarrow{\alpha_p} (A, \iota, \Lambda, \bar{\eta}),$$

with $\eta = \eta' \circ \alpha_p^*$, we obtain g by writing $\eta' = g^{-1} \circ \eta_0 \circ \alpha^{p*}$. Then we regard α_p as an isogeny from A_0 to A to embed $M(A)$ into $M(A_0)$. Via η_0, we map $\mathrm{Aut}(A_0, \iota_0, \Lambda_0)$
206 into $G(\mathbb{A}_f^p)$, and if we replace α by $\alpha \circ \gamma$, $\gamma \in \mathrm{Aut}(A_0, \iota_0, \Lambda_0)$, then g is replaced by $\gamma^{-1}g$. Moreover, $M(A)$ is replaced by $\gamma^{-1}M(A)$, where it is important to note that $\gamma^{-1}M(A)$ depends only on the p-primary component of γ.

We first show that the mapping just defined is a bijection from $\mathscr{I}_K$ to $Y(K)$. The theory of Dieudonné modules immediately shows that every element of $G(\mathbb{A}_f^p)/K^p \times Y_p$ defines an isogeny α, so surjectivity is clear. Conversely, if α_1 and α_2 are two isogenies with the same image in $Y(K)$, we obtain two elements in $G(\mathbb{A}_f^p)/K^p \times Y_p$ that differ by an element $\alpha \in \mathrm{Aut}(A_0, \iota_0, \Lambda_0)$. Let $\gamma = \frac{1}{n} \cdot \gamma'$, where γ' is an endomorphism of the abelian variety A_0 (and n is a power of p). If we replace α_1 by $\alpha_1 \circ \gamma'$ and α_2 by $\alpha_2 \circ n$, we may assume that the elements of $G(\mathbb{A}_f^p)/K^p \times Y_p$ defined by α_1 and α_2 are identical. Let

$$\varphi_1 = \alpha_2\alpha_1^{-1}, \quad \text{and} \quad \varphi_2 = \alpha_1^{-1}\alpha_2.$$

If we can show that φ_1 and φ_2 are morphisms in the category of abelian varieties up to prime-to-p isogeny and not just isogenies, then they define mutually inverse *isomorphisms* between $(A_1, \iota_1, \Lambda_1)$ and $(A_2, \iota_2, \Lambda_2)$, which are also compatible with the level structures. Let (with a new n) $\varphi_1 = \frac{1}{n} \cdot \varphi_1'$ with $\varphi_1' \in \mathrm{Hom}(A_1, A_2)$. On the Dieudonné modules, we obtain an inclusion

$$\varphi_1'(M_2) \subset M_1.$$

But since, by assumption, the elements of Y_p defined by α_1 and α_2 are identical, we have $\varphi_1'(M_2) \subset n \cdot M_1$, and thus φ_1' factors through the quotient of A_1 by its n-torsion points, so φ_1 is a morphism. Similarly, one shows that φ_2 is a morphism. In the constructed bijection, the action of Φ on $\mathscr{I}_K$ corresponds to the operator on $Y(K)$ that acts on the Y_p-component and replaces the lattice M by the lattice F^rM. This follows immediately from the fact that the relative Frobenius $\mathrm{Fr} : A \to A^{(p)}$ defines a map on the Dieudonné modules, $M^{(p)} \to M$, which identifies the lattice $M^{(p)}$ with FM (see [Dem], p. 63).

Next, we want to compare the sets $Y(K)$ and $X_\phi(K)$. After choosing a D-linear L_0-symplectic similarity $M_{0\mathbb{Q}} \cong V \otimes \mathfrak{k}$, the group $G(\mathfrak{k}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on M_0. (Here, V is again a D-module.) The set of lattices $M \subset M_{0\mathbb{Q}}$ satisfying conditions (i) and (ii) before 6.5 forms a homogeneous space under $G(\mathfrak{k})$ and can be identified with $\mathscr{X} = G(\mathfrak{k}) \cdot x^0$ (existence of a normal form). Due to the nature of the identification of $\mathscr{P}$ with $\mathscr{M}$, $\xi_p = \phi \circ \zeta_p$ determines the rational Dieudonné module $(M_{0\mathbb{Q}}, F)$ up to isomorphism. This means that there exists an element $h_1 \in G(\mathfrak{k})$ such that

$$F = \operatorname{ad} h_1(F_\phi),$$

where $F_\phi = \xi'(d_\sigma)$ is the "Frobenius" constructed before (5.d). Here, $d_\sigma \in W_{L_n/\mathbb{Q}_p}$ denotes a lift of the Frobenius element in $\operatorname{Gal}(L_n/\mathbb{Q}_p)$, and ξ' denotes the homomorphism from the Weil group $W_{L_n/\mathbb{Q}_p}$ to $G(\mathfrak{k}) \rtimes \operatorname{Gal}(\mathfrak{k}/\mathbb{Q}_p)$ used in the construction of 207
$\xi'_p = \operatorname{ad} h(\xi_p) : \mathscr{D}_{L_n}^{L_n} \to \mathscr{G}_G$. In the above equation, the right hand side is of course to be interpreted as the operator on $M_{0\mathbb{Q}}$ defined by the element there. We show that the condition

$$\operatorname{inv}(x^1, x^2) = \mu,$$

where x^i corresponds to the lattice M^i, is equivalent to the inclusion

$$pM^1 \subset M^2 \subset M^1,$$

such that $\operatorname{Tr}(x \mid M^1/M^2) = t(x)$, $x \in \mathcal{O}_D$.

