AN ALTERNATIVE DESCRIPTION OF THE DRINFELD p-ADIC HALF-PLANE

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Abstract. We show that the Deligne formal model of the Drinfeld p-adic halfplane relative to a local field F represents a moduli problem of polarized O_F-modules with an action of the ring of integers in a quadratic extension E of F. The proof proceeds by establishing a comparison isomorphism with the Drinfeld moduli problem. This isomorphism reflects the accidental isomorphism of $\mathrm{SL}_2(F)$ and $\mathrm{SU}(C)(F)$ for a two-dimensional split hermitian space C for E/F.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers O_F, uniformizer π, and residue field k of characteristic p with q elements. The Drinfeld half-plane Ω_F associated to F is the rigid-analytic variety over F,

$$\Omega_F = \mathbb{P}^1_F \smallsetminus \mathbb{P}^1(F)\,.$$

We denote by $\hat{\Omega}_F$ Deligne's formal model of Ω_F, cf. Drinfeld [2]. This is a formal scheme over $\operatorname{Spf} O_F$ with generic fiber Ω_F. The formal scheme $\hat{\Omega}_F$ has semi-stable reduction and has a special fiber which is a union of projective lines over k. There is a projective line for each homothety class of O_F-lattices Λ in F^2, and any two lines, corresponding to the homothety classes of lattices Λ and Λ', meet if and only if the vertices of the Bruhat-Tits tree $\mathcal{B}(\mathrm{PGL}_2, F)$ associated to Λ and Λ' are joined by an edge, i.e., the dual graph of the special fiber of $\hat{\Omega}_F$ can be identified with $\mathcal{B}(\mathrm{PGL}_2, F)$.

Let $\breve{\Omega}_F = \hat{\Omega}_F \times_{\operatorname{Spf} O_F} \operatorname{Spf} \breve{O}_F$ be the base change of $\hat{\Omega}_F$ to the ring of integers $\breve{O}_F$ in the completion of the maximal unramified extension $\breve{F}$ of F. Drinfeld [2] proved that $\breve{\Omega}_F$ represents the following functor $\mathcal{M}$ on the category $\mathrm{Nilp}_{\breve{O}_F}$ of $\breve{O}_F$-schemes S such that $\pi\mathcal{O}_S$ is a locally nilpotent ideal. The functor $\mathcal{M}$ associates to S the set of isomorphism classes of triples (X, ι_B, ϱ). Here X is a formal O_F-module of dimension 2 and F-height 4 over S, and $\iota_B : O_B \longrightarrow \mathrm{End}(X)$ is an action of the ring of integers in the quaternion division algebra B over F satisfying the *special condition*, cf. [1]. Over the algebraic closure $\bar{k}$ of k, there is, up to O_B-linear isogeny, precisely one such object which we denote by $\mathbb{X}$, or $(\mathbb{X}, \iota_{\mathbb{X}})$. The final entry ϱ in a triple (X, ι_B, ϱ) is a O_B-linear quasi-isogeny

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S} \tag{1.1}$$

of height zero. Here $\bar{S} = S \times_{\operatorname{Spec} \breve{O}_F} \operatorname{Spec} \bar{k}$. We refer to ρ as a framing for our fixed framing object $(\mathbb{X}, \iota_{\mathbb{X}})$. Note that no polarization data is included in a triple (X, ι_B, ϱ). However, the following result of Drinfeld provides the automatic existence of polarizations on special formal O_B-modules, [1], p.138.

Proposition 1.1. (*Drinfeld*): *Let $\Pi \in O_B$ be a uniformizer such that $\Pi^2 = \pi$ is a uniformizer of F, and consider the involution $b \longmapsto b^* = \Pi\, b'\, \Pi^{-1}$ of B, where $b \longmapsto b'$ denotes the main involution.*

a) On $\mathbb{X}$ there exists a principal polarization $\lambda^0_{\mathbb{X}} : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ with associated Rosati involution $b \longmapsto b^*$. Furthermore, $\lambda^0_{\mathbb{X}}$ is unique up to a factor in $O_F^\times$.

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b) Fix $\lambda^0_{\mathbb{X}}$ as in a). Let[1] $(X, \iota, \varrho) \in \mathcal{M}(S)$*, where* $S \in \mathrm{Nilp}_{\breve{O}_F}$*. On* X *there exists a unique principal polarization* $\lambda^0_X : X \xrightarrow{\sim} X^\vee$ *making the following diagram commutative,*

$$\begin{tikzcd} X \times_S \bar{S} \arrow[r, "\lambda^0_X"] \arrow[d, "\varrho"'] & X^\vee \times_S \bar{S} \\ \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \bar{S} \arrow[r, "\lambda^0_{\mathbb{X}}"] & \mathbb{X}^\vee \times_{\mathrm{Spec}\, \bar{k}} \bar{S} \, . \arrow[u, "\varrho^\vee"'] \end{tikzcd}$$

In this paper we show that, at least when the residue characteristic $p \neq 2$, the formal scheme $\mathcal{M} \simeq \breve{\Omega}_F$ is also the solution of certain other moduli problems on $\mathrm{Nilp}_{\breve{O}_F}$, whose definition we now describe.

Let E/F be a quadratic extension with ring of integers O_E and nontrivial Galois automorphism $\alpha \mapsto \bar{\alpha}$. Fix an F-embedding $E \to B$.

(a) When E/F is unramified, we find $\delta \in O_E$ such that $\delta^2 \in O_F^\times \setminus O_F^{\times,2}$, and we choose a uniformizer Π of O_B such that $\Pi\alpha\Pi^{-1} = \bar{\alpha}$, $\forall \alpha \in O_E$, and with $\Pi^2 = \pi$ a uniformizer of O_F. We denote by $k' = O_E/\Pi O_E$ the residue field of E.

(b) When E/F is ramified, there exists a unit $\zeta \in O_B^\times$ which generates O_B as an O_E-algebra and which normalizes E, i.e., such that $\alpha \longmapsto \zeta\alpha\zeta^{-1}$ is the non-trivial element in $\mathrm{Gal}(E/F)$. We choose[2] a uniformizer Π of O_E with $\Pi^2 = \pi \in O_F$, which also serves as a uniformizer of O_B.

From now on, we assume that $p \neq 2$ in the ramified case.

Let $\mathcal{N}_E$ be the functor on $\mathrm{Nilp}_{\breve{O}_F}$ that associates to S the set of isomorphism classes $\mathcal{N}_E(S)$ of quadruples $(X, \iota, \lambda, \varrho)$, where X is a formal O_F-module of dimension 2 over S and $\iota : O_E \longrightarrow \mathrm{End}(X)$ is an action of the ring of integers of E satisfying the *Kottwitz condition*

$$\mathrm{char}_{\mathcal{O}_S}(T, \iota(\alpha) \mid \mathrm{Lie} X) = (T - \alpha) \cdot (T - \bar{\alpha}) \, , \qquad \forall \alpha \in O_E \, . \tag{1.2}$$

The polynomial $T^2 - (\alpha + \bar{\alpha})T + \alpha\bar{\alpha} \in O_F[T]$ on the right side is considered as a polynomial in $\mathcal{O}_S[T]$ via the structure map $O_F \subset \breve{O}_F \to \mathcal{O}_S$. The third entry λ is a polarization

$$\lambda : X \longrightarrow X^\vee$$

such that the corresponding Rosati involution $*$ satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for all $\alpha \in O_E$. In addition, we impose the following condition:

(λ.a) If E/F is unramified, we ask that $\mathrm{Ker}\, \lambda$ be an $O_E/\pi O_E$-group scheme over S of order $|O_E/\pi O_E|$. In other words, $\mathrm{Ker}\, \lambda$ is a k'-group scheme of height one, in the sense of Raynaud [11].

(λ.b) If E/F is ramified, we ask that λ be a principal polarization.

Finally, ϱ is again a framing, (1.1), as in the Drinfeld moduli problem. This requires the choice of a suitable framing object $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ over $\bar{k}$ defined as follows. Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be the framing object for Drinfeld's functor, and let ι be the restriction of $\iota_{\mathbb{X}} : O_B \longrightarrow \mathrm{End}(\mathbb{X})$ to O_E. We equip $\mathbb{X}$ with a principal polarization $\lambda^0_{\mathbb{X}}$ as in Drinfeld's Proposition 1.1, relative to our choice of uniformizer Π. Then we let

$$\lambda_{\mathbb{X}} = \begin{cases} \lambda^0_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\Pi\delta) & \text{when } E/F \text{ is unramified,} \\ \lambda^0_{\mathbb{X}} & \text{when } E/F \text{ is ramified.} \end{cases}$$

[1] Here and elsewhere we will sometimes abuse notation and write $\mathcal{M}(S)$ for the category of objects (X, ι, ϱ) over S rather than the set of their isomorphism classes.

[2] When $p = 2$, this restricts the possibilities for E/F.

We take $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ as a framing object for $\mathcal{N}_E$.

For a quadruple $(X, \iota, \lambda, \varrho)$, where ϱ is a quasi-isogeny of height zero, (1.1), we require that, locally on $\bar{S}$, $\varrho^*(\lambda_{\mathbb{X}})$ and $\lambda \times_S \bar{S}$ differ by a scalar in $O_F^{\times}$, a condition which we write as

$$\lambda \times_S \bar{S} \sim \varrho^*(\lambda_{\mathbb{X}}). \tag{1.3}$$

Finally, two quadruples $(X, \iota, \lambda, \varrho)$ and $(X', \iota', \lambda', \varrho')$ are isomorphic if there exists an O_E-linear isomorphism $\alpha : X \xrightarrow{\sim} X'$ with $\varrho' \circ (\alpha \times_S \bar{S}) = \varrho$ and such that $\alpha^*(\lambda')$ differs locally on S from λ by a scalar in $O_F^{\times}$.

By [10], the functor $\mathcal{N}_E$ is representable by a formal scheme, formally locally of finite type over $\operatorname{Spf} \breve{O}_F$, which we also denote by $\mathcal{N}_E$.

