A geometric approach to the fundamental lemma
for unitary groups

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0. Introduction

Let n_1, n_2 be two positive integers, let $G = U(n_1 + n_2)$ be "the" unramified unitary group in $n_1 + n_2$ variables over a non archimedean local field F and let H be the elliptic endoscopic group $U(n_1) \times U(n_2)$ for G. Let K be a hyperspecial compact open subgroup of $G(F)$ and let K^H be a hyperspecial compact open subgroup of $H(F)$. Then, for any regular semisimple element $\gamma \in G(F)$ which comes from an elliptic semisimple element δ in $H(F)$ Langlands and Shelstad ([La-Sh]) have defined the κ-orbital integral $O_\gamma^\kappa(1_K)$, the stable orbital integral $SO_\delta^H(1_{K^H})$ and the transfer factor $\Delta(\gamma, \delta)$ and they have conjectured that

$$(*) \qquad\qquad SO_\delta^H(1_{K^H}) = \Delta(\delta, \gamma) O_\gamma^\kappa(1_K).$$

This relation is called the fundamental lemma for the pair (G, H). It has been proved by Labesse and Langlands in the particular case $n_1 = n_2 = 1$ ([La-La]) and by Kottwitz in the particular case $n_1 = 1$ and $n_2 = 2$ ([Ko]).

In this paper we restrict ourselves to the case where F is of equal characteristic $p > 0$ and we consider the relation $(*)$ from a geometric point of view. The restriction to the equal characteristic case is more or less equivalent to considering the unequal characteristic case under the hypothesis that the residual characteristic is "large enough with respect to the element δ". In the equal characteristic case the geometric interpretation of the orbital integrals is straightforward and does not involve Witt vector schemes. Moreover we restrict ourselves to the "totally ramified case" : we fix totally ramified separable extensions E_1 and E_2 of F of degree n_1 and n_2 respectively, we embed the corresponding elliptic torus $T = E_1'^1 \times E_2'^1$ as a maximal torus in both G and in H and we take as elements γ and δ the images by these embeddings of some element $(\gamma_1, \gamma_2) \in T$. Here E_i' is the unramified quadratic extension of E_i and $E_i'^1 \subset E_i'^\times$ is the subgroup of elements of norm 1 with respect to E_i.

We use the standard computation of the orbital integrals as numbers of selfdual lattices which are fixed by unitary automorphisms in certain hermitian vector spaces (see [Ko]). In this computation the κ-orbital integral appears as the difference between the number of selfdual lattices for two different hermitian forms Φ^+ and Φ^- on $E_1' \oplus E_2'$ which are

fixed by the multiplication by (γ_1, γ_2). The stable orbital integral is equal to the product of the number of selfdual lattices in E_1' and in E_2' which are fixed by the multiplication by γ_1 and γ_2 respectively.

The transfer factor $\Delta(\delta, \gamma)$ has been computed by Waldspurger for classical groups. In our case we simply have

$$\Delta(\delta, \gamma) = (-1)^r q^{-r}$$

where q is the number of elements in the residue field k and r denotes the valuation of the resultant of the minimal polynomials of γ_1 and γ_2.

We now explain the contents of this paper. In a first step (Part I) we construct schemes $\mathcal{X}^+$, $\mathcal{X}^-$ and $\mathcal{Y}_1$, $\mathcal{Y}_2$ over the residue field k whose k-rational points are in bijection with the sets of lattices in question. We thus have

$$O_\gamma^\kappa(1_K) = |\mathcal{X}^+(k)| - |\mathcal{X}^-(k)|$$

and

$$SO_\delta^H(1_{K^H}) = |\mathcal{Y}_1(k)| \cdot |\mathcal{Y}_2(k)|.$$

It turns out that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are projective schemes (closed subschemes of Grassmannians), whereas $\mathcal{X}^+$ and $\mathcal{X}^-$ are only locally of finite type. However $\mathcal{X}^+$ and $\mathcal{X}^-$ carry natural actions of $\mathbb{Z}$ such that the quotients $\mathcal{X}^+/\mathbb{Z}$ and $\mathcal{X}^-/\mathbb{Z}$ are representable by projective schemes over k. Of course, this construction may be viewed as a special case of the construction of Kazhdan and Lusztig [Ka-Lu].

The closed subschemes $\mathcal{X}^+$ and $\mathcal{X}^-$ contain canonical closed subschemes which are projective schemes and contain all k-rational points. In order to take into account the sign $(-1)^r$ of the transfer factor it is convenient to denote these subschemes $\mathcal{X} \subset \mathcal{X}^+$ and $\mathcal{X}' \subset \mathcal{X}^-$ if r is even, and $\mathcal{X}' \subset \mathcal{X}^+$ and $\mathcal{X} \subset \mathcal{X}^-$ if r is odd. The geometric version of the conjecture of Langlands and Shelstad establishes a close relation between the number of points of the schemes introduced above for any finite extension of k. Namely, we conjecture that for any extension k_f of finite degree f of k,

$$(**) \qquad |\mathcal{X}(k_f)| - |\mathcal{X}'(k_f)| = q^{fr} \cdot |\mathcal{Y}_1(k_f)| \cdot |\mathcal{Y}_2(k_f)|.$$

By the above remarks, the relation $(*)$ is the particular case $f = 1$ of $(**)$.

The main result of this paper (Part II) is the proof of this conjecture for extensions of even degree of k, i.e. extensions of the quadratic extension k' of k. For this we construct a partition (in fact, two such partitions, interchanged by the Frobenius morphism over k) of $\mathcal{X}^\pm \otimes_k k'$ into locally closed subsets which are vector bundles of rank r over $(\mathcal{Y}_1 \times \mathcal{Y}_2) \otimes_k k'$. This last assertion is proved by interpreting our lattices as coherent modules on germs of singular curves contained in $\mathrm{Spec}(k'[[T_1, T_2]])$ and using the interpretation of r as the intersection multiplicity of these curves. At this point we make use of a fundamental result of Deligne on intersection multiplicities "with weights". We then construct a closed embedding (in fact, two such embeddings interchanged by the Frobenius morphism over

k) of $\mathcal{X}' \otimes_k k'$ into $\mathcal{X} \otimes_k k'$ such that the complement is one piece of the above partition of $\mathcal{X}^{\pm} \otimes_k k'$. Our main result follows now by a simple counting argument.

In the final part (Part III) we explain a possible approach to the descent from k' to k. Whereas the theory over k' is essentially elementary, we envisage the use of ℓ-adic cohomology for this descent problem. Briefly put, even though the vector bundle structure on the strata of $\mathcal{X}^{\pm} \otimes_k k'$ and the closed embeddings of $\mathcal{X}' \otimes_k k'$ into $\mathcal{X} \otimes_k k'$ definitely do not descend to k, the structure in ℓ-adic cohomology that these data induce should descend. We cannot prove this, but we show that this approach works at least in the simple case of $U(1,1)$.

Contents

PART I

1. Hermitian forms

Let F be a non archimedean local field of equal characteristic $p > 2$ and let F' be an unramified quadratic extension of F. We denote by $\mathcal{O}_F$ and $\mathcal{O}_{F'}$ the rings of integers of F and F' and we fix a uniformizing parameter ϖ_F of F and therefore of F'. We denote by σ the non trivial element of the Galois group $\mathrm{Gal}(F'/F) \cong \mathrm{Gal}(k'/k)$ where k and k' are the residue fields of F and F'.

We have the isomorphism

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} F^\times/\mathrm{N}_{F'/F}F'^\times, \ \varepsilon \mapsto \varpi_F^\varepsilon \mathrm{N}_{F'/F}F'^\times$$

where $\mathrm{N}_{F'/F} : F'^\times \to F^\times$ is the norm map. Therefore, for each finite dimensional F'-vector space V and each $\varepsilon = 0, 1$ there exists a non degenerate hermitian form on V with discriminant $\varpi_F^\varepsilon \mathrm{N}_{F'/F}F'^\times$. Any two such forms are equivalent.

Let E be a totally ramified separable finite extension of F. The tensor product $E' = E \otimes_F F'$ is an unramified field extension of E. We denote by $\mathcal{O}_E$ and $\mathcal{O}_{E'}$ the rings of integers of E and E' and we denote again by σ the automorphism $1 \otimes \sigma$ of E'. We denote by $\delta_{E/F}$ the exponent of the discriminant ideal $\mathfrak{d}_{\mathcal{O}_E/\mathcal{O}_F} \subset \mathcal{O}_F$ of $\mathcal{O}_E$ over $\mathcal{O}_F$.

For each $\alpha \in E^\times$ we define a non degenerate hermitian form $\Phi_{(\alpha)}$ on the F'-vector space E' by

$$\Phi_{(\alpha)}(e'_1, e'_2) = \mathrm{tr}_{E'/F'}(\alpha e_1'^\sigma e'_2).$$

Its discriminant is equal to $\mathrm{N}_{E/F}(\alpha)\varpi_F^{\delta_{E/F}}\mathrm{N}_{F'/F}F'^\times$ and the dual of the $\mathcal{O}_{F'}$-lattice $\mathcal{O}_{E'}$ with respect to that hermitian form

$$(\mathcal{O}_{E'})^{\perp(\alpha)} \overset{\mathrm{dfn}}{=\!=} \{e' \in E' \mid \Phi_{(\alpha)}(e', e'') \in \mathcal{O}_{F'}, \ \forall e'' \in \mathcal{O}_{E'}\},$$

is equal to $\alpha^{-1}\varpi_E^{-\delta_{E/F}}\mathcal{O}_{E'}$ where ϖ_E is a uniformizing parameter of E (cf. [Se] Ch. III, §3). As the norm map $\mathrm{N}_{E/F} : E^\times \to F^\times$ induces an isomorphism from $E^\times/\mathcal{O}_E^\times$ onto $F^\times/\mathcal{O}_F^\times$, for any $\alpha^+ \in \varpi_E^{-\delta_{E/F}}\mathcal{O}_{E'}^\times$ (resp. $\alpha^- \in \varpi_E^{1-\delta_{E/F}}\mathcal{O}_{E'}^\times$) the hermitian form $\Phi_{(\alpha^+)}$ (resp. $\Phi_{(\alpha^-)}$) has discriminant 1 (resp. ϖ_F) modulo $\mathrm{N}_{F'/F}F'^\times$ and the dual of the $\mathcal{O}_{F'}$-lattice $\mathcal{O}_{E'}$ with respect to that hermitian form is equal to $\mathcal{O}_{E'}$ (resp. $\varpi_E^{-1}\mathcal{O}_{E'}$). We fix once for all such an element α^+ (resp. α^-) and we set

$$\Phi_{E'}^\pm \overset{\mathrm{dfn}}{=\!=} \Phi_{(\alpha^\pm)}.$$

2. Statement of the Langlands-Shelstad conjecture

We closely follow [Ko] (in this paper Kottwitz has proved Conjecture 2.2 below in the particular case $n_1 = 1$ and $n_2 = 2$ but for an arbitrary local field F).

We fix two totally ramified separable finite extensions E_1 and E_2 of F of degrees n_1 and n_2.

We denote by E_1' and E_2' the unramified quadratic field extensions $E_1 F'$ and $E_2 F'$ of E_1 and E_2. We denote by $\mathcal{O}_{E_1}$, $\mathcal{O}_{E_2}$, $\mathcal{O}_{E_1'}$ and $\mathcal{O}_{E_2'}$ the rings of integers of E_1, E_2, E_1' and E_2'. We fix uniformizing parameters ϖ_{E_1} and ϖ_{E_2} of E_1 and E_2 and therefore of E_1' and E_2'.

We set $E' = E_1' \oplus E_2'$. It is a F'-vector space of dimension $n_1 + n_2$. We endow E' with the non degenerate hermitian forms

$$\Phi^+ = \Phi_{E_1'}^+ \oplus \Phi_{E_2'}^+$$

and

$$\Phi^- = \Phi_{E_1'}^- \oplus \Phi_{E_2'}^-.$$

These two forms are equivalent as their discriminants are 1 and ϖ_F^2 modulo $N_{F'/F} F'^\times$. Therefore we can find $g \in GL_{F'}(E')$ such that

$$\Phi^-(e', e'') = \Phi^+(ge', ge'') \qquad (\forall e', e'' \in E').$$

We fix $\gamma_1 \in E_1'^\times$ and $\gamma_2 \in E_2'^\times$ such that $\gamma_1 \gamma_1^\sigma = \gamma_2 \gamma_2^\sigma = 1$, so that γ_i is a unit in the ring $\mathcal{O}_{E_i'}$. We assume that $E_i' = F'[\gamma_i]$, i.e. the minimal polynomial $P_i(T) \in F'[T]$ of γ_i has degree n_i. We assume moreover that the polynomials $P_1(T)$ and $P_2(T)$ are separable and prime with respect to each other. Then the diagonal element $(\gamma_1, \gamma_2) \in GL_{F'}(E')$ may be simultaneously viewed as an elliptic regular semisimple element γ^+ in the unitary group

$$G(F) \overset{\mathrm{dfn}}{=\!=\!=} U(E', \Phi^+) = gU(E', \Phi^-)g^{-1} \subset GL_{F'}(E'),$$

as an elliptic regular semisimple element γ^- in the unitary group

$$U(E', \Phi^-) \subset GL_{F'}(E')$$

and as an elliptic (G, H)-regular semisimple element δ in the endoscopic group

$$H(F) = U(E_1', \Psi_1) \times U(E_2', \Psi_2) \subset GL_{F'}(E')$$

of $G(F)$ where we have set

$$\Psi_i = \Phi_{E_i'}^+.$$

The elements γ^+ and $g\gamma^- g^{-1}$ of $G(F)$ are conjugate in $GL_{F'}(E')$ but are not conjugate in $G(F)$. The conjugacy class of δ in $H(F)$ is equal to its stable conjugacy class (an

element of $U(E'_i, \Psi_i) \subset GL_{F'}(E'_i)$ is stably conjugate to γ_i if and only if it has the same minimal polynomial as γ_i).

Let K be the hyperspecial maximal compact open subgroup of $G(F)$ which fixes the $\mathcal{O}_{F'}$-lattice $\mathcal{O}_{E'_1} \oplus \mathcal{O}_{E'_2}$ of E' (this lattice is selfdual for the hermitian form Φ^+) and let K^H be the hyperspecial maximal compact open subgroup of $H(F)$ which fixes the same lattice. The κ-orbital integral $O^\kappa_\gamma(1_K)$ is equal to the difference ([Ko])

$$O^\kappa_\gamma(1_K) = |\{\mathcal{L}' \subset E' \mid \mathcal{L}'^{\perp^+} = \mathcal{L}' \text{ and } (\gamma_1, \gamma_2)\mathcal{L}' = \mathcal{L}'\}|$$
$$-|\{\mathcal{L}' \subset E' \mid \mathcal{L}'^{\perp^-} = \mathcal{L}' \text{ and } (\gamma_1, \gamma_2)\mathcal{L}' = \mathcal{L}'\}|$$

where the $\mathcal{L}'$'s are $\mathcal{O}_{F'}$-lattices and where $(\cdot)^{\perp^\pm}$ denotes the duality for such lattices with respect to the hermitian form $\Phi^\pm$. Similarly the (stable) orbital integral $SO^H_\delta(1_{K^H})$ is equal to the product

$$SO^H_\delta(1_{K^H}) = |\{\mathcal{M}'_1 \subset E'_1 \mid \mathcal{M}'^{\perp_1}_1 = \mathcal{M}'_1 \text{ and } \gamma_1 \mathcal{M}'_1 = \mathcal{M}'_1\}|$$
$$\times |\{\mathcal{M}'_2 \subset E'_2 \mid \mathcal{M}'^{\perp_2}_2 = \mathcal{M}'_2 \text{ and } \gamma_2 \mathcal{M}'_2 = \mathcal{M}'_2\}|.$$

where the $\mathcal{M}'_i$'s are $\mathcal{O}_{F'}$-lattices and where $(\cdot)^{\perp_i}$ denotes the duality for such lattices with respect to the hermitian form Ψ_i.