Clearly, it suffices to find two lattices M^1, M^2 satisfying the above condition, and such that for the corresponding points $x^i \in \mathscr{X}$, the invariant is equal to μ. Let T' be a Cartan subgroup of G defined over $\mathbb{Q}_p$ that splits over $\mathfrak{k}$, and let $\mu' \in X_*(T')$ be a cocharacter conjugate to μ. We choose in the apartment corresponding to T' the special point x^1 corresponding to $T'(\mathbb{Z}_p)$ ([T], §3.6.1), which corresponds to a lattice M^1. Then $T'(\mathcal{O}_\mathfrak{k})$ acts on M^1 and $T'(\mathbb{F})$ acts on M^1/pM^1, and the resulting representation is the reduction modulo p of the representation of T' on $V_{\mathbb{Q}_p}$. We set

$$M^2 = p \cdot M^1 + \sum_{\langle \mu', \chi \rangle = 0} \mathbb{F}_\chi \subset M^1,$$

where $\mathbb{F}_\chi$ is the character space corresponding to χ. Since μ' is conjugate to μ, the characters appearing in M^1/M^2 are precisely those characters χ that appear in $V_\mathbb{C}$ and for which $\langle \mu, \chi \rangle = 1$. Thus,

$$\operatorname{Tr}(x \mid M^1/M^2) = \operatorname{Tr}(x \mid V_h^{1,0}) \pmod p$$

for $x \in O_D$. It is also clear that $x^2 = p^{\mu'} x^1$, so

$$\operatorname{inv}(x^1, x^2) = -\operatorname{inv}(x^2, x^1) = \mu$$

(cf. example before 5.2). We obtain that for $M \in Y(K)$,

$$\operatorname{inv}(M, FM) = \mu.$$

It follows that $h_1 : \mathscr{X} \to \mathscr{X}$ establishes a bijection between X_p and Y_p, which transforms the operator Φ into the action of Frobenius, and that $g \mapsto h_1 \cdot g \cdot h_1^{-1}$ defines an isomorphism between J_ϕ and $\operatorname{Aut}(M_{0\mathbb{Q}}, \iota_0, \Lambda_0)$. From the definition of the motivic gerbe, it follows that we can identify I_ϕ with $\operatorname{Aut}(A_0, \iota_0, \Lambda_0)$, as well as the action of I_ϕ on $X^p/K^p \times X_p$ with the action of $\operatorname{Aut}(A_0, \iota_0, \Lambda_0)$ on $G(\mathbb{A}_f^p)/K^p \times Y_p$, provided we choose the base point of X^p/K^p determined by the l-adic cohomology of A_0 and identify X_p with Y_p and J_ϕ with $\operatorname{Aut}(M_{0\mathbb{Q}}, \iota_0, \Lambda_0)$ in the manner described above. We obtain the desired bijection between $X_\phi(K)$ and $Y(K)$.

7. Two examples

208 The first example will be a z-extension in the sense of Kottwitz [K1],

$$1 \to A \to G' \to G \to 1,$$

such that the conjecture of §5 cannot hold simultaneously for both G and G'. In this example, the derived groups of G' and G cannot both be simply connected. Below we will construct a group G'' over $\mathbb{Q}$ and a map $h'' : \mathbb{S} \to G''_{\mathbb{R}}$ with the following properties:

(7.a) (G'', h'') defines a Shimura variety.

(7.b) Let Z'' be the center of G''. Then

$$H^1(\mathbb{Q}, Z'') = 0 \quad \text{and} \quad H^1(\mathbb{Q}_v, Z'') = 0$$

for each prime v of $\mathbb{Q}$.

(7.c) Let Z''_{der} be the center of G''_{der}. Then the map

$$H^2(\mathbb{Q}, Z''_{\mathrm{der}}) \to \prod_v H^2(\mathbb{Q}_v, Z''_{\mathrm{der}})$$

is injective.

(7.d) There exists a finite algebraic group U over $\mathbb{Q}$ and an embedding $u'' : U \to Z''_{\mathrm{der}}$ such that the map

$$H^2(\mathbb{Q}, U) \to \prod_v H^2(\mathbb{Q}_v, U)$$

is not injective.

We embed U in an induced torus A

$$u : U \to A$$

and set

$$G' = A \times G''/(u \times u'')(U), \quad G = G''/u''(U).$$

Then G' is a z-extension of G by A as above, and

$$G'_{\mathrm{der}} = G''_{\mathrm{der}}, \quad G_{\mathrm{der}} = G''_{\mathrm{der}}/u''(U).$$

Let $\bar{G}$ be an inner twist of G, and $\bar{G}'$ and $\bar{G}''$ the corresponding twistings of G' and G''.

209 Next, we define a cocycle g'_σ with values in $\bar{G}'$, whose image in $\bar{G}$ is locally trivial everywhere and globally trivial in G_{ab}, but whose image in G'_{ab} is not trivial. Since A is an induced torus, we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{Q}, A/U) \arrow[r] \arrow[d] & H^2(\mathbb{Q}, U) \arrow[r] \arrow[d] & H^2(\mathbb{Q}, A) \arrow[d, hook] \\
0 \arrow[r] & \prod_v H^1(\mathbb{Q}_v, A/U) \arrow[r] & \prod_v H^2(\mathbb{Q}_v, U) \arrow[r] & \prod_v H^2(\mathbb{Q}_v, A).
\end{tikzcd}$$