Now suppose that $(X, \iota_B, \varrho) \in \mathcal{M}(S)$. Let ι be the restriction of ι_B to O_E. By Proposition 1.1, X is equipped with a unique principal polarization λ_X^0, satisfying the conditions of that proposition relative to our choice of Π. When E/F is unramified, the Rosati involution of λ_X^0 induces the trivial automorphism on O_E, and the element $\Pi\delta$ is Rosati invariant. When E/F is ramified, the Rosati involution of λ_X^0 induces the nontrivial Galois automorphism on O_E. We let

$$\lambda_X = \begin{cases} \lambda_X^0 \circ \iota_B(\Pi\delta) & \text{when } E/F \text{ is unramified,} \\ \lambda_X^0 & \text{when } E/F \text{ is ramified.} \end{cases}$$

Then it is easy to see that $(X, \iota, \lambda_X, \varrho)$ is an object of $\mathcal{N}_E(S)$.

Our main result is the following

Theorem 1.2. *Assume that $p \neq 2$ when E/F is ramified. The morphism of functors on* $\mathrm{Nilp}_{\breve{O}_F}$ *given by $(X, \iota_B, \varrho) \mapsto (X, \iota, \lambda_X, \varrho)$ induces an isomorphism of formal schemes*

$$\eta : \mathcal{M} \xrightarrow{\sim} \mathcal{N}_E \,.$$

There is an action of

$$G = \{g \in \operatorname{End}^0_{O_B}(\mathbb{X}) \mid \det(g) = 1\} \simeq \mathrm{SL}_2(F)$$

on $\mathcal{M}$, via $g : (X, \iota_B, \varrho) \mapsto (X, \iota_B, g \circ \varrho)$. Similarly, there is an action of a special unitary group $\mathrm{SU}(C)(F)$ on $\mathcal{N}_E$, where C is a hermitian space of dimension 2 over E. In the unramified case, C is defined before (2.2), and the action of $\mathrm{SU}(C)(F)$ in (2.9). In the ramified case, C is defined before Lemma 3.2, and the action is defined in an analogous way. The isomorphism η in Theorem 1.2 is compatible with these actions; more precisely, Proposition 1.1 implies that any $g \in G$ preserves $\lambda_{\mathbb{X}}$ and can therefore be considered as an element of $\mathrm{SU}(C)(F)$, and the isomorphism η is compatible with this identification.

Drinfeld's theorem now implies the following characterization of $\breve{\Omega}_F$. First we point out that the moduli problem $\mathcal{N}_E$ can be defined without reference to the Drinfeld moduli problem, cf. section 5. Again we assume that $p \neq 2$ when E/F is ramified.

Corollary 1.3. *The formal scheme $\breve{\Omega}_F$ represents the functor $\mathcal{N}_E$ on* $\mathrm{Nilp}_{\breve{O}_F}$. *In particular, the formal scheme $\mathcal{N}_E$ is adic over* $\operatorname{Spf} \breve{O}_F$*, i.e., a uniformizer of $\breve{O}_F$ generates an ideal of definition.*

Since the unramified and ramified cases are structurally rather different, we will treat them separately. It should be noted however that, in both cases, the proof eventually boils down to an analogue of the beautiful trick of Drinfeld that is the basis for the proof of Proposition 1.1.

Theorem 1.2 is obviously a manifestation of the exceptional isomorphism $\mathrm{PU}_2(E/F) \simeq \mathrm{PGL}_2$ of algebraic groups over F. In particular, it does not generalize to Drinfeld half-spaces of higher dimension. It would be interesting to find other exceptional isomorphisms between RZ-spaces of PEL-type.

In a companion paper [5] we introduce and study, for E/F unramified and any integers r, n with $0 < r < n$, moduli spaces $\mathcal{N}_E^{[r]}(1, n-1)$ of formal O_E-modules of signature $(1, n-1)$ and mild level structure analogous to that occurring in this paper. The present case corresponds to $n = 2$ and $r = 1$. We expect these spaces to provide a useful tool in the study of the special cycles in the moduli spaces $\mathcal{N}(1, n-1)$ considered in [4] and [14], and, in particular, in the computation of arithmetic intersection numbers, cf. [12] for the case $n = 3$. For E/F ramified and any integer $n \geq 2$, moduli spaces analogous to $\mathcal{N}_E$ are studied in [15], with results analogous to [13, 14].

We excluded the case $p = 2$ when E/F is ramified to keep this paper as simple as possible. We are, however, convinced that a suitable formulation of Theorem 1.2 holds even in this case.

In [6], we use the results of this paper to establish new cases of p-adic uniformization for certain Shimura varieties attached to groups of unitary similitudes for binary hermitian forms over totally real fields.

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Notation. For a finite extension F of $\mathbb{Q}_p$, with ring of integers O_F, fixed uniformizer π, and residue field k. We write $W_{O_F}(R)$ for the ring of relative Witt vectors of an O_F-algebra R, cf. [2], §1. If $F = \mathbb{Q}_p$, then $W_{O_F}(R) = W(R)$ is the usual Witt ring. If R is a k-algebra with structure map $\alpha : k \to R$, then $W(R)$ is an algebra over $W(k) = O_{F^t}$, where F^t is the maximal unramified extension of $\mathbb{Q}_p$ in F. In this case, the natural homomorphism $O_F \otimes_{O_{F^t},\alpha} W(R) \longrightarrow W_{O_F}(R)$ is an isomorphism if R is a perfect ring. For example, $\breve{O}_F = W_{O_F}(\bar{k})$.

Formal O_F-modules of F-height n over $\bar{k}$ are described by their *relative* Dieudonné modules, which are free $\breve{O}_F$-modules of rank n equipped with a σ^{-1}-linear operator V and a σ-linear operator F with $VF = FV = \pi$. Here σ denotes the relative Frobenius automorphism in $\mathrm{Aut}(\breve{F}/F)$.

The relation between the (absolute) Dieudonné module $(\tilde{M}, \tilde{V})$ of the underlying p-divisible group of a formal O_F-module and its relative Dieudonné module (M, V) is described as follows, cf. [RZ], Prop. 3.56. On $\tilde{M}$, there is an action of

$$O_F \otimes_{\mathbb{Z}_p} W(\bar{k}) = \prod_{\alpha : k \to \bar{k}} O_F \otimes_{O_{F^t},\alpha} W(\bar{k}),$$

where the index set is the set of $\mathbb{F}_p$-embeddings $\alpha : k \longrightarrow \bar{k}$, and a resulting decomposition

$$\tilde{M} = \bigoplus\nolimits_{\alpha : k \to \bar{k}} \tilde{M}^\alpha .$$

Then the relative Dieudonné module is

$$\left(M = \tilde{M}^{\alpha_0}, V = \tilde{V}^f\right),$$

where $\tilde{M}^{\alpha_0}$ denotes the summand corresponding to the fixed embedding of k into $\bar{k}$ and where $f = |F^t : \mathbb{Q}_p| = |k : \mathbb{F}_p|$.

2. The case when E/F is unramified.

We will prove the following proposition.

Proposition 2.1. *Let $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(S)$. There exists a unique principal polarization λ_X^0 on X with Rosati involution inducing the trivial automorphism on O_E and such that*

$$\lambda_X \times_S \bar{S} = (\lambda_X^0 \times_S \bar{S}) \circ \varrho_X^*(\iota_{\mathbb{X}}(\Pi)). \tag{2.1}$$

Once this is shown, the endomorphism $\beta_X = (\lambda_X^0)^{-1} \circ \lambda_X$ of X satisfies the identity

$$\beta_X \times_S \bar{S} = \varrho_X^*(\iota_{\mathbb{X}}(\Pi)),$$

on $X \times_S \bar{S}$ and thus defines the action of Π on X in a functorial way. Since $O_B = O_E[\Pi]$, we obtain an extension of the action of O_E to O_B. The resulting O_B-module structure on X is special, since this can be tested after restricting the action to the ring of integers in an unramified quadratic subfield of B, cf. [1], Ch. II, §2. Hence this construction defines a morphism of functors in the opposite direction, $\mathcal{N}_E \longrightarrow \mathcal{M}$, and it is easy to see that this is the desired inverse to the morphism in Theorem 1.2.

It remains to prove Proposition 2.1. To this end, we first have to establish some properties of the formal scheme $\mathcal{N}_E$. We fix an embedding of E into $\breve{F}$ and hence, equivalently, an embedding of the residue field $k' = O_E/\pi O_E$ into $\bar{k} = \breve{O}_F/\pi\breve{O}_F$, the residue field of $\breve{F}$.

Let

$$N = M(\mathbb{X}) \otimes_{\breve{O}_F} \breve{F}$$

be the rational relative Dieudonné module [1], Ch. II, §1. Then N is a 4-dimensional $\breve{F}$-vector space equipped with operators V and F, where the first one is σ^{-1}-linear, and the second σ-linear, σ denoting the relative Frobenius automorphism in $\mathrm{Aut}(\breve{F}/F)$. Moreover, $VF = FV = \pi$. Since E has been identified with a subfield of $\breve{F}$, the action ι of O_E determines a $\mathbb{Z}/2$-grading

$$N = N_0 \oplus N_1,$$

such that $\deg V = \deg F = 1$. The polarization $\lambda_{\mathbb{X}}$ determines a non-degenerate $\breve{F}$-bilinear alternating pairing

$$\langle\, ,\, \rangle : N \times N \longrightarrow \breve{F},$$

such that N_0 and N_1 are maximal isotropic subspaces. The slopes of the σ^2-linear operator $\tau = \pi V^{-2}|N_0$ are all zero and hence, setting $C = N_0^{\tau}$, we have

$$N_0 = C \otimes_E \breve{F}.$$

Furthermore, the restriction of the form

$$h(x,y) = \pi^{-1}\delta^{-1}\langle x, Fy\rangle \tag{2.2}$$

defines a E/F-hermitian form h on C. Using the fact that the polarization $\lambda_{\mathbb{X}}$ has the form (2.12), it follows easily that C has isotropic vectors, i.e., is split.

