Φ-MODULES AND COEFFICIENT SPACES

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Introduction

This paper is inspired by Kisin's article [Ki1], in which he studies deformations of Galois representations of a local p-adic field which are defined by finite flat group schemes. The result of Kisin most relevant to our paper is his construction of a kind of resolution of the formal deformation space of the given Galois representation, by constructing a scheme which classifies all finite flat group schemes giving rise to the deformed Galois representation. Our purpose here is to globalize Kisin's construction.

Let K be a finite extension of $\mathbb{Q}_p$, with residue field k. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Then K_0 is the fraction field of the ring of Witt vectors $W = W(k)$. Let π be a uniformizer of K and $E(u) \in W[u]$ the Eisenstein polynomial that π satisfies. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. Set $\mathfrak{S}$ for the ring of formal power series $W[[u]]$. Let $\phi : \mathfrak{S} \to \mathfrak{S}$ be such that $\phi|W$ is the Frobenius automorphism and with $\phi(u) = u^p$.

Kisin's construction is based on the existence of a fully faithful exact functor from a suitable category of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a ϕ-linear endomorphism Φ to the category of finite flat (commutative) group schemes of p-power rank over $\mathrm{Spec}\,(\mathcal{O}_K)$. This in turn was inspired by work of Breuil [Br] who gave a similar but more complicated description of such group schemes. A variant of this functor also works with coefficients: if R is a $\mathbb{Z}_p$-algebra with finitely many elements, then there is a similar functor from a suitable category of $\mathfrak{S} \otimes_{\mathbb{Z}_p} R$-modules $\mathfrak{M}$ with ϕ-linear endomorphism Φ to the category of finite flat group schemes of p-power rank with R-action over $\mathrm{Spec}\,\mathcal{O}_K$.

Let K_∞/K be the extension obtained by adjoining a compatible system of p^n-power roots of π, and let $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$ be its absolute Galois group. Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$, a complete discrete valuation ring, with uniformizer p and residue field $k((u)) = k[[u]][1/u]$. Then there exists an equivalence of categories between the category of finitely generated $\mathcal{O}_{\mathcal{E}}$-modules M equipped with an isomorphism $\Phi : \phi^*(M) \to M$ and the category of continuous representations of G_{K_∞} in $\mathbb{Z}_p$-modules, and this is compatible with the previous functor via the restriction functor from G_K-representations to G_{K_∞}-representations. Again there is also a variant for representations with values in a finite coefficient $\mathbb{Z}_p$-algebra R.

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Our basic idea in this paper is to formally forget about Galois representations and finite flat group schemes and simply consider the modules themselves, without any finiteness conditions on R. More precisely, for any $\mathbb{Z}_p$-algebra R set $R_W = W \otimes_{\mathbb{Z}_p} R$ and extend the endomorphism ϕ of $W((u)) = W[[u]][1/u]$ to $R_W((u)) = W((u))\hat{\otimes}_{\mathbb{Z}_p} R$ by the identity on the second factor. We define the fpqc-stack $\mathcal{C}_{d,K}$ by giving its values on $\mathbb{Z}_p$-algebras R as the groupoid of pairs $(\mathfrak{M}, \Phi)$, where $\mathfrak{M}$ is a finitely generated $R_W[[u]]$-module which is free of rank d locally fpqc on $\operatorname{Spec} R$ and where

$$\Phi : \phi^*\mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u]$$

is an isomorphism of $R_W((u))$-modules such that $E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$.

We also introduce the fpqc-stack $\mathcal{R}_d$ with values in a $\mathbb{Z}_p$-algebra R the groupoid of pairs (M, Φ), where M is a finitely generated $R_W((u))$-module which is locally fpqc on $\operatorname{Spec} R$ free of rank d as $R_W((u))$-module, and where $\Phi : \phi^*(M) \xrightarrow{\sim} M$.

There is an obvious morphism

$$\theta : \mathcal{C}_{d,K} \to \mathcal{R}_d$$

sending $(\mathfrak{M}, \Phi)$ to $(\mathfrak{M}[1/u], \Phi)$.

Our main results concern the algebraicity of the previous construction. Let $\widehat{\mathcal{C}}_{d,K}$ be the p-adic completion of the stack $\mathcal{C}_{d,K}$, i.e. its restriction to p-nilpotent $\mathbb{Z}_p$-algebras. Then

$$\widehat{\mathcal{C}}_{d,K} = \varinjlim_a \mathcal{C}_{d,K} \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathbb{Z}/p^a\mathbb{Z} \ .$$

Our main result shows that this presents $\widehat{\mathcal{C}}_{d,K}$ as an inductive 2-limit of Artin stacks of finite type over $\operatorname{Spec} \mathbb{Z}/p^a\mathbb{Z}$. Furthermore, the singularities of $\widehat{\mathcal{C}}_{d,K}$ are modeled by local models.

Theorem 0.1. *(i) For each a, the stack $\mathcal{C}_{d,K} \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathbb{Z}/p^a\mathbb{Z}$ on $\mathbb{Z}/p^a\mathbb{Z}$-algebras is representable by an Artin stack $\mathcal{C}^a_{d,K}$ of finite type over $\operatorname{Spec} \mathbb{Z}/p^a\mathbb{Z}$. The inductive limit $\varinjlim_a \mathcal{C}^a_{d,K}$ is the formal p-adic completion $\widehat{\mathcal{C}}_{d,K}$ of $\mathcal{C}_{d,K}$.*

(ii) There is a "local model" diagram

$$\begin{array}{ccc} & \widetilde{\widehat{\mathcal{C}}_{d,K}} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\varphi} \\ \widehat{\mathcal{C}}_{d,K} & & \widehat{M}_{d,K} \ , \end{array}$$

in which π is a principal homogeneous space under the positive loop group L^+G of $G = \operatorname{Res}_{W/\mathbb{Z}_p}(GL_d)$ completed along its special fiber, and in which φ is formally smooth. Here $M_{d,K}$ is the projective $\mathbb{Z}_p$-scheme parametrizing all $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-submodules of $(\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_S)^d$ which are locally direct summands as $\mathcal{O}_S$-modules, and $\widehat{M}_{d,K}$ denotes its completion along the special fiber.

(iii) For each a, set $\mathcal{R}_d^a = \mathcal{R}_d \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{Z}/p^a\mathbb{Z}$*. The morphism* $\theta^a : \mathcal{C}_{d,K}^a \to \mathcal{R}_d^a$ *given by reducing* θ *modulo* $p^a\mathbb{Z}$ *is representable and proper; hence* $\hat{\theta} : \widehat{\mathcal{C}}_{d,K} \to \widehat{\mathcal{R}}_d$ *is an inductive limit of representable and proper morphisms.*

The fibers of θ over a finite field $\mathbb{F}$ are interesting projective subvarieties of the affine Grassmannian of the group $G = \operatorname{Res}_{W/\mathbb{Z}_p}(GL_d)$, which we call *Kisin varieties*. More precisely, we define variants of $\mathcal{C}_{d,K}$ and $M_{d,K}$ depending on a co-character μ of G and define Kisin varieties associated to (G, A, μ), where $A \in G(\mathbb{F}((u)))$ defines the given $\mathbb{F}$-valued point in $\mathcal{R}_d$. They are the analogues, for the kind of Frobenius involved here, of the affine Deligne-Lusztig varieties appearing in the isocrystal context, cf., eg. [GHKR]. The study of these varieties was begun by Kisin in [Ki1], in the case $d = 2$ and $W = \mathbb{Z}_p$. In a companion paper to ours, E. Hellmann extends Kisin's results (again for $d = 2$ and $W = \mathbb{Z}_p$). In Hellmann's paper, one of the main tools is the Bruhat-Tits building of GL_2. We show here how the Bruhat-Tits building can be used in general to gain a qualitative overview of Kisin varieties.

The other extreme to the fiber over $\mathbb{F}$ of $\widehat{\mathcal{C}}_{d,K}$ is its fiber over $\mathbb{Q}_p$. Here we construct a kind of period map of stacks over the category of adic formal schemes locally of finite type over $\operatorname{Spf}(\mathbb{Z}_p)$,

$$\Pi(\mathcal{X}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \to \mathfrak{D}_{d,K}(\mathcal{X}^{\mathrm{rig}}) \ ,$$

where $\mathfrak{D}_{d,K}$ is the stack over the category of rigid-analytic spaces over $\mathbb{Q}_p$ parametrizing filtered Φ-modules (both the filtration and Φ vary!). To determine the image of the period map seems one of the major challenges in the theory. We conjecture that the image should be given somewhat analogously to Hartl's *admissible set* in [Ha].

As is apparent from the above, the theory developed here bears many similarities to the theory of period spaces for p-divisible groups in [RZ], but there are also substantial differences. The stack $\widehat{\mathcal{C}}_{d,K}$ is analogous to one of the period spaces of p-divisible groups in [RZ], but unlike these it is adic over $\operatorname{Spec}(\mathbb{Z}_p)$ (p generates an ideal of definition); the local model diagram looks formally just like the corresponding one in [RZ]; Kisin varieties are the analogues of affine Deligne-Lusztig varieties, and the stack $\mathfrak{D}_{d,K}$ plays a role similar to the Grassmannian containing the period space of [RZ]. In [RZ], the base scheme of the p-divisible groups is variable; here the base scheme $\operatorname{Spec}(\mathcal{O}_K)$ is constant, but the coefficients are variable.

We now explain the lay-out of the paper. In section 1 we explain by analogy on the classical theory of unit root crystals the spaces/stacks we encounter. In section 2 we prove our main technical result, which states that the stack $\mathcal{C}_d$, which associates to $\mathbb{Z}_p$-algebras R the groupoid of locally free $R_W[[u]]$-modules of rank d with Φ-module structure, can be presented as an inductive limit of Artin stacks of finite type over $\mathbb{Z}_p$. In section 3 we fix the local field as above and prove the main theorem stated above. In section 4 we indicate the relation to the deformation spaces of Galois representations which is at the origin of Kisin's theory. In section 5 we construct and discuss the period morphism. In the final section 6 we discuss Kisin varieties and their analysis through Bruhat-Tits buildings.

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1. Motivation: Unit crystals and Galois representations

1.a. **Unit root crystals.** Let k be a finite field of characteristic $p > 0$; for simplicity, we will assume that $k = \mathbb{F}_p$. Let X a variety over k. Denote by $\phi : X \to X$ the Frobenius $\phi(a) = a^p$ for $a \in \mathcal{O}_X$. Suppose that S is a k-scheme, and let $\phi_S = \phi \times id_S : X \times S \to X \times S$. Consider pairs $(\mathcal{M}, F)$ consisting of a locally free $\mathcal{O}_{X\times S}$-coherent sheaf $\mathcal{M}$ of rank d on $X \times S$ and an isomorphism

$$F : \phi_S^* \mathcal{M} \xrightarrow{\sim} \mathcal{M} \ .$$

As S varies, these pairs form an fpqc stack $FM_X^{d,et}$ over $\operatorname{Spec} k$. In fact, $FM_X^{d,et}$ is an Artin stack locally of finite type over k. Indeed, let $Fib_{X/k}^d$ be the Artin stack locally of finite type over k, whose values in a k-scheme S is the groupoid of locally free $\mathcal{O}_{X\times S}$-modules of rank d, and let $\widetilde{Fib}_{X/k}^d$ be the GL_d-torsor over $Fib_{X/k}^d$, consisting of a locally free $\mathcal{O}_{X\times S}$-modules $\mathcal{M}$ of rank d and a basis $\iota : \mathcal{M} \xrightarrow{\sim} \mathcal{O}_{X\times S}^d$. Then there is an action of GL_d on the product $\widetilde{Fib}_{X/k}^d \times GL_d$ via

$$g : \quad (\mathcal{M}, \iota, A) \mapsto (\mathcal{M}, g^{-1}\cdot \iota, g^{-1}\cdot A \cdot \phi(g)) \ .$$

This presents $FM_X^{d,et}$ as a quotient of $\widetilde{Fib}_{X/k}^d \times GL_d$ by GL_d, and hence $FM_X^{d,et}$ is an Artin stack locally of finite type, as claimed.

Suppose $S = \operatorname{Spec}(\Lambda)$ with $|\Lambda| < \infty$. Then by Katz [Ka], 4.1, cf. also [E-K], there is a bijective correspondence between pairs $(\mathcal{M}, F)$ over $\operatorname{Spec}(\Lambda)$ and étale sheaves of Λ-modules on X. In the case $\Lambda = \mathbb{F}_p$, this correspondence is obtained via push-out from the injection $GL_d(\mathbb{F}_p) \to GL_d$ which induces an equivalence of categories between the category of $GL_d(\mathbb{F}_p)$-torsors on X (for the étale topology) and the category of GL_d-torsors P on X with an isomorphism $\phi^*(P) \xrightarrow{\sim} P$.

We want to think of $FM_X^{d,et}$ as a "coefficient space" for p-torsion representations of $\pi_1(X, \bar\eta)$. However, it seems that global questions on these spaces (i.e., when S is not a local Artin ring) have not been studied much in the literature. For instance, are there non-constant morphisms of projective k-schemes into $FM_X^{d,et}$? What is the dimension of $FM_X^{d,et}$? Etc. The only result we are aware of is Laszlo's construction [La] of a projective curve X of genus 2 over the field with 2 elements, a projective curve S over a finite extension k' of $\mathbb{F}_2$ and a locally free coherent $\mathcal{O}_{X\times S}$-module $\mathcal{M}$ of arbitrary rank with an isomorphism $F : (\phi_S^2)^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$.

1.b. **Variants.** We mention here some variants of the above theory. Let G be a reductive group over $\mathbb{F}_p$. Then we can consider the fpqc stack $FM_X^{G,et}$ of pairs (P, F), where P is a G-torsor on $X \times S$ and where $F : \phi_S^*(P) \xrightarrow{\sim} P$. For $G = GL_d$, we recover the stack considered above.

We may also consider "meromorphic Frobenius structures", as follows. Assuming X to be irreducible, with generic point $\eta(X)$, consider the fpqc stack FM_X^d of pairs $(\mathcal{M}, F)$ with $\mathcal{M}$ a locally free $\mathcal{O}_{X\times S}$-coherent sheaf of rank d on $X \times S$ and

$$F : \phi_S^*\mathcal{M} \to \mathcal{M}$$

a homomorphism such that $F_{|\eta(X)\times S}$ is an isomorphism.

One may also control the degeneracy of the meromorphic Frobenius structure. For instance, let X be a curve. Then we may consider triples $(\mathcal{M}, F, x)$ with $(\mathcal{M}, F)$ in FM_X^d and $x : S \to X$ such that $Coker(F)$ is supported on the graph $\Gamma_x \subset X \times S$ and is annihilated by the power of the ideal sheaf $I_{\Gamma_x}^e$ for some fixed $e \geq 1$. Denoting the corresponding stack by $FM_{X,e}^d$, there is a morphism (the "pole morphism"),

$$p : FM_{X,e}^d \to X$$

Similarly we can obtain a construction that resembles shtuka, but the Frobenius is "on the other factor". Namely, assume that X is a curve as before. Consider $(\mathcal{M}, \mathcal{M}', F, F', x, y)$ with $\mathcal{M}$, $\mathcal{M}'$ locally free $\mathcal{O}_{X\times S}$-coherent sheaves of rank d on $X \times S$ and homomorphisms

$$\phi_S^*\mathcal{M} \xrightarrow{F} \mathcal{M}' \xleftarrow{F'} \mathcal{M}$$

such that $Coker(F)$, resp. $Coker(F')$ is supported on the graph of $x : S \to X$, resp. $y : S \to X$. Again we can ask that $Coker(F)$, resp. $Coker(F')$, satisfy some additional property.

Another variant is obtained by replacing the variety X by the spectrum of the completed local ring at a closed point, or by its fraction field.

All these "spaces"/stacks seem interesting geometric objects.

2. Spaces of Kisin-Breuil modules

Fix a finite field k of characteristic p and denote by $\phi(a) = a^p$ the Frobenius automorphism of k. We will denote by $W = W(k)$ the ring of Witt vectors of k and by $\phi : W \to W$ the unique lifting of Frobenius.