The map in the last column is injective. Consequently, the class $\{a_{\varrho,\sigma}\} \in H^2(\mathbb{Q}, U)$, which is locally trivial everywhere and whose existence we required in (7.d), is the image of a class in $H^1(\mathbb{Q}, A/U)$. Let $\sigma \mapsto a'_\sigma \in A$ be a lift of a representative of this class. The coboundary of this cochain is a 2-cocycle with values in Z''_{der}, which is locally trivial everywhere and thus trivial, according to (7.c). Let $\{z_\sigma\}$ be a cochain whose coboundary is the inverse of the coboundary of $\{a'_\sigma\}$. Then,

$$\sigma \mapsto g'_\sigma = a'_\sigma \cdot z_\sigma$$

is a cocycle with values in $A \cdot Z''_{\mathrm{der}}/U \subset \bar{G}'$. Its image g_σ in $\bar{G} = \bar{G}''/U$ is the image of z_σ. Since $G_{\mathrm{ab}} = \bar{G}''/\bar{G}''_{\mathrm{der}}$, its class in G_{ab} is obviously trivial. Locally,

$$z_\sigma = v_\sigma \cdot u_\sigma,$$

where v_σ is a cocycle with values in Z''_{der} and $u_\sigma \in U$. Consequently, g_σ is also the image of v_σ and is thus trivial by (7.b). Therefore, the class $\{g_\sigma\}$ is locally trivial everywhere in $\bar{G}$.

We define h' as $1 \times h''$, and define h as the composition of h' and the projection from G' to G. Then, (G', h') and (G, h) are Shimura varieties. The natural map $\mathscr{G}_{G'} \to \mathscr{G}_G$ assigns to each homomorphism $\phi' : \mathscr{Q} \to \mathscr{G}_{G'}$ a homomorphism $\phi : \mathscr{Q} \to \mathscr{G}_G$.

Lemma 7.1. *Two homomorphisms ϕ'_1 and ϕ'_2, which on the kernel induce the same homomorphism to G'_{ab}, and give rise to equivalent homomorphisms ϕ_1 and ϕ_2, are equivalent.*

The homomorphisms ϕ_1 and ϕ_2 are equal. The following sequence is exact:

$$1 \to U \to G' \to G'_{\mathrm{ab}} \times G.$$

From the assumptions, it follows easily that ϕ_1 and ϕ_2 induce identical homomorphisms on the kernel. The statement then follows from $H^1(\mathbb{Q}, A) = 0$.

We have already seen (Remark before 5.3) that $\mathrm{Sh}_{K'}(G', h')$ is a surjective cov- 210
ering space of $\mathrm{Sh}_K(G, h)$ with the fiber

$$A(\mathbb{Q})\backslash \alpha^{-1}(K)/K'.$$

Next, we want to show that if ϕ' and ϕ are admissible, where ϕ is induced by ϕ', the set

$$Y_{\phi'} = I_{\phi'} \backslash X'_p \times X'^p / K'^p$$

is a surjective covering of the corresponding set X_ϕ with the same fiber. Thus, the conjecture of §5 can only hold for G' and G simultaneously, at least in a compatible way, if every admissible homomorphism ϕ comes from an admissible homomorphism ϕ'. We will later find an admissible ϕ' such that ϕ' is rational on the kernel and $I = G'$. Then, $\mathscr{I}_{\phi'} = \bar{G}'$ and $\mathscr{I}_\phi = \bar{G}$ are inner twistings of G' and G. Let $\{g'_\sigma\}$ be the cocycle constructed above and $\{g_\sigma\}$ its image in $\bar{G}$. Then

$$g \cdot \phi : \mathscr{Q} \to \mathscr{G}_G$$

is admissible. In fact, because the image of $\{g_\sigma\}$ in G_{ab} is trivial, the validity of condition (5.a) is preserved under the transition from ϕ to $g \cdot \phi$, and since $\{g_\sigma\}$ is locally trivial everywhere, conditions (5.b), (5.c), (5.d) are also preserved. The homomorphism $g' \cdot \phi'$ gives rise to $g \cdot \phi$ and is not admissible because the image of g_σ in G'_{ab} is not trivial. If there were an admissible homomorphism $\widetilde{\phi}'$ that gives ϕ, it would induce the same homomorphism on the kernel as $g' \cdot \phi'$, and thus, by Lemma 7.1, would coincide with $g' \cdot \phi'$, which is impossible.

To prove the above statement, we specify that we take G' and G to be quasi-split over $\mathbb{Q}_p$, and that K'_p and K_p are chosen to be hyperspecial. The torus A splits over an unramified extension. We may assume that the restriction of ϕ' to the kernel is defined over $\mathbb{Q}$. Then, $I_{\phi'} = \mathscr{I}_{\phi'}(\mathbb{Q})$ and $I_\phi = \mathscr{I}_\phi(\mathbb{Q})$, and the sequence

$$1 \to A \to \mathscr{I}_{\phi'} \to \mathscr{I}_\phi \to 1$$

is exact. Hence,

$$I_{\phi'} \to I_\phi$$

is surjective. Similarly, one concludes that $X'^p \to X^p$ is surjective. Since $G'(\mathfrak{k}) \to G(\mathfrak{k})$ is also surjective, $\mathscr{X}' \to \mathscr{X}$ is surjective. Here,

$$\mathscr{X}' \subset \mathscr{B}(G', \mathfrak{k}) = \mathscr{B}(A, \mathfrak{k}) \times \mathscr{B}(G'', \mathfrak{k}).$$