Let $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(\bar{k})$. The quasi-isogeny ϱ_X can be used to identify the rational relative Dieudonné module of X with N. Then the relative Dieudonné module of X can be viewed as an $\breve{O}_F$-lattice M in N such that

(a) $M = M_0 \oplus M_1$, where $M_i = M \cap N_i$, $i = 0, 1$,
(b) $\pi M_0 \subset VM_1 \subset M_0$, and $\pi M_1 \subset VM_0 \subset M_1$,
(c) $M_0 \subset (M_1)^\vee \subset \pi^{-1}M_0$, and $M_1 \subset (M_0)^\vee \subset \pi^{-1}M_1$,

where all inclusions in (b) and (c) are strict, and where we have set

$$M_i^\vee = \{x \in N_{i+1} \mid \langle x, M_i\rangle \subset \breve{O}_F\}.$$

For an $\breve{O}_F$-lattice L in N_0, set

$$L^\sharp = \{x \in N_0 \mid h(x, L) \subset \breve{O}_F\},$$

and note that $L^{\sharp\sharp} = \tau(L)$. We use the same notation for O_E-lattices in C. Recall from (the analogous situation in) [13] that an O_E-lattice Λ in C is a *vertex lattice* of type t if

$$\pi\Lambda \subset \Lambda^\sharp \overset{t}{\subset} \Lambda.$$

In our present case, as follows from the next lemma, there are vertex lattices of type 0, with $\Lambda^\sharp = \Lambda$, and of type 2, with $\Lambda^\sharp = \pi\Lambda$.

We associate to $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(\bar{k})$ the two $\breve{O}_F$-lattices in N_0,

$$A = V(M_1)^\sharp \quad , \quad B = M_0\,. \tag{2.3}$$

Lemma 2.2. *The above construction gives a bijection between $\mathcal{N}_E(\bar{k})$ and the set of pairs of $\breve{O}_F$-lattices (A, B) in N_0 such that there is a square of inclusions with all quotients of dimension 1 over $\bar{k}$,*

$$\begin{array}{ccc} B & \subset & A \\ \cup & & \cup \\ A^\sharp & \subset & B^\sharp\,. \end{array}$$

Here the lower line is the dual of the upper line. □

Corollary 2.3. *Either $B = B^\sharp$ or $A^\sharp = \pi A$ (or both). In the first case $B = \tau(B)$ is of the form $B = \Lambda_0 \otimes_{O_E} \breve{O}_F$, with Λ_0 a vertex lattice of type 0 in C. In the second case $A = \tau(A)$ is of the form $A = \Lambda_1 \otimes_{O_E} \breve{O}_F$, with Λ_1 a vertex lattice of type 2 in C.*

Proof. The case when $B = B^\sharp$ is clear. If $B \neq B^\sharp$, then $\pi A \subset B \cap B^\sharp$ and thus these lattices must coincide due to the equality of their indices in A. Similarly, $B \cap B^\sharp = A^\sharp$. Thus, $A^\sharp = \pi A$, so that $A^\tau = A^{\sharp\sharp} = \pi^{-1} \cdot A^\sharp = \pi^{-1}\pi A = A$. □

If $B = B^\sharp$, with associated self-dual vertex lattice Λ_0, then we obtain an injective map

$$\mathbb{P}(\pi^{-1}\Lambda_0/\Lambda_0)(\bar{k}) \longrightarrow \mathcal{N}_E(\bar{k}) \tag{2.4}$$

by associating to any line $\ell \subset (\pi^{-1}\Lambda_0/\Lambda_0) \otimes_{k'} \bar{k}$ the pair (A, B), where $B = \Lambda_0 \otimes_{O_E} \breve{O}_F$ and where A is the inverse image of ℓ in $\pi^{-1}B$. Note that this construction induces a bijection between the set of those special pairs (A, B) with $B = \Lambda_0 \otimes_{O_E} \breve{O}_F$ and $A^\sharp = \pi A$ and

$$\{\, \ell \in \mathbb{P}(\pi^{-1}\Lambda_0/\Lambda_0)(k') \mid \ell \text{ isotropic with respect to } h_{\Lambda_0} \,\}\,. \tag{2.5}$$

Here h_{Λ_0} is the induced k'/k-hermitian form on $\pi^{-1}\Lambda_0/\Lambda_0$, obtained by reducing $h(x, y)$ modulo π. Note that the set (2.5) has $q + 1$ elements.

If $A^\sharp = \pi A$, with associated vertex lattice Λ_1 of type 2, we obtain an injective map

$$\mathbb{P}(\Lambda_1/\pi\Lambda_1)(\bar{k}) \longrightarrow \mathcal{N}_E(\bar{k}) \tag{2.6}$$

by associating to any line $\ell \subset (\Lambda_1/\pi\Lambda_1) \otimes_{k'} \bar{k}$ the pair (A, B) with $A = \Lambda_1 \otimes_{O_E} \breve{O}_F$ and B the inverse image of ℓ in A. In this case, the construction induces a bijection between the set of those special pairs (A, B) with $A = \Lambda_1 \otimes_{O_E} \breve{O}_F$ and with $B = B^\sharp$, and

$$\{\, \ell \in \mathbb{P}(\Lambda_1/\pi\Lambda_1)(k') \mid \ell \text{ isotropic with respect to } h_{\Lambda_1} \,\}\,. \tag{2.7}$$

Here h_{Λ_1} is the k'/k-hermitian form on $\Lambda_1/\pi\Lambda_1$ obtained by reducing $\pi h(x, y)$ modulo π. Again, this set has $q + 1$ elements. The proof of the following result will be given in section 4.

Proposition 2.4. *The maps (2.4) and (2.6) are induced by morphisms of schemes*[3] *over* $\operatorname{Spec} \bar{k}$,

$$\mathbb{P}(\Lambda_0/\pi\Lambda_0) \longrightarrow (\mathcal{N}_E)_{\mathrm{red}}\ , \ \ \textit{resp.}\ \ \mathbb{P}(\Lambda_1/\pi\Lambda_1) \longrightarrow (\mathcal{N}_E)_{\mathrm{red}}\,. \tag{2.8}$$

These morphisms present $(\mathcal{N}_E)_{\mathrm{red}}$ as a union of projective lines, each corresponding to a vertex lattice in C. In this way the dual graph of $(\mathcal{N}_E)_{\mathrm{red}}$ is identified with the Bruhat-Tits tree $\mathcal{B}(\mathrm{PU}(C))$, compatible with the actions of $\mathrm{SU}(C)(F)$.

[3] Here, as elsewhere in the paper, $(\mathcal{N}_E)_{\mathrm{red}}$ denotes the underlying reduced scheme of the formal scheme $\mathcal{N}_E$.

Here the special unitary group

$$G = \{g \in \mathrm{End}^0_{O_E}(\mathbb{X}) \mid g^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}, \det(g) = 1\} = \mathrm{SU}(C)(F),$$

acts on the formal scheme $\mathcal{N}_E$ by

$$g : (X, \iota, \lambda_X, \varrho_X) \mapsto (X, \iota, \lambda_X, g \circ \varrho_X). \tag{2.9}$$

Proof of Proposition 2.1. To construct the principal polarization λ^0_X, we imitate Drinfeld's proof of Lemma 4.2 in [1]. Starting with an object $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(S)$, there is a unique polarization $\lambda_{X^\vee}$ of $X^\vee$ such that $\lambda_{X^\vee} \circ \lambda_X = [\pi]_X$ (multiplication by π). The Rosati involution corresponding to $\lambda_{X^\vee}$ induces the non-trivial F-automorphism on O_E, and $\lambda_{X^\vee}$ has degree q^2 with kernel killed by π. Hence $(X^\vee, \iota^\vee, \lambda_{X^\vee})$ satisfies the conditions imposed on the objects of $\mathcal{N}_E(S)$. To obtain an object of $\mathcal{N}_E(S)$, we still have to define the quasi-isogeny $\varrho_{X^\vee}$. For this we take the quasi-isogeny of height 0 defined by

$$\varrho_{X^\vee} = \iota_{\mathbb{X}}(\Pi) \circ \varrho_X \circ (\lambda_X \times_S \bar{S})^{-1}, \tag{2.10}$$

which is O_E-linear as required. Next we check condition (1.3). To do this, writing $[\Pi] = \iota_{\mathbb{X}}(\Pi)$ and noting that

$$\lambda_{\mathbb{X}}^{-1} \circ [\Pi]^\vee \circ \lambda_{\mathbb{X}} = [\Pi], \tag{2.11}$$

we compute

$$\begin{aligned} \varrho^*_{X^\vee}(\lambda_{\mathbb{X}}) \circ (\lambda_X \times_S \bar{S}) &= (\lambda_X \times_S \bar{S})^{-1} \circ \varrho^\vee_X \circ [\Pi]^\vee \circ \lambda_{\mathbb{X}} \circ [\Pi] \circ \varrho_X \\ &= [\pi] \circ (\lambda_X \times_S \bar{S})^{-1} \circ \varrho^*_X(\lambda_{\mathbb{X}}) \\ &\sim [\pi] \end{aligned}$$

which implies that

$$\varrho^*_{X^\vee}(\lambda_{\mathbb{X}}) \sim \lambda_{X^\vee} \times_S \bar{S},$$

as required.

We therefore have associated to an object $(X, \iota, \lambda_X, \varrho_X)$ of $\mathcal{N}_E(S)$ a new object $(X^\vee, \iota^\vee, \lambda_{X^\vee}, \varrho_{X^\vee})$ in a functorial way. Note that, if we apply the same construction to $(X^\vee, \iota^\vee, \lambda_{X^\vee}, \varrho_{X^\vee})$, and write ϱ'_X for the resulting framing for $(X^\vee)^\vee = X$, we have

$$\varrho'_X = [\Pi] \circ ([\Pi] \circ \varrho_X \circ (\lambda_X \times_S \bar{S})^{-1}) \circ (\lambda_{X^\vee} \times_S \bar{S})^{-1} = \varrho_X.$$

Thus, we obtain an involutive automorphism j of the formal $\breve{O}_F$-scheme $\mathcal{N}_E$.