Let R be a commutative $\mathbb{Z}_p$-algebra and set $R_W = W \otimes_{\mathbb{Z}} R$. We extend ϕ in a R-linear way to R_W and denote this extension also by ϕ. We also denote by ϕ the endomorphism ϕ of $R_W((u)) = W((u))\hat{\otimes}_{\mathbb{Z}}R$ given by

$$\phi(\sum_i a_i u^i) = \sum_i \phi(a_i) u^{pi} \ .$$

2.a. We define now various stacks of modules with Frobenius structure.

Let us consider the stack $\mathcal{C}_d$ such that $\mathcal{C}_d(R)$ is the groupoid of $R_W[[u]]$-Φ-modules $(\mathfrak{M}, \Phi)$: These are by definition pairs of a $R_W[[u]]$-module $\mathfrak{M}$ which is locally on R (for the fpqc topology) $R_W[[u]]$-free of rank d and a $R_W((u))$-module isomorphism

$$\Phi : \phi^*\mathfrak{M}[1/u] = R_W((u)) \otimes_{\phi, R_W[[u]]} \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}[1/u] = R_W((u)) \otimes_{R_W[[u]]} \mathfrak{M} .$$

It is easy to see that $\mathcal{C}_d$ is a stack for the fpqc topology.

Next, consider the stack $\mathcal{R}_d$ which is such that $\mathcal{R}_d(R)$ is the groupoid of pairs (M, Φ) of $R_W((u))$-modules M which are fpqc locally on R free of rank d, together with a $R_W((u))$-linear isomomorphism

$$\Phi : \phi^* M := R_W((u)) \otimes_{\phi, R_W((u))} M \to M .$$

Again it is easy to see that $\mathcal{R}_d$ gives a stack for the fpqc topology. Write

$$\theta : \mathcal{C}_d \to \mathcal{R}_d \ ; \quad (\mathfrak{M}, \Phi) \mapsto (\mathfrak{M}[1/u], \Phi)$$

for the forgetful morphism.

Fix an integer $m \geq 0$. Let us consider the stack $\mathcal{C}_{m,d}$ such that $\mathcal{C}_{m,d}(R)$ is the groupoid of $R_W[[u]]$-Φ-modules $(\mathfrak{M}, \Phi)$ as above that satisfy the additional hypothesis

$$u^m\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset u^{-m}\mathfrak{M} . \tag{2.1}$$

Once again, $\mathcal{C}_{m,d}$ gives a stack for the fpqc topology. The natural morphism $\mathcal{C}_{m,d} \to \mathcal{C}_d$ is a representable closed immersion.

If d is fixed we will often write $\mathcal{C}$, $\mathcal{R}$, $\mathcal{C}_m$ instead of $\mathcal{C}_d$, $\mathcal{R}_d$ and $\mathcal{C}_{m,d}$.

2.b. For simplicity, we will set $G = \mathrm{Res}_{W/\mathbb{Z}_p} GL_d$. Set

$$\begin{aligned} LG(R) &:= GL_d(R_W((u))) \ , \\ L^+G(R) &:= GL_d(R_W[[u]]) \ , \\ LG^{\leq m}(R) &:= \{ A \in GL_d(R_W((u))) \mid A, A^{-1} \in u^{-m} M_d(R_W[[u]]) \ \} \ . \end{aligned}$$

Hence $L^+G = LG^{\leq 0}$. Note that the functor

$$R \mapsto LG^{\leq m}(R)$$

is represented by a scheme $LG^{\leq m}$ (which is infinite dimensional). Let $(\mathfrak{M}, \Phi) \in \mathcal{C}_m(R)$ such that $\mathfrak{M}$ is a free $R_W[[u]]$-module. By picking a $R_W[[u]]$-basis of $\mathfrak{M}$, we can write Φ as multiplication by $A \in LG^{\leq m}(R)$. Changing the basis by $g \in GL_d(R_W[[u]])$ amounts to changing A to $g^{-1} \cdot A \cdot \phi(g)$. Therefore, we can write

$$\mathcal{C}_{m,d} = [LG^{\leq m} /_\phi \, L^+G] \tag{2.2}$$

where the quotient $/_\phi$ is via the right action of $L^+G(R) = GL_d(R_W[[u]])$ by ϕ-conjugation by $A \star g = g^{-1} \cdot A \cdot \phi(g)$.

Similarly, we can write

$$\mathcal{C}_d = [LG/_\phi\, L^+G]\ ,\quad \mathcal{R}_d = [LG/_\phi\, LG]. \tag{2.3}$$

In fact, we can consider the fpqc stack $\tilde{\mathcal{C}}_d$ defined as follows: $\tilde{\mathcal{C}}_d(R)$ is the groupoid of pairs $((\mathfrak{M}, \Phi), \alpha)$ of $R_W[[u]]$-Φ-modules $(\mathfrak{M}, \Phi)$ together with an $R_W[[u]]$-module isomorphism

$$\alpha : R_W[[u]]^d \xrightarrow{\sim} \mathfrak{M}.$$

The stack $\tilde{\mathcal{C}}_d$ is represented by the ind-scheme LG and the forgetful morphism

$$\pi : \tilde{\mathcal{C}}_d \to \mathcal{C}_d$$

is a L^+G-torsor.

2.b.1. Denote by $\mathcal{F}_G = LG/L^+G$ the affine Grassmannian of $R_W[[u]]$-"lattices" in $R_W((u))^d$. (Here by $R_W[[u]]$-lattice we mean a locally on R free $R_W[[u]]$-submodule L of $R_W((u))^d$ such that $L \otimes_{R[[u]]} R((u)) = R_W((u))^d$.) The fpqc quotient $\mathcal{F}_G = LG/L^+G$ is represented by an ind-scheme which is ind-projective over $\mathbb{Z}_p$. For $m \geq 0$, let $\mathcal{F}_G^{\leq m}$ be the projective subscheme of $\mathcal{F}_G$ parametrizing $R_W[[u]]$-lattices L

$$u^m R_W[[u]]^d \subset L \subset u^{-m} R_W[[u]]^d.$$

(This is a finite union of Schubert varieties in the affine Grassmannian.) Set $U_0(R) = L^+G(R) = GL_d(R_W[[u]])$ and define for $n \geq 1$ the principal congruence subgroup U_n of level n by $U_n(R) = I + u^n \cdot M_d(R_W[[u]])$. The subgroup scheme U_n is normal in L^+G and the quotient L^+G/U_n is represented by the smooth finite type group scheme $\mathcal{G}_n$ given by the Weil restriction of GL_d from $W[[u]]/(u^n)$ to $\mathbb{Z}_p$ (so that $\mathcal{G}_n(R) = GL_d(R_W[[u]]/(u^n))$. Note that under the action of L^+G on $\mathcal{F}_G^{\leq m}$ the subgroup U_{2m} acts trivially, and hence the action factors through $\mathcal{G}_{2m}$.

Theorem 2.1. *a) For $m \geq 1$, $\mathcal{C}_m = [LG^{\leq m}/_\phi L^+G]$ is an Artin stack of finite type over $\mathbb{Z}_p$. We can write $\mathcal{C}$ as a direct 2-limit*

$$\mathcal{C} = \varinjlim_{m} \mathcal{C}_m$$

and so $\mathcal{C}$ is an "ind-Artin stack of ind-finite type over $\mathbb{Z}_p$".

b) There is a formally smooth morphism

$$q : \mathcal{C} \to [L^+G\backslash \mathcal{F}_G] = [L^+G\backslash LG/L^+G]\ .$$

In fact, q is given as the limit of formally smooth morphisms

$$q_m : \mathcal{C}_m \to [L^+G\backslash \mathcal{F}_G^{\leq m}] = [L^+G\backslash LG^{\leq m}/L^+G]\ .$$

The composition of q_m with the natural morphism $[L^+G\backslash \mathcal{F}_G^{\leq m}] \to [\mathcal{G}_{2m}\backslash \mathcal{F}_G^{\leq m}]$ is a smooth morphism of Artin stacks of finite type,

$$\bar{q}_m : \mathcal{C}_m \to [\mathcal{G}_{2m}\backslash \mathcal{F}_G^{\leq m}]\ .$$

The relative dimension of $\bar{q}_m$ is equal to $2md^2$.

Proof. The group $LG(R)$ is a topological group with topology described by the neighborhoods U_n of the identity $I = I_d$. We have

$$LG(R) = \bigcup_{m \geq 0} LG^{\leq m}(R) \ .$$

Suppose now that A is in $LG^{\leq m}(R)$. For any integer $n \geq 0$ and for all A' in the neighborhood (coset)

$$\{A' \mid A' \cdot A^{-1} \in U_n(R)\} = U_n(R)\, A$$

of A, we have $A' \in LG^{\leq m}(R)$ also.

Proposition 2.2. *Suppose $n > 2m/(p-1)$.*

1) For each $g \in U_n(R)$, $A \in LG^{\leq m}(R)$, we can write $g^{-1} \cdot A \cdot \phi(g) = H^{-1} \cdot A$ with a unique $H = H(g, A) \in U_n(R)$.

2) Conversely, for each $A \in LG^{\leq m}(R)$ and $h \in U_n(R)$, there is a unique $g \in U_n(R)$ such that $A \star g = g^{-1} \cdot A \cdot \phi(g) = h^{-1} \cdot A$. [1]

Proof. Let us first prove (1). Write $g^{-1} = I + u^n X$ with $X \in M_d(R_W[[u]])$. Then $\phi(g) = I + u^{pn} Y$, with $Y \in M_d(R_W[[u]])$. Now

$$g^{-1} \cdot A \cdot \phi(g) \cdot A^{-1} = (I + u^n X) \cdot A \cdot (I + u^{pn} Y) \cdot A^{-1} =$$

$$= (I + u^n X) \cdot (I + u^{pn} A Y A^{-1}).$$

Observe that $AYA^{-1} \in u^{-2m} M_d(R_W[[u]])$ and $pn - 2m > n$. Hence, for

$$H^{-1} = (I + u^n X) \cdot (I + u^{pn} A Y A^{-1})$$

we obtain $g \cdot A \cdot \phi(g)^{-1} = H^{-1} \cdot A$. The element H is uniquely determined from g and A by $g \cdot A \cdot \phi(g)^{-1} = H^{-1} \cdot A$.

The statement (2) is little trickier. First we show that if such a g exists it is uniquely determined by h and A. It is enough to assume $g \cdot A \cdot \phi(g)^{-1} = A$ with $g \in U_n(R)$ and $A \in LG^{\leq m}(R)$ and show $g = 1$. Write $g = I + u^n X$, $\phi(g) = I + u^{pn} \phi(X)$. We have

$$(I + u^n X) \cdot A = A \cdot (I + u^{pn} \phi(X))$$

which gives

$$u^n X \cdot A = u^{pn} A \cdot \phi(X) \ ,$$

i.e.,

$$X_0 + X_1 u + X_2 u^2 + \cdots = u^{(p-1)n} A \cdot (X_0 + X_1 u^p + X_2 u^{2p} + \cdots) A^{-1}$$

Note that $A \cdot X_i \cdot A^{-1} \in u^{-2m} M_d(R[[u]])$. Since $(p-1)n - 2m > 0$, we obtain $X_0 = 0$ which implies $g \in U_{n+1}(R)$. An induction finishes the proof of uniqueness.

[1] The fact that two elements A and A' in $GL_d(\bar{\mathbb{F}}_p((u)))$ which are u-adically close, are ϕ-conjugate is also used by Caruso in [Ca]. The analogous fact for classical Dieudonné modules is also true.

Now we will show that such a g exists. Let $A' = h^{-1}A$. Set $A_0 = A$, $h_0 = h$ and define h_i and A_i inductively by

$$A_i = h_{i-1}^{-1} \cdot A_{i-1} \cdot \phi(h_{i-1}) \ , \quad A' = h_i^{-1} \cdot A_i \ . \tag{2.4}$$

Set $\kappa(i) := p^i n - 2(1 + p + \cdots + p^{i-1})m$ with $\kappa(0) = n$. Note that under our assumption $n > 2m/(p-1)$, the function $\kappa(i)$ strictly increases with $i \geq 0$.

The existence of g will now follow from

Lemma 2.3. *1) We have $h_i \in U_{\kappa(i)}(R)$ and so $\lim_{i\to\infty} h_i = I$.*

2) Let $g_i = \prod_{j=0}^{i} h_j$. Then the limit $g = \lim_{i\to\infty} g_i$ exists and belongs to $U_n(R)$.

Proof. We will prove (1) by induction. It is true by our hypothesis when $i = 0$. The equalities (2.4) imply

$$h_i = A' \cdot \phi(h_{i-1}) \cdot A'^{-1}.$$

By the induction hypothesis $h_{i-1} \in U_{\kappa(i-1)}(R)$ so

$$\phi(h_{i-1}) = I + \phi(u^{\kappa(i-1)}X) = I + u^{p\cdot\kappa(i-1)}\phi(X)$$

with $X \in M_d(R_W[[u]])$. Since $A' \in LG^{\leq m}(R)$, we have

$$A' \cdot \phi(X) \cdot A'^{-1} \in u^{-2m} M_d(R_W[[u]]),$$

and so

$$h_i = A' \cdot \phi(h_{i-1}) \cdot A'^{-1} = I + u^{p\cdot\kappa(i-1)-2m}Y$$

with $Y \in M_d(R_W[[u]])$. Since $\kappa(i) = p \cdot \kappa(i-1) - 2m$ this completes the proof of (1). Part (2) now follows immediately since from part (1)

$$g_i = \prod_{j=0}^{i} h_j = (I + u^{\kappa(0)}X_0) \cdot (I + u^{\kappa(1)}X_1) \cdot \cdots \cdot (I + u^{\kappa(i)}X_i)$$

with $X_j \in M_d(R_W[[u]])$ and $i \mapsto \kappa(i)$ is strictly increasing. □

Now $h_i^{-1}A_i = g_i^{-1} \cdot A \cdot \phi(g_{i-1})$, hence passing to the limit, we obtain $g^{-1} \cdot A \cdot \phi(g) = A' = h^{-1} \cdot A$ as desired. □

Remark 2.4. Let M be a $R((u))$-module and let $R \to R'$ be a flat extension such that $M' = M \hat{\otimes}_R R' \simeq R'((u))^d$. The module M has a natural topology as a Tate R-module (see [Dr]). The R-lattices of M (i.e R-modules L which are open and such that L/U is finitely generated for every open submodule $U \subset L$) give a basis of open neighborhoods of 0. Multiplication by u on M is topologically nilpotent; i.e given any two R-lattices L, L', there is $N \geq 0$ such that $u^N \cdot L \subset L'$.

Then $\mathcal{G}_M := \mathrm{Aut}_{R((u))}(M)$ has a natural structure of a topological group. To obtain a basis of neighborhoods of the identity I we take a lattice L and for $n >> 0$ we consider

$$U_n(L) = \{g \in \mathcal{G}_M \mid g(L) \subset L, \ \ g_{|L} \equiv I \bmod u^n L\}.$$

We can then show:

Given two isomorphisms $A, A' : \phi^* M \xrightarrow{\sim} M$, there exists an open neighborhood U of the identity in the topological group $\mathcal{G}_M$ such that if $A' \cdot A^{-1} \in U$ then A, A' are ϕ-conjugate by a uniquely determined element of $\mathcal{G}_M$.

The argument is similar as above: Suppose that $A : \phi^* M \xrightarrow{\sim} M$ is an $R((u))$-isomorphism and for $h \in \mathcal{G}_M$ define $h_0 = h$ and inductively

$$h_i = A \cdot \phi^*(h_{i-1}) \cdot A^{-1}.$$

The result follows from the statement: There is an open neighborhood U of I such that for $h \in U$, we have $\lim_{i\to\infty} h_i = I$ and the limit $\tilde{h} = \lim_{i\to\infty} \prod_{0\leq j\leq i} h_j$ exists. Indeed, the arguments above show that this is true when M is a free $R((u))$-module. In general, let $R \to R'$ be a flat homomorphism such that $M' = M\hat{\otimes}_R R' \simeq R'((u))^d$. Consider $h'_i = h_i \otimes 1 \in \mathcal{G}_{M'}$ and let L be a lattice in M. Then $L' = L\hat{\otimes}_R R'$ is a lattice in M'. By the above, there is n such that when $h \in U_n(L)$ (and hence $h' = h \otimes 1 \in U_n(L')$) we have

$$(h_i(x) - x) \otimes 1 = h'_i(x') - x' \in u^{\kappa(i)} L'$$

for a strictly increasing sequence $\kappa(i)$. Since $u^{\kappa(i)} L' \cap M = u^{\kappa(i)} L$ this shows the result.