Let $x' = (g'y^0, g'x^0) \in \mathscr{X}'$ with image in X_p:

$$\mathrm{inv}((gx^0), F(gx^0)) = \mu'' = \mu.$$

211 Then we have

$$\begin{aligned} \mathrm{inv}\big((g'y^0, g'x^0), F(g'y^0, g'x^0)\big) &= \mathrm{inv}\big((g'y^0, g'x^0), (\sigma(g')y^0, Fg'x^0)\big) \\ &= 0 + \mu'', \end{aligned}$$

if $g'y^0$ is rational over $\mathbb{Q}_p$. Since we can always find such a g', $X'_p \to X_p$ is surjective. Now consider the map

$$X'_p \times X'^p \to X_p \times X^p.$$

If $x'_p \times x'^p$ and $\bar{x}'_p \times \bar{x}'^p$ have the same image in Y_ϕ, then for the images in $X_p \times X^p$,

$$\bar{x}_p \times \bar{x}^p = \gamma(x_p \times x^p)k^p, \quad \gamma \in I_\phi, \quad k^p \in K^p.$$

Since $I_{\phi'} \to I_\phi$ is surjective, we can, after changing $\bar{x}'_p \times \bar{x}'^p$ by γ', assume that $\gamma = 1$. Then,

$$\begin{aligned} \bar{x}'_p &= a \cdot x'_p \quad a \in A(\mathbb{Q}_p), \\ \bar{x}'^p &= x'^p g' \quad g' \in \alpha^{-1}(K^p). \end{aligned}$$

Thus, $Y_{\phi'} \to Y_\phi$ is a surjective covering with fiber

$$A(\mathbb{Q})\backslash \alpha^{-1}(K^p) \cdot A(\mathbb{Q}_p)/K'^p \cdot V,$$

where V is the stabilizer of y^0 in $A(\mathbb{Q}_p)$. The statement follows because $K'_p \to K_p$ is surjective, and

$$\alpha^{-1}(K_p)/K'_p \cong A(\mathbb{Q}_p)/V.$$

It remains to construct an example. Let L be a totally imaginary quadratic extension of a totally real number field L_0. Let J be a 4-dimensional Hermitian form with coefficients in L, and let H_J be the corresponding group of unitary similarities:

$$H_J(\mathbb{Q}) = \{A \in \mathrm{GL}(4, L) \mid A^* J A = \lambda \cdot J\}.$$

The center of H_J is $L^\times$. The group G'' will be a product of the form

$$G'' = \prod_{(L^i, L_0^i, J^i)} H_{J^i},$$

and h'' is chosen as usual so that (7.a) holds. Property (7.b) is obvious. To check (7.c), it suffices to consider $G'' = H_J$. Then Z''_{der} is the induced module

$$Z''_{\mathrm{der}} = \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L_0)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mathbb{Z}/4.$$

212 It must be shown that

$$H^2(L_0,\mathbb{Z}/4) \to \prod_v H^2(L_{0v},\mathbb{Z}/4)$$

is injective, or equivalently, by the Poitou-Tate theorem ([Se1], p. II-49), that

$$H^1(L_0,\mu_4) \to \prod_v H^1(L_{0v},\mu_4)$$

is injective. This follows from the Grunewald-Wang theorem ([AT], Chap. 10, Th. 1), because $-1 \in L_0$ is not a square.

Suppose we can find a totally real Galois extension L_0 of $\mathbb{Q}$ and a Galois module U over $\mathrm{Gal}(L_0/\mathbb{Q})$ such that $4U = 0$, and such that the localization map in (7.d) is not injective. Then we can clearly find G'' as above such that U can be embedded into Z''_{der}. Instead of U, we construct

$$V = \mathrm{Hom}(U,\mu_4).$$

We require that $4V = 0$, that V is a $\mathrm{Gal}(L/\mathbb{Q})$-module for a Galois CM-extension L of $\mathbb{Q}$, for which complex conjugation ι acts as -1, and that

$$H^1(\mathbb{Q},V) \to \prod_v H^1(\mathbb{Q}_v,V)$$

is not injective. We imitate the procedure of Serre ([Se1], p. III-39). Let

$$\begin{aligned} L &= \mathbb{Q}(\sqrt{-1},\sqrt{73},\sqrt{89}) = \mathbb{Q}(\sqrt{-1})\cdot\mathbb{Q}(\sqrt{73},\sqrt{89}) \\ &= K\cdot L_0, \end{aligned}$$

with $K = \mathbb{Q}(\sqrt{-1})$. Then,

$$\begin{aligned} \mathrm{Gal}(L/\mathbb{Q}) &= \mathrm{Gal}(L/L_0)\times\mathrm{Gal}(L/K) \\ &\cong \mathbb{Z}/2\times(\mathbb{Z}/2\oplus\mathbb{Z}/2). \end{aligned}$$

It is easy to verify that all decomposition groups are either trivial or cyclic of order 2. This is clear at the archimedean places and the places outside 2, 73, and 89, because at those places, the extension L is unramified. In $\mathbb{Q}_2$, 73 and 89 are squares because $73 \equiv 89 \equiv 1 \mod 8$. Accordingly,

$$\left(\frac{-1}{73}\right) = 1, \quad \left(\frac{89}{73}\right) = \left(\frac{16}{73}\right) = 1$$

and

$$\left(\frac{-1}{89}\right) = 1, \quad \left(\frac{73}{89}\right) = \left(\frac{-16}{89}\right) = 1.$$

Let 213

$$M = \mathrm{Ind}_{\mathrm{Gal}(L/L_0)}^{\mathrm{Gal}(L/\mathbb{Q})}\mu_4.$$

Then M is the set of μ_4-valued functions on $\mathrm{Gal}(L/K)$, and ι acts as -1. Let $\mathrm{Gal}(L/K) = \{1,\varrho_1,\varrho_2,\varrho_3\}$.