Lemma 2.5. *The involution j commutes with the action of $G = \mathrm{SU}(C)(F)$.*

Proof. We use the coordinates introduced on pp. 136-7 of [1], so that $\mathbb{X}$ and $\mathbb{X}^\vee$ are identified with the product $\mathcal{E} \times \mathcal{E}$ for a formal O_F-module $\mathcal{E}$ over $\bar{k}$ of dimension 1 and F-height 2. Then $\mathrm{End}^0(\mathbb{X}) = M_2(B)$ and, for $b \in B$,

$$\iota_{\mathbb{X}}(b) = \begin{pmatrix} b & \\ & \Pi b \Pi^{-1} \end{pmatrix}.$$

Then, for $\beta \in \mathrm{End}^0(\mathbb{X})$, $\beta^\vee = {}^t\beta'$, and our polarizations are given by

$$\lambda^0_{\mathbb{X}} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \text{and} \quad \lambda_{\mathbb{X}} = \begin{pmatrix} & -\Pi\delta \\ \Pi\delta & \end{pmatrix}. \tag{2.12}$$

An easy calculation shows that

$$\mathrm{SL}_2(F) \xrightarrow{\sim} G, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b\Pi \\ \Pi^{-1}c & d \end{pmatrix}, \tag{2.13}$$

and from this it is immediate that G commutes with $\iota_{\mathbb{X}}(\Pi)$. Our claim is now clear from (2.10). $\square$

Now, by Proposition 2.4, the reduced locus of $\mathcal{N}_E$ is a union of projective lines whose intersection behavior is described by the Bruhat-Tits tree of $\mathrm{PGL}_2(F)$. Hence the proof of Lemma 4.5 of [1] shows that any automorphism of the formal $\breve{O}_F$-scheme $\mathcal{N}_E$ which commutes with the action of G is necessarily the identity. Let us recall the argument.

As a first step, one observes that any automorphism of the Bruhat-Tits tree of $\mathrm{PGL}_2(F)$ which commutes with the action of $\mathrm{SL}_2(F)$ is the identity. Hence the automorphism of $\mathcal{N}_E$ stabilizes each irreducible component of $(\mathcal{N}_E)_{\mathrm{red}}$ and fixes all intersection points of irreducible components; it follows that the induced automorphism of $(\mathcal{N}_E)_{\mathrm{red}}$ is the identity. Next one observes that the restriction of the automorphism to the first infinitesimal neighbourhood of $(\mathcal{N}_E)_{\mathrm{red}}$ corresponds to a vector field on $(\mathcal{N}_E)_{\mathrm{red}}$ which vanishes at all intersection points of irreducible components; it follows that this restriction has to be trivial. Now an induction shows that the restriction of the automorphism to all higher infinitesimal neighbourhoods of $(\mathcal{N}_E)_{\mathrm{red}}$ is trivial, and hence that the automorphism is trivial.

We conclude that $j = \mathrm{id}$, and thus there is an isomorphism $(X, \iota, \lambda_X, \varrho_X) \xrightarrow{\sim} (X^\vee, \iota^\vee, \lambda_{X^\vee}, \varrho_{X^\vee})$. In particular, we obtain an isomorphism $\alpha : X \xrightarrow{\sim} X^\vee$ such that

$$\varrho_X = \varrho_{X^\vee} \circ (\alpha \times_S \bar{S}) = [\Pi] \circ \varrho_X \circ (\lambda_X \times_S \bar{S})^{-1} \circ (\alpha \times_S \bar{S}).$$

Hence

$$\alpha \times_S \bar{S} = (\lambda_X \times_S \bar{S}) \circ \varrho_X^{-1} \circ [\Pi]^{-1} \circ \varrho_X,$$

and this characterizes α uniquely. Now, locally on $\bar{S}$, there is an element $\nu \in O_F^\times$ such that

$$(\lambda_X \times_S \bar{S}) = [\nu] \circ \varrho_X^\vee \circ \lambda_{\mathbb{X}} \circ \varrho_X,$$

and so

$$\alpha \times_S \bar{S} = [\nu] \circ \varrho_X^\vee \circ \lambda_{\mathbb{X}} \circ [\Pi]^{-1} \circ \varrho_X.$$

This implies that

$$\begin{aligned} \alpha^\vee \times_S \bar{S} &= \varrho_X^\vee \circ ([\Pi]^{-1})^\vee \circ \lambda_{\mathbb{X}} \circ \varrho_X \circ [\nu]^\vee \\ &= [\nu] \circ \varrho_X^\vee \circ \lambda_{\mathbb{X}} \circ [\Pi]^{-1} \circ \varrho_X \\ &= \alpha \times_S \bar{S}, \end{aligned}$$

where we have used (2.11) and the O_F-linearity of ϱ_X and $\lambda_{\mathbb{X}}$. Then, by rigidity, $\alpha^\vee = \alpha$, so that $\lambda_X^0 = \alpha$ is a polarization of X satisfying (2.1). $\square$

3. The case when E/F is ramified.

In this case, recall that we have fixed an element $\zeta \in O_B^\times$ such that $\alpha \mapsto \zeta\alpha\zeta^{-1}$ is the non-trivial Galois automorphism of E/F and that we have also fixed a uniformizer Π of O_E with $\Pi^2 = \pi$, which we use as the uniformizer of O_B. Recall that the Rosati involution of $\lambda_{\mathbb{X}} = \lambda_{\mathbb{X}}^0$ is $b \mapsto b^*$ and note that

$$\zeta^* = -\Pi\zeta\Pi^{-1} = \zeta \cdot (-\Pi'\,\Pi^{-1}) = \zeta.$$

Finally, note that the inverse different of E/F is

$$\partial_{E/F}^{-1} = (2\Pi)^{-1} O_E = \Pi^{-1} O_E,$$

since in this section we assume that $p \neq 2$.

The proof of Theorem 1.2 in the ramified case is based on the following analogue of Proposition 2.1.

Proposition 3.1. *Let $(X, \iota_X, \lambda_X, \varrho_X) \in \mathcal{N}_E(S)$. There exists a unique principal polarization λ_X^0 on X with Rosati involution inducing the trivial automorphism on O_E and such that*

$$\lambda_X \times_S \bar{S} = (\lambda_X^0 \times_S \bar{S}) \circ \varrho_X^*(\iota_{\mathbb{X}}(\zeta)). \tag{3.1}$$

To prove this proposition, we again need to establish some properties of the formal scheme $\mathcal{N}_E$. Let

$$N = M(\mathbb{X}) \otimes_{\breve{O}_F} \breve{F}$$

be the rational relative Dieudonné module of $\mathbb{X}$. Then N is a 4-dimensional $\breve{F}$-vector space equipped with operators V and F with $VF = FV = \pi$, and an endomorphism Π commuting with V and F and such that $\Pi^2 = \pi \cdot \mathrm{id}_N$. The polarization $\lambda_{\mathbb{X}}$ determines a non-degenerate alternating pairing

$$\langle\ ,\ \rangle : N \times N \longrightarrow \breve{F}$$

such that $\Pi = -\Pi^*$ for the adjoint Π^* of Π with respect to $\langle\ ,\ \rangle$. Hence we may consider N as a 2-dimensional vector space over $\breve{E} = E \otimes_F \breve{F}$. Choose an element $\delta \in \breve{O}_F$ with $\delta^2 \in O_F^\times \smallsetminus O_F^{\times,2}$, and define an $\breve{E}/\breve{F}$-hermitian form h on N by

$$h(x, y) = \delta(\langle \Pi x, y\rangle + \Pi \cdot \langle x, y\rangle)\,.$$

The reason for the twist by δ will be clear in a moment. Note that

$$\langle x, y\rangle = \mathrm{Tr}_{\breve{E}/\breve{F}}((2\Pi\delta)^{-1} \cdot h(x, y))\,.$$

This implies that, for a $\breve{O}_E$-lattice M in N, we have $M^\vee = M^\sharp$, where

$$M^\vee = \{x \in N \mid \langle x, M\rangle \subset \breve{O}_F\}\,,$$

and

$$M^\sharp = \{x \in N \mid h\,(x, M) \subset \breve{O}_E\}\,.$$

The slopes of the σ-linear operator $\tau = \Pi V^{-1}$ are all zero, and hence, setting $C = N^\tau$, we have

$$N = C \otimes_E \breve{E}\,,$$

where C is a 2-dimensional vector space over E. Since $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ and $\delta^\sigma = -\delta$,

$$h(Fx, y) = -h(x, Vy)^\sigma\,.$$

Therefore,

$$h(\tau x, \tau y) = -h(\Pi x, F^{-1}V^{-1}\Pi y)^{\sigma^{-1}} = h(x, y)^{\sigma^{-1}}\,,$$

and hence h induces an E/F-hermitian form on C. This explains the twist by δ in the definition of h. Transposing from [13], a vertex lattice of type t in C is a lattice Λ with

$$\Pi\Lambda \subset \Lambda^\sharp \overset{t}{\subset} \Lambda.$$

As in the unramified case, the form (2.12) of the polarization $\lambda_{\mathbb{X}}$ implies that C is isotropic, and hence split. In our present case, note that there are vertex lattices of type 0, with $\Lambda^\sharp = \Lambda$, and of type 2, with $\Lambda^\sharp = \Pi\Lambda$.

Let $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(\bar{k})$. Then the relative Dieudonné module of X can be viewed as an $\breve{O}_E$-lattice M in N such that

(a) $\Pi^2 M \subset VM \subset M$, with successive quotients of length 2 over $\breve{O}_E$,

(b) $M^\sharp = M$.

Lemma 3.2. *(i) The lattice $M + \tau(M)$ is always τ-stable.*
(ii) If M is τ-stable, then M is of the form $M = \Lambda_0 \otimes_{O_E} \breve{O}_E$ for a vertex lattice Λ_0 in C with $\Lambda_0^\sharp = \Lambda_0$.
(iii) If M is not τ-stable, then

$$M + \tau(M) = \Lambda_1 \otimes_{O_E} \breve{O}_E\,,$$

for a vertex lattice Λ_1 in C with $\Lambda_1^\sharp = \Pi\Lambda_1$.