We now continue with the proof of Theorem 2.1 (a). Recall

$$\mathcal{C}_m = [LG^{\leq m}/_\phi\, L^+G]$$

Let $n > 2m/(p-1)$. Recall the normal subgroup U_n of L^+G and its smooth finite type quotient $\mathcal{G}_n$. Consider the quotient stack $[LG^{\leq m}/_\phi\, U_n]$. Proposition 2.2 implies that $[LG^{\leq m}/_\phi\, U_n]$ coincides with the quotient $X^{\leq m}_{n,d} := [LG^{\leq m}/U_n]$ by the free translation action of U_n on $LG^{\leq m}$. The quotient $X^{\leq m}_{n,d}$ is represented by a scheme of finite type over $\mathbb{Z}_p$. This can be seen as follows.

Recall that the quotient $X^{(m)}_{0,d} = [LG^{\leq m}/L^+G]$ is represented by the closed subscheme $\mathcal{F}^{\leq m}_G$ of the affine Grassmannian $\mathcal{F}_G = LG/L^+G$ that parametrizes lattices L such that $u^m L_0 \subset L \subset u^{-m} L_0$, where $L_0 = R_W[[u]]^d$. The natural map

$$p^{\leq m} : X^{\leq m}_{n,d} \to X^{\leq m}_{0,d} = \mathcal{F}^{\leq m}_G$$

is represented by the $\mathcal{G}_n$-torsor that parametrizes pairs (L, α) where L is a lattice as above and $\alpha : L_0/u^n L_0 \xrightarrow{\sim} L/u^n L$ is an $R_W[[u]]/(u^n)$-isomorphism.

Combining the above, we now see that

$$\mathcal{C}_m \simeq [X^{\leq m}_{n,d}/_\phi\, \mathcal{G}_n]\ , \tag{2.5}$$

where the quotient is for the action of the smooth group scheme $\mathcal{G}_n$ on $X^{\leq m}_{n,d}$ which is induced by ϕ-conjugation. This (right) action of $\mathcal{G}_n$ on $X^{\leq m}_{n,d}$ can be explicitly described as follows: Let $\gamma \in \mathcal{G}_n(R)$ which we can lift to $g \in L^+G(R) = GL_d(R_W[[u]])$ and consider the point $x = (L, \alpha) \in X^{\leq m}_{n,d}(R)$ given through the matrix A by $L = L_0 \cdot A$, $\alpha = A \bmod u^n$. (Here the elements of L_0 are viewed as row vectors.) Then $x \star \gamma$ is the point of $X^{\leq m}_{n,d}$ that corresponds to the matrix $g^{-1}\cdot A\cdot\phi(g)\in LG^{\leq m}(R)$. Observe that if $n'>n>2m/(p-1)$, the natural morphism $X^{\leq m}_{n',d}\to X^{\leq m}_{n,d}$ induces an isomorphism

$$[X^{\leq m}_{n',d}/_\phi\,\mathcal{G}_{n'}]\xrightarrow{\sim}[X^{\leq m}_{n,d}/_\phi\,\mathcal{G}_n\,]\,. \tag{2.6}$$

It follows from (2.5) and the above that $\mathcal{C}_m$ is an algebraic (Artin) stack of finite type over $\mathbb{Z}_p$ of dimension equal to the dimension of the scheme $\mathcal{F}^{\leq m}_{G}$. It is clear that we can represent $\mathcal{C}$ as the 2-limit of the algebraic stacks $\mathcal{C}_m$ and so the rest of (a) follows.

For part (b) observe that the quotient description of $\mathcal{C}_m$ implies the existence of

$$q_m:\mathcal{C}_m=[LG^{\leq m}/_\phi\,L^+G]\to[L^+G\backslash\mathcal{F}^{\leq m}_{G}]=[\mathcal{F}^{\leq m}_{G}/L^+G]$$

(here in the last quotient $g\in L^+G$ acts by $L\cdot g=g^{-1}L$). This descends the quotient morphism

$$LG^{\leq m}\to LG^{\leq m}/L^+G=\mathcal{F}^{\leq m}_{G}\ .$$

Now for $n\geq 2m$, U_n acts trivially on $\mathcal{F}^{\leq m}_{G}$ and the action of L^+G on $\mathcal{F}^{\leq m}_{G}$ factors through the quotient $\mathcal{G}_n$. Hence composing q_m with the quotient morphism by U_n, we obtain a morphism $q'_m:\mathcal{C}_m\to[\mathcal{F}^{\leq m}_{G}/\mathcal{G}_n]$. When $n>2m/(p-1)$, the morphism q'_m is given by taking the quotient

$$[X^{\leq m}_{n,d}/_\phi\,\mathcal{G}_n]\to[\mathcal{F}^{\leq m}_{G}/\mathcal{G}_n]$$

of the smooth torsor $X^{\leq m}_{n,d}\to\mathcal{F}^{\leq m}_{G}$ and hence is smooth. It follows that q_m itself is formally smooth. Also the morphism $\bar q_m$ of the statement of part (b), which is given as a composition of q'_m with $[\mathcal{F}^{\leq m}_{G}/\mathcal{G}_n]\to[\mathcal{F}^{\leq m}_{G}/\mathcal{G}_{2m}]$, is also smooth. A straightforward dimension count now gives that the relative dimension of $\bar q_m$ is equal to the (relative) dimension of $\mathcal{G}_{2m}$ over $\mathbb{Z}_p$; this is equal to $2md^2$. □

2.c. We consider now some properties of $\mathcal{R}$, $\theta_m:\mathcal{C}_m\to\mathcal{R}$ and $\theta:\mathcal{C}\to\mathcal{R}$. Recall that, for each $R_W((u))$-module M which is fpqc locally on $S=\mathrm{Spec}\,(R)$ free of rank d, we have the (twisted) affine Grassmannian $Gr_M\to S$ whose A-points for an R-algebra A are given by $A_W[[u]]$-lattices $\mathfrak{M}$ of $M_A=M\hat\otimes_R A$. By [Dr] (Theorem 3.8 and Remark (b) below it), Gr_M is represented by an ind-algebraic space which is ind-proper and of ind-finite presentation over S.

Theorem 2.5. *a) For each $S=\mathrm{Spec}\,(R)\to\mathcal{R}$ which corresponds to a $R_W((u))$-Φ-module (M,Φ), the fiber products*

$$\theta\times_{\mathcal{R}}S:\mathcal{C}\times_{\mathcal{R}}S\to S\ ,\quad \theta_m\times_{\mathcal{R}}S:\mathcal{C}_m\times_{\mathcal{R}}S\to S,$$

are represented by the (twisted) affine Grassmannian $Gr_M\to S$, resp. by a proper algebraic subspace of $Gr_M\to S$.

b) The diagonal morphism $\delta:\mathcal{R}\to\mathcal{R}\times_{\mathbb{Z}_p}\mathcal{R}$ is representable and of finite presentation.

Corollary 2.6. *a) $\theta:\mathcal{C}\to\mathcal{R}$ is ind-representable and ind-proper.*

b) $\theta_m:\mathcal{C}_m\to\mathcal{R}$ is representable, proper and of finite presentation.

Proof. Part (a). The first part of the statement regarding $\theta\times_{\mathcal R} S:\mathcal C\times_{\mathcal R} S\to S$ follows from the definition. Note here that we do not necessarily know that M contains a free $R_W[[u]]$-lattice. However, there is a flat base change $R\to R'$ such that $M'=M\hat\otimes_R R'$ is $R'_W((u))$-free. Then there is a (free) $R'_W[[u]]$-lattice $\mathfrak M'_0$ in M'. We will now show the second part of the statement. Let δ be the smallest integer for which

$$u^\delta\mathfrak M'_0\subset\Phi(\phi^*\mathfrak M'_0)\subset u^{-\delta}\mathfrak M'_0. \tag{2.7}$$

Set $S'=\mathrm{Spec}\,(R')$ with R' as above. Suppose $T=\mathrm{Spec}\,(A)$ is an S-scheme and set $A'=A\otimes_R R'$, $T'=T\times_S S'=\mathrm{Spec}\,(A')$. Let $\mathfrak M$ be an $A'_W[[u]]$-Φ-lattice in $M\hat\otimes_R A'$ that corresponds to an object of $(\mathcal C_m\times_{\mathcal R} S')(T')$. For simplicity, set $\mathfrak M'_{0,A}=\mathfrak M'_0\hat\otimes_{R'}A'$. Then

$$u^m\mathfrak M\subset\Phi(\phi^*\mathfrak M)\subset u^{-m}\mathfrak M\ ,\quad u^N\mathfrak M'_{0,A}\subset\mathfrak M\subset u^{-N}\mathfrak M'_{0,A}, \tag{2.8}$$

for some $N\geq 0$ (we can suppose that N is the smallest integer with this property). Applying Φ to the second chain of inclusions (2.8) gives

$$u^{pN}\Phi(\phi^*\mathfrak M'_{0,A})\subset\Phi(\phi^*\mathfrak M)\subset u^{-pN}\Phi(\phi^*\mathfrak M'_{0,A})$$

and pN is the smallest integer with this property. On the other hand, we have

$$\Phi(\phi^*\mathfrak M)\subset u^{-m}\mathfrak M\subset u^{-m-N}\mathfrak M'_{0,A}\subset u^{-m-N-\delta}\Phi(\phi^*\mathfrak M'_{0,A}),\ \text{and}$$
$$u^{m+N+\delta}\Phi(\phi^*\mathfrak M'_{0,A})\subset u^{m+N}\mathfrak M'_{0,A}\subset u^m\mathfrak M\subset\Phi(\phi^*\mathfrak M).$$

Combining these gives

$$u^{N+m+\delta}\Phi(\phi^*\mathfrak M'_{0,A})\subset\Phi(\phi^*\mathfrak M)\subset u^{-N-m-\delta}\Phi(\phi^*\mathfrak M'_{0,A}).$$

This implies that $pN\leq N+m+\delta$, i.e $N\leq(m+\delta)/(p-1)$, and so

$$u^{[\frac{m+\delta}{p-1}]}\mathfrak M'_{0,A}\subset\mathfrak M\subset u^{-[\frac{m+\delta}{p-1}]}\mathfrak M'_{0,A}.$$

(This is essentially the same argument as in [Ki1] Prop. 2.1.7.) By the above, and the definition [Dr] of the ind-structure on Gr_M, this implies that $\mathcal C_m\times_{\mathcal R} S'$ is represented by a proper S'-scheme; therefore, by descent, $\mathcal C_m\times_{\mathcal R} S$ is an S-proper algebraic space.

Part (b). Suppose that M, N are two $R_W((u))$-Φ–modules of rank d. Consider the functor on R-algebras

$$A\mapsto \mathrm{Isom}_{\mathcal R}(M,N)(A):=\mathrm{Isom}_{\Phi,A_W((u))}(M_A,N_A),$$

where for simplicity we write $M_A=M\hat\otimes_R A$, $N_A=N\hat\otimes_R A$. We will show that this is representable by a scheme of finite presentation over R. This implies then the statement in (b). Using the existence of R_W-lattices in both M and N ([Dr], see Remark 2.4), we can see that the functor that sends R to the $R_W((u))$-linear isomorphisms $M\to N$ is represented by an ind-scheme. It is not hard to see that $\mathrm{Isom}_{\mathcal R}(M,N)$ is represented by a ind-closed ind-subscheme of this ind-scheme. To show that this is actually a scheme of finite presentation we can employ an fpqc base change $R\to R'$ and assume that $M'=M_{R'}$, $N'=N_{R'}$ are given by A, $B\in LG(R')$. By the definitions, the additional condition on the

$R'_W((u))$-linear isomorphism $M' \to N'$ given by $g \in GL_d(R'_W((u)))$ that guarantees that it respects Φ is

$$A = g^{-1} \cdot B \cdot \phi(g), \quad \text{or equivalently,}$$

$$g = B \cdot \phi(g) \cdot A^{-1} \ . \tag{2.9}$$

Suppose that A and B are in $LG^{\leq m}(R')$, $LG^{\leq n}(R')$ respectively. Assume that g is in $LG^{\leq s}(R')$ and $s = s(g)$ is the smallest integer with that property. Then $\phi(g)$ belongs to $LG^{\leq ps}(R')$ and we can see that ps is the smallest integer with this property. The identities above now imply that $\phi(g)$ is in $LG^{\leq s+m+n}(R')$. Therefore $ps \leq s + m + n$ which gives

$$s \leq \frac{m+n}{p-1} \ . \tag{2.10}$$

Let us write out (2.9) explicitly

$$\sum_{i \geq -s} g_i u^i = B \cdot (\sum_{i \geq -s} g_i u^{pi}) \cdot A^{-1} = \sum_{i \geq -s} u^{pi} \cdot (B \cdot g_i \cdot A^{-1}) \tag{2.11}$$

with $g_i \in M_d(R')$. Now consider the matrix identity obtained by comparing the u^a terms of both sides of (2.11) for $a > (m+n)/(p-1)$. We see that this has the form

$$g_a = \sum_{i,k,l} B_l \cdot g_i \cdot A'_k \tag{2.12}$$

with $pi + k + l = a$ and $i \geq -s$, $k \geq -m$, $l \geq -n$ and $A^{-1} = \sum_{k=-m}^{\infty} A'_k u^k$. Since $a > (m+n)/(p-1)$, these inequalities imply that $i < a$. Therefore, all these matrix identities for $a > (m+n)/(p-1)$ amount to determining g_a from g_i for $i < a$. The result now follows. □

Corollary 2.7. *There is a diagram*

$$\begin{array}{ccc} & \mathcal{C} & \\ {}^{\theta}\swarrow & & \searrow^{q} \\ \mathcal{R} & & [L^+G \backslash \mathcal{F}_G] \ , \end{array}$$

where the morphism q is formally smooth and the morphism θ is ind-representable and ind-proper.

2.c.1. Recall that we set $G = \mathrm{Res}_{W/\mathbb{Z}_p} GL_d$. Let K_0 be the fraction field of W and set $f = [K_0 : \mathbb{Q}_p] = [k : \mathbb{F}_p]$. Then, after ordering the elements of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$, we can write

$$G(K_0) = \prod_{i=1}^{f} GL_d(K_0), \quad (\mathcal{F}_G)_W = \prod_{i=1}^{f} (\mathcal{F}_{GL_d})_W \ .$$

Let us set $\nu = (\nu(1), \ldots, \nu(f))$ where for each $i = 1, \ldots, f$, $\nu(i) = (n_1(i), \ldots, n_d(i))$ is a collection of integers with $n_1(i) \geq n_2(i) \geq \cdots \geq n_d(i)$. Let $F \subset K_0$ be the fixed field of the subgroup of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ that fixes ν.

Denote by $u^{\nu(i)}$ the diagonal matrix $\mathrm{diag}(u^{n_1(i)}, \ldots, u^{n_d(i)})$ in $GL_d(W((u)))$ and set

$$u^\nu = (u^{\nu(1)}, \ldots, u^{\nu(f)}) \in LG(W).$$

Suppose that $N = \max\{|n_j(i)|\}$. Let S^0_ν, resp. S_ν, be the corresponding open, resp. projective, affine Schubert variety in $(\mathcal{F}_G^{\leq N})_W \subset (\mathcal{F}_G)_W$ which is given as the image of $L^+G \cdot u^\nu \cdot L^+G$, resp. the Zariski closure of that image. By descent, we see that this is defined over the integers W' of the reflex field F. We set $\mathcal{C}_\nu$ for the Artin stack over W' which is the inverse image of S_ν under q; this is a closed substack of $(\mathcal{C}_N)_{W'}$. If $n_d(i) \geq 0$ for all i, then for $S = \mathrm{Spec}\,(R)$, the groupoid $\mathcal{C}_\nu(S)$ is given by $R_W[[u]]$-Φ-modules $(\mathfrak{M}, \Phi)$ of rank d such that $\Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$ and such that the action of u on $Coker(\Phi) = \mathfrak{M}/\Phi(\phi^*\mathfrak{M})$ has elementary divisors $u^{\nu'}$ with ν' which satisfies $\nu'(i) \leq \nu(i)$ in the usual ordering, for all i; $\mathcal{C}_\nu$ is a closed substack of $\mathcal{C}_N$.