As the polynomials $P_1(T), P_2(T) \in F'[T]$ are prime with respect to each other and have coefficients in $\mathcal{O}_{F'}$ their resultant $\mathrm{Res}(P_1, P_2)$ is a non zero element in $\mathcal{O}_{F'}$. Let us denote by

(2.1) $$r = r(\gamma_1, \gamma_2) \geq 0$$

its order. Let us recall that, up to a sign, we have

$$\mathrm{Res}(P_1, P_2) = \prod_{k_1=0}^{n_1-1} \prod_{k_2=0}^{n_2-1} (\gamma_1^{(k_1)} - \gamma_2^{(k_2)})$$

where $\gamma_i = \gamma_i^{(0)}, \ldots, \gamma_i^{(n_i-1)}$ are the roots of $P_i(T)$ in some algebraic closure of F' containing E'_1 and E'_2.

CONJECTURE 2.2 (Langlands-Shelstad). — *Under the above hypotheses we have*

$$O^\kappa_\gamma(1_K) = \varepsilon(\gamma_1, \gamma_2) q^r SO^H_\delta(1_{K^H})$$

where $\varepsilon(\gamma_1, \gamma_2)$ is some sign and q is the number of elements in the residue field k.

Remark : Waldspurger has computed the sign of the Langlands-Shelstad transfer factor at any regular semisimple element $\gamma \in G(F)$ which is close enough to the identity. In particular he has proved that

$$\varepsilon(\gamma_1, \gamma_2) = (-1)^r$$

(private communication). This is exactly the sign which arises from our geometric approach (cf. Theorem 4.2 and §8).

3. Orbital integrals as numbers of rational points of k-schemes

Our first goal is to introduce k-schemes $\mathcal{X}^+$, $\mathcal{X}^-$ and $\mathcal{Y}_i$ $(i = 1, 2)$ such that

$$O_\gamma^\kappa(1_K) = |\mathcal{X}^+(k)| - |\mathcal{X}^-(k)|$$

and

$$SO_\delta^H(1_{K^H}) = |\mathcal{Y}_1(k)| \cdot |\mathcal{Y}_2(k)|.$$

DEFINITION 3.1. — *Let R be a commutative k'-algebra. A $(R \otimes_{k'} \mathcal{O}_{F'})$-lattice in a finite dimensional F'-vector space V is a $(R \otimes_{k'} \mathcal{O}_{F'})$-submodule L of $R \otimes_{k'} V$ such that there exist $\mathcal{O}_{F'}$-lattices $L_0 \subset L_1$ in V having the following properties :*

(i) $R \otimes_{k'} L_0 \subset L \subset R \otimes_{k'} L_1$,

(ii) *the R-module $L/(R \otimes_{k'} L_0)$ is locally a direct factor of the free R-module* $(R \otimes_{k'} L_1)/(R \otimes_{k'} L_0)$.

If R is a commutative k-algebra, a $(R \otimes_k \mathcal{O}_{F'})$-lattice $\mathcal{L}$ in a finite dimensional F'-vector space V is by definition a $((R \otimes_k k') \otimes_{k'} \mathcal{O}_{F'})$-lattice in V.

If R is a k'-algebra and if L_1 and L_2 are two $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices in a finite dimensional F'-vector space V we set

$$[L_1 : L_2] = \mathrm{rk}_R(L_1/(R \otimes_{k'} L_3)) - \mathrm{rk}_R(L_1/(R \otimes_{k'} L_3))$$

where L_3 is any $\mathcal{O}_{F'}$-lattice in V such that $R \otimes_{k'} L_3$ is contained in both lattices L_1 and L_2. It is a locally constant function on $\mathrm{Spec}(R)$ with integral values. If R is a k-algebra and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two $(R \otimes_k \mathcal{O}_{F'})$-lattices we define $[\mathcal{L}_1 : \mathcal{L}_2]$ by simply replacing R by $R \otimes_k k'$ in the above definition.

If V is a finite dimensional F'-vector space which is equipped with a non degenerate hermitian form Φ and if R is a commutative k-algebra, we define the dual $(R \otimes_k \mathcal{O}_{F'})$-lattice $\mathcal{L}^\perp$ of a $(R \otimes_k \mathcal{O}_{F'})$-lattice $\mathcal{L}$ in the obvious way. In particular, if we have $R \otimes_k \mathcal{L}_0 \subset \mathcal{L} \subset R \otimes_k \mathcal{L}_1$ as in Definition 3.1 we have

$$R \otimes_k \mathcal{L}_1^\perp \subset \mathcal{L}^\perp \subset R \otimes_k \mathcal{L}_0^\perp$$

and, if we identify $(R \otimes_k \mathcal{L}_0^\perp)/(R \otimes_k \mathcal{L}_1^\perp)$ with the dual of the free $(R \otimes_k k')$-module $(R \otimes_k \mathcal{L}_1)/(R \otimes_k \mathcal{L}_0)$,

$$\mathcal{L}^\perp/(R \otimes_k \mathcal{L}_1^\perp) \subset (R \otimes_k \mathcal{L}_0^\perp)/(R \otimes_k \mathcal{L}_1^\perp)$$

is the orthogonal of the $\mathcal{L}/(R \otimes_k \mathcal{L}_0) \subset (R \otimes_k \mathcal{L}_1)/(R \otimes_k \mathcal{L}_0)$.

If R is a commutative k'-algebra and if L (resp. M_i) is a $(R \otimes_{k'} \mathcal{O}_{F'})$-lattice in E' (resp. E_i') we define the index of L (resp. M_i) as the locally constant function

$$\mathrm{ind}(L) = [L : R \otimes_{k'} (\mathcal{O}_{E_1'} \oplus \mathcal{O}_{E_2'})] : \mathrm{Spec}(R) \to \mathbb{Z}$$

(resp.

$$\mathrm{ind}(M_i) = [M_i : R \otimes_{k'} \mathcal{O}_{E_i'}] : \mathrm{Spec}(R) \to \mathbb{Z}).$$

If R is a commutative k-algebra and if $\mathcal{L}$ (resp. $\mathcal{M}_i$) is a $(R \otimes_k \mathcal{O}_{F'})$-lattice in E' (resp. E_i') we define the index of $\mathcal{L}$ (resp. $\mathcal{M}_i$) by replacing R by $R \otimes_k k'$ in the above definitions. We then have

$$\mathrm{ind}(\mathcal{L}^{\perp^+}) = -\mathrm{ind}(\mathcal{L})$$

and

$$\mathrm{ind}(\mathcal{L}^{\perp^-}) = -\mathrm{ind}(\mathcal{L}) + 2$$

where $\perp^\pm$ is the duality for lattices with respect to the hermitian form $\Phi^\pm$ (resp.

$$\mathrm{ind}(\mathcal{M}_i^{\perp_i}) = -\mathrm{ind}(\mathcal{M}_i)$$

where $\perp_i$ is the duality for lattices with respect to the hermitian form Ψ_i).

For each commutative k-algebra R we now set

(3.2.1) $$\mathcal{X}^\pm(R) = \{\mathcal{L}^\pm \mid (\mathcal{L}^\pm)^{\perp^\pm} = \mathcal{L}^\pm \text{ and } (1 \otimes (\gamma_1, \gamma_2))\mathcal{L}^\pm = \mathcal{L}^\pm\}$$

and

(3.2.2) $$\mathcal{Y}_i(R) = \{\mathcal{M}_i \mid \mathcal{M}_i^{\perp_i} = \mathcal{M}_i \text{ and } (1 \otimes \gamma_i)\mathcal{M}_i = \mathcal{M}_i\}$$

where $\mathcal{L}^\pm$ and $\mathcal{M}_i$ are $(R \otimes_k \mathcal{O}_{F'})$-lattices in E' and E_i'. If $\varphi : S \to R$ is a homomorphism of commutative k-algebras we have obvious base change maps $\mathcal{X}^\pm(R) \to \mathcal{X}^\pm(S)$ and $\mathcal{Y}_i(R) \to \mathcal{Y}_i(S)$.

PROPOSITION 3.3. — (i) *The functor $\mathcal{X}^\pm$ is representable by a k-scheme which is locally of finite type.*

(ii) *The functor $\mathcal{Y}_i$ is representable by a projective k-scheme.*

Let us recall that to give a quasi-projective k-scheme $\mathcal{S}$ is the same as to give the quasi-projective k'-scheme $k' \otimes_k \mathcal{S}$ together with the endomorphism $k' \otimes_k \mathrm{Frob}_{\mathcal{S}}$ where $\mathrm{Frob}_{\mathcal{S}}$ is the Frobenius endomorphism of $\mathcal{S}$ with respect to k (cf. [SGA 1](VIII, 7.6)). Therefore Proposition 3.3 is an immediate consequence of the following proposition :

PROPOSITION 3.4. — (i) *The functor $k' \otimes_k \mathcal{X}^\pm$ (i.e. the restriction of the functor $\mathcal{X}^\pm$ to k'-algebras) is representable by a k'-scheme which is an increasing union of $(k' \otimes_k \mathrm{Frob}_{\mathcal{X}^\pm})$-stable quasi-projective open subsets.*

(ii) *The functor $k' \otimes_k \mathcal{Y}_i$ (i.e. the restriction of the functor $\mathcal{Y}_i$ to k'-algebras) is representable by a projective k'-scheme.*

Remark : This proposition is a particular case of a result of D. Kazhdan and G. Lusztig (cf. [Ka-Lu] §2).

Before proving Proposition 3.4 let us give a description of the functors $k' \otimes_k \mathcal{X}^\pm$ and $k' \otimes_k \mathcal{Y}_i$ which does not involve the hermitian forms $\Phi^\pm$ and Ψ_i.

For every finite dimensional F'-vector space V which is equipped with a non degenerate hermitian form Φ we may split the $(k' \otimes_k F)$-vector space $k' \otimes_k V$ into

$$\widetilde{V} \oplus \widetilde{\widetilde{V}}$$

where

$$\widetilde{V} = \{x \in k' \otimes_k V \mid (1 \otimes \alpha')x = (\alpha' \otimes 1)x, \ \forall \alpha' \in k'\}$$

and

$$\widetilde{\widetilde{V}} = \{x \in k' \otimes_k V \mid (1 \otimes \alpha')x = (\alpha'^\sigma \otimes 1)x, \ \forall \alpha' \in k'\}.$$

The $(k' \otimes_k F)$-bilinear form $k' \otimes_k \Phi$ is then given by

$$(k' \otimes_k \Phi)(\widetilde{x}_1 \oplus \widetilde{\widetilde{x}}_1, \widetilde{x}_2 \oplus \widetilde{\widetilde{x}}_2) = \widetilde{\Phi}(\widetilde{\widetilde{x}}_1, \widetilde{x}_2) + \big(\widetilde{\Phi}(\widetilde{\widetilde{x}}_2, \widetilde{x}_1)\big)^{1 \otimes \sigma}$$

for some non degenerate $(k' \otimes_k F)$-bilinear form

$$\widetilde{\Phi} : \widetilde{\widetilde{V}} \times \widetilde{V} \to \{x \in k' \otimes_k F' \mid (1 \otimes \alpha')x = (\alpha' \otimes 1)x, \ \forall \alpha' \in k'\}.$$

The map $\sigma \otimes_k \mathrm{Id}_V : k' \otimes_k V \to k' \otimes_k V$ induces $(\sigma \otimes_k \mathrm{Id}_{F'})$-linear bijections $F : \widetilde{V} \to \widetilde{\widetilde{V}}$ and $G : \widetilde{\widetilde{V}} \to \widetilde{V}$. The maps $G \circ F$ and $F \circ G$ are the Frobenius endomorphisms with respect to F', i.e. the identities of the F'-vector spaces $\widetilde{V}$ and $\widetilde{\widetilde{V}}$. One easily checks that

$$\widetilde{\Phi}(F(\widetilde{x}), G(\widetilde{\widetilde{x}})) = \big(\widetilde{\Phi}(\widetilde{\widetilde{x}}, \widetilde{x})\big)^{\sigma \otimes \sigma}$$

for every $\widetilde{x} \in \widetilde{V}$ and $\widetilde{\widetilde{x}} \in \widetilde{\widetilde{V}}$.

We may identify $k' \otimes_k F$ and $\{x \in k' \otimes_k F' \mid (1 \otimes \alpha')x = (\alpha' \otimes 1)x, \ \forall \alpha' \in k'\}$ with F' by

$$\alpha' \otimes 1 \mapsto \alpha' \text{ and } \sum_i \alpha'_i \otimes b'_i \mapsto \sum_i \alpha'_i b'_i.$$

Similarly we will identify $\widetilde{V}$ and $\widetilde{\widetilde{V}}$ with the F'-vector spaces V and $k' \otimes_{\sigma,k'} V$ by

$$\sum_i \alpha'_i \otimes v_i \mapsto \sum_i \alpha'_i v_i \text{ and } \sum_i \alpha'_i \otimes v_i \mapsto \sum_i \alpha'^{\sigma}_i v_i.$$

Then $\Phi : (k' \otimes_{\sigma,k'} V) \times V \to F'$ is a non degenerate F'-bilinear form and we may identify $k' \otimes_{\sigma,k'} V$ with the F'-linear dual V^* of V. We have σ-linear bijections $F : V \to V^*$ and $G : V^* \to V$. The maps $G \circ F$ and $F \circ G$ are the identities of the F'-vector spaces V and V^*. We have

$$\langle F(x), G(x^*) \rangle = \sigma\big(\langle x^*, x \rangle\big)$$

for every $x \in V$ and $x^* \in V^*$.

LEMMA 3.5. — *Let us denote by $\perp$ the duality (for lattices) with respect to Φ and let us assume moreover that V admits a selfdual $\mathcal{O}_{F'}$-lattice $\mathcal{L}_0$.*

Then, for every commutative k'-algebra R there is a natural bijection between the set of $(R \otimes_k \mathcal{O}_{F'})$-lattices $\mathcal{L}$ in V such that $\mathcal{L}^{\perp} = \mathcal{L}$ and the set of $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices L in V such that $[L : R \otimes_{k'} \mathcal{L}_0] = 0$.

Proof : The relation between $\mathcal{L}$ and L is

$$\mathcal{L} = L \oplus L^{\perp} \subset (R \otimes_{k'} V) \oplus (R \otimes_{k'} V^*) = R \otimes_k V$$

where $L^{\perp}$ is the dual $(R \otimes_{k'} F')$-lattice of L in $R \otimes_{k'} V^*$. $\qquad\square$

If u is a unitary automorphism of (V, Φ) the $(k' \otimes_k F')$-linear automorphism $1 \otimes u$ of $k' \otimes_k V$ is the direct sum $u \oplus ({}^t u)^{-1}$ where ${}^t u$ is the transposed endomorphism of u. Moreover we have the relations

$${}^t u \circ F \circ u = F, \quad u \circ G \circ {}^t u = G.$$

In particular, we can take $(V, \Phi) = (E'_i, \Phi^{\pm}_{E'_i})$, $(V, \Phi) = (E', \Phi^{\pm})$ or $(V, \Phi) = (E'_i, \Psi_i)$ and we get the σ-linear bijections $F^{\pm}_i : E'_i \to E'^*_i$, $G^{\pm}_i : E'^*_i \to E'_i$, $F^{\pm} = F^{\pm}_1 \oplus F^{\pm}_2 : E' \to E'^*$, $G^{\pm} = G^{\pm}_1 \oplus G^{\pm}_2 : E'^* \to E'$, $F_i = F^+_i : E'_i \to E'^*_i$ and $G_i = G^+_i : E'^*_i \to E'_i$.