Let

$$V_0 = \{(\varepsilon_\varrho) \mid \varepsilon_\varrho = \pm 1, \prod_{\varrho\in\mathrm{Gal}(L/K)} \varepsilon_\varrho = 1\}.$$

and

$$V = V_0 \cup V_0\cdot v$$

with

$$v = (\alpha,-\alpha,-\alpha,-\alpha), \quad \alpha\in\mu_4, \alpha\neq\pm 1.$$

Then V is a $\mathrm{Gal}(L/\mathbb{Q})$-module. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
H^0(\mathbb{Q},M) \arrow[r] \arrow[d] & H^0(\mathbb{Q},M/V) \arrow[r] \arrow[d] & H^1(\mathbb{Q},V) \arrow[r] \arrow[d] & H^1(\mathbb{Q},M) \arrow[d, hook] \\
\prod_v H^0(\mathbb{Q}_v,M) \arrow[r] & \prod_v H^0(\mathbb{Q}_v,M/V) \arrow[r] & \prod_v H^1(\mathbb{Q}_v,V) \arrow[r] & \prod_v H^1(\mathbb{Q}_v,M).
\end{tikzcd}$$

Here, according to Grunewald's theorem, the right vertical arrow is injective because $-1 \in L_0$ is not a square. To show that the localization map for V is not injective, it suffices to find an element y in M/V such that for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ it is the residue class of a σ-invariant element $y_\sigma \in M$, but it is not the residue class of a $\mathrm{Gal}(L/\mathbb{Q})$-invariant element (in M). The residue class y of $(1,-1,-1,-1)$ is such an element. In fact, it is not the image of a $\mathrm{Gal}(L/\mathbb{Q})$-invariant element because only $\pm(1,1,1,1)$ are invariant.

(i) If $\sigma = \iota$, then we choose $y_\sigma = (1,-1,-1,-1)$.
(ii) If $\sigma \in \mathrm{Gal}(L/K)$, then we choose $y_\sigma = (\alpha,\alpha,\alpha,\alpha)$.
(iii) If $\sigma = \iota \cdot \varrho, \varrho \in \mathrm{Gal}(L/K)$, then we choose $y = (z_1, z_\varrho, z_\tau, z_{\tau\varrho})$ with

$$z_1 = \alpha, \quad z_\varrho = -\alpha, \quad z_\tau = \alpha, \quad z_{\tau\varrho} = -\alpha.$$

Finally, we need a ϕ' such that $\mathscr{I}_{\phi'}$ is an inner form of G'. We define ϕ' as $\psi_{T',\mu'}$ with an appropriate choice of (T',μ'). Instead of T',μ', it is sufficient to find (T'',μ'') such that the corresponding element $\gamma''_n \in G''(\mathbb{Q})$ is central, and for this, we can assume that G'' is of the form H_J. We then define T'' using an extension K of L of degree 4. It suffices to choose K such that the primes lying above p in K are inert[13] over L, because then the image of γ''_n in G''_{ad} will be elliptic at p, and since it is elliptic at infinity, it is of finite order. We still have the free choices K and p,
214 which should however be unramified in L. We select K of the form $K = L \cdot K_1$, where K_1 is an extension of degree 4 over $\mathbb{Q}$, such that p is totally ramified in K_1. Namely, if $p \equiv 1 \mod 4$, then there is a subgroup of index 4 in $\mathbb{Z}_p^\times$, which defines a totally ramified abelian extension of degree 4 over $\mathbb{Q}_p$, and one could choose a totally real global number field of degree 4 that behaves this way in p. We define J by the equation

$$J(x,y) = \mathrm{Tr}_{K/L} x\bar{y}.$$

We now want to give an example of a group where there are admissible homomorphisms ϕ that are not equivalent to a $\psi_{T,\mu}$. Let E be a totally real extension of degree n of $\mathbb{Q}$ where p remains prime. Let D be a quaternion algebra over E, which is unramified at all infinite places and ramified at p. The group G is the multiplicative group $D^\times$, viewed as an algebraic group over $\mathbb{Q}$. After choosing an isomorphism $G_{\mathbb{R}} \cong \mathrm{GL}(2)^n_{\mathbb{R}}$, we define $h : \mathbb{S} \to G_{\mathbb{R}}$ component-wise as usual:

$$h_i : a + b\sqrt{-1} \in \mathbb{C}^\times \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}).$$

Let K be a purely imaginary quadratic extension of E that is embedded in D at all places outside p, while p splits as $p = \mathfrak{p}_1 \cdot \mathfrak{p}_2$. Let $q = p^m$. As in §3, for sufficiently

[13]It seems that the authors use the term 'inert" (träge) to describe any situation where $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}^e$ for a prime $\mathfrak{p}$ of L and $\mathfrak{P}$ of K.

large m, one constructs a Weil number $\gamma \in K$ for which $\mathrm{ord}_{\mathfrak{p}_i}\gamma/\mathrm{ord}_{\mathfrak{p}_i}q = v_i/2n$, where v_1 and v_2 are two given positive integers with

$$v_1 + v_2 = 2n. \tag{7.e}$$

If $E \neq \mathbb{Q}$, we can choose v_i to be odd (for later purposes) and different from each other. Then $K = E(\gamma)$. Let $T = K^\times$ as an algebraic torus. First, we embed $T(\overline{\mathbb{Q}})$ into $G(\overline{\mathbb{Q}})$, and then we construct an admissible homomorphism $\phi : \mathscr{P} \to \mathscr{G}_G$, where the group $K_p \subset G(\mathbb{Q}_p)$ appearing in condition (5.d) is the unique maximal compact subgroup, such that $\phi(\delta) = \gamma$, i.e., $\phi(\delta_{km}) = \gamma^k$. This gives the sought example, as clearly $\gamma \in I_\phi = K^\times$, which cannot be embedded in $G(\mathbb{Q})$.