The obvious version of Theorem 2.1 holds for the stack $\mathcal{C}_\nu$; it is an Artin stack of finite type over W' smoothly equivalent to the Schubert variety S_ν in the affine Grassmannian $\mathcal{F}_G$. Furthermore, restricting θ to $\mathcal{C}_\nu$ gives

$$\theta_\nu : \mathcal{C}_\nu \to \mathcal{R}_{W'}\ ,$$

which is representable, proper and of finite presentation. (Similarly, we can consider the inverse image $\mathcal{C}^0_\nu$ of S^0_ν under q and the restriction $\theta_\nu : \mathcal{C}^0_\nu \to \mathcal{R}_{W'}$ which is therefore representable and of finite presentation.) Summarizing, we obtain a diagram

$$\begin{array}{ccc} & \mathcal{C}_\nu & \\ {\scriptstyle\theta_\nu}\swarrow & & \searrow{\scriptstyle q} \\ \mathcal{R}_{W'} & & [(L^+G)_{W'}\backslash S_\nu]\ , \end{array} \tag{2.13}$$

where the morphism q is formally smooth and the morphism θ_ν is representable and proper.

2.d. Let us sketch how to generalize the above theory to reductive groups. Let H be any reductive algebraic group scheme H over W. Let us set $G = \mathrm{Res}_{W/\mathbb{Z}_p}(H)$. Instead of $R_W[[u]]$-Φ-modules of rank d we consider H-torsors $\mathcal{T}$ over $R_W[[u]]$ together with a H-isomorphism $\Phi : \phi_S^*(\mathcal{T}[1/u]) \xrightarrow{\sim} \mathcal{T}[1/u]$ (here $S = \mathrm{Spec}\,(R)$). The corresponding fpqc stack $\mathcal{C}_G$ can be viewed as the quotient $[LG/_\phi L^+G]$. Similarly, we can consider the fpqc stack $\mathcal{R}_G$ of H-torsors T over $R_W((u))$ (which are trivial fpqc locally on R), together with a H-isomorphism $\Phi : \phi_S^*(T) \xrightarrow{\sim} T$. The stack $\mathcal{R}_G$ can be viewed as the quotient $[LG/_\phi\, LG]$. The obvious generalizations of Theorems 2.1 and Corollary 2.6 hold in this situation. The proofs are extensions of the above proofs for GL_d after using a faithful representation $H \hookrightarrow GL_N$. For example, the ind-structure is modeled on the ind-scheme structure of

$LG = \lim_m (LG \cap L\mathrm{Res}_{W/\mathbb{Z}_P} GL_N^{\leq m})$. In particular, we again obtain a diagram

$$\begin{array}{ccc} & \mathcal{C}_G & \\ {\scriptstyle\theta}\swarrow & & \searrow{\scriptstyle q} \\ \mathcal{R} & & [L^+G\backslash\mathcal{F}_G] \ , \end{array} \tag{2.14}$$

where the morphism q is formally smooth and the morphism θ is ind-representable and ind-proper.

Suppose we are given a dominant coweight ν of G defined over an unramified extension F of $\mathbb{Q}_p$ with integers W'. Denote by $u^\nu \in LG(F) = G(F((u)))$ the element given as the image of $u \in \mathbb{G}_m(F((u)))$ by the corresponding homomorphism $\mathbb{G}_{mF} \to G_F$. As in §2.c.1, we can define S_ν^0, resp. S_ν, to be the corresponding open, resp. projective, affine Schubert variety in $(\mathcal{F}_G)_{W'}$ which is given as the image, resp. the closure of the image of $(L^+G)_{W'} \cdot u^\nu \cdot (L^+G)_{W'}$.

We set $\mathcal{C}_{G,\nu}$ for the Artin stack over W' which can be defined by descent as the inverse image of S_ν under q. The obvious version of Theorem 2.1 holds for the stacks $\mathcal{C}_{G,\nu}$; they are Artin stacks of finite type over W' smoothly equivalent to Schubert varieties in the affine Grassmannian $\mathcal{F}_G$. Once again, we have a diagram

$$(\mathcal{R}_G)_{W'} \xleftarrow{\theta_\nu} \mathcal{C}_{G,\nu} \xrightarrow{q_\nu} [(L^+G)_{W'}\backslash S_\nu] \ , \tag{2.15}$$

where the morphism q_ν is formally smooth and the morphism θ_ν is representable and proper.

3. p-adic models and local models

In this section, we define the stacks $\mathcal{C}_{d,h,K}$ and prove Theorem 0.1 of the introduction. Let K a finite extension of $\mathbb{Q}_p$ with residue field k and ramification index e. Choose a uniformizer π of K with Eisenstein polynomial $E(u)$ over $K_0 = \mathrm{Fr}(W)$. Then we can write $\mathcal{O}_K \simeq W[[u]]/(E(u))$. Fix $h \geq 1$ (the "height"). For $a \geq 1$, note that u^{ea} vanishes in $W_a[[u]]/(E(u))$ where $W_a = W/p^aW$. Hence, if R is a $\mathbb{Z}/p^a\mathbb{Z}$-algebra, we have

$$u^{eah} R_W[[u]] \subset E(u)^h R_W[[u]]. \tag{3.16}$$

Now consider the category Nil_p of schemes S such that $p^b \cdot \mathcal{O}_S = 0$ for some $b \geq 1$. Such schemes can be viewed as formal schemes over $\mathbb{Z}_p$. We will call set-valued functors on Nil_p which satisfy descent for the fpqc topology "formal spaces". A formal scheme X over $\mathrm{Spf}(\mathbb{Z}_p)$ gives a formal space by sending S in Nil_p to the set of formal scheme morphisms $S \to X$ over $\mathrm{Spf}(\mathbb{Z}_p)$. Also if $\mathcal{S}$ is a fpqc stack over $\mathbb{Z}_p$, we can consider the restriction $\hat{\mathcal{S}}$ to a groupoid over the category Nil_p; we can think of the "formal stack" $\hat{\mathcal{S}}$ as "the formal completion of $\mathcal{S}$ along its fiber over p".

3.a. Consider the functor $M_{d,h,K}$ on schemes over $\mathbb{Z}_p$ that associates to $S = \mathrm{Spec}\,(R)$ the set of $R_W[u]$-submodules

$$\mathcal{E} \subset (R_W[u]/(E(u)^h))^d\ , \tag{3.17}$$

such that both $\mathcal{E}$ and the quotient $(R_W[u]/(E(u)^h))^d/\mathcal{E}$ are R_W-projective with rank locally constant on $\mathrm{Spec}\,(R)$. This functor is represented by a projective scheme over $\mathbb{Z}_p$ (a disjoint sum of closed subschemes of Grassmannians), which we will also denote by $M_{h,K}$. Once again, here and in what follows we will omit the subscript d from the notation. In fact, if in addition $h = 1$, we will also omit h from the notation and simply write M_K. The group scheme

$$\mathrm{Res}_{(W[u]/(E(u)^h))/\mathbb{Z}_p} GL_d$$

over $\mathbb{Z}_p$ acts on $M_{h,K}$.

Suppose that $p^a \cdot R = 0$ for $a \geq 1$. Then $M_{h,K}(R)$ is in bijection with the set of $R_W[[u]]$-modules $\mathcal{L}$ with

$$u^{eah} \cdot R_W[[u]]^d \subset E(u)^h R_W[[u]]^d \subset \mathcal{L} \subset R_W[[u]]^d$$

which are, locally on R, free over $R_W[[u]]$. This gives a functorial injection,

$$M_{h,K}(R) \hookrightarrow \mathcal{F}_G(R) \tag{3.18}$$

and it implies that we can view the formal completion $\hat{M}_{h,K}$ of $M_{h,K}$ along its fiber over p as a subspace of the formal space $\hat{\mathcal{F}}_G$ defined by the affine Grassmannian $\mathcal{F}_G$.

3.b. We now define a groupoid $\mathcal{C}_{d,h,K}$ over $\mathbb{Z}_p$-schemes as follows. Let R be a $\mathbb{Z}_p$-algebra. Then $\mathcal{C}_{d,h,K}(R)$ is given by pairs $(\mathfrak{M}, \Phi)$ of an $R_W[[u]]$-module $\mathfrak{M}$ which is, locally fpqc on $\mathrm{Spec}\,(R)$, free of rank d and a $R_W((u))$-module isomorphism

$$\Phi : \phi^*\mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u], \tag{3.19}$$

such that $E(u)^h\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$. We can see that the groupoid $\mathcal{C}_{d,h,K}$ is an fpqc stack. In what follows, we will omit d from the notation and write $\mathcal{C}_{h,K}$.

We can consider the formal p-adic completion $\widehat{\mathcal{C}}_{h,K}$ of $\mathcal{C}_{h,K}$. This is a fpqc stack over Nil_p defined by considering $\mathcal{C}_{h,K}(R)$ as above for $\mathbb{Z}_p$-algebras R in which p is nilpotent. We can write $\hat{\mathcal{C}}_{h,K}$ as a 2-limit

$$\widehat{\mathcal{C}}_{h,K} := \lim_{\overrightarrow{a}} \mathcal{C}^a_{h,K}$$

where $\mathcal{C}^a_{h,K} := \widehat{\mathcal{C}}_{h,K} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$ is the reduction modulo p^a. Using (3.16) we can see that $\mathcal{C}^a_{h,K}$ is a (closed) substack of $\mathcal{C}_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$. Now suppose that $\mathcal{X}$ is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ such that $p\mathcal{O}_{\mathcal{X}}$ is an ideal of definition. Then, for each $a \geq 1$, $\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$ is a scheme over $\mathbb{Z}/p^a\mathbb{Z}$. We set

$$\widehat{\mathcal{C}}_{h,K}(\mathcal{X}) := \lim_{\overleftarrow{a}} \widehat{\mathcal{C}}_{h,K}(\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z})\ .$$

This allows us to extend $\widehat{\mathcal{C}}_{h,K}$ to a groupoid over the category of adic formal schemes over $\mathrm{Spf}(\mathbb{Z}_p)$. Suppose that R is a Noetherian p-adic ring (i.e a Noetherian $\mathbb{Z}_p$-algebra which is p-adically complete and separated, $R = \varprojlim_a R/p^a R$), and set $\mathcal{X} = \mathrm{Spf}(R)$. Set

$$R_W\{\{u\}\} = \varprojlim_a [(R_W/p^a R_W)((u))] = \left\{ \sum_{i=-\infty}^{+\infty} a_i u^i \mid a_i \in R_W, \lim_{i\to-\infty} a_i = 0 \right\}.$$

We can see that the objects of $\widehat{\mathcal{C}}_{h,K}(\mathrm{Spf}(R))$ are given by pairs $(\mathfrak{M}, \Phi)$ of an $R_W[[u]]$-module $\mathfrak{M}$ which is locally $R_W[[u]]$-free of rank d and a $R_W\{\{u\}\}$-module isomorphism

$$\Phi : \phi^*\mathfrak{M} \otimes_{R_W[[u]]} R_W\{\{u\}\} \xrightarrow{\sim} \mathfrak{M} \otimes_{R_W[[u]]} R_W\{\{u\}\}, \tag{3.20}$$

such that

$$E(u)^h \mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}\ . \tag{3.21}$$

(Note that $E(u)$ is a unit in $W\{\{u\}\}$ and so therefore also in $R_W\{\{u\}\}$.)

3.b.1. For $\mathrm{Spec}\,(R)$ in Nil_p now set

$$LG^{h,K}(R) = \{A \in M_d(R_W[[u]]) \mid A^{-1} \in E(u)^{-h} \cdot M_d(R_W[[u]]) \subset M_d(R_W((u)))\}.$$

This defines a functor on Nil_p. As before, we can write

$$\widehat{\mathcal{C}}_{h,K} = [LG^{h,K}/_\phi\, L^+G].$$

where (by abusing notation) we also denote by L^+G the formal p-adic completion of L^+G. The map

$$A \mapsto A \cdot R_W[[u]]^d \subset R_W((u))^d$$

gives a morphism of formal stacks,

$$q_{h,K} : \widehat{\mathcal{C}}_{h,K} = [LG^{h,K}/_\phi\, L^+G] \to [L^+G\backslash \widehat{M}_{h,K}].$$

3.b.2. Using Proposition 2.2 and the above, we see that if $n(a) > eah/(p-1)$ then

$$[LG^{h,K}/_\phi U_{n(a)}]_{\mathbb{Z}/p^a\mathbb{Z}} \simeq [LG^{h,K}/U_{n(a)}]_{\mathbb{Z}/p^a\mathbb{Z}}.$$

As in the proof of Theorem 2.1 we can see that the quotient stack $[LG^{h,K}/U_{n(a)}]_{\mathbb{Z}/p^a\mathbb{Z}}$ is represented by a torsor $(X^{h,K}_{n(a),d})_{\mathbb{Z}/p^a\mathbb{Z}}$ for the group scheme $(L^+G/U_{n(a)})_{\mathbb{Z}/p^a\mathbb{Z}} = (\mathcal{G}_{n(a)})_{\mathbb{Z}/p^a\mathbb{Z}}$ over $(M_{h,K})_{\mathbb{Z}/p^a\mathbb{Z}}$. Similarly to that proof, we can conclude

$$\mathcal{C}^a_{h,K} \simeq [(X^{h,K}_{n(a),d})_{\mathbb{Z}/p^a\mathbb{Z}}/_\phi\, (\mathcal{G}_{n(a)})_{\mathbb{Z}/p^a\mathbb{Z}}]\ ,$$

and that the morphism

$$q^a_{h,K} : \mathcal{C}^a_{h,K} \to [L^+G\backslash M_{h,K}]_{\mathbb{Z}/p^a\mathbb{Z}}$$

is formally smooth. Hence, the morphism between the formal stacks

$$q_{h,K} : \widehat{\mathcal{C}}_{h,K} \to [L^+G\backslash \widehat{M}_{h,K}] \tag{3.22}$$

is also formally smooth.

Recall the definition of the stack $\mathcal{R} = \mathcal{R}_d$ over $\mathbb{Z}_p$-schemes, cf. §2.a, and the notations $\mathcal{R}^a = \mathcal{R} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$, $\mathcal{C}^a_{h,K} = \mathcal{C}_{h,K} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$. By sending an object $(\mathfrak{M}, \Phi)$ to $(\mathfrak{M}[1/u], \Phi)$, we obtain a morphism of stacks $\theta : \mathcal{C}_{h,K} \to \mathcal{R}$. Note that the morphism $\widehat{\theta} : \widehat{\mathcal{C}}_{h,K} \to \widehat{\mathcal{R}}$ on formal completions is obtained by passing to the limit on the morphisms $\theta^a : \mathcal{C}^a_{h,K} \to \mathcal{R}^a$ which arise by restricting the morphisms

$$\theta_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z} : \mathcal{C}_{eah} \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z} \to \mathcal{R}^a$$

to the closed substacks $\mathcal{C}^a_{h,K}$. This together with Corollary 2.6 implies that the morphisms θ^a are representable and proper.

Our discussion, in the previous two paragraphs gives the proof of Theorem 0.1 of the introduction when $h = 1$.

Remark 3.1. Of course, the above actually shows that the obvious generalization of Theorem 0.1 to $\mathcal{C}_{h,d,K}$ for any $h \geq 1$ is also valid.