Now, for any commutative k'-algebra R we set

(3.6.1) $\qquad X^{\pm}(R) = \{L^{\pm} \mid \text{ind}(L^{\pm}) \equiv \delta^{\pm} \text{ and } (1 \otimes (\gamma_1, \gamma_2))L^{\pm} = L^{\pm}\}$

where $\delta^+ = 0$ and $\delta^- = 1$ and where the $L^{\pm}$'s are $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices in E', and we set

(3.6.2) $\qquad Y_i(R) = \{M_i \mid \text{ind}(M_i) \equiv 0 \text{ and } (1 \otimes \gamma_i)M_i = M_i\}$

where the M_i's are $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices in E'_i. If $\varphi : S \to R$ is a homomorphism of commutative k-algebras we have obvious base change maps $X^{\pm}(R) \to X^{\pm}(S)$ and $Y_i(R) \to Y_i(S)$. We denote by

(3.6.3) $\qquad\qquad\qquad\qquad F_{X^{\pm}} : X^{\pm} \to X^{\pm}$

and

(3.6.4)
$$F_{Y_i} : Y_i \to Y_i$$

the functor endomorphisms which are given on the R-valued points by

$$L^{\pm} \mapsto \left(F_R^{\pm}(L^{\pm})\right)^{\perp^{\pm}} = G_R^{\pm}\!\left((L^{\pm})^{\perp^{\pm}}\right)$$

and

$$M_i \mapsto \left(F_{i,R}(M_i)\right)^{\perp_i} = G_{i,R}\!\left((M_i)^{\perp_i}\right)$$

where $F_R^{\pm} : R \otimes_{k'} E' \to (R \otimes_{k'} E')^*$, $G_R^{\pm} : (R \otimes_{k'} E')^* \to R \otimes_{k'} E'$, $F_{i,R} : R \otimes_{k'} E_i' \to (R \otimes_{k'} E_i')^*$ and $G_{i,R} : (R \otimes_{k'} E_i')^* \to R \otimes_{k'} E_i')$ are the natural σ-linear extensions of $F^{\pm}$, $G^{\pm}$, F_i and G_i.

Then it follows from the above discussion that

LEMMA 3.7. — *We have natural identifications*

$$(k' \otimes_k \mathcal{X}^{\pm}, k' \otimes_k \mathrm{Frob}_{\mathcal{X}^{\pm}}) = (X^{\pm}, F_{X^{\pm}})$$

and

$$(k' \otimes_k \mathcal{Y}_i, k' \otimes_k \mathrm{Frob}_{\mathcal{Y}_i}) = (Y_i, F_{Y_i}).$$

$\square$

Let

(3.8)
$$m_i = m(\gamma_i) \geq 0$$

be the conductor of $\mathcal{O}_{F'}[\gamma_i]$ in $\mathcal{O}_{E_i'}$, i.e. the smallest non negative integer m such that

$$\varpi_{E_i}^m \mathcal{O}_{E_i'} \subset \mathcal{O}_{F'}[\gamma_i] \subset \mathcal{O}_{E_i'}.$$

The proof of Proposition 3.4 is based on the next two lemmas :

LEMMA 3.9. — *Let R be a commutative k'-algebra and let L_i be a $(R \otimes_{k'} \mathcal{O}_{F'})$-lattice in E_i' such that $\gamma_i L_i \subset L_i$ with constant index. Then there exists an integer ℓ such that*

$$\mathrm{ind}(L_i) \leq \ell \leq \mathrm{ind}(L_i) + m_i$$

and

$$\varpi_{E_i}^{m_i - \ell}(R \otimes_{k'} \mathcal{O}_{E_i'}) \subset L_i \subset \varpi_{E_i}^{-\ell}(R \otimes_{k'} \mathcal{O}_{E_i'}).$$

In particular we automatically have

$$\varpi_{E_i}^{m_i - \mathrm{ind}(L_i)}(R \otimes_{k'} \mathcal{O}_{E_i'}) \subset L_i \subset \varpi_{E_i}^{-m_i - \mathrm{ind}(L_i)}(R \otimes_{k'} \mathcal{O}_{E_i'}).$$

Proof : We have

$$\varpi_{E_i}^{m_i} \mathcal{O}_{E_i'} L_i \subset \mathcal{O}_{F'}[\gamma_i] L_i \subset L_i \subset \mathcal{O}_{E_i'} L_i$$

and we only need to check that $\mathcal{O}_{E_i'} L_i$ is equal to $\varpi_{E_i}^{-\ell}(R \otimes_{k'} \mathcal{O}_{E_i'})$ for some integer ℓ.

But, as L_i is a $(R \otimes_{k'} \mathcal{O}_{F'})$-lattice there exist $\mathcal{O}_{F'}$-lattices $\mathcal{L}_0 \subset \mathcal{L}_1$ in E_i' such that $R \otimes_{k'} \mathcal{L}_0 \subset L_i \subset R \otimes_{k'} \mathcal{L}_1$ and we have

$$R \otimes_{k'} (\mathcal{O}_{E_i'} \mathcal{L}_0) \subset \mathcal{O}_{E_i'} L_i \subset R \otimes_{k'} (\mathcal{O}_{E_i'} \mathcal{L}_1)$$

with $\mathcal{O}_{E_i'} \mathcal{L}_0 = \varpi_{E_i}^{-\ell_0} \mathcal{O}_{E_i'}$ and $\mathcal{O}_{E_i'} \mathcal{L}_1 = \varpi_{E_i}^{-\ell_1} \mathcal{O}_{E_i'}$ for some integers $\ell_0 \le \ell_1$. Now the multiplication by ϖ_{E_i} on the free R-module

$$\varpi_{E_i}^{-\ell_1}(R \otimes_{k'} \mathcal{O}_{E_i'})/\varpi_{E_i}^{-\ell_0}(R \otimes_{k'} \mathcal{O}_{E_i'})$$

is a regular nilpotent endomorphism and the R-submodule

$$\mathcal{O}_{E_i'} L_i/\varpi_{E_i}^{-\ell_0}(R \otimes_{k'} \mathcal{O}_{E_i'}) \subset \varpi_{E_i}^{-\ell_1}(R \otimes_{k'} \mathcal{O}_{E_i'})/\varpi_{E_i}^{-\ell_0}(R \otimes_{k'} \mathcal{O}_{E_i'})$$

is locally a direct factor. Therefore this R-submodule is equal to the R-submodule

$$\varpi_{E_i}^{-\ell}(R \otimes_{k'} \mathcal{O}_{E_i'})/\varpi_{E_i}^{-\ell_0}(R \otimes_{k'} \mathcal{O}_{E_i'})$$

for some integer ℓ with $\ell_0 \le \ell \le \ell_1$ and the proof of the lemma is complete. $\qquad\square$

For each point $L^{\pm}$ in $X^{\pm}$ with value in some field extension K of k' let us denote by $B_i^{\pm}$ the intersection of $L^{\pm}$ with $K \otimes_{k'} E_i' \subset K \otimes_{k'} E'$ and by $C_i^{\pm}$ the projection of $L^{\pm}$ on $K \otimes_{k'} E_i'$. Then

$$(3.10.1) \qquad\qquad B_i^{\pm} \subset C_i^{\pm} \subset K \otimes_{k'} E_i'$$

are $(K \otimes_{k'} \mathcal{O}_{F'})$-lattices in E_i' and $L^{\pm}$ is the graph of an isomorphism of $(K \otimes_{k'} \mathcal{O}_{F'})$-modules of finite dimension over K

$$(3.10.2) \qquad\qquad \iota^{\pm} : C_1^{\pm}/B_1^{\pm} \xrightarrow{\sim} C_2^{\pm}/B_2^{\pm}.$$

Moreover, if we set

$$(3.10.3) \qquad b_i^{\pm} = b_i^{\pm}(L^+) := \mathrm{ind}(B_i^{\pm}) \le c_i^{\pm} = c_i^{\pm}(L^+) := \mathrm{ind}(C_i^{\pm})$$

we have

$$(3.10.4) \qquad\qquad b_1^{\pm} + c_2^{\pm} = b_2^{\pm} + c_1^{\pm} = \mathrm{ind}(L^{\pm}) = \delta^{\pm}$$

with $\delta^+ = 0$ and $\delta^- = 1$.

If $L^{\pm}$ is now a point in $X^{\pm}$ with value in some commutative k'-algebra R we define as above integers

$$b_i^{\pm}(\mathfrak{p}) = b_i^{\pm}(\mathrm{Frac}(R/\mathfrak{p}) \otimes_R L^{\pm}) \le c_i^{\pm}(\mathfrak{p}) = c_i^{\pm}(\mathrm{Frac}(R/\mathfrak{p}) \otimes_R L^{\pm})$$

for each prime ideal $\mathfrak{p}$ in R. We thus have functions $b_i^{\pm}, c_i^{\pm} : \mathrm{Spec}(R) \to \mathbb{Z}$ which are easily seen to be semicontinuous : for each integer λ_i the set of points in $\mathrm{Spec}(R)$ such that $b_i^{\pm} \le \lambda_i$ (or equivalently such that $c_j^{\pm} \ge \delta^{\pm} - \lambda_i$ if $\{i, j\} = \{1, 2\}$) is open for the Zariski topology.

LEMMA 3.11. — *Let $L^\pm$ be a point in $X^\pm$ with value in some commutative k'-algebra R. Then the functions $b_i^\pm, c_i^\pm : \mathrm{Spec}(R) \to \mathbb{Z}$ satisfy the inequalities*

$$b_i^\pm \leq c_i^\pm \leq b_i^\pm + r.$$

Proof : We may assume that R is a field extension of k' and therefore we may introduce the lattices $B_i^\pm, C_i^\pm$ and the isomorphism $\iota^\pm$. As $\iota^\pm$ exchanges the multiplications by γ_1 and γ_2 and as $P_1(\gamma_1) = 0$ and $P_2(\gamma_2) = 0$ we have

$$P_2(\gamma_1)C_1^\pm \subset B_1^\pm$$

and

$$P_1(\gamma_2)C_2^\pm \subset B_2^\pm.$$

But $P_2(\gamma_1)$ and $P_1(\gamma_1)$ are of order $r = r(\gamma_1, \gamma_2)$ in E_1' and E_2', so that

$$b_i^\pm \leq c_i^\pm \leq b_i^\pm + r$$

as required. $\qquad\square$

Proof of Proposition 3.4 : Let us begin with Part (ii). It follows from Lemma 3.9 that, for any commutative k'-algebra R, $Y_i(R)$ may be identified with the set of $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices M_i in E_i' such that

$$\varpi_{E_i}^{m_i}(R \otimes_{k'} \mathcal{O}_{F'}) \subset M_i \subset \varpi_{E_i}^{-m_i}(R \otimes_{k'} \mathcal{O}_{F'}),$$

$$\mathrm{rk}_R(M_i/\varpi_{E_i}^{m_i}(R \otimes_{k'} \mathcal{O}_{F'})) = m_i$$

and

$$(1 \otimes \gamma_i)M_i \subset M_i.$$

Therefore, Y_i is representable by a closed k'-subscheme of the Grassmann variety of m_i-planes in the $2m_i$-dimensional k'-vector space

$$\varpi_{E_i}^{-m_i}\mathcal{O}_{F'}/\varpi_{E_i}^{m_i}\mathcal{O}_{F'}$$

and Part (ii) is proved.

The proof of Part (i) is similar to the proof of Part (ii). For each pair of integers (λ_1, λ_2) let us consider the open subfunctor

(3.12.1) $$X^\pm(\lambda_1, \lambda_2) = \{L^\pm \in X^\pm \mid b_1^\pm \leq \lambda_1 \text{ and } b_2^\pm \leq \lambda_2\}.$$

We only need to prove that this open subfunctor is representable by a quasi-projective k'-scheme.

It follows from Lemma 3.11 that $X^\pm(\lambda_1, \lambda_2)$ is contained in the closed subfunctor

$$X^\pm[\delta^\pm - \lambda_2 - r, \delta^\pm - \lambda_1 - r] = \{L^\pm \in X^\pm \mid b_1^\pm \geq \delta^\pm - \lambda_2 - r \text{ and } b_2^\pm \geq \delta^\pm - \lambda_1 - r\}$$

of $X^\pm$. Therefore it is sufficient to prove that, for any pair of integers (μ_1, μ_2) the closed subfunctor

(3.12.2) $$X^\pm[\mu_1, \mu_2] = \{L^\pm \in X^\pm \mid b_1^\pm \geq \mu_1 \text{ and } b_2^\pm \geq \mu_2\}$$

is representable by a projective k'-scheme. But it follows from Lemma 3.9 that the functor $X^\pm[\mu_1, \mu_2]$ is a closed subfunctor of the functor which associates to any commutative k'-algebra R the set of $(R \otimes_{k'} \mathcal{O}_{F'})$-lattices $L^\pm$ in E' such that

$$\varpi_{E_1}^{m_1 - \mu_1}(R \otimes_{k'} \mathcal{O}_{E_1'}) \oplus \varpi_{E_2}^{m_2 - \mu_2}(R \otimes_{k'} \mathcal{O}_{E_2'})$$

$$\subset L^\pm \subset \varpi_{E_1}^{\mu_2 - m_1 - \delta^\pm}(R \otimes_{k'} \mathcal{O}_{E_1'}) \oplus \varpi_{E_2}^{\mu_1 - m_2 - \delta^\pm}(R \otimes_{k'} \mathcal{O}_{E_2'}),$$

$$\mathrm{rk}_R\big(L^\pm / (\varpi_{E_1}^{m_1 - \mu_1}(R \otimes_{k'} \mathcal{O}_{E_1'}) \oplus \varpi_{E_2}^{m_2 - \mu_2}(R \otimes_{k'} \mathcal{O}_{E_2'}))\big) = m_1 + m_2 - \mu_1 - \mu_2 + \delta^\pm$$

and

$$(1 \otimes (\gamma_1, \gamma_2))L^\pm \subset L^\pm.$$

Therefore, if we set $N = m_1 + m_2 - \mu_1 - \mu_2 + \delta^\pm$ the functor $X^\pm[\mu_1, \mu_2]$ is representable by a closed k'-subscheme of the Grassmann variety of N-planes in the $2N$-dimensional k'-vector space

$$(\varpi_{E_1}^{m_1 - \mu_1}\mathcal{O}_{E_1'} \oplus \varpi_{E_2}^{m_2 - \mu_2}\mathcal{O}_{E_2'}) / (\varpi_{E_1}^{\mu_2 - m_1 - \delta^\pm}\mathcal{O}_{E_1'} \oplus \varpi_{E_2}^{\mu_1 - m_2 - \delta^\pm}\mathcal{O}_{E_2'})$$

and Part (ii) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.13 : In fact the two k'-schemes X^+ and X^- are isomorphic. More precisely, there are two k'-scheme isomorphisms

$$\alpha_1, \alpha_2 : X^- \xrightarrow{\sim} X^+$$

given by

$$\alpha_1(L^-) = (\varpi_{E_1} \oplus 1)L^-$$

and

$$\alpha_2(L^-) = (1 \oplus \varpi_{E_2})L^-.$$

Moreover the squares of k'-scheme morphisms

$$
\begin{array}{ccc}
X^- & \xrightarrow{F_{X^-}} & X^- \\
\alpha_1 \downarrow & & \downarrow \alpha_2 \\
X^+ & \xrightarrow{F_{X^+}} & X^+
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X^- & \xrightarrow{F_{X^-}} & X^- \\
\alpha_2 \downarrow & & \downarrow \alpha_1 \\
X^+ & \xrightarrow{F_{X^+}} & X^+
\end{array}
$$

commute.