For the construction of ϕ, we need some cohomological considerations, which we will introduce first. We fix a quadratic extension K' of E, in which p remains prime, and which is embedded in D. Let $L = K' \cdot K$. The following diagram describes the situation and defines the elements of the Galois group:

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash, "\varepsilon"'] \arrow[d, dash, "\nu"] \arrow[dr, dash, "\eta"] & \\
K \arrow[dr, dash] & K'' \arrow[d, dash] & K' \arrow[dl, dash] \\
 & E &
\end{tikzcd}$$

We have $\nu = \varepsilon \cdot \eta = \eta \cdot \varepsilon$. We first consider $D^\times$ and $K^\times$ as algebraic groups over 215
E, and denote them by G^1 and T^1. We embed T^1 in $\mathrm{GL}(2)_K$ such that γ goes to $\begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}$, $s_1 \in K, s_2 = \eta(s_1)$. The action of the Galois group $\mathrm{Gal}(L/E)$ on $\mathrm{GL}(2, L)$ is:

$$\eta_* \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} = \begin{pmatrix} s_2 & \\ & s_1 \end{pmatrix},$$
$$\varepsilon_* \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}.$$

Over K', G^1 is isomorphic to $\mathrm{GL}(2)$. As the twisting cocycle of $\mathrm{Gal}(K'/E)$, we choose the image in the adjoint group of the following cocycle in $\mathrm{GL}(2)$:

$$1 \mapsto 1, \quad \bar{\varepsilon} \mapsto j = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

Here, $\lambda \in E^\times$ is chosen to be totally negative with $\mathrm{ord}_p\lambda = 1$, so that λ is not a norm of an element of K'_p. The twisted action of $\mathrm{Gal}(K'/E)$ is

$$\bar{\varepsilon} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = j \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} j^{-1}.$$

For the twisted action of $\mathrm{Gal}(L/E)$ on γ, we get

$$\varepsilon \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}, \quad \eta \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} = \begin{pmatrix} s_2 & \\ & s_1 \end{pmatrix}. \tag{7.f}$$

The central 2-cocycle corresponding to the above cochain is

$$a_{1,1} = a_{1,\bar{\varepsilon}} = a_{\bar{\varepsilon},1} = 1, \quad a_{\bar{\varepsilon},\bar{\varepsilon}} = \lambda.$$

We consider the following cochain of $\mathrm{Gal}(L/E)$ with values in $G^1(L)$:

$$g_\varepsilon = j = g_\eta, \quad g_1 = g_\nu = 1.$$

Its coboundary is:

$$g_{1,*} = g_{*,1} = 1, \quad g_{\nu,*} = g_{*,\nu} = 1, \quad g_{\varepsilon,\eta} = g_{\eta,\varepsilon} = \lambda.$$

By the above formulas (7.f), we have

$$g_\sigma \sigma(\gamma) g_\sigma^{-1} = \gamma, \quad \sigma \in \mathrm{Gal}(L/E). \tag{7.g}$$

So on the left side of this equation is the usual action of $\mathrm{Gal}(L/E)$ on $T^1(L)$.

216 This formula also holds when g_σ is replaced by $t_\sigma \cdot g$, where $t_\sigma \in T^1$. Therefore, we consider $\{g_{\varrho,\sigma}\}$ as a 2-cocycle with values in T^1. Its class in

$$H^2(L/E, T^1) = H^2(L/K, \mathbb{G}_m) = \ker\big(\mathrm{Br}(K) \to \mathrm{Br}(L)\big)$$

is clearly the image of the twisting cocycle of G^1 under the map from $\mathrm{Br}(E)$ to $\mathrm{Br}(K)$. Since the field K can be embedded locally everywhere outside p into G^1, the class of $\{g_{\varrho,\sigma}\}$ is trivial outside p, while in p it is the image of the nontrivial element of $\mathrm{Br}(E_p)_2$:

$$\mathrm{Br}(E_p)_2 \longrightarrow \mathrm{Br}(K_{\mathfrak{p}_1})_2 \oplus \mathrm{Br}(K_{\mathfrak{p}_2})_2$$

$$\mathbb{Z}/2 \xrightarrow{\text{diagonal}} \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Now, to define the homomorphism ϕ, let $\mathfrak{A}$ be a left representative system of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ modulo $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ that contains 1. Then

$$\mathscr{G}_G = \prod_{\tau \in \mathfrak{A}} \mathrm{GL}(2, \overline{\mathbb{Q}}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where

$$\sigma\big(\prod_\tau A_\tau\big) = \prod_\tau \sigma_\tau(A_{\tau'}), \qquad \tau\sigma = \sigma_\tau \cdot \tau'; \quad \tau, \tau' \in \mathfrak{A}, \quad \sigma_\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/E).$$

On the kernel P, ϕ is defined by:

$$\phi(\delta_k) = (\gamma_\tau^k), \quad \delta_k \in P(L, m)$$

with $\gamma_\tau = \gamma$ for all $\tau \in \mathfrak{A}$. Then, $\phi|_P$ is a homomorphism of tori defined over $\mathbb{Q}$, because $\gamma \in T(\mathbb{Q})$. Thus, $\phi(p_{\varrho,\sigma})$ is a 2-cocycle of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in T.