Remark 3.2. a) The morphism $\theta : \mathcal{C}_{h,K} \to \mathcal{R}$ is ind-representable: Indeed, suppose that we are given a point $\xi : S = \mathrm{Spec}\,(R) \to \mathcal{R}$ corresponding to a module (M, Φ) over $R_W((u))$. Then by Theorem 2.5, the fiber $\mathcal{C} \times_{\mathcal{R},\xi} S \to S$ is ind-represented by an ind-algebraic space. We can see that the subspace $\mathcal{C}_{h,K} \times_{\mathcal{R},\xi} S \hookrightarrow \mathcal{C} \times_{\mathcal{R},\xi} S$ is described by a closed condition and it is a closed ind-algebraic subspace.

b) In general, $\mathcal{C}_{h,K} \times_{\mathcal{R},\xi} S \to S$ is not representable for all $S = \mathrm{Spec}\,(R)$. However, assume that $R \simeq \varprojlim_a R/p^aR$ is a Noetherian p-adic ring and that

$$\hat{\xi} = (\xi^a)\ , \quad \xi^a : \mathrm{Spec}\,(R/p^aR) \to \mathcal{R}^a\ ,$$

is a point of the formal completion $\widehat{\mathcal{R}}$. Assume also that the $R_W\{\{u\}\}$-module $\hat{M}$ which corresponds to $\hat{\xi}$ is free over $R_W\{\{u\}\}$. Fix a basis $R_W\{\{u\}\}^d \simeq \hat{M}$ and set $M = R_W((u))^d \subset \hat{M}$. The affine Grassmannian $Gr_M \to S$ is ind-projective and supports a natural line bundle whose restriction on each closed subscheme is very ample. As above, we can see that for each a, the fiber of the morphism $\theta^a : \mathcal{C}^a_{h,K} \to \mathcal{R}^a$ over ξ^a is representable by a closed (and hence) projective subscheme of $\mathrm{Gr}_M \times_{\mathbb{Z}_p} \mathbb{Z}/p^a\mathbb{Z}$. Varying a, this defines a formal scheme over $\mathrm{Spf}(R)$. As in [Ki3], proof of Prop. 1.3, by using the above ample line bundle on Gr_M, we may algebraicize this p-adic formal scheme over $\mathrm{Spf}(R)$ to a projective scheme $\mathcal{C}_{K,\xi}$ over $\mathrm{Spec}\,(R)$. The result is that, in this case, the fiber $\widehat{\mathcal{C}}_{d,h,K} \times_{\widehat{\mathcal{R}},\hat{\xi}} \hat{S} \to \hat{S}$ between the formal completions is representable by the formal scheme over $\hat{S} = \mathrm{Spf}(R)$ associated to the projective scheme $\mathcal{C}_{K,\xi} \to \mathrm{Spec}\,(R)$. If in addition to the hypotheses above, R/pR is of finite type over $\mathbb{F}_p$, the arguments in the proof of [Ki3] Prop. 1.6.4, show that the morphism $\mathcal{C}_{K,\xi} \to \mathrm{Spec}\,(R)$ induces a closed immersion between the generic fibers.

3.c. Choose a cocharacter

$$\mu : \bar{\mathbb{Q}}_p^\times \to (\mathrm{Res}_{K/\mathbb{Q}_p} GL_d)(\bar{\mathbb{Q}}_p) = \prod_{\psi : K \to \bar{\mathbb{Q}}_p} GL_d(\bar{\mathbb{Q}}_p)$$

defined over $\bar{\mathbb{Q}}_p$ whose conjugacy class is defined over the reflex field E. The projection to the component corresponding to ψ

$$\mu_\psi = pr_\psi \circ \mu : \bar{\mathbb{Q}}_p^\times \to GL_d(\bar{\mathbb{Q}}_p)$$

provides a grading

$$\bar{\mathbb{Q}}_p^d = \bigoplus_{n\in\mathbb{Z}} V_n^\psi \ , \tag{3.23}$$

with $V_n^\psi = \{v \in \bar{\mathbb{Q}}_p^d \mid \mu_\psi(a) = a^n v\}$. Let h_+, resp. h_-, the maximum, resp. minimum value of n (among all the values for all ψ) for which $V_n^\psi \neq (0)$. Set $h = h_+ - h_-$.

We will now define the corresponding local model $M^{\rm loc}_{\mu,K}$ over $\mathcal{O}_E$; it is going to be a projective subscheme of $M_{d,h,K}$ (see 3.a).

First, we define the generic fiber of $M^{\rm loc}_{\mu,K}$ over the reflex field E. Suppose that R is a $\bar{\mathbb{Q}}_p$-algebra, and fix an embedding $\psi : K \to \bar{\mathbb{Q}}_p$, this induces a homomorphism

$$R_W = W \otimes_{\mathbb{Z}_p} R \to R\ ; \quad a \otimes r \mapsto \psi(a) r\ .$$

Elements of $M_{h,K}(R)$ correspond bijectively to $R_W[u]$-modules $\mathcal{M}$ such that

$$E(u)^{h_+} R_W[u]^d \subset \mathcal{M} \subset E(u)^{h_-} R_W[u]^d$$

with graded pieces R_W-projective and with rank locally constant on Spec$\,(R)$. Write

$$\mathrm{Norm}_{K_0/\mathbb{Q}_p}(E(u)) = \prod_{\psi:K\to\bar{\mathbb{Q}}_p} (u - \varpi_\psi) \in \bar{\mathbb{Q}}_p[u]$$

so that $\varpi_\psi = \psi(\pi)$. Using this, we see that we can write $\mathcal{M} = \oplus_\psi \mathcal{M}_\psi$ with $\mathcal{M}_\psi$ a $R[u]$-submodule with

$$(u - \varpi_\psi)^{h_+} R[u]^d \subset \mathcal{M}_\psi \subset (u - \varpi_\psi)^{h_-} R[u]^d.$$

For each such ψ, consider the R-module

$$\mathcal{M}_\psi \cap (u - \varpi_\psi)^j R[u]^d / \mathcal{M}_\psi \cap (u - \varpi_\psi)^{j+1} R[u]^d$$

We ask that for each ψ, $j \in \mathbb{Z}$, this is a projective R-module of rank $\dim(V_j^\psi)$. We can see that this condition defines a locally closed subvariety of $M_{h,K} \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$. This carries an action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$ that allows us to descend it to a subvariety Z of $M_{h,K} \otimes_{\mathbb{Z}_p} E$. By definition, the generic fiber of the local model $M^{\rm loc}_{\mu,K}$ is the Zariski closure $\bar{Z}$ of Z in $M_{h,K} \otimes_{\mathbb{Z}_p} E$. Finally, by definition, the local model $M^{\rm loc}_{\mu,K}$ is the flat closure of $\bar{Z}$ in $M_{h,K} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$.

Observe that, by the above, for each $\mathcal{O}_E$-scheme $S = \mathrm{Spec}\,(R)$ in Nil_p we have

$$M^{\rm loc}_{\mu,K}(R) \hookrightarrow (\mathcal{F}_G \otimes_{\mathbb{Z}_p} \mathcal{O}_E)(R). \tag{3.24}$$

Remark 3.3. Suppose that, for all ψ, we have $n \in \{0, 1\}$ in (3.23). Then μ is miniscule. Assume $h_+ = 1$, $h_- = 0$, which is the typical case. Then $h = 1$, $R_W[u]/(E(u)^h) = \mathcal{O}_K \otimes_{\mathbb{Z}_p} R$, and $M_{h,K}(R)$ is given by $\mathcal{O}_K \otimes_{\mathbb{Z}_p} R$-submodules

$$\mathcal{E} \subset (\mathcal{O}_K \otimes_{\mathbb{Z}_p} R)^d$$

which are locally on R direct summands as R_W-modules. Set $r_\psi = \dim(V_0^\psi)$. The conditions above amount to asking that $\mathrm{rank}_R(\mathcal{E}_\psi) = r_\psi$ and so $M^{\rm loc}_{\mu,K}$ agrees with the local model of [PR1]. If $k = \mathbb{F}_p$, the special fiber $M^{\rm loc}_{\mu,K} \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}_p$ can be identified with the affine Schubert variety S_ν in the affine Grassmannian of GL_d, where the coweight ν is the dual partition to $(r_\psi)_\psi$, i.e., $\nu_i = \sharp\{\psi \mid r_\psi \geq i\}$, cf. [PR1], Thm. 5.4.

3.d. We continue with the above notations of §3.a. By definition we have a closed immersion

$$\widehat{M}^{\rm loc}_{\mu,K} \hookrightarrow \widehat{M}_{h,K}$$

which is equivariant for the natural action of (the formal completion of) L^+G. Using descent and (3.22) we obtain a fpqc stack $\widehat{\mathcal{C}}_{\mu,K}$ over $\mathrm{Nil}_p \cap (\mathrm{Sch}/\mathcal{O}_E)$ together with a formally smooth morphism

$$q_{\mu,K} : \widehat{\mathcal{C}}_{\mu,K} \to [L^+G \backslash \hat{M}^{\rm loc}_{\mu,K}].$$

Indeed, if $S = \mathrm{Spec}\,(R)$ is an $\mathrm{Spec}\,(\mathcal{O}_E)$-scheme in Nil_p then $\widehat{\mathcal{C}}_{\mu,K}(R)$ is the groupoid of pairs $(\mathfrak{M}, \Phi)$ of a $R_W[[u]]$-module $\mathfrak{M}$ which is $R_W[[u]]$-free of rank d (fpqc) locally on R and a $R_W((u))$-module isomorphism

$$\Phi : \phi^*\mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u]\ , \tag{3.25}$$

such that, locally, there is an isomorphism $\alpha : R_W[[u]]^d \xrightarrow{\sim} \mathfrak{M}$ for which the $R_W[[u]]$-lattice $\alpha^{-1}(\Phi(\phi^*\mathfrak{M})) \subset R_W((u))^d$ belongs to the subset $\hat{M}^{\rm loc}_{\mu,K}(R)$ of $(\mathcal{F}_G \otimes_{\mathbb{Z}_p} \mathcal{O}_E)(R)$. As in §3.b.2, we see that there is also a morphism of formal completions

$$\widehat{\theta}_\mu : \widehat{\mathcal{C}}_{\mu,K} \to \widehat{\mathcal{R}}_d \otimes_{\mathbb{Z}_p} \mathcal{O}_E \tag{3.26}$$

which can be obtained as the limit of representable and proper morphisms.

4. Deformations of Galois representations

In this section, we explain an aspect of the connection between the spaces of Φ-modules and the deformation theory of Galois representations as developed by Kisin [Ki1], [Ki3]. We restrict attention to the *flat* or *Barsotti-Tate* case (cf. [Ki1]) This corresponds to the case $h = 1$. For simplicity, we also assume p is odd.

4.a. Galois representations. Suppose that $R = \Lambda$ is a $\mathbb{Z}_p$-algebra with finitely many elements. As in §1 (see also [Fo]), a pair (M, Φ) corresponding to an object of $\mathcal{R}(\Lambda)$ gives an étale Λ-sheaf over $\mathrm{Spec}\,(k((u)))$ which is free of rank d, i.e an equivalence class of a representation

$$\rho_{(M,\Phi)} : \mathrm{Gal}(k((u))^{\rm sep}/k((u))) \to GL_d(\Lambda).$$

As a result, an object $(\mathfrak{M}, \Phi)$ of $\mathcal{C}_d(\Lambda)$ also gives a representation $\rho_{(\mathfrak{M}[1/u],\Phi)}$ of the Galois group $\mathrm{Gal}(k((u))^{\rm sep}/k((u)))$.

Now let $\mathbb{F}$ be a finite field and suppose that

$$\rho : \mathrm{Gal}(k((u))^{\rm sep}/k((u))) \to GL_d(\mathbb{F})$$

is a representation which corresponds to a pair (M_0, Φ_0), and consider the corresponding object $[\rho] : \mathrm{Spec}\,(\mathbb{F}) \to \mathcal{R}$. Denote by $\mathcal{R}_{[\rho]}$ the groupoid over finite local Artinian $\mathbb{Z}_p$-algebras Λ with residue field $\mathbb{F}$, with objects

$$\mathcal{R}_{[\rho]}(\Lambda) = \{(M, \Phi) \in \mathcal{R}(\Lambda), \alpha : (M, \Phi) \otimes_\Lambda \mathbb{F} \xrightarrow{\sim} (M_0, \Phi_0)\}$$

and obvious morphisms. By the above, $\mathcal{R}_{[\rho]}$ is identified with the groupoid of deformations $\mathfrak{D}_\rho$ of the Galois representation ρ.

4.b. Finite flat group schemes. Now let K be a finite extension of $\mathbb{Q}_p$ with residue field k and ramification index e. Choose a uniformizer π of K with Eisenstein polynomial $E(u)$ over $\mathrm{Fr}(W)$. Set $K_\infty = \cup_n K(\pi_n)$, where $\pi_n = \pi^{1/p^n}$ are compatible choices of roots; then the theory of norm fields allows us to identify the Galois groups

$$G_\infty := \mathrm{Gal}(\bar{K}/K_\infty) \xrightarrow{\sim} \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u))) \ ,$$

comp. [Ki1], §1. The following can be derived from [Ki2] Theorem 0.5 by taking into account the functoriality of the Λ-action and the properties of the Breuil-Kisin module functors (for example see [Ki1] §1.2).

Theorem 4.1. *(Kisin) Assume $p > 2$ and let Λ be a $\mathbb{Z}_p$-algebra with finitely many elements. There is an equivalence between the groupoid of finite flat commutative group schemes $\mathcal{G}$ with an action of Λ (i.e "Λ-module schemes") over $\mathcal{O}_K$ such that $\mathcal{G}(\bar{\mathcal{O}}_K) \simeq \Lambda^d$, and the groupoid of pairs $(\mathfrak{M}, \Phi)$ of $\Lambda_W[[u]]$-Φ-modules with the following properties:*

a) $\mathrm{Coker}(\Phi)$ *is annihilated by $E(u)$,*

b) $\mathfrak{M}[1/u]$ is $\Lambda_W((u))$-free of rank d,

c) $\mathfrak{M}$ is a $W[[u]]$-module of projective dimension 1, i.e., equivalently by [Ki2]*, Lemma (2.3.2), $\mathfrak{M}$ is an iterated extension of free $k[[u]]$-modules.*

Under this equivalence, the restriction of

$$\rho_{\mathcal{G}} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}_\Lambda(\mathcal{G}(\bar{\mathcal{O}}_K))$$

to $G_\infty \simeq \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$ is isomorphic to $\rho_{(\mathfrak{M}[1/u],\Phi)}(1)$ [twist by the cyclotomic character].

Note that property (c) is automatically satisfied when $p \cdot \Lambda = (0)$. We will also consider the groupoid of modules as above with the additional property

d) $\mathfrak{M}$ is $\Lambda_W[[u]]$-free.

Lemma 4.2. *Let A be a local Noetherian ring with residue field l and suppose that $M \subset A((u))^d$ is a finitely generated $A[[u]]$-module such that $M[1/u] = A((u))^d$ and $M \otimes_A l \simeq l[[u]]^d$. Then $M \simeq A[[u]]^d$.*

Proof. By Nakayama's lemma, there is a surjective $A[[u]]$-homomorphism $\phi : A[[u]]^d \to M$. Then $\phi[1/u] : A((u))^d \to M[1/u] = A((u))^d$ is also a surjection which then has to be bijective. This implies that ϕ is also injective. □

Write $\Lambda \otimes_{\mathbb{Z}_p} W = \prod_j \Lambda_j$ with Λ_j Artin local. An application of the lemma to Λ_j shows that the additional property (d) is satisfied if and only if $\mathfrak{M} \otimes_\Lambda \mathbb{F} \simeq (\mathbb{F} \otimes k)[[u]]^d$.

4.c. Suppose that $(A, \mathfrak{m})$ is an Artin local Noetherian ring with finite residue field $\mathbb{F}$. A representation

$$\rho : \mathrm{Gal}(\bar{K}/K) \to GL_d(A)$$

is called *flat* [Ra] if the corresponding $\mathbb{Z}_p[\mathrm{Gal}(\bar{K}/K)]$-module is isomorphic to the twist by (-1) of the module obtained by the Galois action on the $\mathbb{Z}_p$-module of $\mathcal{O}_{\bar{K}}$-points of some commutative finite flat group scheme over $\mathcal{O}_K$. This notion extends to the more general situation that $(A, \mathfrak{m})$ is a complete local Noetherian ring with finite residue field $\mathbb{F}$. In this case, the representation

$$\rho : \mathrm{Gal}(\bar{K}/K) \to GL_d(A)$$

is *flat* iff, for all $n \geq 1$, the representation obtained by reducing ρ modulo $\mathfrak{m}^n$ is flat, cf. [Ra].