Remark 3.14 : Let G_i be the Grassmann variety of m_i-planes in the $2m_i$-dimensional k'-vector space

$$W_i = \varpi_{E_i}^{-m_i} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i} \mathcal{O}_{E_i'}$$

and let N_i be the regular nilpotent endomorphism of W_i which is induced by the multiplication by ϖ_{E_i}. In the course of the proof of Proposition 3.4 we have identified Y_i with a closed subscheme of G_i.

Let Z_i be the closed subscheme of G_i of m_i-planes $D_i \subset W_i$ such that

$$N_i^{m_i+j}(D_i) \subset D_i$$

for every non negative integer j. Then, as $\varpi_{E_i}^{m_i} \mathcal{O}_{E_i'} \subset \mathcal{O}_{F'}[\gamma_i]$ by definition of the conductor m_i, we have

$$Y_i \subset Z_i.$$

The structure of Z_i is much simpler than that of Y_i. In fact, it follows from Lemma 3.15 below that either $m_i = 0$ and $Z_i = G_i = \mathrm{Spec}(k')$, or $m_i > 0$ and

$$Z_i = \bigcup_{j=1}^{m_i-1} Z_{i,j}$$

where

$$Z_{i,j} = \{D_i \subset W_i \mid \dim(D_i) = m_i \text{ and } \mathrm{Ker}(N_i^j) \subset D_i \subset \mathrm{Ker}(N_i^{m_i+j})\}.$$

Moreover, if $m_i > 0$ the irreducible components of Z_i are exactly the $Z_{i,j}$ for $j = 1, \ldots, m_i - 1$ and the intersection of any two of these irreducible components is again a Grassmann variety.

LEMMA 3.15. — *Let E be a finite dimensional k'-vector space which is equipped with a regular nilpotent endomorphism N. We denote by e the dimension of E. Let m be a non negative integer and let F be a k'-vector subspace of E which is N^{m+j}-stable for every $j \geq 0$. Let ℓ be a positive integer.*

If F is not contained in $\mathrm{Im}(N^\ell)$ it automatically contains $\mathrm{Im}(N^{m+\ell-1})$ and therefore its dimension is at least $e - m - \ell + 1$ (and even at least $e - m - \ell + 2$ if $m \geq 1$ as $\mathrm{Im}(N^{m+\ell-1}) \subset \mathrm{Im}(N^\ell)$ in this case).

If F does not contain $\mathrm{Ker}(N^\ell)$ it is automatically contained in $\mathrm{Ker}(N^{m+\ell-1})$ and therefore its dimension is at most $m + \ell - 1$ (and even at most $m + \ell - 2$ if $m \geq 1$ as $\mathrm{Ker}(N^{m+\ell-1}) \supset \mathrm{Ker}(N^\ell)$ in this case).

Proof : If $F \not\subset \mathrm{Im}(N^\ell) = \mathrm{Ker}(N^{e-\ell})$ there exists $x \in F$ such that $N^{e-\ell}(x) \neq 0$. Let j be the unique non negative integer such that $N^{e-\ell+j}(x) \neq 0$ and $N^{e-\ell+j+1}(x) = 0$. Then we may consider the vector subspace of E generated by $N^{m+j}(x), N^{m+j+1}(x), \ldots,$

$N^{e-\ell+j}(x)$. It is contained in F by hypothesis and it is easily seen to be equal to $\mathrm{Ker}\big(N^{e-m-\ell+1}\big)$. The first assertion of the lemma follows.

The second assertion of the lemma is another formulation of the first one. $\qquad\square$

4. Statement of the main theorem

On the k'-scheme $X^{\pm}$ we have a translation automorphism $\tau^{\pm}: X^{\pm} \to X^{\pm}$ which is given by

$$\tau^{\pm}(L^{\pm}) = (\varpi_{E_1} \oplus \varpi_{E_2}^{-1})L^{\pm}$$

and which satisfies the anticommutation relation

$$\tau^{\pm} \circ F_{X^{\pm}} = F_{X^{\pm}} \circ (\tau^{\pm})^{-1}.$$

In fact, with the notation of Remark 3.13 we have

$$\tau^{+} = \alpha_1 \circ \alpha_2^{-1}$$

and

$$\tau^{-} = \alpha_2^{-1} \circ \alpha_1.$$

In particular τ^{-} and τ^{+} are exchanged by the isomorphisms α_1 and α_2.

Remark : The action of $\mathbb{Z}$ on $X^{\pm}$ generated by the translation automorphism $\tau^{\pm}$ has been introduced by D. Kazhdan and G. Lusztig in [Ka-Lu] §2.

Let us first assume that $r = 2r'$ is even. Then the projective closed subscheme (cf. (3.12.2))

$$\begin{aligned} X^{+}[-r', -r'] &= \{L^{+} \in X^{+} \mid b_1^{+} \geq -r' \text{ and } b_2^{+} \geq -r'\} \\ &= \{L^{+} \in X^{+} \mid -r' \leq b_i^{+} \leq c_i^{+} \leq r'\} \end{aligned}$$

may be viewed as a "fundamental domain" for the translation τ^{+} on the k'-scheme X^{+} : for each integer n we have

$$\begin{aligned} (\tau^{+})^{n} X^{+}[-r', -r'] &= X^{+}[-r' - n, -r' + n] \\ &= \{L^{+} \in X^{+} \mid -r' - n \leq b_1^{+} \leq c_1^{+} \leq r' - n\} \\ &= \{L^{+} \in X^{+} \mid -r' + n \leq b_2^{+} \leq c_2^{+} \leq r' + n\}, \end{aligned}$$

so that

$$X^{+} = \bigcup_{n \in \mathbb{Z}} (\tau^{+})^{n} X^{+}[-r', -r']$$

and the intersection

$$X^{+}[-r', -r'] \cap (\tau^{+})^{n} X^{+}[-r', -r'] = X^{+}[-\inf(r', r' + n), -\inf(r', r' - n)]$$

is empty if $|n| > r$. Moreover, as

$$b_i^+(F_{X^+}L^+) = -c_i^+(L^+)$$

all the fixed points by F_{X^+} in X^+ are contained in $X^+[-r', -r']$ and we have

(4.1.1) $$|\mathcal{X}^+(k)| = \left|(X^+[-r', -r'])^{F_{X^+}}\right|.$$

Similarly, as

$$b_i^-(F_{X^-}L^-) = 1 - c_i^-(L^-)$$

all the fixed points by F_{X^-} in X^- are contained in

$$X^-[1 - r', 1 - r'] = \{L^- \in X^- \mid 1 - r' \leq b_i^- \leq c_i^- \leq r'\}$$

and we have

(4.1.2) $$|\mathcal{X}^-(k)| = \left|(X^-[1 - r', 1 - r'])^{F_{X^-}}\right|.$$

Let us now assume that $r = 2r' + 1$ is odd. Then the projective closed subscheme (cf. (3.12.2))

$$\begin{aligned} X^-[-r', -r'] &= \{L^- \in X^- \mid b_1^- \geq -r' \text{ and } b_2^- \geq -r'\} \\ &= \{L^- \in X^- \mid -r' \leq b_i^- \leq c_i^- \leq 1 + r'\} \end{aligned}$$

may be viewed as a "fundamental domain" for the translation τ^- on the k'-scheme X^- : for each integer n we have

$$\begin{aligned} (\tau^-)^n X^-[-r', -r'] &= X^-[-r' - n, -r' + n] \\ &= \{L^- \in X^- \mid -r' - n \leq b_1^- \leq c_1^- \leq 1 + r' - n\} \\ &= \{L^- \in X^- \mid -r' + n \leq b_2^- \leq c_2^- \leq 1 + r' + n\}, \end{aligned}$$

so that

$$X^- = \bigcup_{n \in \mathbb{Z}} (\tau^-)^n X^-[-r', -r']$$

and the intersection

$$X^-[-r', -r'] \cap (\tau^-)^n X^-[-r', -r'] = X^-[-\inf(r', r' + n), -\inf(r', r' - n)]$$

is empty if $|n| > r$. Moreover, as

$$b_i^-(F_{X^-}L^-) = 1 - c_i^-(L^-)$$

all the fixed points by F_{X^-} in X^- are contained in $X^-[-r', -r']$ and we have

(4.1.3) $$|\mathcal{X}^-(k)| = \left|(X^-[-r', -r'])^{F_{X^-}}\right|.$$

Similarly, as

$$b_i^+(F_{X^+}L^+) = -c_i^+(L^+)$$

all the fixed points by F_{X^+} in X^+ are contained in

$$X^+[-r', -r'] = \{L^+ \in X^+ \mid -r' \leq b_i^+ \leq c_i^+ \leq r'\}$$

and we have

(4.1.4) $$|\mathcal{X}^+(k)| = \left|(X^+[-r', -r'])^{F_{X^+}}\right|.$$

Theorem 4.2. — *Let us set*

$$(X, F_X) = \left(X^+[-r', -r'], F_{X+} \big| X^+[-r', -r']\right)$$
$$(X', F_{X'}) = \left(X^-[1 - r', 1 - r'], F_{X-} \big| X^-[1 - r', 1 - r']\right)$$

if $r = 2r'$ *is even and*

$$(X, F_X) = \left(X^-[-r', -r'], F_{X-} \big| X^-[-r', -r']\right)$$
$$(X', F_{X'}) = \left(X^+[-r', -r'], F_{X+} \big| X^+[-r', -r']\right)$$

if $r = 2r' + 1$ *is odd, so that* X *and* X' *are projective* k'-*schemes and that* F_X *and* $F_{X'}$ *are the Frobenius endomorphisms for some* k-*rational structures on* X *and* X' *respectively.*
   Then, for every even positive integer f *we have*

$$\left| X^{F_X^f} \right| - \left| X'^{F_{X'}^f} \right| = q^{fr} \cdot \left| Y_1^{F_{Y_1}^f} \right| \cdot \left| Y_2^{F_{Y_1}^f} \right|.$$

The proof of Theorem 4.2 will be given in §7.

Remark : Taking into account the relations

$$X^{F_X} = \mathcal{X}^\pm(k) \quad \text{and} \quad X'^{F_{X'}} = \mathcal{X}^\mp(k)$$

where we have put $\pm = +$ if r is even and $\pm = -$ if r is odd (cf. (4.1.1) to (4.1.4)), the Langlands-Shelstad conjecture is equivalent to the analogous statement of Theorem 4.2 for $f = 1$. An obvious extrapolation of the Langlands-Shelstad conjecture is that the equalities in the statement of Theorem 4.2 should hold for every positive odd integer f.

PART II

5. Stratifications of the k'-scheme $X^\pm$

With the notation of (3.10.1) to (3.10.4) a point $L^\pm$ in $X^\pm$ with values in some field extension K of k' may be given as a triple

$$(B_1^\pm \subset C_1^\pm, B_2^\pm \subset C_2^\pm, \iota^\pm)$$

where $B_i^\pm$ and $C_i^\pm$ are $(K \otimes_{k'} \mathcal{O}_{F'})$-lattices in E_i' such that

$$b_1^\pm + c_2^\pm = b_2^\pm + c_1^\pm = \delta^\pm$$

with $\delta^+ = 0$ and $\delta^- = 1$, and where

$$\iota^\pm : C_1^\pm / B_1^\pm \xrightarrow{\sim} C_2^\pm / B_2^\pm$$

is an isomorphism of $(K \otimes_{k'} \mathcal{O}_{F'})$-modules (of finite dimension over K) which exchanges the automorphisms induced by the multiplications by γ_1 and γ_2. It may also be given as a triple

$$(B_1^\pm, C_2^\pm, f_1^\pm)$$

(resp.

$$(C_1^\pm, B_2^\pm, f_2^\pm))$$

where $B_1^\pm$ and $C_2^\pm$ (resp. $C_1^\pm$ and $B_2^\pm$) are as before and where

$$f_1^\pm : C_2^\pm \to (K \otimes_{k'} E_1')/B_1^\pm$$

(resp.

$$f_2^\pm : C_1^\pm \to (K \otimes_{k'} E_2')/B_2^\pm)$$

is a morphism of $(K \otimes_{k'} \mathcal{O}_{F'})$-modules which exchanges the automorphisms induced by the multiplications by γ_1 and γ_2.

For $i = 1, 2$ and each integer j let us denote by

$$U_{i,j}^\pm$$

the locally closed subset of the k'-scheme $X^\pm$ which is defined by the condition

$$b_i^\pm(L^\pm) = j.$$

For $i = 1, 2$ the family $(U^\pm_{i,j})_{j \in \mathbb{Z}}$ of locally closed subsets is a partition of $X^\pm$. For each integer j we have k'-scheme morphisms

$$\pi^\pm_{1,j} : U^\pm_{1,j} \to Y_1 \times_{k'} Y_2$$

and

$$\pi^\pm_{2,j} : U^\pm_{2,j} \to Y_1 \times_{k'} Y_2$$

which are defined by

$$\pi^\pm_{1,j}(L^\pm) = (\varpi^j_{E_1} B^\pm_1, \varpi^{\delta^\pm - j}_{E_2} C^\pm_2)$$

and

$$\pi^\pm_{2,j}(L^\pm) = (\varpi^{\delta^\pm - j}_{E_1} C^\pm_1, \varpi^j_{E_2} B^\pm_2).$$

Remark : The above k'-scheme morphisms are nothing else than the restriction of the map q_N introduced by Kazhdan and Lusztig in [Ka-Lu] §5.

It is obvious that

$$(\tau^\pm)^k(U^\pm_{i,j}) = U^\pm_{i,j-k}$$

where $\tau^\pm$ is the translation automorphism introduced in Section 4 and that

$$\pi^\pm_{i,j} = \pi^\pm_{i,j-k} \circ (\tau^\pm)^k.$$

It is also obvious that

$$U^+_{i,j} = \alpha_i(U^-_{i,j+1})$$

where $\alpha_i : X^- \overset{\sim}{\longrightarrow} X^+$ is the isomorphism introduced in Remark 3.13 and that

$$\pi^+_{i,j} \circ \alpha_i = \pi^-_{i,j+1}.$$

Moreover we have

$$F_{X^\pm}(U^\pm_{1,j}) = U^\pm_{2,j} \quad \text{and} \quad F_{X^\pm}(U^\pm_{2,j}) = U^\pm_{1,j}$$

and if we simply denote by F_U the k'-morphisms $U^\pm_{1,j} \to U^\pm_{2,j}$ and $U^\pm_{2,j} \to U^\pm_{1,j}$ which are induced by $F_{X^\pm}$, the squares of k'-scheme morphisms

$$
\begin{array}{ccc}
U^\pm_{1,j} & \overset{F_U}{\longrightarrow} & U^\pm_{2,j} \\
\pi^+_{1,j} \downarrow & & \downarrow \pi^+_{2,j} \\
Y_1 \times_{k'} Y_2 & \underset{F_{Y_1} \times F_{Y_2}}{\longrightarrow} & Y_1 \times_{k'} Y_2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
U^\pm_{2,j} & \overset{F_U}{\longrightarrow} & U^\pm_{1,j} \\
\pi^+_{2,j} \downarrow & & \downarrow \pi^+_{1,j} \\
Y_1 \times_{k'} Y_2 & \underset{F_{Y_1} \times F_{Y_2}}{\longrightarrow} & Y_1 \times_{k'} Y_2
\end{array}
$$

commute.