On the generators p_ϱ of $\mathscr{P}$, we define ϕ tentatively by

$$\phi(p_\varrho) = \prod_\tau g_\tau(\varrho) \times \varrho$$

with $g_\tau(\varrho) = g_{\varrho\tau}$. By (7.g), we have

$$\begin{aligned} \phi(p_\varrho) \cdot \phi(\delta) \cdot \phi(p_\varrho)^{-1} &= \prod_\tau g_\tau(\varrho) \cdot \varrho_\tau(\gamma) \cdot g_\tau(\varrho)^{-1} \\ &= \prod_\tau \gamma_\tau = \phi(\delta). \end{aligned}$$

This identity remains true even if $g_\tau(\varrho)$ is replaced by $t_\tau(\varrho) \cdot g_\tau(\varrho)$ with $t_\tau \in T^1$, and we reserve the right to make this change.

For ϕ to be a homomorphism, we must also have:

$$g_\tau(\varrho) \cdot \varrho_\tau(g_{\tau'}(\sigma)) = \phi(p_{\varrho,\sigma}) \cdot g_\tau(\varrho\sigma).$$

217 Since $(\varrho\sigma)_\tau = \varrho_\tau \sigma_\tau$, we have

$$g_\tau(\varrho)\cdot\varrho_\tau(g_{\tau'}(\sigma))\cdot g_\tau(\varrho\sigma)^{-1} = g_{\varrho_\tau}\cdot\varrho_\tau(g_{\sigma_{\tau'}})\cdot g^{-1}_{\varrho_\tau\cdot\sigma_{\tau'}}.$$

We must now compare this 2-cocycle of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ with values in T^1, which we already identified earlier, with the 2-cocycle $\phi(p_{\varrho,\sigma})$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in T. According to Shapiro's Lemma, $\phi(p_{\varrho,\sigma})$ is cohomologous to an induced cocycle with values in T^1, namely the cocycle $\phi^1(p_{\varrho,\sigma})$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, where $\phi^1 : P \to T^1$ maps δ to γ. By construction of $\mathscr{P}$, this cocycle is trivial outside p and at the archimedean places. Since K is totally imaginary, the Brauer group at infinity is trivial, so the cocycle is also trivial at the infinite places. To determine the class of the cocycle at the place p, we can replace ϕ by $\xi_p = \phi\circ\zeta_p$ and then project onto the first component. Let L' be a Galois extension of $\mathbb{Q}$ that contains L. We can assume that L' is unramified at p. Let $\mathfrak{p}$ be an extension of $\mathfrak{p}_1$ to L', and let $[L'_{\mathfrak{p}} : \mathbb{Q}_p] = r$. Then ξ_p is defined on $\mathscr{D}^{L_r}_{L_r}$. If we introduce the fundamental cocycle a_{σ^i,σ^j} in the usual way (where σ is the Frobenius element), we have $a_{\sigma^i,\sigma^j} = 1$, $0 \le i, j < r, i + j < r$; $a_{\sigma^i,\sigma^j} = p^{-1}$, $0 \le i, j < r, i + j \ge r$, and

$$\xi^1_p(p^{-1}) = \nu_2(p^{-1}),$$

where ν_2 is defined by

$$\left|\prod_{\sigma\in\mathrm{Gal}(L'_{\mathfrak{p}}/\mathbb{Q}_p)}\sigma\lambda(\gamma)\right|_p = q^{\nu_2(\lambda)},\quad \lambda\in X^*(T^1).$$

By the definition of γ, the left hand side is equal to

$$q^{-\frac{(\lambda_1 v_1+\lambda_2 v_2)r}{2n}},$$

where λ_1 and λ_2 have an obvious meaning. The cocycle over E_p can therefore be defined on $\mathrm{Gal}(K'_p/E_p)$, namely by

$$\varepsilon \mapsto p^{-(\lambda_1 v_1+\lambda_2 v_2)}.$$

Since v_1 and v_2 are odd, we get the image in $H^2(K'_p/E_p, T^1)$ of the nontrivial cocycle in $H^2(K'_p/E_p, \mathbb{G}_m)$.

Thus, we see that the two cocycles being compared are locally equivalent everywhere and hence equivalent. After correcting $g_\tau(\varrho)$ with suitable $t_\tau(\varrho)$, the desired homomorphism ϕ is defined. It is important to note that the equivalence class of ϕ is independent of the choice of correction factors (t_τ), because $H^1(\mathbb{Q}, T) = 0$.

We still need to prove that ϕ is admissible. It is immediately checked that ϕ_{ab} and $\psi_{G_{\mathrm{ab}},\mu_{\mathrm{ab}}}$ agree on the kernel. (This follows from (7.e).) Since $H^1(\mathbb{Q}, \mathbb{G}_m) = 0$, they indeed agree. Outside of p and ∞, $\mathscr{P}$ is trivial, so $\phi\circ\zeta_l$ defines a cocycle with values in G. Since $H^1(\mathbb{Q}_l, G) = 0$, $\phi\circ\zeta_l$ is equivalent to the canonical neutralization of $\mathscr{G}_G$.

The composition $\phi\circ\zeta_\infty$ is, on the kernel $\mathbb{C}^\times$ of $\mathscr{W}$, equal to the diagonal homo- 218
morphism $\mathbb{C}^i \to G(\mathbb{C}) = \prod \mathrm{GL}(2, \mathbb{C})$. The corresponding cohomology class in

$$H^1(\mathbb{R}, G_{\mathrm{ad}}) \cong H^2(\mathbb{R}, R_{E/\mathbb{Q}}\mathbb{G}_m) \cong (\mathbb{Z}/2)^n$$

was determined earlier. Since λ was taken to be totally negative, it is the diagonal element $(-1, \ldots, -1)$ of the group on the right hand side. The homomorphism ξ_∞ determines the same cohomology class. Since the center of G has trivial H^1, it follows that ξ_∞ and $\phi\circ\zeta_\infty$ are equivalent.