Consider the morphism of formal stacks $\theta_K : \widehat{\mathcal{C}}_K := \widehat{\mathcal{C}}_{1,K} \to \widehat{\mathcal{R}}$. Suppose that $(A, \mathfrak{m})$ is a complete local Noetherian ring with finite residue field $\mathbb{F}$. Let $\xi = (\xi_n)_{n\geq 1} \in \widehat{\mathcal{R}}(A)$ be an A-valued object of $\widehat{\mathcal{R}}$, where for each $n \geq 1$, ξ_n is in $\widehat{\mathcal{R}}(A/\mathfrak{m}^n)$. The 2-fiber product $\xi_n \times_{\widehat{\mathcal{R}}} \widehat{\mathcal{C}}_K$ is representable by a projective scheme $\mathcal{C}_{K,\xi_n} \to \mathrm{Spec}\,(A/\mathfrak{m}^n)$. In the limit, we obtain a formal scheme $\mathcal{C}_{K,\xi}$ over $\mathrm{Spf}(A)$. The argument in [Ki3], Cor. 1.5.1 (or see Remark 3.2 (b)) shows that this is algebraizable to a projective scheme

$$\mathcal{C}_{K,\xi} \to \mathrm{Spec}\,(A).$$

Denote by A^K the quotient of A that corresponds to the scheme theoretic image of this morphism. We obtain

$$\xi_K : \mathrm{Spec}\,(A^K) \to \mathrm{Spec}\,(A) \to \widehat{\mathcal{R}}.$$

Proposition 4.3. *With the above notations, assume in addition that A is Artinian. Then*

$$\rho_{\xi_K} : G_\infty = \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u))) \to GL_d(A^K)$$

extends to a representation of $\mathrm{Gal}(\bar{K}/K)$ *which is flat.*

Proof. For simplicity, set $B = A^K$. Then there is $B \hookrightarrow B'$ with B' a B-algebra of finite type such that $\mathcal{C}_K$ affords a B'-valued point ζ' that lifts ξ_B. Denote by (M_B, Φ) the $B_W((u))$-module that corresponds to ξ_B. Since B is Artinian, $M_B \simeq B_W((u))^d$. Giving the point ζ' amounts to giving a $B'_W[[u]]$-projective module $\mathfrak{M}'$ of rank d in $M' = M_B \hat{\otimes}_B B'$ which satisfies

$$E(u)\mathfrak{M}' \subset \Phi(\phi^*\mathfrak{M}') \subset \mathfrak{M}'.$$

Now set $\mathfrak{M} := \mathfrak{M}' \cap M \subset M'$; this is a $B_W[[u]]$-module which is Φ-stable. The proof of [Ki1] Prop. 2.1.4 applies to show that $\mathfrak{M}$ satisfies properties (a), (b) and (c) (with $\Lambda = B$). Therefore, the $\mathbb{Z}_p[G_\infty]$-module given by $\rho(1)$ is given by a finite flat group scheme over $\mathcal{O}_K$ as desired. (However, note here that, as pointed out by Kisin, $\mathfrak{M}$ does not have to be $B_W[[u]]$-free.) □

4.c.1. Assume now that $(A, \mathfrak{m})$ is a complete local Noetherian ring with finite residue field $\mathbb{F}$ and that in addition:

1) the A-valued representation ρ_ξ of G_∞ which corresponds to ξ extends to a representation of $\mathrm{Gal}(\bar{K}/K)$, and

2) the $\mathbb{F}$-valued representation of $\mathrm{Gal}(\bar{K}/K)$ which is obtained by reducing ρ_ξ modulo $\mathfrak{m}$ is flat.

By [Ra] there is a quotient $A \to A^{\mathrm{fl}}$ such that for each $A \to B$ with B a local Artinian A-algebra with residue field $\mathbb{F}$, the representation obtained by composing with $GL_d(A) \longrightarrow GL_d(B)$ is flat if and only if $A \to B$ factors as $A \to A^{\mathrm{fl}} \to B$. Using the above proposition one can see that $A \to A^K$ factors as

$$A \to A^{\mathrm{fl}} \to A^K.$$

Remark 4.4. In general, it is not clear whether we should expect that $A^{\mathrm{fl}} \to A^K$ is an isomorphism. The issue is the following: Consider a deformation ρ of $\rho_0 = \rho_\xi$ modulo $\mathfrak{m}$ over a finite Artin local $\mathbb{Z}_p$-algebra Λ with residue field $\mathbb{F}$. It corresponds to a $\Lambda_W((u))$-Φ-module M. By assumption, the $(\mathbb{F} \otimes_{\mathbb{Z}_p} W)((u))$-$\Phi$-module $M_0 = M \otimes_\Lambda \mathbb{F}$ which corresponds to ρ_0 contains a $(\mathbb{F} \otimes_{\mathbb{Z}_p} W)[[u]]$-$\Phi$-submodule $\mathfrak{M}_0$ with $E(u)\mathfrak{M}_0 \subset \Phi(\phi^*\mathfrak{M}_0) \subset \mathfrak{M}_0$. We can easily see that $\mathfrak{M}_0 \simeq (\mathbb{F} \otimes_{\mathbb{Z}_p} W)[[u]]^d$. Assume now that ρ is also flat; this implies that M contains a $\Lambda_W[[u]]$-Φ-module $\mathfrak{M}$ with $\mathfrak{M}[1/u] = M$ that satisfies properties (a), (b) and (c). The problem is that if $e > p-1$, we cannot expect that $\mathfrak{M}$ is a deformation of $\mathfrak{M}_0$ (so that we can apply Lemma 4.2). The question is: Is there is some $\mathbb{Z}_p$-algebra C that contains Λ and a $C_W[[u]]$-Φ-module $\mathfrak{M}_C$ with $\mathfrak{M}_C[1/u] = M \otimes_\Lambda C$ which is in addition $C_W[[u]]$-projective and such that $\mathfrak{M} = M \cap \mathfrak{M}_C$? Our discussion implies

Proposition 4.5. *In the situation of Remark 4.4, assume that $e \leq p-1$. Then $A^{\mathrm{fl}} \simeq A^K$.*

5. Coefficient domains and a period morphism

5.a. Fix d, the local field K and $h \geq 1$. We define the stack in groupoids $\mathcal{D}_{d,h,K}$ over schemes over $\mathbb{Q}_p$ which is described as follows:

If R is a $\mathbb{Q}_p$-algebra, then the objects of $\mathcal{D}_{d,h,K}(R)$ are triples $(D, \Phi, \mathrm{Fil}^\bullet)$ where

- D is a $R \otimes_{\mathbb{Q}_p} K_0$-module which is, locally on R, free of rank d,
- $\Phi : D \to D$ is an $\mathrm{Id} \otimes_{\mathbb{Q}_p} \phi$-linear automorphism,
- $\mathrm{Fil}^\bullet$ is a exhausting, decreasing filtration of $D_K := D \otimes_{K_0} K$ by $R \otimes_{\mathbb{Q}_p} K$-modules which are locally direct summands and satisfy $\mathrm{Fil}^0 = D_K$, $\mathrm{Fil}^{h+1} = (0)$.

We can see that $\mathcal{D}_{d,h,K}$ is a fpqc stack over $\mathbb{Q}_p$. Locally we can choose a basis of D; this allows us to write the stack as a quotient

$$\mathcal{D}_{d,h,K} = [(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,h,K})/_{(\phi,\cdot)} \mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d]. \tag{5.27}$$

Here $\mathrm{Gr}_{d,h,K}$ is the Grassmannian of filtrations as above of length $h+1$ on a vector space of dimension d over K. (Notice here that we are not yet prescribing dimensions for the graded pieces $\mathrm{Fil}^i/\mathrm{Fil}^{i+1}$; in particular, $\mathrm{Gr}_{d,h,K}$ is not necessarily connected.) The symbol $(\phi,\cdot)$ is supposed to remind us that the action of the group $\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d$ on the product is by ϕ-conjugation on the first factor and by translation on the second. It follows from this description that $\mathcal{D}_{d,h,K}$ is an Artin stack, smooth of finite type over $\mathbb{Q}_p$.

Similarly, suppose that $\mu : \bar{\mathbb{Q}}_p^\times \to (\mathrm{Res}_{K/\mathbb{Q}_p} GL_d)(\bar{\mathbb{Q}}_p)$ is a coweight as in §3.d before. Assume that μ is defined over the reflex field E and, for simplicity, assume $h_- = 0$ so that $h = h_+$. We define the stack in groupoids $\mathcal{D}_{\mu,K}$ over schemes over E which is described as follows: If R is an E-algebra, then the objects of $\mathcal{D}_{\mu,K}(R)$ are triples $(D, \Phi, \mathrm{Fil}^\bullet)$ that correspond to objects of $\mathcal{D}_{d,h,K}(R)$ as above with the additional property

- The filtration $\mathrm{Fil}^\bullet$ is of type μ in the sense that the base change of the graded piece $\mathrm{Fil}^j/\mathrm{Fil}^{j+1}$ under $\mathrm{id}\otimes\psi : R\otimes_{\mathbb{Q}_p} K \to R\otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$ has rank equal to $\dim(V_j^\psi)$ for each ψ and each j (see (3.23)).

Once again $\mathcal{D}_{\mu,K}$ is a fpqc stack over E which is an Artin stack, smooth of finite type over E. We can write

$$\mathcal{D}_{\mu,K} = [(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d)_E \times_E \mathrm{Gr}_{\mu,K}/_{(\phi,\cdot)} (\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d)_E]. \tag{5.28}$$

Here $\mathrm{Gr}_{\mu,K}$ is the Grassmannian of filtrations as above of type μ on $(E\otimes_{\mathbb{Q}_p} K)^d$.

Remark 5.1. We can also consider the following "rigid" variants: $\mathfrak{D}_{d,h,K} = \mathcal{D}^{\mathrm{rig}}_{d,h,K}$ is the category fibered in groupoids over the category of rigid spaces over $\mathbb{Q}_p$ which is defined as follows. If $\mathfrak{X}$ is a rigid space, then $\mathfrak{D}_{d,h,K}(\mathfrak{X})$ is the groupoid of pairs $(D, \Phi, \mathrm{Fil}^\bullet)$ with D a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}\otimes_{\mathbb{Q}_p} K_0$-modules over $\mathfrak{X}$ which is locally free of rank d, Φ an $1\otimes_{\mathbb{Q}_p}\phi$-linear isomorphism of D, and $\mathrm{Fil}^\bullet$ a filtration of $D\otimes_{K_0} K$ (of length h, as above) by coherent $\mathcal{O}_{\mathfrak{X}}\otimes_{\mathbb{Q}_p} K$-sheaves over $\mathfrak{X}$ with locally free graded pieces. Here we are implicitly using that descent of coherent modules under fpqc morphisms of rigid spaces is effective, cf. [BG]. Similarly, we can define $\mathfrak{D}_{\mu,K}$ etc.

5.a.1. Now suppose that $\mathcal{A}$ is a p-adic ring. Set $A = \mathcal{A}[1/p]$. Let $(\mathfrak{M}, \Phi)$ a $\mathcal{A}_W[[u]]$-Φ-module which corresponds to an object of $\widehat{\mathcal{C}}_{d,h,K}(\mathcal{A})$. Consider

$$D = (\mathfrak{M}/u\mathfrak{M})[1/p]$$

with the $1\otimes\phi$-endomorphism given by Φ mod u; we can easily see that this endomorphism is bijective and that D is an $A\otimes_{\mathbb{Q}_p} K_0$-module which is projective of rank d.

Similarly to [Ki2], [Ki3], we set

$$\mathcal{O}_A := \varprojlim_n (\mathcal{A}_W[[u, u^n/p]][1/p]) \subset A_W[[u]] = (A\otimes_{\mathbb{Q}_p} K_0)[[u]];$$

in particular

$$\mathcal{O} = \mathcal{O}_{\mathbb{Q}_p} := \varprojlim_n (W[[u, u^n/p]][1/p]) \subset K_0[[u]]$$

is the ring of rigid analytic functions on the open disk $\mathbb{U}$ of radius 1 over K_0. (The inverse limit is under the maps given by $u^{n'}/p \mapsto u^{n'-n} \cdot (u^n/p)$.)

We have inclusions $W[[u]][1/p] \hookrightarrow \mathcal{O}$, $\mathcal{A}_W[[u]][1/p] \hookrightarrow \mathcal{O}_A$. The endomorphism ϕ has a unique continuous extension to $\mathcal{O}$ and $\mathcal{O}_A$. Set $\mathcal{M} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$; the Φ-structure on $\mathfrak{M}$ induces

$$\Phi : \phi^*(\mathcal{M}) \to \mathcal{M}.$$

This map is injective and we have $E(u)^h \mathcal{M} \subset \Phi(\phi^*(\mathcal{M})) \subset \mathcal{M}$, and $D = \mathcal{M}/u\mathcal{M}$. Set

$$\lambda = \prod_{n=0}^{\infty} \phi^n(E(u)/E(0)) \in \mathcal{O}.$$

For each m, let $r(m)$ be the smallest integer such that $em < p^{r(m)}$ and consider

$$\mathcal{O}_{\mathcal{A},e,m} := \mathcal{A}_W[[u, u^{p^{r(m)}}/p^m]][1/p].$$

There is a ring homomorphism $\mathcal{O}_A \to \mathcal{O}_{\mathcal{A},e,m}$. Since $|\pi| = p^{-1/e}$, we can see that $u \mapsto \pi$ gives $\mathcal{O}_{\mathcal{A},e,m} \to A \otimes_{\mathbb{Q}_p} K$ which induces an isomorphism

$$\mathcal{O}_{\mathcal{A},e,m}/(E(u)) \xrightarrow{\sim} A \otimes_{\mathbb{Q}_p} K. \tag{5.29}$$

Recall $\mathcal{M} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$. As in [Ki3] Lemma (2.2) we see that there is a unique ϕ-compatible A_W-linear map

$$\xi : D \to \mathcal{M}$$

with the following properties:

1) The reduction modulo u of ξ is the identity.
2) The induced map $\xi : D \otimes_{A \otimes_{\mathbb{Q}_p} K_0} \mathcal{O}_A \to \mathcal{M}$ is injective and has cokernel killed by λ^h.
3) For any sufficiently large m, the induced map

$$\xi \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} : D \otimes_{A \otimes_{\mathbb{Q}_p} K_0} \mathcal{O}_{\mathcal{A},e,m} \to \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} = \mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_{\mathcal{A},e,m}$$

is injective and its image is equal to that of the map

$$\phi^*(\mathcal{M}) \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m} \to \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_{\mathcal{A},e,m}$$

induced by Φ.

Indeed, the construction of $\xi : D \to \mathcal{M}$ in [Ki3] works verbatim when $\mathfrak{M}$ is $\mathcal{A}_W[[u]]$-free. (The assumptions that $\mathcal{A}$ is complete, local and Noetherian are not needed for our version of the construction. We can choose $\mathfrak{M}$ and $\mathfrak{M}/u\mathfrak{M}$ to play the roles of the modules denoted by $\mathfrak{M}_A^\circ$ and D_A° in loc. cit.) The general case is obtained by gluing, using the uniqueness of ξ in the free case. To check claims (2) and (3) we can argue as in [Ki3]. (Note that loc. cit. Lemma (2.2.1) is also valid with essentially the same proof even when $\mathcal{A}$ is not Noetherian.)