6. The vector bundle structure

THEOREM 6.1. — *For $i = 1, 2$ and each $j \in \mathbb{Z}$ the morphism*

$$\pi_{i,j}^{\pm} : U_{i,j}^{\pm} \to Y_1 \times_{k'} Y_2$$

is a rank r vector bundle.

Proof : First of all it is not difficult to see that $\pi_{i,j}^{\pm}$ has a natural structure of generalized vector bundle structure in the sense of Grothendieck, the fiber of which through a K-rational point $L^{\pm} \in X^{\pm}$ is the vector space

$$\mathrm{Hom}_{\mathcal{O}_{F'}, \gamma}(C_2^{\pm}, (K \otimes_{k'} E_1')/B_1^{\pm})$$

of $\mathcal{O}_{F'}$-linear maps $f_1^{\pm} : C_2^{\pm} \to (K \otimes_{k'} E_1')/B_1^{\pm}$ which exchange the automorphisms induced by the multiplications by γ_2 and γ_1 if $i = 1$, and the vector space

$$\mathrm{Hom}_{\mathcal{O}_{F'}, \gamma}(C_1^{\pm}, (K \otimes_{k'} E_2')/B_2^{\pm})$$

of $\mathcal{O}_{F'}$-linear maps $f_2^{\pm} : C_1^{\pm} \to (K \otimes_{k'} E_2')/B_2^{\pm}$ which exchange the automorphisms induced by the multiplications by γ_1 and γ_2 if $i = 2$.

Therefore it suffices to show that the rank of this generalized vector bundle is constant of value r. But this is a direct consequence of the following proposition. □

PROPOSITION 6.2. — *For $i = 1, 2$ let M_i be a $(K \otimes_{k'} \mathcal{O}_{F'})$-lattice in E_i' where K is some field extension of k'.*

We consider $\mathcal{O}_{F'}[\gamma_1]$ and $\mathcal{O}_{F'}[\gamma_2]$ as quotients of the 2-dimensional regular local ring $\mathcal{O}_{F'}[[T]]$ by sending T onto γ_1 and γ_2 respectively.

Then the dimension of the K-vector space

$$\mathrm{Hom}_{K \otimes_{k'} \mathcal{O}_{F'}[[T]]}(M_2, (K \otimes_{k'} E_1')/M_1)$$

is equal to r.

Proof : For simplicity we will only consider the case $K = k'$, the proof for K arbitrary being the same. In order to prove the proposition it is sufficient to prove that the complex

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, E_1'/M_1)$$

is concentrated in degree 0 and that its Euler-Poincaré characteristic is equal to r.

(a) We have a distinguished triangle

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1) \to R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, E_1') \to R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, E_1'/M_1) \to$$

in the derived category of k'-vector spaces. Moreover $R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, E_1')$ is zero in this derived category as the endomorphism

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(P_2(\gamma_2), E_1') = R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, P_2(\gamma_1))$$

of this complex is at the same time zero and an isomorphism. We thus have an isomorphism

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, E_1'/M_1) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1)[1]$$

in the derived category.

The complex

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1)$$

is concentrated in degrees 1 and 2 as $\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1)$ is obviously zero and as $\mathcal{O}_{F'}[[T]]$ is a regular local ring of dimension 2. To prove the proposition it is thus sufficient to prove that

$$\mathrm{Ext}^2_{\mathcal{O}_{F'}[[T]]}(M_2, M_1) = (0)$$

and that the Euler-Poincaré characteristic of $R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1)$ is equal to $-r$.

(b) Let us take a more geometric point of view. Let S be the germ of surface $\mathrm{Spec}(\mathcal{O}_{F'}[[T]])$ and, for $\alpha = 1, 2$ let

$$\iota_\alpha : C_\alpha = \mathrm{Spec}(\mathcal{O}_{F'}[\gamma_\alpha]) \hookrightarrow S$$

be the germ of curve with equation $P_\alpha(T) = 0$. By hypothesis these two curves are integral and the support of their intersection is the origin $s = (\varpi_F = 0, T = 0)$ of S. For $\alpha = 1, 2$ the $\mathcal{O}_{F'}[\gamma_\alpha]$-module M_α defines a torsion free coherent $\mathcal{O}_{C_\alpha}$-module of generic rank 1 that we will also denote by M_α. Clearly we have

$$R\mathrm{Hom}_{\mathcal{O}_{F'}[[T]]}(M_2, M_1) = R\mathrm{Hom}_{\mathcal{O}_S}(\iota_{2,*}M_2, \iota_{1,*}M_1)$$

and the support of $R\mathcal{H}om_{\mathcal{O}_S}(\iota_{2,*}M_2, \iota_{1,*}M_1)$ is $\{s\}$.

(c) Let us now show that

$$\mathrm{Ext}^2_{\mathcal{O}_S}(\iota_{2,*}M_2, \iota_{1,*}M_1) = (0).$$

Let $\widetilde{M_2}$ be the saturated module

$$\widetilde{M_2} = \mathcal{O}_{E_2'}M_2 \subset \mathcal{O}_{E_2'}$$

From the geometric point of view $\widetilde{M_2}$ is nothing else than the free $\mathcal{O}_{\widetilde{C}_2}$-module of rank one $\pi_2^* M_2$ where $\pi_2 : \widetilde{C}_2 \to C_2$ is the normalization of the curve C_2. We have an injective map of $\mathcal{O}_S$-modules

$$\iota_{2,*} M_2 \hookrightarrow \iota_{2,*} \pi_{2,*} \widetilde{M_2}$$

and then a surjective map of k'-vector spaces

$$\mathrm{Ext}^2_{\mathcal{O}_S}(\iota_{2,*} \pi_{2,*} \widetilde{M_2}, \iota_{1,*} M_1) \twoheadrightarrow \mathrm{Ext}^2_{\mathcal{O}_S}(\iota_{2,*} M_2, \iota_{1,*} M_1)$$

(all the $\mathrm{Ext}^3_{\mathcal{O}_S}$'s are zero as S is a regular surface). It is thus sufficient to prove that

$$\mathrm{Ext}^2_{\mathcal{O}_S}(\iota_{2,*} \pi_{2,*} \widetilde{M_2}, \iota_{1,*} M_1) = (0).$$

But, by Grothendieck duality (cf. [Ha]) this last Ext group is isomorphic to

$$\mathrm{Ext}^2_{\mathcal{O}_{\widetilde{C}_2}}(\widetilde{M_2}, L(\iota_2 \circ \pi_2)^! \iota_{1,*} M_1)$$

and as $\widetilde{M_2}$ is a free $\widetilde{\mathcal{O}}_{\widetilde{C}_2}$-module of rank 1 it is isomorphic to

$$\mathcal{H}^2 L(\iota_2 \circ \pi_2)^! \iota_{1,*} M_1.$$

As S is a regular surface and C_2 is a Cartier divisor on S, the dualizing complexes K_S and $K_{\widetilde{C}_1}$ are of the form $\omega_S[2]$ and $\omega_{\widetilde{C}_2}[1]$ where ω_S and $\omega_{\widetilde{C}_2}$ are invertible modules and $L(\iota_2 \circ \pi_2)^! \iota_{1,*} M_1$ is isomorphic to $L(\iota_2 \circ \pi_2)^* \iota_{1,*} M_1[-1]$. As this last complex is obviously concentrated in degrees ≤ 1 the assertion follows.

(d) Finally, thanks to a result of Deligne (see Proposition 6.3 and its corollary below) it follows that the Euler-Poincaré characteristic of $R\mathrm{Hom}_{\mathcal{O}_S}(\iota_{2,*} M_2, \iota_{1,*} M_1)$ is equal to

$$(-1)^{\dim(C_1)} \mathrm{mult}(C_2, C_1)$$

where $\mathrm{mult}(C_2, C_1)$ is the intersection multiplicity of C_2 and C_1 on S. By definition this intersection multiplicity is equal to r. This concludes the proof of Proposition 6.2.   □

PROPOSITION 6.3 (Deligne). — *Let (X, x) be a germ of smooth variety over a field k. For simplicity we assume that x is rational over k. Let $\iota_1 : Y_1 \hookrightarrow X$ and $\iota_2 : Y_2 \hookrightarrow X$ be two integral closed subschemes of X such that $\dim(Y_1) + \dim(Y_2) = \dim(X)$ and that $(Y_1 \cap Y_2)_{\mathrm{red}} = \{x\}$. Let K_1 and K_2 be two bounded complexes of flat quasi-coherent $\mathcal{O}_X$-modules. We assume that all the cohomology sheaves of K_1 (resp. K_2) are coherent $\mathcal{O}_X$-modules with support in Y_1 (resp. Y_2).*

Then $\Gamma(X, K_1 \otimes_{\mathcal{O}_X} K_2)$ is a bounded complex of k-vector spaces with finite dimensional cohomology and its Euler-Poincaré characteristic

$$\sum_n (-1)^n \dim_k \Gamma(X, \mathcal{H}^n(K_1 \otimes_{\mathcal{O}_X} K_2))$$

is equal to

$$m(Y_1, Y_2)\mathrm{rk}_{\eta_1}(K_1)\mathrm{rk}_{\eta_2}(K_2)$$

where $m(Y_1, Y_2)$ is the intersection multiplicity of Y_1 and Y_2, i.e.

$$m(Y_1, Y_2) = \sum_n (-1)^n \dim_k \mathrm{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y_1,x}, \mathcal{O}_{Y_2,x}),$$

and where, for $\alpha = 1, 2$

$$\mathrm{rk}_{\eta_\alpha}(K_\alpha) = \sum_n (-1)^n \dim_{\kappa(\eta_\alpha)} \mathcal{H}^n(K_\alpha)_{\eta_\alpha}$$

is the rank of K_α at the generic point η_α of Y_α.

Proof: See [SGA4$\frac{1}{2}$] [Cycle] Thm. 2.3.8 (iii). Deligne's argument can be easily modified to avoid the use of ℓ-adic cohomology. $\square$

COROLLARY 6.4. — *Let us assume moreover that Y_2 is a complete intersection in X. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two coherent modules on Y_1 and Y_2 respectively with generic rank r_1 and r_2. Then the complex of k-vector spaces*

$$R\mathrm{Hom}_{\mathcal{O}_X}(\iota_{2,*}\mathcal{M}_2, \iota_{1,*}\mathcal{M}_1)$$

has bounded and finite dimensional cohomology and its Euler-Poincaré characteristic is equal to

$$(-1)^{\dim(Y_1)} m(Y_2, Y_1) r_1 r_2.$$

Proof : For $\alpha = 1, 2$ let

$$K_\alpha \to \iota_{\alpha,*}\mathcal{M}_\alpha \to 0$$

be a finite resolution of the $\mathcal{O}_X$-module $\iota_{\alpha,*}\mathcal{M}_\alpha$ by free $\mathcal{O}_X$-modules of finite rank. Obviously the complexes K_1 and K_2 satisfy the hypotheses of the above proposition and $\mathrm{rk}_{\eta_\alpha}(K_\alpha) = r_\alpha$.

In the derived category of k-vector spaces the complex

$$R\mathrm{Hom}_{\mathcal{O}_X}(\iota_{2,*}\mathcal{M}_2, \iota_{1,*}\mathcal{M}_1)$$

is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_X}(K_2, K_1) = \Gamma(X, K_2^\vee \otimes_{\mathcal{O}_X} K_1)$$

where $K_2^\vee = \mathrm{Hom}_{\mathcal{O}_X}(K_2, \mathcal{O}_X)$ is also a bounded complex of free $\mathcal{O}_X$-modules of finite rank. As X is local and smooth over k, $\mathcal{O}_X[\dim(X)]$ is a dualizing complex for X and as Y_2 is a complete intersection in X, $\mathcal{O}_{Y_2}[\dim(Y_2)]$ is a dualizing complex for Y_2. Therefore, by Grothendieck duality we have

$$R\mathcal{H}om_{\mathcal{O}_X}(\iota_{2,*}\mathcal{M}_2, \mathcal{O}_X[\dim(X)]) = \iota_{2,*}R\mathcal{H}om_{\mathcal{O}_{Y_2}}(\mathcal{M}_2, \mathcal{O}_{Y_2}[\dim(Y_2)])$$

and thus $K_2^\vee$ is isomorphic to $\iota_{2,*}R\mathcal{H}om_{\mathcal{O}_{Y_2}}(\mathcal{M}_2, \mathcal{O}_{Y_2})[-\dim(Y_1)]$ in the derived category of $\mathcal{O}_X$-modules. In particular

$$\mathrm{rk}_{\eta_2}(K_2) = (-1)^{-\dim(Y_1)} r_2.$$

Now the corollary is an immediate consequence of the proposition. $\square$

7. Proof of the main theorem

Let us define two closed embeddings

(7.1)
$$i_1, i_2 : X' \hookrightarrow X$$

by

$$i_1(L^-) = \alpha_2(L^-) = (1 \oplus \varpi_{E_2})L^-$$

and

$$i_2(L^-) = \alpha_1(L^-) = (\varpi_{E_1} \oplus 1)L^-$$

if r is even and by

$$i_1(L^+) = \alpha_1^{-1}(L^+) = (\varpi_{E_1}^{-1} \oplus 1)L^+$$

and

$$i_2(L^+) = \alpha_2^{-1}(L^+) = (1 \oplus \varpi_{E_2}^{-1})L^+$$

if r is odd.

By definition we have

$$U_1 := X - i_1(X') = U_{1,-r'}^+ \subset X \subset X^+$$

and

$$U_2 := X - i_2(X') = U_{2,-r'}^+ \subset X \subset X^+$$

if r is even and

$$U_1 := X - i_1(X') = U_{1,-r'}^- \subset X \subset X^-$$

and

$$U_2 := X - i_2(X') = U_{2,-r'}^- \subset X \subset X^-$$

if r is odd. Therefore we have projections

(7.2)
$$\pi_1 : U_1 \to Y_1 \times_{k'} Y_2 \text{ and } \pi_2 : U_2 \to Y_1 \times_{k'} Y_2$$

which are both rank r vector bundles by Theorem 6.1 ($\pi_i = \pi_{i,-r'}^+$ if r is even and $\pi_i = \pi_{i,-r'}^-$ if r is odd).

Theorem 4.2 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Part III

8. Descent from k' to k

Up to this point our approach to the fundamental lemma was essentially elementary in that it was based on counting points of algebraic varieties. However to treat the extensions of odd degree of k and thus the original statement of Langlands and Shelstad we envisage the use of ℓ-adic cohomology and the Grothendieck fixed point formula (cf. [Gr]). Indeed whereas all the structures considered so far are only defined over k' we predict that their ℓ-adic cohomology descends to k. Even though we cannot prove this assertion we will now explain more precisely its meaning.