Proof. Note that, for any lattice L, $\tau(L)^\sharp = \tau(L^\sharp)$. Then, when $\tau(M) = M$, our claim (ii) is immediate. Next suppose that M is not τ-stable, and note that

$$VM \overset{1}{\subset} VM + \Pi M \overset{1}{\subset} M,$$

since Π induces a nilpotent operator on M/VM. Thus, $M \overset{1}{\subset} M + \tau(M)$, and we obtain a diagram of inclusions of index 1,

$$\begin{array}{ccc} M & \overset{1}{\subset} & M + \tau(M) \\ \cup & & \cup \\ M \cap \tau(M) & \overset{1}{\subset} & \tau(M) \end{array}$$

The remaining indices must also be 1, since M and $\tau(M)$ have the same index in any $\breve{O}_E$-lattice containing them. Now

$$(M + \tau(M))^\sharp = M^\sharp \cap \tau(M^\sharp) = M \cap \tau(M)\,. \tag{3.2}$$

Suppose that $M+\tau(M)$ is τ-stable. Then so is its dual $M\cap\tau(M)$. The inclusion $\Pi\tau(M) \subset M\cap\tau(M)$ follows from the condition $\Pi^2 M \subset VM$. On the other hand, applying τ^{-1} and using the τ-invariance of $M \cap \tau(M)$, we obtain $\Pi M \subset M \cap \tau(M)$. Hence $\Pi(M + \tau(M)) \subset M \cap \tau(M)$ and this inclusion is an equality (compare indices in $M + \tau(M)$), i.e. $(M + \tau(M))^\sharp = \Pi(M + \tau(M))$. This proves (iii).

Finally, to show that $M + \tau(M)$ is always τ-invariant, we choose a vector $e_0 \in N$ that is τ-invariant and isotropic. After scaling by a suitable power of Π if necessary, we may assume that $e_0 \in M$ is primitive. Since $M^\sharp = M$, there is a vector $e_1 \in M$ such that $h(e_0, e_1) = 1$. Note that $h(e_1, e_1) = a \in \breve{O}_F$ and the $\breve{O}_E$-lattice $[e_0, e_1]$ spanned by e_0 and e_1 is unimodular and hence coincides with M. Now, since $h(e_0, \tau(e_1)) = h(\tau(e_0), \tau(e_1)) = 1$, we have $\tau(e_1) = \alpha e_0 + e_1$, where $\alpha \in \breve{E}$. But now $M + \tau(M) = [e_0, e_1, \alpha e_0]$ and

$$\tau(M) + \tau^2(M) = [e_0, \tau(e_1), \sigma(\alpha)e_0] = [e_0, e_1, \alpha e_0] = M + \tau(M),$$

as claimed. □

Lemma 3.3. *(i) For Λ_1 a vertex lattice in C with $\Lambda_1^\sharp = \Pi\Lambda_1$, there is an injective map*

$$i_{\Lambda_1} : \mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k}) \longrightarrow \mathcal{N}_E(\bar{k}) \tag{3.3}$$

defined by associating to any line $\ell \subset (\Lambda_1/\Pi\Lambda_1) \otimes \bar{k}$ the lattice M which is the inverse image of ℓ in $\Lambda_1 \otimes_{O_E} \breve{O}_E$.
(ii) The lattices M coming from points in $\mathbb{P}(\Lambda_1/\Pi\Lambda_1)(k)$ are precisely the τ-invariant points in the image of i_{Λ_1}. There are $q + 1$ such points.
(iii) For each vertex lattice $\Lambda_0 = \Lambda_0^\sharp$ of type 0, the corresponding τ-invariant point of $\mathcal{N}_E$ lies in the image of precisely two such i_{Λ_1}'s.

Proof. For M the inverse image of ℓ, condition (a) is easily checked. To check condition (b), i.e., that $M = M^\sharp$, let $e \in \Lambda_1$ be a preimage of a basis vector for the line ℓ. Then

$$h(e, M) = h(e, \breve{O}_E e + \Pi\Lambda_1) \subset \breve{O}_E h(e, e) + \breve{O}_E \subset \breve{O}_E,$$

since

$$h(e,e) \in \Pi^{-1}\breve{O}_E \cap \breve{F} = \breve{O}_F.$$

Thus $M \subset M^\sharp$, and they must coincide as they both have index 1 in $\Lambda_1 \otimes_{O_E} \breve{O}_E$. Now the assertion (ii) is immediate from the construction.

Finally, suppose that Λ_0 is a type 0 vertex lattice. Then the hermitian form h induces a non-degenerate *symmetric* bilinear form[4] on $\Lambda_0/\Pi\Lambda_0$ with values in $k = O_E/\Pi O_E$. This form is isotropic and there are precisely 2 isotropic lines ℓ_1 and ℓ_1' in $\Lambda_0/\Pi\Lambda_0$. Let Λ_1 (resp. Λ_1') be the O_E-lattice in C such that $\Pi\Lambda_1$ is the inverse image of ℓ_1 (resp. ℓ_1') in Λ_0. Then $\Pi\Lambda_1 = \Lambda_1^\sharp$, $\Pi\Lambda_1' = (\Lambda_1')^\sharp$, and Λ_1 and Λ_1' are the only type 2 vertex lattices Λ such that the point in $\mathcal{N}_E(\bar{k})$ corresponding to Λ_0 lies in the image of i_Λ. $\square$

The following result will be proved in section 4.

Proposition 3.4. *The map (3.3) is induced by a morphism of schemes over* $\operatorname{Spec}\bar{k}$*,*

$$i_{\Lambda_1} : \mathbb{P}(\Lambda_1/\Pi\Lambda_1) \longrightarrow (\mathcal{N}_E)_{\mathrm{red}}\,. \tag{3.4}$$

These morphisms present $(\mathcal{N}_E)_{\mathrm{red}}$ as a union of projective lines, each corresponding to a vertex lattice in C of type 2. The points of intersection of these projective lines are in bijection with the vertex lattices in C of type 0, and two projective lines, corresponding to Λ_1, resp. Λ_1', intersect if and only if there is a vertex lattice Λ_0 of type 0 such that $\Lambda_0 \subset \Lambda_1$ and $\Lambda_0 \subset \Lambda_1'$.

In this way the dual graph of $(\mathcal{N}_E)_{\mathrm{red}}$ is identified with the Bruhat-Tits tree $\mathcal{B}(\mathrm{PU}(C))$, compatible with the actions of $\mathrm{SU}(C)(F)$*.*

Here it should be pointed out that the vertices in the Bruhat-Tits tree $\mathcal{B}(\mathrm{PU}(C))$ correspond to the vertex lattices of type 2 (the maximal parahoric subgroups of $\mathrm{SU}(C)(F)$ are exactly the stabilizers of vertex lattices of type 2); the edges in the Bruhat-Tits tree correspond to the vertex lattices of type 0 (the Iwahori subgroups are exactly the stabilizers in $\mathrm{SU}(C)(F)$ of vertex lattices of type 0), cf. [9], Remark 2.35.

Remark 3.5. This is in analogy to the unramified case studied in [13], [14] and [4], but different. In that case the maximal parahorics are exactly the stabilizers of vertex lattices. The strata correspond to the *maximal* parahoric subgroups and the simplicial structure of the building accounts for the incidence combinatorics of the strata. The strata of maximal dimension correspond to the maximal parahorics to vertex lattices of maximum type.

Proof of Proposition 3.1. The argument is analogous to the proof of Proposition 2.1. Starting with an object $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(S)$, define a principal polarization $\lambda_{X^\vee}$ of $X^\vee$ by

$$\lambda_{X^\vee} \circ \lambda_X = [\zeta^2],$$

so that the Rosati involution corresponding to $\lambda_{X^\vee}$ induces the non-trivial F-automorphism on O_E. Again, to obtain an object of $\mathcal{N}_E$, we have to define the quasi-isogeny $\varrho_{X^\vee}$. For this we take the quasi-isogeny of height 0 defined by

$$\varrho_{X^\vee} = \iota_{\mathbb{X}}(\zeta) \circ \varrho_X \circ (\lambda_X \times_S \bar{S})^{-1}, \tag{3.5}$$

which is O_E-linear as required.

Thus, we obtain an involutive automorphism j of the formal $\breve{O}_F$-scheme $\mathcal{N}_E$. An analogous calculation to that in the unramified case shows that j commutes with $G = \mathrm{SU}(C)(F)$ and hence $j = 1$. Thus, there is an O_E-linear isomorphism $\alpha : X \to X^\vee$ such that

$$\varrho_X \circ ((\alpha^{-1} \circ \lambda_X) \times_S \bar{S}) = \iota_{\mathbb{X}}(\zeta) \circ \varrho_X.$$

[4]Recall that $p \neq 2$.

The same argument as before shows that $\alpha^\vee = \alpha$, so that $\lambda_X^0 = \alpha$ is the desired polarization □

Proof. Now we may finish the proof of Theorem 1.2 in the ramified case. Let $(X, \iota, \lambda_X, \varrho_X) \in \mathcal{N}_E(S)$, and consider the automorphism

$$\beta_X = (\lambda_X^0)^{-1} \circ \lambda_X\,,$$

so that β_X induces the automorphism $\varrho_X^*(\iota_{\mathbb{X}}(\zeta))$ on $X \times_S \bar{S}$. Hence β_X extends the action of O_E to $O_B = O_E[\zeta]$, so that X is an O_B-module in a functorial way. We claim that X is a special formal O_B-module. It suffices to prove this in each geometric fiber of X. But then it follows from the flatness of $\mathcal{N}_E$, cf. Lemma 3.6. □

Lemma 3.6. *$\mathcal{N}_E$ is flat over* $\operatorname{Spf} \breve{O}_F$.

Proof. This follows from the theory of local models. In the case at hand, $\mathcal{N}_E$ is modeled on the $\breve{O}_F$-scheme $M_{1,1}$ of [7], Definition 3.7 (i.e. has complete local rings isomorphic to complete local rings appearing in $M_{1,1}$). However, the scheme $M_{1,1}$ has semi-stable reduction, cf. [7], Thm. 4.5., b).