As a result of (2) and (3), and by using (5.29) to reduce modulo $(E(u))$, we obtain an isomorphism

$$D\otimes_{K_0}K\xrightarrow{\sim}\phi^*(\mathcal{M})\otimes_{\mathcal{O}_A}(A\otimes_{\mathbb{Q}_p}K)\simeq\Phi(\phi^*\mathfrak{M}[1/p])\otimes_{\mathcal{A}_W[[u]][1/p]}(A\otimes_{\mathbb{Q}_p}K). \tag{5.30}$$

(In the last tensor product, we use $\mathcal{A}_W[[u]][1/p]\to\mathcal{A}_W[[u]][1/p]/(E(u))\xrightarrow{\sim}A\otimes_{\mathbb{Q}_p}K$.) We can see that the isomorphism (5.30) is independent of the choice of m.

5.a.2. Recall that $\mathbb{U}$ denotes the rigid open unit disk over K_0. If I is a subinterval of $[0,1)$, we set $\mathbb{U}(I)$ for the admissible open subspace of points with absolute value in I. Set $\mathcal{O}_I=\Gamma(\mathbb{U}(I),\mathcal{O}_{\mathbb{U}(I)})$ so that $\mathcal{O}=\mathcal{O}_{[0,1)}$. We denote by $\phi:\mathbb{U}\to\mathbb{U}$ the "Frobenius" morphism which corresponds to $\phi:\mathcal{O}\to\mathcal{O}$ as before.

We can consider the category $\mathfrak{C}_{d,h,K}$ fibered in groupoids over $\mathbb{Q}_p$-rigid spaces which is defined as follows. Let $\mathfrak{X}$ be a rigid space over $\mathbb{Q}_p$ and consider $\mathfrak{X}\times\mathbb{U}$ with the partial Frobenius $\phi:=\mathrm{id}\times\phi:\mathfrak{X}\times\mathbb{U}\to\mathfrak{X}\times\mathbb{U}$. Then, by definition, $\mathfrak{C}_{d,h,K}(\mathfrak{X})$ is the groupoid of pairs $(\mathcal{M},\Phi)$ where $\mathcal{M}$ is a coherent sheaf over $\mathfrak{X}\times\mathbb{U}$ which is locally on $\mathfrak{X}$ free of rank d, and $\Phi:\phi^*\mathcal{M}\to\mathcal{M}$ an injective homomorphism with cokernel annihilated by $E(u)^h$.

Denote by $i:\mathfrak{X}_{K_0}\hookrightarrow\mathfrak{X}\times\mathbb{U}$ the inclusion $i(x)=(x,0)$ and by $p:\mathfrak{X}\times\mathbb{U}\to\mathfrak{X}_{K_0}$ the projection. If $(\mathcal{M},\Phi)$ is an object of $\mathfrak{C}_{d,K}(\mathfrak{X})$ we set $D=i^*\mathcal{M}$. This is a coherent sheaf on $\mathfrak{X}_{K_0}$ which is locally free of rank d; the morphism $\Phi:\phi^*\mathcal{M}\to\mathcal{M}$ induces a ϕ-linear isomorphism $\Phi:D\to D$.

Proposition 5.2. *There is a (unique) Φ-compatible morphism of sheaves of $\mathcal{O}_{\mathfrak{X}_{K_0}}$-modules $\xi:D\to p_*(\mathcal{M})$ such that*

1) $i^*\xi$ is the identity,

2) the induced morphism $p^*\xi:p^*D\to\mathcal{M}$ is injective and has cokernel annihilated by λ^h,

3) If $r\in(|\pi|,|\pi|^{1/p})$, then the image of the restriction $p^*\xi_{[0,r)}$ to $\mathfrak{X}\times\mathbb{U}[0,r)$ coincides with the image of $\Phi_{[0,r)}:\phi^*\mathcal{M}_{[0,r)}\to\mathcal{M}_{[0,r)}$.

Proof. When $\mathfrak{X}=\mathrm{Sp}(\mathbb{Q}_p)$ is a point, this is [Ki2] Lemma 1.2.6. Note that there is at most one Φ-compatible $\xi:D\to p^*(\mathcal{M})$ that satisfies property (1). Indeed, if ξ, ξ' are two such morphisms we have $\mathrm{Im}(\xi-\xi')\subset u\cdot p^*(\mathcal{M})$. The Φ-compatibility gives $\Phi\cdot(\xi-\xi')=(\xi-\xi')\cdot\Phi$. Hence, since $\Phi:D\to D$ is an isomorphism, we obtain inductively $\mathrm{Im}(\xi-\xi')\subset u^s\cdot p^*(\mathcal{M})$ for all $s\geq 0$. This implies that $\xi=\xi'$. To show the existence of ξ we suppose first that A is a Tate $\mathbb{Q}_p$-algebra and that $\mathfrak{X}=\mathrm{Sp}(A)$ is the corresponding affinoid rigid space. Then $\mathcal{O}_A$ is the ring of rigid analytic functions $\mathcal{O}_{\mathrm{Sp}(A)\times\mathbb{U}}$ on the product $\mathrm{Sp}(A)\times\mathbb{U}$ and $\mathcal{M}$ is given by an $\mathcal{O}_A$-module as in the previous paragraph. There is a p-adic ring $\mathcal{A}$ which is topologically of finite presentation (tfp) over $\mathbb{Z}_p$ and p-torsion free such that $A=\mathcal{A}[1/p]$. The arguments of [Ki2] Lemma 1.2.6, [Ki3] Lemma (2.2) (see also the previous paragraph) extend to this case to construct ξ that satisfies all the required properties. The result in the case of a general rigid space $\mathfrak{X}$ follows by the affinoid case above by gluing using the uniqueness of ξ. □

Suppose now that r is in $(|\pi|, |\pi|^{1/p})$. Then $\mathcal{O}_{\mathfrak{X}\times\mathbb{U}[0,r)}/(E(u)) \simeq \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K$. As a result of (2) and (3), we obtain an isomorphism

$$p^* D_{[0,r)} \xrightarrow{\sim} \Phi(\phi^*\mathcal{M})_{[0,r)} \tag{5.31}$$

which by reducing modulo $(E(u))$ gives an isomorphism

$$D \otimes_{K_0} K \xrightarrow{\sim} \Phi(\phi^*\mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X}\times\mathbb{U}[0,r)}} (\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K) \tag{5.32}$$

of coherent $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Q}_p} K$-sheaves over $\mathfrak{X}$.

5.b. In what follows, we will assume that $h = 1$.

Let $\mathcal{A}$ be a p-adic ring and set $A = \mathcal{A}[1/p]$. Suppose that $(\mathfrak{M}, \Phi)$ is an $\mathcal{A}_W[[u]]$-Φ-module which corresponds to an object of $\widehat{\mathcal{C}}_{d,K}(\mathcal{A})$. We will define an object $D(\mathfrak{M}, \Phi)$ of $\mathcal{D}_{d,K}(A)$ by following the construction of [Ki2], [Ki3]. Let $D = (\mathfrak{M}/u\mathfrak{M})[1/p]$ with its Φ-structure be as in §5.a.1. It remains to define the filtration $\mathrm{Fil}^\bullet$ on $D \otimes_{K_0} K$. Since $h = 1$, we have

$$E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}\ .$$

The module $\Phi(\phi^*\mathfrak{M}[1/p])$ is filtered

$$E(u)\Phi(\phi^*\mathfrak{M}[1/p]) \subset E(u)\mathfrak{M}[1/p] \subset \Phi(\phi^*\mathfrak{M}[1/p])\ .$$

Hence, we can filter the $A \otimes_{\mathbb{Q}_p} K$-module $D \otimes_{K_0} K \simeq \Phi(\phi^*\mathfrak{M}[1/p]) \otimes_{\mathcal{A}_W[[u]][1/p]} (A \otimes_{\mathbb{Q}_p} K)$ via (5.30) by taking the image of this filtration, i.e we set

$$\begin{aligned} \mathrm{Fil}^2 &= (0), \\ \mathrm{Fil}^1 &= E(u)\mathfrak{M}[1/p] \bmod E(u)\Phi(\phi^*\mathfrak{M}[1/p]), \\ \mathrm{Fil}^0 &= D \otimes_{K_0} K\ . \end{aligned}$$

Since $E(u)$ is not a zero divisor in $\mathcal{A}_W[[u]][1/p]$, we can see (cf. [Ki3] 2.6.1 (1)) that the quotient $\mathfrak{M}[1/p]/\Phi(\phi^*\mathfrak{M}[1/p])$ is a finitely generated projective $A \otimes_{\mathbb{Q}_p} K$-module. We conclude that Fil^1 is a finitely generated projective $A \otimes_{\mathbb{Q}_p} K$-module which is locally a direct summand of $D \otimes_{K_0} K$. Hence, $(D, \Phi, \mathrm{Fil}^\bullet)$ gives an object of $\mathcal{D}_{d,K}(A)$ which we will denote by $D(\mathfrak{M}, \Phi)$. This gives a functor of groupoids,

$$\underline{D}(\mathcal{A}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \to \mathcal{D}_{d,K}(A).$$

5.b.1. Similarly, if $(\mathcal{M}, \Phi)$ is an object of $\mathfrak{C}_{d,K}(\mathfrak{X})$ for a rigid space $\mathfrak{X}$, we consider $D = i^*\mathcal{M}$ with its ϕ-linear isomorphism $\Phi : D \to D$ as above. We also have

$$E(u)\Phi(\phi^*\mathcal{M}) \subset E(u)\mathcal{M} \subset \Phi(\phi^*\mathcal{M})$$

(A filtration of coherent sheaves over $\mathfrak{X} \times \mathbb{U}$.) As above, we can use this and (5.32) to produce a filtration

$$(0) = \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset \mathrm{Fil}^0 = D \otimes_{K_0} K$$

of the coherent sheaf $D \otimes_{K_0} K$ over $\mathfrak{X}_{K_0}$. We can see that the triple $(D, \Phi, \mathrm{Fil}^\bullet)$ gives an object of $\mathfrak{D}_{d,K}(\mathfrak{X})$. Since $(\mathcal{M}, \Phi) \mapsto (D, \Phi, \mathrm{Fil}^\bullet)$ is functorial, this defines a functor of groupoids, $\mathfrak{C}_{d,K}(\mathfrak{X}) \to \mathfrak{D}_{d,K}(\mathfrak{X})$. This functor is compatible with descent, hence we obtain a morphism of stacks over the category of rigid spaces,

$$\underline{D} : \mathfrak{C}_{d,K} \to \mathfrak{D}_{d,K}. \tag{5.33}$$

Similarly, if μ is a miniscule cocharacter with $h_- = 0$, $h_+ = 1$ and with reflex field E as in Remark 3.3, we can define the category $\mathfrak{C}_{\mu,K}$ fibered in groupoids over E-rigid spaces by requiring the cokernels $\mathcal{M}/\Phi(\phi^*\mathcal{M})$ to have a filtration "of type μ". In this case, the morphism $\underline{D}$ sends $\mathfrak{C}_{\mu,K}$ to $\mathfrak{D}_{\mu,K}$.

5.b.2. It follows from [Ki2] Theorem (1.2.15) that the functor $\underline{D}(\mathrm{Sp}(L))$ gives an equivalence $\mathfrak{C}_{d,K}(\mathrm{Sp}(L)) \xrightarrow{\sim} \mathfrak{D}_{d,K}(\mathrm{Sp}(L))$ for any finite extension $L/\mathbb{Q}_p$. To briefly explain the construction of the inverse functor we need some notation: Denote by σ the isomorphism $\mathcal{O} \to \mathcal{O}$ given by applying Frobenius (only) to the coefficients of the power series. Denote by x_n the point of $\mathbb{U}$ that corresponds to the irreducible polynomial $E(u^{p^n})$ and let $\hat{\mathcal{O}}_{\mathbb{U},x_n}$ the complete local ring of $\mathbb{U}$ at x_n. Notice that the function $\sigma^{-n}(\lambda) \in \mathcal{O}$ has a simple zero at x_n. Now consider the composite map

$$\mathcal{O} \otimes_{K_0} D \xrightarrow{\sigma^{-n}\otimes\Phi^{-n}} \mathcal{O} \otimes_{K_0} D \to \hat{\mathcal{O}}_{\mathbb{U},x_n} \otimes_{K_0} D = \hat{\mathcal{O}}_{\mathbb{U},x_n} \otimes_K D_K,$$

where in the first arrow $\Phi^{-n} : D \to D$ makes sense since $\Phi = \Phi_D$ is bijective. By the above, this induces a map

$$i_n : \mathcal{O}[\lambda^{-1}] \otimes_{K_0} D \to \hat{\mathcal{O}}_{\mathbb{U},x_n}[(u - x_n)^{-1}] \otimes_K D_K \ .$$

Now suppose we are given an object $(D, \Phi, \mathrm{Fil}^\bullet)$ over L. Kisin constructs a Φ-module $(\mathcal{M}, \Phi) = \mathcal{M}(D, \Phi, \mathrm{Fil}^\bullet)$ over $\mathcal{O}_L = \mathcal{O}_{\mathrm{Sp}(L)\times\mathbb{U}}$ by taking

$$\mathcal{M} = \bigcap_{n\geq 0} i_n^{-1}(\mathrm{Fil}^1 \otimes_K (u - x_n)^{-1}\hat{\mathcal{O}}_{\mathbb{U},x_n} + D_K \otimes_K \hat{\mathcal{O}}_{\mathbb{U},x_n}) \tag{5.34}$$

and setting $\Phi : \phi^*\mathcal{M} \to \mathcal{M}$ to be the restriction of

$$1 \otimes \Phi_D : \phi^*(\mathcal{O}[\lambda^{-1}] \otimes_{K_0} D) \to \mathcal{O}[\lambda^{-1}] \otimes_{K_0} D \ .$$

Note that by definition, we have

$$\mathcal{O} \otimes_{K_0} D \subset \mathcal{M} \subset \lambda^{-1}\mathcal{O} \otimes_{K_0} D \ . \tag{5.35}$$

Observe that by its construction, $\mathcal{M}$ is a closed $\mathcal{O}$-submodule of $\lambda^{-1}\mathcal{O} \otimes_{K_0} D$ and so by [Ki2] Lemma 1.1.4, $\mathcal{M}$ is finite free over $\mathcal{O}$.

5.b.3. This construction extends to the case that L is a complete rank-1 valued field: Let R° be a p-adic valuation ring of rank 1 with $L = R^\circ[1/p]$. Then $\mathcal{O}_L = \mathcal{O}_{[0,1)}$ is the ring of rigid functions on the open unit disk over $L \otimes_{\mathbb{Q}_p} K_0$. For simplicity, set $L_{K_0} = L \otimes_{\mathbb{Q}_p} K_0$, $L_K = L \otimes_{\mathbb{Q}_p} K$. Consider a triple $(D, \Phi, \mathrm{Fil}^\bullet)$ as above. We can construct a Φ-module $\mathcal{M}$ over $\mathcal{O}_{[0,1)}$ by (5.34) as above that satisfies

$$\mathcal{O}_{[0,1)} \otimes_{L_{K_0}} D \subset \mathcal{M} \subset \lambda^{-1}\mathcal{O}_{[0,1)} \otimes_{L_{K_0}} D \ . \tag{5.36}$$

By [Gr], V, Rem. 3°, p. 87, $\mathcal{O}_{[0,1)}$ is a product of Prüfer domains. We can see that when an integral power of r is in the set $|L|$ the restriction $\mathcal{M}_{|[0,r]}$ of $\mathcal{M}$ to the closed disk $[0,r]$ is given by a finitely generated torsion free $\mathcal{O}_{[0,r]}$-module which is free (since $\mathcal{O}_{[0,r]}$ is a product of p.i.ds). Using [Gr], V, Thm. 1, p. 83, we can see that $\mathcal{M}$ is a projective finitely generated $\mathcal{O}_{[0,1)}$-module. As such it is a direct sum of a free module with a projective module $\mathcal{L}$ of rank 1 and $\mathcal{L} \simeq \det(\mathcal{M})$. We can see that $\det(\mathcal{M}) = \lambda^{-a}\mathcal{O}_{[0,1)}$ with $a = \dim_{L_K}(\mathrm{Fil}^1)$; therefore $\mathcal{L}$ and hence $\mathcal{M}$ is finite free over $\mathcal{O}_{[0,1)}$.