In §3 we have introduce the projective k'-schemes Y_1 and Y_2 together with the Frobenius endomorphisms F_{Y_1} and F_{Y_2} relative to k. We set

$$(Y, F_Y) = (Y_1, F_{Y_1}) \times_{k'} (Y_2, F_{Y_2}).$$

In the statement of Theorem 4.2 we have introduced the projective k'-schemes X and X' together with the Frobenius endomorphisms F_X and $F_{X'}$ relative to k. In the course of the proof of Theorem 4.2 we have introduced the closed embeddings $i_1 : X' \hookrightarrow X$ and $i_2 : X' \hookrightarrow X$ and the projections $\pi_1 : U_1 := X - i_1(X') \hookrightarrow Y$ and $\pi_2 : U_2 := X - i_2(X') \hookrightarrow Y$. We have shown that π_1 and π_2 have a natural structure of rank r vector bundle. Let us denote by

$$j_1 : U_1 \hookrightarrow X \quad \text{and} \quad j_2 : U_2 \hookrightarrow X$$

the obvious open embeddings.
Then the diagrams

(8.1)

$$
\begin{array}{ccccc}
X' & \overset{i_1}{\hookrightarrow} & X & \overset{j_1}{\longleftarrow} & U_1 \\
& & & & \downarrow \pi_1 \\
& & & & Y
\end{array}
$$

$$
\begin{array}{ccccc}
X' & \overset{i_2}{\hookrightarrow} & X & \overset{j_2}{\longleftarrow} & U_2 \\
& & & & \downarrow \pi_2 \\
& & & & Y
\end{array}
$$

are exchanged by the Frobenius endomorphisms $F_{X'}$, F_X and F_Y : there are unique k'-scheme morphisms $F_U : U_1 \longrightarrow U_2$ and $F_U : U_2 \longrightarrow U_1$ such that the diagrams

$$
\begin{array}{ccccccc}
X' & \xrightarrow{i_1} & X & \xleftarrow{j_1} & U_1 & \xrightarrow{\pi_1} & Y \\
F_{X'}\downarrow & & \downarrow F_X & & \downarrow F_U & & \downarrow F_Y \\
X' & \xrightarrow{i_2} & X & \xleftarrow{j_2} & U_2 & \xrightarrow{\pi_2} & Y
\end{array}
$$

(8.2)

$$
\begin{array}{ccccccc}
X' & \xrightarrow{i_2} & X & \xleftarrow{j_2} & U_2 & \xrightarrow{\pi_2} & Y \\
F_{X'}\downarrow & & \downarrow F_X & & \downarrow F_U & & \downarrow F_Y \\
X' & \xrightarrow{i_1} & X & \xleftarrow{j_1} & U_1 & \xrightarrow{\pi_1} & Y
\end{array}
$$

are commutative.

Let us fix a prime number ℓ which is not equal to the characteristic of k. For any quasi-projective (resp. projective) variety Z over k' we set

$$
R\Gamma_{\mathrm{c}}(Z) = R\Gamma_{\mathrm{c}}(\overline{k} \otimes_{k'} Z, \mathbb{Q}_\ell)
$$

(resp.

$$
R\Gamma(Z) = R\Gamma(\overline{k} \otimes_{k'} Z, \mathbb{Q}_\ell)).
$$

We consider the following morphisms of distinguished triangles in the derived category $D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Spec}(k'), \mathbb{Q}_\ell)$

$$
\begin{array}{ccccccc}
R\Gamma_{\mathrm{c}}(U_2) & \xrightarrow{j_{2,!}} & R\Gamma(X) & \xrightarrow{i_2^*} & R\Gamma(X') & \xrightarrow{\partial_2} & R\Gamma_{\mathrm{c}}(U_2)[1] \\
F_U^*\downarrow & & F_X^*\downarrow & & \downarrow F_{X'}^* & & \downarrow F_U^*[1] \\
R\Gamma_{\mathrm{c}}(U_1) & \xrightarrow{j_{1,!}} & R\Gamma(X) & \xrightarrow{i_1^*} & R\Gamma(X') & \xrightarrow{\partial_1} & R\Gamma_{\mathrm{c}}(U_1)[1]
\end{array}
$$

and

$$
\begin{array}{ccccccc}
R\Gamma_{\mathrm{c}}(U_1) & \xrightarrow{j_{1,!}} & R\Gamma(X) & \xrightarrow{i_1^*} & R\Gamma(X') & \xrightarrow{\partial_1} & R\Gamma_{\mathrm{c}}(U_1)[1] \\
F_U^*\downarrow & & F_X^*\downarrow & & \downarrow F_{X'}^* & & \downarrow F_U^*[1] \\
R\Gamma_{\mathrm{c}}(U_2) & \xrightarrow{j_{2,!}} & R\Gamma(X) & \xrightarrow{i_2^*} & R\Gamma(X') & \xrightarrow{\partial_2} & R\Gamma_{\mathrm{c}}(U_2)[1]
\end{array}
$$

(cf. [SGA4] Exp. XVII, 5.1.16).

We also consider the morphisms

$$\pi_1^! : R\Gamma(Y)[-2r](-r) \longrightarrow R\Gamma_c(U_1)$$

and

$$\pi_2^! : R\Gamma(Y)[-2r](-r) \longrightarrow R\Gamma_c(U_2)$$

in that derived category which are induced by the trace morphisms

$$\mathrm{tr}_1 : R\pi_{1,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[-2r](-r) \quad \text{and} \quad \mathrm{tr}_2 : R\pi_{2,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[-2r](-r)$$

(cf. [SGA 4] Exp. XVIII, Thm. 2.9).

PROPOSITION 8.3. — *The morphisms $\pi_1^!$ and $\pi_2^!$ are both isomorphisms and the squares in $D_c^b(\mathrm{Spec}(k'), \mathbb{Q}_\ell)$ which are induced by the right squares of the diagrams (8.2),*

$$
\begin{array}{ccc}
R\Gamma(Y)[-2r](-r) & \xrightarrow{\ \pi_2^!\ } & R\Gamma_c(U_2) \\
{\scriptstyle F_Y^*[-2r](-r)}\downarrow & & \downarrow{\scriptstyle F_U^*} \\
R\Gamma(Y)[-2r](-r) & \xrightarrow[\ \pi_1^!\ ]{} & R\Gamma_c(U_1)
\end{array}
$$

and

$$
\begin{array}{ccc}
R\Gamma(Y)[-2r](-r) & \xrightarrow{\ \pi_1^!\ } & R\Gamma_c(U_1) \\
{\scriptstyle F_Y^*[-2r](-r)}\downarrow & & \downarrow{\scriptstyle F_U^*} \\
R\Gamma(Y)[-2r](-r) & \xrightarrow[\ \pi_2^!\ ]{} & R\Gamma_c(U_2)
\end{array}
$$

are commutative.

Proof : As π_1 and π_2 are both rank r vector bundles (cf. Theorem 6.1) the trace morphisms tr_1 and tr_2, and therefore the induced morphisms $\pi_1^!$ and $\pi_2^!$, are all isomorphisms (cf. [SGA 4] Exp. XVIII, Thm. 2.9).

The commutative right squares of the diagrams (8.2) induce isomorphisms

$$f_1 : F_Y^* R\pi_{2,!}\mathbb{Q}_\ell \xrightarrow{\sim} R\pi_{1,!}\mathbb{Q}_\ell$$

and

$$f_2 : F_Y^* R\pi_{1,!}\mathbb{Q}_\ell \xrightarrow{\sim} R\pi_{2,!}\mathbb{Q}_\ell$$

in $D_c^b(Y, \mathbb{Q}_\ell)$. The composed isomorphisms $f_1 \circ F_Y^*(f_2)$ and $f_2 \circ F_Y^*(f_1)$ are the natural lifts of the Frobenius endomorphisms of Y with respect to k'. But, as the trace

morphisms $\mathrm{tr}_1 : R\pi_{1,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[-2r](-r)$ and $\mathrm{tr}_2 : R\pi_{2,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[-2r](-r)$ are both isomorphisms, we may view f_1 and f_2 as automorphisms of $\mathbb{Q}_\ell[-2r](-r)$. It follows that f_1 and f_2 are induced by the multiplication on the constant sheaf $\mathbb{Q}_\ell$ by locally constant functions $\varphi_1 : Y \to \mathbb{Q}_\ell$ and $\varphi_2 : Y \to \mathbb{Q}_\ell$ such that the products $\varphi_1 F_Y^*(\varphi_2)$ and $\varphi_2 F_Y^*(\varphi_1)$ are both the constant function with value q^{2r}.

To conclude the proof of the proposition it is sufficient to check that the functions φ_1 and φ_2 are both constant with value q^r. But this a direct consequence of the base change theorem in etale cohomology ([SGA 4] Exp. XVII, Thm. 5.2.6) and the next lemma. $\square$

LEMMA 8.4. — *Let E_1 and E_2 be two vector spaces of the same finite dimension e over the finite field k'. Let $F : E_1 \to E_2$ be a σ-linear bijective map. Let us view E_1 and E_2 as affine k'-schemes and F as a finite k'-scheme morphism. Then we have*

$$\mathrm{tr}_1 \circ F^* = q^e \mathrm{tr}_2$$

where $\mathrm{tr}_i : H_c^{2e}(\overline{k} \otimes_{k'} E_i, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathbb{Q}_\ell(-e)$ is the trace morphism and

$$F^* : H_c^{2e}(\overline{k} \otimes_{k'} E_2, \mathbb{Q}_\ell) \to H_c^{2e}(\overline{k} \otimes_{k'} E_1, \mathbb{Q}_\ell)$$

is induced by F.

Proof : The degree of the k'-scheme morphism $F : E_1 \to E_2$ is q^e. $\square$

It is now clear that the Langlands-Shelstad conjecture, and more generally the equality

$$\left| X^{F_X^f} \right| - \left| X'^{F_{X'}^f} \right| = q^{fr} \cdot \left| Y_1^{F_{Y_1}^f} \right| \cdot \left| Y_2^{F_{Y_1}^f} \right|$$

for every positive odd integer f, would be an immediate consequence of the Grothendieck fixed point formula (cf. [Gr]) and the following conjecture :

CONJECTURE 8.5. — *The two distinguished triangles*

$$R\Gamma(Y)[-2r](-r) \xrightarrow{(\pi_1^!)^{-1} \circ j_{1,!}} R\Gamma(X) \xrightarrow{i_1^*} R\Gamma(X') \xrightarrow{\partial_1} R\Gamma(Y)[1-2r](-r)$$

and

$$R\Gamma(Y)[-2r](-r) \xrightarrow{(\pi_2^!)^{-1} \circ j_{2,!}} R\Gamma(X) \xrightarrow{i_2^*} R\Gamma(X') \xrightarrow{\partial_2} R\Gamma(Y)[1-2r](-r)$$

are identical.

To conclude this section we will discuss a possible homotopy argument for proving the equality of the restriction maps i_1^* and i_2^*.

The k'-schemes X, X' and Y_i are naturally embedded into Grassmann varieties $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{H}_i$.

More precisely, let $\mathcal{G}$ be the Grassmann variety of $(m_1 + m_2 + r)$-planes in the $2(m_1 + m_2 + r)$-dimensional k'-vector space $V_1 \oplus V_2$ where we have set

$$
V_i = \begin{cases} \varpi_{E_i}^{-m_i - r'} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i + r'} \mathcal{O}_{E_i'} & \text{if } r = 2r' \text{ is even,} \\[2mm] \varpi_{E_i}^{-m_i - r' - 1} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i + r'} \mathcal{O}_{E_i'} & \text{if } r = 2r' + 1 \text{ is odd,} \end{cases}
$$

let $\mathcal{G}'$ be the Grassmann variety of $(m_1 + m_2 + r - 1)$-planes in the $2(m_1 + m_2 + r - 1)$-dimensional k'-vector space $V_1' \oplus V_2'$ where we have set

$$
V_i' = \begin{cases} \varpi_{E_i}^{-m_i - r'} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i + r' - 1} \mathcal{O}_{E_i'} & \text{if } r = 2r' \text{ is even,} \\[2mm] \varpi_{E_i}^{-m_i - r'} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i + r'} \mathcal{O}_{E_i'} & \text{if } r = 2r' + 1 \text{ is odd,} \end{cases}
$$

and let $\mathcal{H}_i$ be the Grassmann variety of m_i-planes in the $2m_i$-dimensional k'-vector space

$$
W_i = \varpi_{E_i}^{-m_i} \mathcal{O}_{E_i'} / \varpi_{E_i}^{m_i} \mathcal{O}_{E_i'}.
$$

The multiplications by ϖ_F and γ_i induce an endomorphism ν_i and an automorphism u_i of V_i (resp. V_i', resp. W_i). Then, if we set

$$
\nu = \nu_1 \oplus \nu_2 \quad \text{and} \quad u = u_1 \oplus u_2
$$

we have the obvious identifications :

$$
X = \{L \in \mathcal{G} \mid \nu(L) \subset L, \ u(L) = L \text{ and } b_1 \geq m_1, b_2 \geq m_2\}
$$

where $b_1 = \dim(L \cap (V_1 \oplus (0)))$ and $b_2 = \dim(L \cap ((0) \oplus V_2))$,

$$
X' = \{L' \in \mathcal{G}' \mid \nu(L') \subset L', \ u(L') = L' \text{ and } b_1' \geq m_1, b_2' \geq m_2\}
$$

where $b_1' = \dim(L' \cap (V_1' \oplus (0)))$ and $b_2' = \dim(L' \cap ((0) \oplus V_2'))$, and

$$
Y_i = \{M_i \in \mathcal{H}_i \mid \nu_i(M_i) \subset M_i \text{ and } u_i(M_i) = M_i\}.
$$

CONJECTURE 8.6. — *The restriction maps*

$$
R\Gamma(\mathcal{G}) \to R\Gamma(X),
$$

$$
R\Gamma(\mathcal{G}') \to R\Gamma(X')
$$

and

$$
R\Gamma(\mathcal{H}_i) \to R\Gamma(Y_i)
$$

induce epimorphisms on the cohomology.

Remark 8.7 : It follows from Conjecture 8.6 that the cohomology complexes $R\Gamma(X)$, $R\Gamma(X)$ and $R\Gamma(X')$ should have all their odd cohomology groups equal to (0) and that, for every even integer n all the eigenvalues of Frobenius acting on the n-th cohomology group should be equal to $q^{\frac{n}{2}}$.

In particular the two boundary maps $\partial_1, \partial_2 : R\Gamma(X') \to R\Gamma(Y)[1-2r](-r)$ should be zero and thus equal.

For each flag

$$F = ((0) \subset F^2 \subset F^1 \subset V_1 \oplus V_2)$$

of vector subspaces of $V_1 \oplus V_2$ with F^1 of codimension 1 and F^2 of dimension 1 and for each isomorphism ι from the vector space $V_1' \oplus V_2'$ onto the vector space F^1/F^2, we have an obvious closed embedding

$$i_{F,\iota} : \mathcal{G}' \hookrightarrow \mathcal{G}.$$

As the k'-scheme of pairs (F, ι) is connected, the restriction maps

$$i_{F,\iota}^* : R\Gamma(\mathcal{G}) \to R\Gamma(\mathcal{G}')$$

are all equal. We will simply denote by i^* the common value of these restriction maps.

Let us denote by N_i the regular nilpotent endomorphism of V_i which is induced by the multiplication by ϖ_{E_i}. By definition we have

$$V_i' = \begin{cases} \mathrm{Coker}(N_i) \longleftarrow V_i & \text{if } r \text{ is even} \\ \mathrm{Im}(N_i) \longhookrightarrow V_i & \text{if } r \text{ is odd} \end{cases}$$

Moreover the endomorphism N_i of V_i induces an isomorphism of $\mathrm{Coker}(N_i)$ onto $\mathrm{Im}(N_i)$ and we may identify these two subquotients of V_i. If

$$F = ((0) \subset \mathrm{Ker}(N_1) \oplus (0) \subset V_1 \oplus \mathrm{Im}(N_2) \subset V_1 \oplus V_2)$$

and if ι is the isomorphism

$$V_1' \oplus V_2' \xrightarrow{\sim} (V_1 \oplus \mathrm{Im}(N_2))/(\mathrm{Ker}(N_1) \oplus (0))$$

which is induced by the above identifications, the closed embedding $i_{F,\iota}$ maps X' into X and extends the closed embedding $i_1 : X' \hookrightarrow X$. Similarly, if

$$F = ((0) \subset (0) \oplus \mathrm{Ker}(N_2) \subset \mathrm{Im}(N_1) \oplus V_2 \subset V_1 \oplus V_2)$$

and ι is the isomorphism

$$V_1' \oplus V_2' \xrightarrow{\sim} (\mathrm{Im}(N_1) \oplus V_2)/((0) \oplus \mathrm{Ker}(N_2))$$

which is induced by the above identifications, the closed embedding $i_{F,\iota}$ maps X' into X and extends the closed embedding $i_2 : X' \hookrightarrow X$. In particular, for $i = 1, 2$, we have a commutative diagram

$$
\begin{array}{ccc}
R\Gamma(\mathcal{G}) & \xrightarrow{i^*} & R\Gamma(\mathcal{G}') \\
\downarrow & & \downarrow \\
R\Gamma(X) & \xrightarrow[i_i^*]{} & R\Gamma(X')
\end{array}
$$

and the equality $i_1^* = i_2^*$ follows from Conjecture 8.6.