Note that the *naive local model* $M_{1,1}$ coincides with the local model associated to the triple

$$(\mathrm{U}_2(E/F), \mu_{(1,1)}, K_{\Lambda_0})\,,$$

where $\mathrm{U}_2(E/F)$ denotes the (quasi-split) unitary group of size 2 for E/F, and $\mu_{(1,1)}$ the co-character of signature $(1, 1)$, and K_{Λ_0} the parahoric subgroup stabilizing the standard selfdual lattice (this is in fact the Iwahori subgroup, cf. [9], Remark 2.35). □

4. Proofs of Propositions 2.4 and 3.4

In this section, we use the method introduced in [14] to establish the existence of morphisms (2.8) and (3.4) inducing the maps (2.4), (2.6) and (3.3) on points. Since most of the arguments of loc. cit. go over without much change, we just sketch the main steps, focusing on the variations needed, for example, in the treatment of the polarizations.

4.1. **The unramified case.** We need to define subschemes $\mathcal{N}_{E,\Lambda}$ of $\mathcal{N}_E$ associated to vertices of type 0 and 2.

For a vertex lattice[5] Λ of type 0, i.e., $\Lambda = \Lambda^\sharp$, or of type 2, i.e., $\Lambda^\sharp = \pi\Lambda$, we define a pair of Dieudonné lattices $M_\Lambda^\pm$ in the isocrystal N as follows. Let

$$M_\Lambda^- = M_{\Lambda,0}^- \oplus M_{\Lambda,1}^- = \begin{cases} \Lambda \oplus V\Lambda, & \text{for } \Lambda \text{ of type } 0, \\ \pi\Lambda \oplus V\Lambda, & \text{for } \Lambda \text{ of type } 2, \end{cases} \tag{4.1}$$

and let

$$M_\Lambda^+ = (M_\Lambda^-)^\vee = \{x \in N \mid \langle x, M_\Lambda^- \rangle \subset \breve{O}_F\} \tag{4.2}$$

be its dual. A short calculation shows that

$$M_\Lambda^+ = \pi^{-1} M_\Lambda^-. \tag{4.3}$$

Note that $V(M_{\Lambda,1}^-) = V^2\Lambda = \pi\Lambda$, since Λ is stable under $\tau = \pi V^{-2} = FV^{-1}$. Thus $M_\Lambda^\pm$ is stable under both F and V and has signature $(2, 0)$ for Λ of type 0 (i.e., $(M_\Lambda^\pm / V M_\Lambda^\pm)_1 = (0)$) and signature

[5] Here $\Lambda = \Lambda_0 \otimes_{O_F} \breve{O}_F$ where Λ_0 is a vertex lattice of type 0 or 2 in C.

$(0,2)$ for Λ of type 2 (i.e., $(M_\Lambda^\pm/VM_\Lambda^\pm)_0=(0)$). Let $X_\Lambda^\pm$ be the formal O_E-module over $\bar{k}$ with relative Dieudonné module $M_\Lambda^\pm$, and let

$$\varrho_\Lambda^\pm : X_\Lambda^\pm \longrightarrow \mathbb{X}$$

be the quasi-isogeny determined by the inclusion of $M_\Lambda^\pm$ into $N=N(\mathbb{X})$. Let $\mathrm{nat}_\Lambda : X_\Lambda^- \longrightarrow X_\Lambda^+$ be the isogeny induced by the inclusion of M_Λ^- into M_Λ^+. Of course, by (4.3), we have an isomorphism $X_\Lambda^+ \xrightarrow{\sim} X_\Lambda^-$ so that nat_Λ is just $[\pi]$, but, to avoid confusion, we will not make this identification.

By (4.2), there is an isomorphism $i_\Lambda : (X_\Lambda^-)^\vee \xrightarrow{\sim} X_\Lambda^+$ such that the diagram

(4.4)
$$\begin{array}{ccccc} X_\Lambda^- & \xrightarrow{\mathrm{nat}} & X_\Lambda^+ & \overset{i_\Lambda^{-1}}{\xrightarrow{\sim}} & (X_\Lambda^-)^\vee \\ \varrho_\Lambda^- \downarrow & & \varrho_\Lambda^+ \downarrow & & \uparrow (\varrho_\Lambda^-)^\vee \\ \mathbb{X} & = & \mathbb{X} & \underset{\lambda_{\mathbb{X}}}{\longrightarrow} & \mathbb{X}^\vee \end{array}$$

commutes. Here note that, under the identification $N(\mathbb{X}) \xrightarrow{\sim} N(\mathbb{X}^\vee)$ induced by $\lambda_{\mathbb{X}}$ and the identification of $N(\mathbb{X}^\vee)$ with $N((X_\Lambda^-)^\vee)$ induced by $(\varrho_\Lambda^-)^\vee$, the lattice $M((X_\Lambda^-)^\vee)$ in $N((X_\Lambda^-)^\vee)$ is identified with the dual lattice $(M_\Lambda^-)^\vee = M_\Lambda^+$ in $N(\mathbb{X})$. We let

$$\varrho_\Lambda^{+*} = i_\Lambda \circ (\varrho_\Lambda^-)^\vee :\ \mathbb{X}^\vee \longrightarrow X_\Lambda^+.$$

In analogy with [14], we define a subfunctor $\mathcal{N}_{E,\Lambda}$ of $\mathcal{N}_E \times_{\breve{O}_F} \bar{k}$ as follows. For a scheme S over $\bar{k}$ and a collection $(X, \iota_X, \lambda_X, \varrho_X)$ giving a point of $\mathcal{N}_E(S)$, define quasi-isogenies

$$\begin{aligned} \varrho_{\Lambda,X}^- &= \varrho_X^{-1} \circ (\varrho_\Lambda^-)_S :\ (X_\Lambda^-)_S \ \longrightarrow\ X \\ \varrho_{\Lambda,X}^{+*} &= (\varrho_\Lambda^{+*})_S \circ ((\varrho_X)^\vee)^{-1} :\ X^\vee \ \longrightarrow\ (X_\Lambda^+)_S. \end{aligned}$$

Since M_Λ^+/M_Λ^- is a $\bar{k}$-vector space of dimension 4 and since ϱ_X has height 0, it follows from (4.4) that $\varrho_{\Lambda,X}^-$ and $\varrho_{\Lambda,X}^{+*}$ have F-height 1.

Definition 4.1. For a scheme S over $\bar{k}$, let $\mathcal{N}_{E,\Lambda}(S)$ be the subset of $\mathcal{N}_E(S)$ corresponding to collections $(X, \iota_X, \lambda_X, \varrho_X)$ for which $\varrho_{\Lambda,X}^-$ is an isogeny.

Lemma 4.2. *$\varrho_{\Lambda,X}^-$ is an isogeny if and only if $\varrho_{\Lambda,X}^{+*}$ is an isogeny.*

Proof. Note that $\varrho_{\Lambda,X}^-$ is an isogeny if and only if $(\varrho_{\Lambda,X}^-)^\vee$ is. But

$$(\varrho_{\Lambda,X}^-)^\vee = (\varrho_\Lambda^-)_S^\vee \circ (\varrho_X^\vee)^{-1} = (i_\Lambda^{-1})_S \circ (i_\Lambda \circ (\varrho_\Lambda^-)^\vee)_S \circ (\varrho_X^\vee)^{-1} = (i_\Lambda^{-1})_S \circ \varrho_{\Lambda,X}^{+*}.$$

□

As in [14], Lemmas 4.2 and 4.3, we have the following two results.

Lemma 4.3. *(i) $\mathcal{N}_{E,\Lambda}$ is representable by a projective scheme over $\bar{k}$.*
(ii) The inclusion of functors $\mathcal{N}_{E,\Lambda} \hookrightarrow \mathcal{N}_E$ is a closed immersion.

Proof. The proof is the same as that of Lemma 4.2 of [14]. □

For an algebraically closed extension $\mathbf{k}$ of $\bar{k}$, and an $\breve{O}_F$-lattice L, let $L_{\mathbf{k}} = L \otimes_{\breve{O}_F} W_{O_F}(\mathbf{k})$. Here we view $\breve{O}_F = W_{O_F}(\bar{k})$ so that $W_{O_F}(\mathbf{k})$ is canonically an $\breve{O}_F$-algebra.

Lemma 4.4. *For $x \in \mathcal{N}_E(\mathbf{k})$, let $M \subset N_{\mathbf{k}}$ be the corresponding relative Dieudonné module, and let $(A : B)$ be the associated square of lattices in $(N_{\mathbf{k}})_0$. Let Λ be a vertex lattice. The following are equivalent:*
(i) $x \in \mathcal{N}_{E,\Lambda}(\mathbf{k})$.

(ii) $(M_\Lambda^-)_{\mathbf{k}} \subset M$.
(iii) $M^\vee \subset (M_\Lambda^+)_{\mathbf{k}}$.
(iv) If Λ *is of type* 0, *then* $B = B^\sharp = \Lambda_{\mathbf{k}}$ *and* x *is in the image of the map*

$$\mathbb{P}(\pi^{-1}\Lambda/\Lambda)(\mathbf{k}) \longrightarrow \mathcal{N}_E(\mathbf{k}). \tag{4.5}$$

(v) If Λ *is of type* 2, *then* $A = \Lambda_{\mathbf{k}}$ *and* x *is in the image of the map*

$$\mathbb{P}(\Lambda/\pi\Lambda)(\mathbf{k}) \longrightarrow \mathcal{N}_E(\mathbf{k}). \tag{4.6}$$

Proof. Let $(X, \iota_X, \lambda_X, \varrho_X)$ be a collection over $\mathbf{k}$ with isomorphism class x and note that the relative Dieudonné module $M = M(X)$ is identified with a submodule of $N_{\mathbf{k}}$ via ϱ_X. Then $\varrho_{\Lambda,X}^-$ is an isogeny if and only if $(M_\Lambda^-)_{\mathbf{k}} \subset M$ and this is equivalent to $M^\vee \subset (M_\Lambda^-)_{\mathbf{k}}^\vee = (M_\Lambda^+)_{\mathbf{k}}$. This proves the equivalence of (i), (ii), and (iii).