The condition at p is more complicated to verify. The building of G is a product

$$\mathscr{B}(G,\mathfrak{k}) = \mathscr{B}(G_{\mathrm{der}},\mathfrak{k}) \times X_*(G_{\mathrm{ab}}) \otimes \mathbb{R},$$

and it is the same as the building of $R_{E_p/\mathbb{Q}_p}\mathrm{GL}(2)$, albeit with the twisted action. On $X_*(G_{\mathrm{ab}}) \otimes \mathbb{R}$, this action agrees with the usual one, while on the building $\mathscr{B}(R_{E_p/\mathbb{Q}_p}\mathrm{SL}(2),\mathfrak{k})$, the twisted action of σ is given by

$$\sigma(x^0,\dots,x^{n-1}) = (x^1,\dots,x^{n-1}, j\cdot\sigma_*^n(x^0)).$$

Here the x^i are in the building of $\mathrm{SL}(2)$. Let

$$x_0 = (x_0^0,\dots,x_0^{n-1}) \in \mathscr{B}(G,\mathfrak{k})$$

be the vertex corresponding to the maximal compact subgroup of the integral elements of $\mathrm{GL}(2,E)$. Then it is clear that $\sigma^{2n}(x_0) = x_0$, and if we denote the translates of x_0 under the Galois group by $x_i = \sigma^i(x_0)$, the x_i for $i \in \mathbb{Z}/2n$ form the vertices of a polysimplex that is invariant under $\mathrm{Gal}(\mathfrak{k}/\mathbb{Q}_p)$. We consider its factor in $\mathscr{B}(G_{\mathrm{der}},\mathfrak{k})$. It is easy to see that its center of mass is the unique fixed point of the Galois action on the building of G_{der}. By the Fixed Point Theorem of Bruhat and Tits, the building $\mathscr{B}(G_{\mathrm{der}},\mathbb{Q}_p)$ consists of this single point. Since we had chosen K_p to be the well-defined maximal compact subgroup of $G(\mathbb{Q}_p)$, i.e., $K_p \cap G_{\mathrm{der}}(\mathbb{Q}_p) = G_{\mathrm{der}}(\mathbb{Q}_p)$, and because the stabilizer of a chamber in a simply connected group fixes all points in the chamber, we have

$$K_p = \{g \in G(\mathbb{Q}_p) \mid g\cdot x_i = x_i,\ i \in \mathbb{Z}/2n\}.$$

Thus, we have represented K_p in the form used in §5 for the formulation of the admissibility conditions. Next, we calculate the operator F. Let $v_1 = 2a+1$ and $v_2 = 2b+1$, so $a+b = n-1$. We choose the fundamental class for the unramified extension $L_{2n} = L_{\mathfrak{p}_1}$ of degree $2n$ as in the example before 5.2. We send the element $d_\sigma = 1\times\sigma \in \mathscr{D}_{L_{2n}}^{L_{2n}}$ to

$$F = \left(\begin{pmatrix} p & \\ & 1\end{pmatrix},\dots,\begin{pmatrix} 1 & \\ & p\end{pmatrix},\dots,\begin{pmatrix} 1 & \\ & p\end{pmatrix}, j\right)\times\sigma.$$

219 Here the matrix $\begin{pmatrix} p & \\ & 1\end{pmatrix}$ occurs a times and $\begin{pmatrix} 1 & \\ & p\end{pmatrix}$ occurs b times. On the kernel, we define the homomorphism

$$\nu : \lambda \mapsto \lambda_1 v_1 + \lambda_2 v_2.$$

To verify that a homomorphism $\widetilde{\xi} : \mathscr{D}^{L_{2n}} \to \mathscr{G}_G$ is defined in this way, it suffices to show that $F^{2n} = \nu(p^{-1})$ in $G(\mathbb{Q}_p) = G^1(E_p)$. By using the fact that both σ^n and the element j act by swapping the entries of a diagonal matrix,

$$\sigma^n\begin{pmatrix} s_1 & \\ & s_2\end{pmatrix} = \begin{pmatrix} s_2 & \\ & s_1\end{pmatrix} = j\begin{pmatrix} s_1 & \\ & s_2\end{pmatrix}j^{-1},$$

we obtain $F^n = \begin{pmatrix} p^a & \\ & p^b\end{pmatrix} j \times \sigma^n \in G^1(E_p)\times\sigma^n$. Hence we have

$$F^{2n} = \begin{pmatrix} p^{2a+1} & \\ & p^{2b+1}\end{pmatrix} = \begin{pmatrix} p^{v_1} & \\ & p^{v_2}\end{pmatrix} = \nu(p^{-1}).$$

The localized homomorphism $\phi \circ \zeta_p$ agrees with $\widetilde{\xi}$ on the kernel. Therefore, they differ by a 1-cocycle with values in T, and since $H^1(\mathbb{Q}_p, T) = 0$, they are equivalent. This implies that the image of d_σ is the operator F defined in §5. Clearly we have

$$\begin{aligned} \operatorname{inv}(\sigma x_i, F x_i) &= \operatorname{inv}\left(x_{i+1}, \left(\begin{pmatrix} p & \\ & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \\ & p \end{pmatrix}, j\right) \cdot x_{i+1}\right) \\ &= (\mu, \ldots, \mu). \end{aligned}$$

Hence X_p contains the point $(x_0, \ldots, x_{2n-1})$ and is non-empty. Therefore, ϕ also satisfies the admissibility condition (5.d).

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