9. $U(1,1)$

In this section we assume that $n_1 = n_2 = 1$. Replacing (γ_1, γ_2) by $(1, \gamma_1^{-1}\gamma_2)$ we may also assume that $\gamma_1 = 1$. Then $\gamma_2 - 1$ is of valuation r in F'.

Let K be a field extension of k'. If $L^{\pm}$ is a $(K \otimes_{k'} \mathcal{O}_{F'})$-lattice in $F' \oplus F'$ of index $\delta^{\pm}$ ($\delta^+ = 0$ and $\delta^- = 1$) we necessarily have

$$
B_i^{\pm} = \varpi_F^{-b_i^{\pm}}(K \otimes_{k'} \mathcal{O}_{F'}) \subset C_i^{\pm} = \varpi_F^{-c_i^{\pm}}(K \otimes_{k'} \mathcal{O}_{F'}) \subset K \otimes_{k'} F'
$$

for some integers $b_i^{\pm} \leq c_i^{\pm}$ such that

$$
b_1^{\pm} + c_2^{\pm} = b_2^{\pm} + c_1^{\pm} = \delta^{\pm}.
$$

Therefore the conditions

$$
(1 \otimes (\gamma_1 \oplus \gamma_2))L^{\pm} = L^{\pm}
$$

and

$$
c_1^{\pm} - b_1^{\pm} = c_2^{\pm} - b_2^{\pm} \leq r
$$

are equivalent.

If $r = 2r'$ is even it follows that

$$
X = X^+[-r', -r']
$$

is simply the k'-scheme of $\mathcal{O}_{F'}$-lattices L^+ in $F' \oplus F'$ satisfying the conditions

$$
\begin{cases}
\varpi_F^{r'}\mathcal{O}_{F'} \oplus \varpi_F^{r'}\mathcal{O}_{F'} \subset L^+ \subset \varpi_F^{-r'}\mathcal{O}_{F'} \oplus \varpi_F^{-r'}I_{F'} \\
\mathrm{ind}(L^+) = 0,
\end{cases}
$$

and that

$$
X' = X^-[1 - r', 1 - r']
$$

is simply the k'-scheme of $\mathcal{O}_{F'}$-lattices L^- in $F' \oplus F'$ satisfying the conditions

$$\begin{cases} \varpi_F^{r'-1}\mathcal{O}_{F'} \oplus \varpi_F^{r'-1}\mathcal{O}_{F'} \subset L^- \subset \varpi_F^{-r'}\mathcal{O}_{F'} \oplus \varpi_F^{-r'}\prime_{F'} \\ \mathrm{ind}(L^-) = 1. \end{cases}$$

Similarly, if $r = 2r'+1$ is odd it follows that

$$X = X^-[-r', -r']$$

is simply the k'-scheme of $\mathcal{O}_{F'}$-lattices L^- in $F' \oplus F'$ satisfying the conditions

$$\begin{cases} \varpi_F^{r'}\mathcal{O}_{F'} \oplus \varpi_F^{r'}\mathcal{O}_{F'} \subset L^+ \subset \varpi_F^{-r'-1}\mathcal{O}_{F'} \oplus \varpi_F^{-r'-1}\prime_{F'} \\ \mathrm{ind}(L^-) = 1, \end{cases}$$

and that

$$X' = X^+[-r', -r']$$

is simply the k'-scheme of $\mathcal{O}_{F'}$-lattices L^+ in $F' \oplus F'$ satisfying the conditions

$$\begin{cases} \varpi_F^{r'}\mathcal{O}_{F'} \oplus \varpi_F^{r'}\mathcal{O}_{F'} \subset L^+ \subset \varpi_F^{-r'}\mathcal{O}_{F'} \oplus \varpi_F^{-r'}\prime_{F'} \\ \mathrm{ind}(L^+) = 0. \end{cases}$$

If $r = 2r'$ is even (resp. $r = 2r'+1$ is odd) let us denote by V the r-dimensional k'-vector space

$$\varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'}$$

(resp.

$$\varpi_F^{-r'-1}\mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'})$$

and by N the nilpotent endomorphism of V which is induced by the multiplication by ϖ_F. Then X may be identified with the k'-scheme of k'-vector subspaces

$$A \subset V \oplus V$$

satisfying the conditions

$$\begin{cases} (N \oplus N)(A) \subset A \\ \dim(A) = r. \end{cases}$$

Similarly, if $r = 2r'$ is even (resp. $r = 2r'+1$ is odd) let us denote by V' the $(r-1)$-dimensional k'-vector space

$$\varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'-1}\mathcal{O}_{F'}$$

(resp.

$$\varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'})$$

and by N' the nilpotent endomorphism of V' which is induced by the multiplication by ϖ_F. Then X' may be identified with the k'-scheme of k'-vector subspaces

$$A' \subset V' \oplus V'$$

satisfying the conditions

$$\begin{cases} (N' \oplus N')(A') \subset A' \\ \dim(A) = r - 1. \end{cases}$$

Moreover under these identifications the closed embeddings i_1 and i_2 defined in (7.1) may be described in the following way. Let us consider the diagram

$$
\begin{array}{ccc}
 & V \oplus V & \\
 \nearrow^{1 \oplus e} & & \nwarrow^{e \oplus 1} \\
V \oplus V' & & V' \oplus V \\
 \searrow_{p \oplus 1} & & \swarrow_{1 \oplus p} \\
 & V' \oplus V' &
\end{array}
$$

where $e : V' \hookrightarrow V$ is the embedding

$$\varpi_F^{-r'} \mathcal{O}_{F'}/\varpi_F^{r'-1}\mathcal{O}_{F'} \hookrightarrow \varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'}$$

which is induced by the multiplication by ϖ_F (resp. the canonical embedding

$$\varpi_F^{-r'} \mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'} \hookrightarrow \varpi_F^{-r'-1}\mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'})$$

and where $p : V \longrightarrow\!\!\!\!\!\rightarrow V'$ is the canonical projection

$$\varpi_F^{-r'} \mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'} \longrightarrow\!\!\!\!\!\rightarrow \varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'-1}\mathcal{O}_{F'}$$

(resp. the projection

$$\varpi_F^{-r'} \mathcal{O}_{F'}/\varpi_F^{r'}\mathcal{O}_{F'} \longrightarrow\!\!\!\!\!\rightarrow \varpi_F^{-r'}\mathcal{O}_{F'}/\varpi_F^{r'-1}\mathcal{O}_{F'}$$

which is induced by the multiplication by ϖ_F). Let us remark that e identifies V' with $\text{Im}(N)$ and N' with the restriction of N to its image, and that p identifies V' with $\text{Coker}(N)$ and N' with the regular nilpotent endomorphism induced by N on its cokernel. Then

$$i_1(A') = (1 \oplus e)((p \oplus 1)^{-1}(A'))$$

and

$$i_2(A') = (e \oplus 1)((1 \oplus p)^{-1}(A')).$$

THEOREM 9.1. — *The two restriction maps*

$$i_1^*, i_2^* : R\Gamma(X) \to R\Gamma(X')$$

are equal.

Proof : As we have explained at the end of Section 8 it is sufficient to prove that the restriction maps

$$R\Gamma(\mathcal{G}) \to R\Gamma(X)$$

where $\mathcal{G}$ is the Grassmann variety of r-planes in $V \oplus V$, and

$$R\Gamma(\mathcal{G}') \to R\Gamma(X')$$

where $\mathcal{G}'$ is the Grassmann variety of $(r-1)$-planes in $V' \oplus V'$, induce epimorphisms on the cohomology. But this has been proved by Hotta and Shimomura as a consequence of a result of Spaltenstein (cf. [Ho-Sh] Lemma 8.1 and its proof). $\qquad\square$

COROLLARY 9.2. — *Conjecture 8.5 holds in the case $n_1 = n_2 = 1$. In particular, for any finite extension k_f of degree f of k we have the identity*

$$|\mathcal{X}^+(k_f)| - |\mathcal{X}^-(k_f)| = (-1)^r q^{rf} \cdot |\mathcal{Y}(k_f)|$$

with $|\mathcal{Y}(k_f)| = 1$.

Proof : We know already that the two restriction maps i_1^* and i_2^* are equal.

As Y is clearly equal to $\mathrm{Spec}(k')$ ($m_1 = m_2 = 0$) it follows from Section 7 that $U_i = X - i_i(X')$ is a standard affine space of dimension r over k'. But X' is isomorphic to the variety X after having replaced r by $r-1$. Therefore an obvious induction on r shows that X is a disjoint union of standard affine spaces, one for each dimension between 0 and r. Consequently either the source or the target of ∂_i is zero and we get

$$\partial_1 = \partial_2 = 0.$$

Finally let us consider the two Gysin maps $j_{1,!}$ and $j_{2,!}$. The only degree where these two maps can be non zero is the top degree $2r$. In this degree the ℓ-adic cohomology groups with compact supports of U_1, U_2 and X can be all canonically identified with $\mathbb{Q}_\ell(-r)$. Under these identifications $(\pi_1^!) \circ j_{1,!}$ and $(\pi_2^!) \circ j_{2,!}$ become the identity of $\mathbb{Q}_\ell(-r)$ and are thus equal. $\qquad\square$

In the $U(1,1)$ case the closed embeddings $i_1, i_2 : X' \hookrightarrow X$ are homotopic. We will conclude this section by constructing such an homotopy and giving another proof of Theorem 9.1. We introduce the projective k'-scheme $\widetilde{\widetilde{X}}$ of partial flags

$$L \subset A \subset H \subset V \oplus V$$

of vector subspaces such that L, A and H are of dimensions 1, r and $2r - 1$ respectively, and are stable by $N \oplus N$. Obviously, for each point $L \subset A \subset H$ in $\widetilde{\widetilde{X}}$ we have

$$L \subset \mathrm{Ker}(N) \oplus \mathrm{Ker}(N)$$

and

$$H \supset \mathrm{Im}(N) \oplus \mathrm{Im}(N).$$

By forgetting either A or (L, H) we get two k'-scheme morphisms

(9.3)
$$
\begin{array}{ccc}
\widetilde{\widetilde{X}} & \xrightarrow{\ \tilde{g}\ } & X \\
{\scriptstyle \tilde{f}}\big\downarrow & & \\
S \times S' & &
\end{array}
$$

where S is the projective line of lines L in the two dimensional k'-vector space $\mathrm{Ker}(N) \oplus \mathrm{Ker}(N)$ and S' is the projective line of hyperplanes H in $V \oplus V$ which contain the codimension 2 subspace $\mathrm{Im}(N) \oplus \mathrm{Im}(N)$. We have a closed embedding

$$S \hookrightarrow S \times S', \ L \mapsto (L, (N \oplus N)^{-(r-1)}(L)).$$

Let us denote by

(9.4)
$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ g\ } & X \\
{\scriptstyle f}\big\downarrow & & \\
S & &
\end{array}
$$

the inverse image of the diagram (9.3) by that closed embedding.

In S we have two k'-rational points s_1 and s_2 given by the lines $\mathrm{Ker}(N) \oplus (0)$ and $(0) \oplus \mathrm{Ker}(N)$ in $\mathrm{Ker}(N) \oplus \mathrm{Ker}(N)$. The restriction of g to the fiber of f over s_i is an isomorphism onto $i_i(X')$. We will denote by $\tilde{i}_i : X' \hookrightarrow \widetilde{X}$ the obvious lifting of i_i to $\widetilde{X}$ with image that fiber. More generally, for any point s in S, corresponding to a line L in $\mathrm{Ker}(N) \oplus \mathrm{Ker}(N)$, the restriction of g to the fiber of f over s is an isomorphism onto the closed subset

$$\{A \in X \mid L \subset A \subset (N \oplus N)^{-(r-1)}(L)\}$$

of X.

PROPOSITION 9.5. — (i) *The projective morphism g is birational. More precisely it is an isomorphism over the complementary subset in X of the closed subset*

$$i_1(X') \cap i_2(X') = \{A \in X \mid \mathrm{Ker}(N) \oplus \mathrm{Ker}(N) \subset A \subset \mathrm{Im}(N) \oplus \mathrm{Im}(N)\}$$

and its restriction to this closed subset is a trivial fibration in projective lines.

(ii) *The projective morphism f is a locally trivial fibration for the Zariski topology, with all its fibers isomorphic to X'. More precisely, if $\{i,j\} = \{1,2\}$ the restriction of f to the Zariski open subset $S - \{s_i\} \subset S$ is isomorphic to the canonical projection $(S - \{s_i\}) \times X' \to S - \{s_i\}$ by an isomorphism which exchanges the closed embeddings*

$$\tilde{i}_j : X' \hookrightarrow f^{-1}(S - \{s_i\})$$

and

$$X' \cong \{s_j\} \times X' \subset (S - \{s_i\}) \times X'.$$

Moreover the gluing datum between the above trivializations of f over the open subsets $S - \{s_1\}$ and $S - \{s_2\}$ is given by a morphism

$$S - \{s_1, s_2\} \to \mathrm{Aut}(X')$$

which factors through the canonical morphism from the centralizer of $N' \oplus N'$ in $\mathrm{GL}(V' \oplus V')$ to the automorphism group of X'.

Proof : Part (i) is obvious.

Let us prove Part (ii). The open subset $S - \{s_1\} \subset S$ may be identified with the affine line of linear maps $\varphi : \mathrm{Ker}(N) \to \mathrm{Ker}(N)$. For example s_2 corresponds to $\varphi = 0$. The fiber of f over a point φ is the closed subvariety

$$X(\varphi) = \{A \in X \mid L(\varphi) \subset A \subset H(\varphi)\}$$

of X where

$$L(\varphi) = \{(\varphi(x_2), x_2) \mid x_2 \in \mathrm{Ker}(N)\}$$

and

$$H(\varphi) = \{(x_1, x_2) \in V \oplus V \mid N^{r-1}(x_1) = \varphi \circ N^{r-1}(x_2)\}.$$

Let us fix a cyclic vector $v \in V$ for N. Then $N^{r-1}(v)$ is a basis of $\mathrm{Ker}(N)$ and any linear map φ as above maps $N^{r-1}(v)$ onto $\lambda(\varphi)N^{r-1}(v)$ for a unique scalar $\lambda(\varphi)$. Let us denote by $\widetilde{\varphi}$ the unique endomorphism of V such that

$$\widetilde{\varphi}(v) = \lambda(\varphi)v$$

and

$$N \circ \widetilde{\varphi} = \widetilde{\varphi} \circ N.$$

Obviously $\widetilde{\varphi}$ extends φ and we may identify $H(\varphi)/L(\varphi)$ with $\mathrm{Im}(N) \oplus \mathrm{Im}(N) = V' \oplus V'$ by sending $(x_1, x_2) \in H(\varphi)$ onto

$$(x_1 - \widetilde{\varphi}(x_2), N(x_2)).$$

As this identification exchanges the endomorphisms which are induced by $N \oplus N$ it induces an isomorphism from $X(\varphi)$ onto X'. Letting φ vary we get an isomorphism from the restriction of f to $S - \{s_1\}$ onto the canonical projection $(S - \{s_1\}) \times X' \to S - \{s_1\}$. Clearly this isomorphism exchanges the closed embeddings $\tilde{i}_2 : X' \hookrightarrow f^{-1}(S - \{s_1\})$ and $X' \cong \{s_2\} \times X' \subset (S - \{s_1\}) \times X'$.