To prove the equivalence of (iv), first suppose that Λ is of type 0 and that a point $x \in \mathcal{N}_{E,\Lambda}(\mathbf{k})$ is given with associated square $(A : B)$. Note that condition (ii) implies that $\Lambda_{\mathbf{k}} \subset B = M_0$. Taking duals with respect to h, we have

$$B^\sharp \subset \Lambda_{\mathbf{k}}^\sharp = \Lambda_{\mathbf{k}} \subset B,$$

and this implies that $B^\sharp = B = \Lambda_{\mathbf{k}}$. It follows that x is in the image of the map (2.4). Conversely, if $x \in \mathcal{N}_E(\mathbf{k})$ corresponds to a square $(A : B)$ with $B = B^\sharp = \Lambda_{\mathbf{k}}$, then $\Lambda_{\mathbf{k}} = ((M_\Lambda^-)_0)_{\mathbf{k}} = B = M_0$ and

$$((M_\Lambda^-)_1)_{\mathbf{k}} \subset M_1 \iff \tau V(((M_\Lambda^-)_1)_{\mathbf{k}}) \subset \tau V(M_1).$$

But, since $A = V(M_1)^\sharp$, we have $\tau V(M_1) = A^\sharp$, whereas $\tau V((M_\Lambda^-)_1) = \tau V^2(\Lambda) = \pi\Lambda = \pi B \subset A^\sharp$. This gives the inclusion (ii).

Next, to prove the equivalence of (v), suppose that Λ is of type 2 and that a point $x \in \mathcal{N}_{E,\Lambda}(\mathbf{k})$ is given with associated square $(A : B)$. Then, applying τV to the inclusion $(M_\Lambda^-)_1 \subset M_1$, we obtain $(\Lambda^\sharp)_{\mathbf{k}} = \pi\Lambda_{\mathbf{k}} \subset A^\sharp$ and hence, in turn, $\Lambda_{\mathbf{k}} = \tau(\Lambda_{\mathbf{k}}) = \tau(A)$. Thus $A = \Lambda_{\mathbf{k}}$ and $\pi\Lambda_{\mathbf{k}} \subset B \subset \Lambda_{\mathbf{k}}$, so that x is in the image of the map (2.6). Conversely, if x is in the image of this map and $A = \Lambda_{\mathbf{k}}$, then $((M_\Lambda^-)_0)_{\mathbf{k}} = \pi\Lambda_{\mathbf{k}} \subset B = M_0$ and

$$\tau V(((M_\Lambda^-)_1)_{\mathbf{k}}) = \pi\Lambda_{\mathbf{k}} = A^\sharp = \tau V(M_1),$$

so that condition (ii) holds. $\square$

Next, we follow the method of [14] sections 4.6 and 4.7 to define a morphism

$$\mathcal{N}_{E,\Lambda} \longrightarrow \mathbb{P}(\Lambda/\pi\Lambda). \tag{4.7}$$

If S is a scheme over $\bar{k}$, let $X \mapsto D(X)$ be the functor from p-divisible groups over S to locally free $\mathcal{O}_S$-modules assigning to a p-divisible group X over S the Lie algebra $D(X)$ of its universal vector extension. This functor is compatible with base change. If an action of O_E on X is given, then $D(X)$ and $\mathrm{Lie}(X)$ are $O_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules. Note that for $(X, \iota_X, \lambda_X, \rho_X)$ defining an S-valued point of $\mathcal{N}$, the ranks of the locally free $\mathcal{O}_S$-modules $D(X)$, resp. $\mathrm{Lie}(X)$, are $4[F : \mathbb{Q}_p]$, resp. 2.

Recall that the isogeny $\mathrm{nat}_\Lambda : X_\Lambda^- \to X_\Lambda^+$ induced by the inclusion $M_\Lambda^- \subset M_\Lambda^+$ of relative Dieudonné modules has $\ker(\mathrm{nat}_\Lambda) = X_\Lambda^-[\pi]$ and this finite flat group scheme over $\bar{k}$ comes equipped with an action of $O_E/\pi O_E$. The corresponding unitary Dieudonné space, [14], is

$$\mathbb{B}_\Lambda := \ker D(\mathrm{nat}_\Lambda) \simeq \tilde{M}(X_\Lambda^+)/\tilde{M}(X_\Lambda^-),$$

where $\tilde{M}(X_\Lambda^+)$ and $\tilde{M}(X_\Lambda^-)$ denote the ordinary Dieudonné modules of the p-divisible groups X_Λ^+ and X_Λ^-. Then $\mathbb{B}_\Lambda$ is a $\bar{k}$-vector space of dimension $4[k : \mathbb{F}_p]$. The action of $k = O_F/\pi O_F$ on $\mathbb{B}_\Lambda$ induces a direct sum decomposition into 4-dimensional $\bar{k}$-subspaces

$$\mathbb{B}_\Lambda = \bigoplus_\alpha \mathbb{B}_\Lambda^\alpha, \tag{4.8}$$

where the index set is the set of $\mathbb{F}_p$-embeddings $\alpha : k \longrightarrow \bar{k}$.

The relation between the ordinary Dieudonné module and the relative Dieudonné module of a formal O_F-module is described in [10], Prop. 3.56, comp. also the notation section. From this description it follows that

$$\mathbb{B}_\Lambda^{\alpha_0} \simeq M_\Lambda^+/M_\Lambda^- = \pi^{-1}M_\Lambda^-/M_\Lambda^-, \tag{4.9}$$

where $\alpha_0 : k \longrightarrow \bar{k}$ denotes the distinguished embedding.

Lemma 4.5. *Let R be a $\bar{k}$-algebra and let $(X, \iota_X, \lambda_X, \varrho_X)$ correspond to a point of $\mathcal{N}_{E,\Lambda}(R)$. Let*

$$\varrho_{\Lambda,R} = \varrho_{\Lambda,X}^{+*} \circ \lambda_X \circ \varrho_{\Lambda,X}^- : (X_\Lambda^-)_R \longrightarrow (X_\Lambda^+)_R.$$

Then, Zariski locally on $\operatorname{Spec} R$*, $\varrho_{\Lambda,R}$ is the base change to R of the morphism* $\mathrm{nat} : X_\Lambda^- \to X_\Lambda^+$*, up to a scalar in $O_F^\times$.*

Proof. This follows from (1.3), diagram (4.4), and the definitions. ☐

We have the following special case of Corollary 4.7 in [14].

Proposition 4.6. *For a scheme S over $\bar{k}$ and p-divisible groups X, Y_1 and Y_2 over S, let $\phi_i : X \to Y_i$ be isogenies such that* $\ker(\phi_1) \subset \ker(\phi_2) \subset X[\pi]$*. Then* $\ker(D(\phi_1))$ *is locally a direct summand of* $\ker(D(\phi_2))$*, and the formation of* $\ker(D(\phi_i))$ *commutes with base change.* ☐

Let $(X, \iota_X, \lambda_X, \varrho_X) \in \mathcal{N}_\Lambda(\operatorname{Spec} R)$, and consider

$$E(X) := \ker(D(\varrho_{\Lambda,X}^-)).$$

Since $\varrho_{\Lambda,X}^-$ is O_F-linear, $E(X)$ is equipped with an action of $k \otimes_{\mathbb{F}_p} R$, and hence can be decomposed compatibly with the decomposition (4.8),

$$E(X) = \bigoplus\nolimits_\alpha \; E(X)^\alpha. \tag{4.10}$$

By Proposition 4.6, $E(X)$ is a locally direct summand of

$$\ker(D((\mathrm{nat}_\Lambda)_R) = \ker(D(\mathrm{nat}_\Lambda)) \otimes_{\bar{k}} R = \mathbb{B}_\Lambda \otimes_{\bar{k}} R,$$

and hence $E(X)^{\alpha_0}$ is a direct summand of $\mathbb{B}_\Lambda^{\alpha_0} \otimes_{\bar{k}} R$. Since $\varrho_{\Lambda,X}^-$ is O_E-linear, $E(X)^{\alpha_0}$ is stable under the action of $O_E/\pi O_E$ and there is a further decomposition

$$E(X)^{\alpha_0} = E(X)_0^{\alpha_0} \oplus E(X)_1^{\alpha_0},$$

compatibly with the analogous decomposition into free R-modules of rank 2,

$$\mathbb{B}_\Lambda^{\alpha_0} \otimes_{\bar{k}} R = \big((\mathbb{B}_\Lambda^{\alpha_0})_0 \otimes_{\bar{k}} R\big) \oplus \big((\mathbb{B}_\Lambda^{\alpha_0})_1 \otimes_{\bar{k}} R\big).$$

First suppose that Λ is of type 0. By (4.9) we have $(\mathbb{B}_\Lambda^{\alpha_0})_0 = \pi^{-1}\Lambda/\Lambda$, while we have an isomorphism

$$\tau V : (\mathbb{B}_\Lambda^{\alpha_0})_1 \xrightarrow{\sim} \Lambda/\pi\Lambda.$$

In the case where $R = \mathbf{k}$ is an algebraically closed field containing $\bar{k}$, and X corresponds to a square $(A : B)$, we have $M_0 = B = \Lambda_{\mathbf{k}}$, as above, and $\tau V(M_1) = A^\sharp$. Then,

$$E(X)^{\alpha_0} = \ker(D(\varrho_{\Lambda,X}^-))^{\alpha_0} \simeq \big((M_\Lambda^-)_{\mathbf{k}} \cap \pi M(X)\big)/\pi(M_\Lambda^-)_{\mathbf{k}},$$

so that $E(X)_0^{\alpha_0} = 0$ and

$$\tau V : E(X)_1^{\alpha_0} \xrightarrow{\sim} A^\sharp/\pi\Lambda_{\mathbf{k}}$$

corresponds to a line in $\Lambda_{\mathbf{k}}/\pi\Lambda_{\mathbf{k}}$. Thus, for general R, the component $E(X)_1^{\alpha_0}$ in

$$(\mathbb{B}_\Lambda^{\alpha_0})_1 \otimes_{\bar{k}} R = \Lambda/\pi\Lambda \otimes_{\bar{k}} R$$

is a locally direct summand of rank 1 and hence defines a point of $\mathbb{P}(\Lambda/\pi\Lambda)(R)$.

Next suppose that Λ is of type 2. Then $(\mathbb{B}_\Lambda^{\alpha_0})_0 = \Lambda/\pi\Lambda$ and

$$\tau V : (\mathbb{B}_\Lambda^{\alpha_0})_1 \xrightarrow{\sim} \Lambda/\pi\Lambda.$$

Again in the case where $R = \mathbf{k}$ is an algebraically closed field containing $\bar{k}$ and X corresponds to a square $(A : B)$, we have $M_0 = B$ and

$$\tau V(M_1) = A^\sharp = \Lambda_{\mathbf{k}}^\sharp = \pi\Lambda_{\mathbf{k}} = \tau V(((M_\Lambda^-)_{\mathbf{k}})_1).$$

Then,

$$E(X)^{\alpha_0} = \ker(D(\varrho_{\Lambda,X}^-))^{\alpha_0} = \big((M_\Lambda^-)_{\mathbf{k}} \cap \pi M(X)\big)/(\pi M_\Lambda^-)_{\mathbf{k}},$$

so that $E(X)_1^{\alpha_0} = 0$ and

$$E(X)_0^{\alpha_0} \xrightarrow{\sim} B/\pi\Lambda_{\mathbf{k}}$$

corresponds to a line in $\Lambda_{\mathbf{k}}/\pi\Lambda_{\mathbf{k}}$. Then, for general R, we associate to X the locally direct summand $E(X)_0^{\alpha_0}$ of rank 1 in $\Lambda/\pi\Lambda \otimes_{\bar{k}} R$.

Thus, for Λ of either type, we have constructed a map

$$\mathcal{N}_{E,\Lambda}(R) \longrightarrow \mathbb{P}(\Lambda/\pi\Lambda)(R).$$

This construction is functorial and commutes with base change and hence defines the morphism (4.7). The argument of the proof of Theorem 4.8 in [14] implies that this morphism is an isomorphism, and that its inverse induces the map (2.4) when Λ is of type 0, and the map (2.6) when Λ is of type 2.

4.2. **The ramified case.** Let Λ be a vertex lattice of type 2 in N, so that $\Lambda^\sharp = \Pi\Lambda$, and we define relative Dieudonné lattices $M_\Lambda^\pm$ by $M_\Lambda^+ = \Lambda$ and $M_\Lambda^- = \Pi\Lambda = \Lambda^\sharp$. Recall that, in this case, $\tau = \Pi V^{-1}$ so that $V\Lambda = \Pi\Lambda$. Again $M_\Lambda^+ = (M_\Lambda^-)^\vee$ and we have associated p-divisible groups $X_\Lambda^\pm$ and quasi-isogenies $\varrho_\Lambda^\pm : X_\Lambda^\pm \longrightarrow \mathbb{X}$. There is again an isomorphism $i_\Lambda : (X_\Lambda^-)^\vee \xrightarrow{\sim} X_\Lambda^+$ and an isogeny $\mathrm{nat}_\Lambda : X_\Lambda^- \longrightarrow X_\Lambda^+$ as in the diagram (4.4). In the present case, there is an isomorphism $X_\Lambda^- \xrightarrow{\sim} X_\Lambda^+$ such that nat_Λ coincides with $[\Pi]$. In particular, $\ker(\mathrm{nat}_\Lambda) = X_\Lambda^-[\Pi]$, and the corresponding Dieudonné space is

$$\mathbb{B}_\Lambda := \ker D(\mathrm{nat}_\Lambda) = \tilde{M}(X_\Lambda^+)/\tilde{M}(X_\Lambda^-),$$

a $\bar{k}$-vector space of dimension $2[k : \mathbb{F}_p]$.

As before, define

$$\varrho_\Lambda^{+*} = i_\Lambda \circ (\varrho_\Lambda^-)^\vee : \mathbb{X}^\vee \longrightarrow X_\Lambda^+.$$

For a point $(X, \iota_X, \lambda_X, \varrho_X)$ in $\mathcal{N}_E(S)$, let

$$\varrho_{\Lambda,X}^- = \varrho_X^{-1} \circ (\varrho_\Lambda^-)_S \qquad \text{and} \qquad \varrho_{\Lambda,X}^{+*} = (\varrho_\Lambda^{+*})_S \circ (\varrho_X^\vee)^{-1}.$$

Then the definition of $\mathcal{N}_{E,\Lambda}$ and Lemmas 4.2 and 4.3 are the same as in the unramified case.

Next suppose that $\mathbf{k}$ is an algebraically closed field containing k and that a point $x \in \mathcal{N}_E(\mathbf{k})$ is given with corresponding relative Dieudonné lattice $M = M^\sharp$ in $N_{\mathbf{k}}$. The equivalence of conditions (i), (ii), and (iii) in Lemma 4.4 are again immediate and amount to the inclusions

$$\Pi\Lambda_{\mathbf{k}} \overset{1}{\subset} M \overset{1}{\subset} \Lambda_{\mathbf{k}}. \tag{4.11}$$

It is clear that (4.11) is, in turn, equivalent to x being in the image of the map (3.2) from $\mathbb{P}(\Lambda/\Pi\Lambda)(\mathbf{k})$. This gives the analogue of Lemma 4.4.

Next suppose that $x \in \mathcal{N}_{E,\Lambda}(R)$ for a $\bar{k}$-algebra R. Then

$$\varrho_{\Lambda,X}^{+*} \circ \lambda_X \circ \varrho_{\Lambda,X}^- : (X_\Lambda^-)_R^\vee \longrightarrow (X_\Lambda^+)_R$$

satisfies

$$\varrho^{+*}_{\Lambda,X} \circ \lambda_X \circ \varrho^{-}_{\Lambda,X} \sim (\mathrm{nat}_\Lambda)_R.$$

As in the unramified case,

$$E(X) := \ker D(\varrho^{-}_{\Lambda,X})$$

is locally a direct summand of

$$\ker D(\mathrm{nat}_{\Lambda,R}) = \ker D(\mathrm{nat}_\Lambda) \otimes_{\bar{k}} R = \mathbb{B}_\Lambda \otimes_{\bar{k}} R.$$

The decomposition into free R-modules of rank 2 under the action of $k \otimes_{\mathbb{F}_p} R$,

$$\mathbb{B}_\Lambda \otimes_{\bar{k}} R = \big(\bigoplus\nolimits_\alpha \ \mathbb{B}^\alpha_\Lambda\big) \otimes_{\bar{k}} R,$$

induces a corresponding decomposition

$$E(X) = \bigoplus\nolimits_\alpha \ E(X)^\alpha,$$

where $E(X)^{\alpha_0}$ is of rank 1. Since $\mathbb{B}^{\alpha_0}_\Lambda \simeq \Lambda/\Pi\Lambda$, the direct summand $E(X)^{\alpha_0}$ corresponds to a point in $\mathbb{P}(\Lambda/\Pi\Lambda)(R)$. Thus, we have defined a map

$$\mathcal{N}_{E,\Lambda}(R) \longrightarrow \mathbb{P}(\Lambda/\Pi\Lambda)(R)$$

functorial in R and compatible with base change. Again the arguments of [14] show that the morphism $\mathcal{N}_{E,\Lambda} \longrightarrow \mathbb{P}(\Lambda/\Pi\Lambda)$ is an isomorphism, whose inverse induces the map (3.3) on $\bar{k}$-valued points.

5. Concluding remarks

When formulating the moduli problem $\mathcal{N}_E$, we must choose a framing object $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$. In the body of the paper, this framing object arose from the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of the Drinfeld moduli problem, together with the chosen embedding of O_E into O_B. Recall that the framing object of the Drinfeld moduli problem is unique up to a O_B-linear isogeny, and in fact $\mathbb{X}$ is supersingular, in the sense that the slopes of the F-isocrystal defined by $\mathbb{X}$ are 1/2, with multiplicity 4, cf. [1].

If we allow ourselves to choose the framing object $(\mathbb{X}, \iota, \lambda)$ without reference to the Drinfeld moduli problem, then other moduli problems arise for a 2-dimensional E/F-hermitian space and a parahoric polarization type. There are four possibilities:

a) E/F unramified, λ a principal polarization.

b) E/F unramified, $\mathrm{Ker}\,\lambda$ a $O_E/\pi O_E$-group scheme of height 1.

c) E/F ramified, λ a principal polarization.

d) E/F ramified, $\mathrm{Ker}\,\lambda = X[\Pi]$, where $\Pi \in O_E$ denotes a uniformizer.

In cases a), b) and d), the framing object is unique up to an O_E-linear isogeny that preserves the polarization up to a scalar in $O_F^\times$.

Case by case we have the following facts:

a) This case leads to a formally smooth formal moduli scheme (of relative dimension 1 over O_E) with reduced locus a single point; the corresponding hermitian space C of dimension 2 is non-split, comp. [13, 4].

b) This case is discussed above. It leads to a flat non-smooth formal moduli scheme; the corresponding hermitian space C is split.

c) In this case, one choice of framing object arises from the Drinfeld framing object, and the resulting moduli problem is the one discussed above. It leads to a flat non-smooth formal moduli scheme; the corresponding hermitian space C is split.

A second choice of framing object arises by again taking $\mathbb{X} = \mathcal{E} \times \mathcal{E}$, as in the proof of Lemma 2.5, with $\iota(a) = \mathrm{diag}(a, a)$. The polarization λ is now given as $\mathrm{diag}(u_0, u_1)$, for $u_0, u_1 \in O_F^\times$ with $-u_0u_1 \notin \mathrm{Nm}\, E^\times$. This choice ensures that the corresponding hermitian space C is anisotropic. The moduli scheme is flat non-smooth with reduced locus a single point. Indeed, the theory of local models can be used to show that we have semi-stable reduction at this unique point, cf. [9], Remark 2.35 (the Iwahori case).

When $F = \mathbb{Q}_p$, these two choices of framing objects can be distinguished by their crystalline discriminants, cf. [6], given by -1 (resp. $+1$) for the first (resp., second) choice.

d) Now the framing object is $\mathbb{X} = \mathcal{E} \times \mathcal{E}$ with $\iota(a) = \mathrm{diag}(a, \bar{a})$, and the polarization is given by $\lambda_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(\Pi\zeta)$. In this case one can show, again using the theory of local models, that the formal moduli scheme is formally smooth of relative dimension 1 over O_E. The corresponding hermitian space is non-split[6].

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[6] The other hypothetical possibility, when the hermitian space is split, is not interesting, since the corresponding moduli scheme consists only of isolated points in characteristic p.