Similarly we identify $S - \{s_2\}$ with the affine line of linear maps $\psi : \mathrm{Ker}(N) \to \mathrm{Ker}(N)$ and we construct an isomorphism from the restriction of f to $S - \{s_2\}$ onto the canonical projection $(S - \{s_2\}) \times X' \to S - \{s_2\}$. With obvious notations it is given by sending $(x_1, x_2) \in H(\psi)$ onto

$$(N(x_1), x_2 - \widetilde{\psi}(x_1)).$$

Over $S - \{s_1, s_2\}$ the gluing datum between the two trivializations of f is given by the morphism

$$S - \{s_1, s_2\} \to \mathrm{Aut}(X')$$

which sends $\varphi = \psi^{-1}$ onto the automorphism of X' which is induced by the automorphism

$$(x_1, x_2) \mapsto (-\widetilde{\varphi}(x_2), \widetilde{\varphi}^{-1}(x_1) + N'(x_2))$$

of $V \oplus V$. $\qquad\qquad\square$

Another proof of Theorem 9.1 : It is sufficient to check that the two restriction maps

$$\tilde{i}_1^*, \tilde{i}_2^* : R\Gamma(\widetilde{X}) \to R\Gamma(X')$$

are equal (we have $i_i^* = \tilde{i}_i^* \circ g^*$). By the Leray spectral sequence we have

$$R\Gamma(\widetilde{X}) = R\Gamma(\overline{k} \otimes_{k'} S, Rf_*\mathbb{Q}_\ell).$$

But the proposition implies that

(i) the complex $Rf_*\mathbb{Q}_\ell$ is constant with value $R\Gamma(X')$ on the open subsets $S - \{s_1\}$ and $S - \{s_2\}$ of S and we can choose the trivializations in such way that $\tilde{i}_1^*$ and $\tilde{i}_2^*$ are both induced by the identity of $R\Gamma(X')$,

(ii) the gluing datum for $Rf_*\mathbb{Q}_\ell$ on $S - \{s_1, s_2\}$ is induced by the identity of $R\Gamma(X')$ as the centralizer of $N' \oplus N'$ in $\mathrm{GL}(V' \oplus V')$ is connected.

The theorem follows. $\qquad\qquad\square$

10. Remarks and examples

(10.1) In general the conductors m_1, m_2 and the order of the resultant r are difficult to compute. Nevertheless there are some interesting particular cases where they admit simple formulae.

In general we have

$$m_i = v_{E'_i}(P'_i(\gamma_i)) - \delta(E'_i/F')$$

and

$$r = v_{E'_i}(P_j(\gamma_i)).$$

Here $v_{E'_i} : E'_i \to \mathbb{Z} \cup \{+\infty\}$ is the discrete valuation of E'_i, $P'_i(T)$ is the derivative of the minimal polynomial $P_i(T)$ of γ_i over F', $\delta(E'_i/F')$ is the order of the different of the totally ramified extension E'_i/F' and $\{i, j\} = \{1, 2\}$ (cf. [Se] Ch. III, §6, Corollaire 1). Moreover, if n_i is prime to the characteristic of k we have

$$\delta(E'_i/F') = n_i - 1$$

(cf. [Se] Ch. III, §6, Proposition 13).

Therefore, if we denote by a_i the constant term of the expansion of γ_i as power series in ϖ_{E_i} with coefficients in k' and we put

$$v_i = v_{E'_i}(\gamma_i - a_i)$$

we have :

(i) $m_i = 0$ and $r = v_{E'_j}(\gamma_j - \gamma_i)$ if $n_i = 1$,

(ii) $m_i = (n_i - 1)(v_i - 1)$ if $n_i > 1$ is prime to the characteristic of k and v_i is prime to n_i,

(ii) $r \geq \mathrm{Inf}(n_1 v_2, n_2 v_1)$ with equality if $n_1 v_2 \neq n_2 v_1$.

(10.2) *The case $r = 0$.* In this case the k'-schemes X^+ and X^- are not connected : they are disjoint sums

$$X^+ = \coprod_{n \in \mathbb{Z}} X^+[-n, n] \quad \text{and} \quad X^- = \coprod_{n \in \mathbb{Z}} X^-[1 - n, n]$$

where each component $X^+[-n, n]$ or $X^-[1 - n, n]$ is isomorphic to $Y = Y_1 \times_{k'} Y_2$. We have $X = X^+[0, 0] = Y$ with $F_X = F_Y$ and $X' = X^-[1, 1] = \emptyset$, and Conjecture 8.5 and the Langlands-Shelstad conjecture are obvious.

(10.3) *The case $n_1 = 1 < n_2$ and $r > 0$.* In this case, we have $m_1 = 0$ and $Y_1 = \mathrm{Spec}(k')$, we may assume that $\gamma_1 = 1$ and thus $\gamma_2 - 1$ is of order r in E'_2.

On $Y = Y_2$ we have the rank r vector bundles

$$\pi_1 : U_1 \to Y$$

and

$$\pi_2 : U_2 \to Y$$

defined by

$$\pi_2^{-1}(M_2) = (M_2/(\gamma_2 - 1)M_2)^\vee$$

where $(M_2/(\gamma_2 - 1)M_2)^\vee$ is the dual of the r-dimensional vector space $M_2/(\gamma_2 - 1)M_2$ and

$$\pi_2^{-1}(M_2) = (\gamma_2 - 1)^{-1}M_2/M_2$$

respectively. For $i = 1, 2$ and for every integer j the rank r-vector bundle $\pi_{i,j}^\pm : U_{i,j}^\pm \to Y$ which has been defined in Section 5 is isomorphic to $\pi_i : U_i \to Y$, the isomorphism being given by

$$(B_1^\pm, C_2^\pm, f_1^\pm) \mapsto (M_2 = \varpi_{E_2}^{c_2^\pm} C_2^\pm, x_1^\vee)$$

where $x_1^\vee$ is induced by the image of the k'-linear form

$$F'/B_1^\pm \to k', \quad \alpha \mapsto \mathrm{Res}(\alpha \varpi_F^{b_1^\pm - 1} \mathrm{d}\varpi_F)$$

by the transpose of the morphism

$$f_1^\pm \circ \varpi_{E_2}^{-c_2^\pm} : M_2 \xrightarrow{\sim} C_2^\pm \longrightarrow F'/B_1^\pm,$$

and

$$(B_2^\pm, C_1^\pm, f_2^\pm) \mapsto (M_2 = \varpi_{E_2}^{b_2^\pm} B_2^\pm, x_2)$$

where x_2 is the image of the vector

$$f_2^\pm(\varpi_F^{-c_1^\pm}) \in E_2'/B_2^\pm$$

by the isomorphism $\varpi_{E_2}^{b_2^\pm} : E_2'/B_2^\pm \xrightarrow{\sim} E_2'/M_2$.

In the next examples we assume that n_1 and n_2 are prime to the characteristic of k. We choose the uniformizers ϖ_F and ϖ_{E_i} in such way that
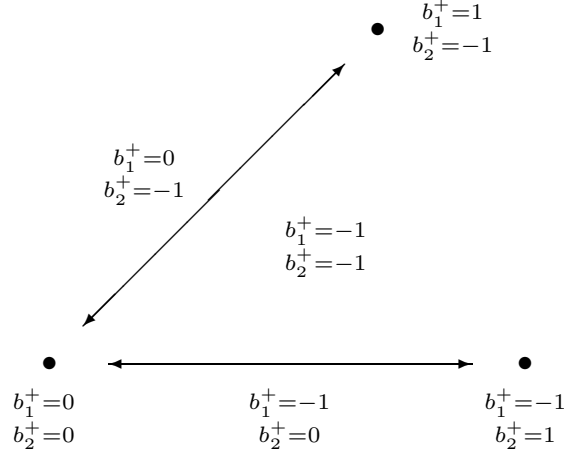
$$\varpi_F = \alpha_i \varpi_{E_i}^{n_i}$$

in E_i' for some $\alpha_i \in k'$.
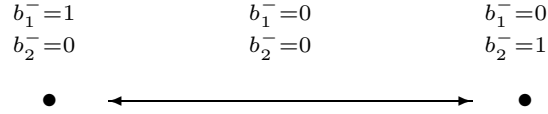
(10.4) *The case $n_1 = 1$, $\gamma_1 = 1$, $n_2 = 3$, $r = 2$ and* $\mathrm{Char}(k) > 3$. In this case $\gamma_2 - 1$ is of order 2 in E_2' and $m_2 = 2$. The k'-scheme $Y_2 = Z_2$ is a projective line (cf. Remark 3.14).

Then the stratification of $X = X^+[-1, -1]$ by the values of b_1^+ and b_2^+ can be symbolically represented by the following diagram

$$
\begin{array}{c}
\bullet \quad \begin{array}{l} b_1^+ = 1 \\ b_2^+ = -1 \end{array} \\[2em]
\begin{array}{l} b_1^+ = 0 \\ b_2^+ = -1 \end{array} \qquad \begin{array}{l} b_1^+ = -1 \\ b_2^+ = -1 \end{array} \\[2em]
\bullet \qquad \longleftrightarrow \qquad \bullet \\
\begin{array}{l} b_1^+ = 0 \\ b_2^+ = 0 \end{array} \qquad \begin{array}{l} b_1^+ = -1 \\ b_2^+ = 0 \end{array} \qquad \begin{array}{l} b_1^+ = -1 \\ b_2^+ = 1 \end{array}
\end{array}
$$

and the stratification of $X' = X^-[0, 0]$ by the values of b_1^- and b_2^- can be symbolically represented by the following diagram

$$
\begin{array}{ccc}
\begin{array}{l} b_1^- = 1 \\ b_2^- = 0 \end{array} & \begin{array}{l} b_1^- = 0 \\ b_2^- = 0 \end{array} & \begin{array}{l} b_1^- = 0 \\ b_2^- = 1 \end{array} \\[1em]
\bullet & \longleftrightarrow & \bullet
\end{array}
$$

In these representations the closed embeddings i_1 (resp. i_2) are given by

$$(b_1^-, b_2^-) \mapsto (b_1^+, b_2^+) = (b_1^-, b_2^- - 1)$$

(resp.

$$(b_1^-, b_2^-) \mapsto (b_1^+, b_2^+) = (b_1^- - 1, b_2^-)).$$

Let V_2 be the 6-dimensional k'-vector space

$$V_2 = \varpi_{E_2}^{-3} \mathcal{O}_{E_i'} / \varpi_{E_2}^3 \mathcal{O}_{E_i'}$$

and let N_2 and ν_2 be the nilpotent endomorphisms of V_2 which are induced by the multiplication by ϖ_{E_2} and ϖ_F, and let u_2 be the automorphism of V_2 which is induced by the multiplication by γ_2.

The stratum $(b_1^+ = -1, b_2^+ = -1)$ is isomorphic to the k'-scheme of triples

$$(B_2 \subset C_2, x_2)$$

where B_2 and C_2 are vector subspaces of V_2 of dimension 2 and 4 respectively such that

$$\mathrm{Ker}(N_2) \subset B_2 \subset \mathrm{Ker}(N_2^3) \subset C_2 \subset \mathrm{Ker}(N_2^5)$$

and

$$(\gamma_2 - 1)(C_2) = B_2,$$

and where x_2 is a vector in C_2/B_2 such that

$$\nu_2(x_2) \neq 0.$$

Obviously the k'-scheme of pairs $(B_2 \subset C_2)$ as above is a projective line over k' with some marked point $\infty := (\mathrm{Ker}(N_2^2) \subset \mathrm{Ker}(N_2^4))$. For a given pair $(B_2 \subset C_2)$ there exist vectors $x_2 \in C_2/B_2$ such that $\nu_2(x_2) \neq 0$ if and only if $(B_2 \subset C_2)$ is not equal to ∞. Moreover, if this holds, the scheme of $x_2 \in C_2/B_2$ such that $\nu_2(x_2) \neq 0$ is isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m$.

The stratum $(b_1^+ = 0, b_2^+ = -1)$ is isomorphic to the k'-scheme of triples

$$(B_2 \subset C_2, x_2)$$

where B_2 and C_2 are vector subspaces of V_2 of dimension 2 and 3 respectively, such that

$$\mathrm{Ker}(N_2) \subset B_2 \subset \mathrm{Ker}(N_2^3)$$

and

$$\mathrm{Ker}(N_2^2) \subset C_2 \subset \mathrm{Ker}(N_2^4)$$

(the condition

$$(\gamma_2 - 1)(C_2) \subset B_2$$

is automatic), and where x_2 is a non zero vector in C_2/B_2. If $B_2 \neq \mathrm{Ker}(N_2^2)$ we necessarily have

$$C_2 = B_2 + \mathrm{Ker}(N_2^2) = \mathrm{Ker}(N_2^3)$$

and if $B_2 = \mathrm{Ker}(N_2^2)$, C_2 may vary freely in the projective line

$$\mathrm{Ker}(N_2^2) \subset C_2 \subset \mathrm{Ker}(N_2^4).$$

Similarly, if $C_2 \neq \mathrm{Ker}(N_2^3)$ we necessarily have

$$B_2 = C_2 \cap \mathrm{Ker}(N_2^3) = \mathrm{Ker}(N_2^2).$$

and if $C_2 = \mathrm{Ker}(N_2^3)$, B_2 may vary freely in the projective line

$$\mathrm{Ker}(N_2) \subset B_2 \subset \mathrm{Ker}(N_2^3).$$

Therefore the k'-scheme of pairs $(B_2 \subset C_2)$ as above may be obtained by gluing the two projective lines over k'

$$\mathrm{Ker}(N_2) \subset B_2 \subset \mathrm{Ker}(N_2^3)$$

and

$$\mathrm{Ker}(N_2^2) \subset C_2 \subset \mathrm{Ker}(N_2^4)$$

along their marked points $B_2 = \mathrm{Ker}(N_2^2)$ and $C_2 = \mathrm{Ker}(N_2^3)$ respectively, and the stratum $(b_1^+ = 0, b_2^+ = -1)$ is a $\mathbb{G}_\mathrm{m}$-torsor on this k'-scheme.

The stratum $(b_1^+ = -1, b_2^+ = 0)$ is isomorphic to the stratum $(b_1^+ = 0, b_2^+ = -1)$.

The stratum $(b_1^+ = 0, b_2^+ = 0)$ is equal to the k'-scheme of vector subspaces $B_2 = C_2$ of dimension 3 of V_2 such that

$$\mathrm{Ker}(N_2^2) \subset B_2 = C_2 \subset \mathrm{Ker}(N_2^4)$$

and is thus a projective line.

Similarly the stratum $(b_1^+ = 1, b_2^+ = -1)$ and $(b_1^+ = -1, b_2^+ = 1)$ are projective lines over k'.

We may summarize this discussion by saying that the set $X(k')$ has

$$q'(q'-1)q' + (q'-1)(2q'+1) + (q'-1)(2q'+1) + (q'+1) + (q'+1) + (q'+1) = q'^3 + 3q'^2 + q' + 1$$

elements and that the set $X'(k')$ has

$$(q'-1)(2q'+1) + (q'+1) + (q'+1) = 2q'^2 + q' + 1$$

elements, so that

$$|X(k')| - |X'(k')| = q'^2(q'+1).$$

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