5.b.4. The constructions of the two previous paragraphs are compatible in the following sense: Suppose that the p-adic ring $\mathcal{A}$ is topologically of finite presentation over $\mathbb{Z}_p$. Then $A = \mathcal{A}[1/p]$ is a Tate algebra and we can consider the affinoid rigid space $\mathrm{Sp}(A)$. Recall that $\mathrm{Sp}(A)$ is the "generic fiber" $\mathrm{Spf}(\mathcal{A})^{\mathrm{rig}}$ of $\mathrm{Spf}(\mathcal{A})$ in the sense of Raynaud. An object $(\mathfrak{M}, \Phi)$ of $\widehat{\mathcal{C}}_{d,K}(\mathcal{A})$ gives an object $(\mathcal{M}, \Phi)$ of $\mathfrak{C}_{d,K}(\mathrm{Sp}(A))$ by taking $\mathcal{M}$ to be the coherent sheaf with global sections $\mathfrak{M} \otimes_{\mathcal{A}_W[[u]]} \mathcal{O}_A$. This gives a functor $\widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \to \mathfrak{C}_{d,K}(\mathrm{Sp}(A))$. The diagram

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) & \longrightarrow & \mathfrak{C}_{d,K}(\mathrm{Sp}(A)) \\ {\scriptstyle \underline{D}(\mathcal{A})}\downarrow & & \downarrow{\scriptstyle \underline{D}(\mathrm{Sp}(A))} \\ \mathcal{D}_{d,K}(A) & \longrightarrow & \mathfrak{D}_{d,K}(\mathrm{Sp}(A)) \ , \end{array}$$

commutes up to natural equivalence. (Here the lower horizontal arrow is given by sending the A-module D to the corresponding coherent sheaf over $\mathrm{Sp}(A)$.) The diagonal arrow

$$\Pi(\mathcal{A}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{A}) \longrightarrow \mathfrak{D}_{d,K}(\mathrm{Sp}(A))$$

(obtained as the composition of the top followed by the right downward arrow) is by definition, the period functor for $\mathcal{A}$. It globalizes as follows.

Suppose that $\mathcal{X}$ is an adic formal scheme which is locally of finite type over $\mathbb{Z}_p$ (hence $p\,\mathcal{O}_{\mathcal{X}}$ is an ideal of definition), and denote by $\mathfrak{X} = \mathcal{X}^{\mathrm{rig}}$ the corresponding rigid space given by its generic fiber, comp. [RZ], Prop. 5.3. The construction $(\mathfrak{M}, \Phi) \mapsto (\mathcal{M}, \Phi)$ above generalizes to give a functor $\widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \to \mathfrak{C}_{d,K}(\mathcal{X}^{\mathrm{rig}})$. Its composition with the functor $\omega(\mathcal{X}^{\mathrm{rig}}) : \mathfrak{C}_{d,K}(\mathcal{X}^{\mathrm{rig}}) \to \mathfrak{D}_{d,K}(\mathcal{X}^{\mathrm{rig}})$ above gives the *period functor*

$$\Pi(\mathcal{X}) : \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \longrightarrow \mathfrak{D}_{d,K}(\mathcal{X}^{\mathrm{rig}}) \ . \tag{5.37}$$

It is localizing in the following sense. Let $\mathcal{X} = \bigcup_i \mathcal{U}_i$ be an open covering of the formal scheme $\mathcal{X}$. This induces an admissible open covering of the associated rigid-analytic spaces,

$$\mathcal{X}^{\rm rig} = \bigcup_i \mathcal{U}_i^{\rm rig} \ ,$$

comp. [RZ], Prop. 5.3. Then the corresponding diagram of 2-cartesian rows, with vertical arrows the period morphisms, is commutative,

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{d,K}(\mathcal{X}) \to & \prod_i \widehat{\mathcal{C}}_{d,K}(\mathcal{U}_i) & \rightrightarrows \prod_{i,j} \widehat{\mathcal{C}}_{d,K}(\mathcal{U}_i \cap \mathcal{U}_j) \\ \downarrow & \downarrow & \downarrow \\ \mathfrak{D}_{d,K}(\mathcal{X}^{\rm rig}) \to & \prod_i \mathfrak{D}_{d,K}(\mathcal{U}_i^{\rm rig}) & \rightrightarrows \prod_{i,j} \mathfrak{D}_{d,K}(\mathcal{U}_i^{\rm rig} \cap \mathcal{U}_j^{\rm rig}) \ . \end{array}$$

5.b.5. Let us now assume that R° is a complete rank one valuation ring with residue field equal to $\bar{\mathbb{F}}_p$ and set $L = R^\circ[1/p]$ for the corresponding complete rank-1 valued field. Recall $\mathcal{O}_L = \mathcal{O}_{[0,1)}$ is the ring of rigid functions on the open unit disk over L_{K_0}. Denote by $\mathcal{O}_L^{\rm R}$ the corresponding Robba ring

$$\mathcal{O}_L^{\rm R} := \lim_{r \to 1^-} \mathcal{O}_{(r,1)} \ .$$

This can be identified with the set of Laurent power series $\sum_{n\in\mathbb{Z}} a_n u^n$ with coefficients in L_{K_0} that converge in some open annulus $r < |u| < 1$. The ring $\mathcal{O}_L^{\rm R}$ is equipped with a Frobenius endomorphism $\phi : \mathcal{O}_L^{\rm R} \to \mathcal{O}_L^{\rm R}$ which restricts to $\phi : \mathcal{O}_L \to \mathcal{O}_L$. Let $\mathcal{O}_L^{\rm int}$ the subring of $\mathcal{O}_L^R$ consisting of those Laurent power series $\sum_{n\in\mathbb{Z}} a_n u^n$ with $a_n \in R^\circ \otimes_{\mathbb{Z}_p} W$, for all $n \in \mathbb{Z}$. By [Ke2] Prop. 3.5.5, $\mathcal{O}_L^{\rm int}$ is a henselian local ring with maximal ideal given by the set of series with $|a_n| < 1$, for all $n \in \mathbb{Z}$, and residue field $\bar{\mathbb{F}}_p((u))$. Notice that $E(u)$ is a unit in $\mathcal{O}_L^{\rm int}$ and ϕ preserves $\mathcal{O}_L^{\rm int}$.

Suppose now that $\mathcal{N}$ is a finite free rank d Φ-module over $\mathcal{O}_L^{\rm R}$ with $\Phi : \phi^*\mathcal{N} \xrightarrow{\sim} \mathcal{N}$ an isomorphism. We will say that $(\mathcal{N}, \Phi)$ is *purely of slope zero* if there is a finite free rank d $\mathcal{O}_L^{\rm int}$-submodule $\mathcal{N}^{\rm int} \subset \mathcal{N}$ such that:

i) $\mathcal{N}^{\rm int} \otimes_{\mathcal{O}_L^{\rm int}} \mathcal{O}_L^{\rm R} = \mathcal{N}$, and

ii) $\Phi_{|\mathcal{N}^{\rm int}}$ induces an isomorphism $\phi^*\mathcal{N}^{\rm int} \xrightarrow{\sim} \mathcal{N}^{\rm int}$.

When $L/\mathbb{Q}_p^{\rm unr}$ is finite, this is equivalent to asking that $(\mathcal{N}, \Phi)$ is purely of slope zero in the sense of Kedlaya [Ke1].

As in the previous paragraph, we have the period functor

$$\Pi(R^\circ) : \widehat{\mathcal{C}}_{d,K}(R^\circ) \to \mathcal{D}_{d,K}(L) \simeq \mathfrak{D}_{d,K}({\rm Sp}(L)) \ .$$

By 5.b.3 we can associate to an object $(D, \Phi, {\rm Fil}^\bullet)$ of $\mathcal{D}_{d,K}(L)$ a Φ-module $(\mathcal{M}, \Phi) = \mathcal{M}(D, \Phi, {\rm Fil}^\bullet)$ over $\mathcal{O}_L$.

Conjecture 5.3. *(i) The object $(D, \Phi, {\rm Fil}^\bullet)$ of $\mathfrak{D}_{d,K}({\rm Sp}(L))$ is in the image of the period functor $\Pi(R^\circ)$, i.e is of the form $\Pi(R^\circ)(\mathfrak{M}, \Phi)$ for some $(\mathfrak{M}, \Phi) \in \widehat{\mathcal{C}}_{d,K}(R^\circ)$, if and only if $\mathcal{M}(D, \Phi, {\rm Fil}^\bullet) \otimes_{\mathcal{O}_L} \mathcal{O}_L^{\rm R}$ is purely of slope zero.*

(ii) There exists an $\mathbb{Q}_p$-analytic subspace (in the sense of Berkovich)

$$(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K})^{\mathrm{an}} \subset \mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K}$$

invariant under $\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d$ *such that the fiber over* L *of the stack quotient*

$$[(\mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d \times_{\mathbb{Q}_p} \mathrm{Gr}_{d,K})^{\mathrm{an}} /_{(\phi,\cdot)} \mathrm{Res}_{K_0/\mathbb{Q}_p} GL_d]$$

parametrizes the image points of Π *over* L.

There is an obvious variant of this conjecture involving a minuscule cocharacter μ with $h_- = 0, h_+ = 1$.

6. Kisin varieties and Bruhat-Tits buildings

6.a. We return to the set-up and notations of §2.c.1. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$.

6.a.1. For simplicity, set $L = \mathbb{F} \otimes k$. Recall $G = \mathrm{Res}_{W/\mathbb{Z}_p} GL_d$. Suppose now that $A \in G(\mathbb{F}((u))) = GL_d(L((u)))$ and consider the corresponding $L((u))$-Φ-module $M_A = (L((u))^d, A \cdot \phi)$ which gives an object of $\mathcal{R}(\mathbb{F})$. Choices A, A' that are ϕ-conjugate, i.e $A' = g^{-1} \cdot A \cdot \phi(g)$ with $g \in GL_d(L((u)))$, give isomorphic modules. By the above, the fiber product $\{M_A\} \times_{\mathcal{R}} \mathcal{C}_\nu$ is represented over $\mathbb{F}$ by a projective subscheme of the affine Grassmannian $\mathcal{F}_G$ of $L[[u]]$-lattices in $L((u))^d$. We denote this subscheme by $\mathcal{C}_\nu(A)$. Similarly, we can consider the fiber product $\{M_A\} \times_{\mathcal{R}} \mathcal{C}^0_\mu$ which is a locally closed subscheme of $\mathcal{C}_\nu(A)$; we denote this subscheme by $\mathcal{C}^0_\nu(A)$. This can be thought of as an inseparable analogue of an affine Deligne-Lusztig variety. We call $\mathcal{C}^0_\nu(A)$ the *Kisin variety* associated to (G, A, ν), and $\mathcal{C}_\nu(A)$ the corresponding *closed Kisin variety.*

Concretely, for every finite extension $\mathbb{F}'$ of $\mathbb{F}$, the $\mathbb{F}'$-points of the Kisin variety $\mathcal{C}^0_\nu(A)$ are given by

$$\mathcal{C}^0_\nu(A)(\mathbb{F}') = \{g \cdot (\mathbb{F}' \otimes k)[[u]]^d \mid g^{-1} \cdot A \cdot \phi(g) \in G(\mathbb{F}'[[u]]) \cdot u^\nu \cdot G(\mathbb{F}'[[u]])\}.$$

The points $\mathcal{C}_\nu(A)(\mathbb{F}')$ parametrize finite flat commutative group schemes $\mathcal{G}$ with $\mathbb{F}'$-action over $\mathcal{O}_K$ which have "Hodge type $\leq \nu$" and are such that the restriction of the Galois representation $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}_{\mathbb{F}'}(\mathcal{G}(\bar{\mathcal{O}}_K))$ to the Galois group G_∞ corresponds to the Φ-module given by A.

6.a.2. The above construction extends to the set-up of a general reductive group $G = \mathrm{Res}_{W/\mathbb{Z}_p} H$ described in §2.d. If ν is a dominant coweight of G and $A \in LG(\mathbb{F}) = G(\mathbb{F}((u)))$, we can define as above $\mathcal{C}^0_{\nu,G}$, $\mathcal{C}_{\nu,G}$ and the Kisin variety $\mathcal{C}^0_{\nu,G}(A)$, and closed Kisin variety $\mathcal{C}_{\nu,G}(A)$. However, the relation with Galois representations of $\mathrm{Gal}(\bar{K}/K)$ or finite group schemes is not so clear in this general case.

6.b. We now explain how the Bruhat-Tits building can help to get an overview of a Kisin variety.

6.b.1. For simplicity, we assume that $k = \mathbb{F}_p$, $W = \mathbb{Z}_p$ and that $H = G$ is a split Chevalley group over $\mathbb{Z}_p$. In the rest of this section, the symbol W is free again, and will be reserved for Weyl groups. Let T be a maximal split torus of G. We will identify the cocharacter groups $X_* = X_*(T_{\mathbb{Q}_p}) = X_*(T_{\mathbb{F}_p}) = X_*(T_{\mathbb{F}_p((u))})$. Suppose that C is a choice of a positive closed Weyl chamber in the vector space

$$V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\mathcal{B} = \mathcal{B}(\mathbb{F}((u))$ be the Bruhat-Tits building of G over $\mathbb{F}((u))$. This is a metric space with equivariant distance function $d : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$. We have the "refined" Weyl distance function $\delta : \mathcal{B} \times \mathcal{B} \to C$ which is defined as follows, cf. [KLM], section 5.1: Let x, $y \in \mathcal{B}$ and suppose that $\mathcal{A}$ is an apartment that contains both x and y. Let $\delta_{\mathcal{A}}(x, y)$ be the unique representative in C of the vector $y - x \in V$ and set $\delta(x, y) = \delta_{\mathcal{A}}(x, y)$. (This is independent of the choice of apartment $\mathcal{A}$.) The function δ is translation G-equivariant and satisfies the triangle inequality, cf.[KLM], Remark 3.33, (ii),

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z) \tag{6.38}$$

for the order that extends the usual order on dominant coweights. Also,

$$\delta(x, y) = \delta(y, x)^* \ , \tag{6.39}$$

where $v \mapsto v^* = w_0(-v)$ is the usual involution of C defined by the longest element w_0 of the finite Weyl group W.

6.b.2. Consider now the homomorphism $\phi : \mathbb{F}((u)) \to \mathbb{F}((u))$, given $\phi(a) = a$ if $a \in \mathbb{F}$, $\phi(u) = u^p$. We will show that it induces a map $\phi : \mathcal{B} \to \mathcal{B}$ with the following properties:

(a) the image of any apartment under ϕ is an apartment,

(b) we have $\phi(g) \cdot \phi(x) = \phi(g \cdot x)$ for any $g \in G(\mathbb{F}((u)))$, $x \in \mathcal{B}$.

(c) For x, $y \in \mathcal{B}$, we have

$$d(\phi(x), \phi(y)) = p \cdot d(x, y), \quad \delta(\phi(x), \phi(y)) = p \cdot \delta(x, y).$$

(d) The map $\phi : \mathcal{B} \to \mathcal{B}$ takes maps geodesics to geodesics; i.e., if $[x, y] \subset \mathcal{B}$ is the geodesic in $\mathcal{B}$ joining x and y, then the image $\phi([x, y])$ is the geodesic $[\phi(x), \phi(y)]$ joining $\phi(x)$ and $\phi(y)$.

(e) The map ϕ has a unique fixed point, i.e., there is a unique $y_0 \in \mathcal{B}$ such that $\phi(y_0) = y_0$. The point y_0 is a special vertex in $\mathcal{B}$.

Indeed, consider the vertex y_0 of $\mathcal{B}$ which is fixed under the subgroup $G(\mathbb{F}[[u]])$. Let $\mathcal{A}_0$ be the apartment in $\mathcal{B}$ that corresponds to a constant maximal torus $T = T_0 \otimes_{\mathbb{F}} \mathbb{F}((u))$ with $T_0 \subset G_{\mathbb{F}}$; then y_0 belongs to $\mathcal{A}_0$ and this choice of base point allows us to identify the affine space $\mathcal{A}_0$ with $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Scaling by p on V now gives a well-defined map $\phi_0 : \mathcal{A}_0 \to \mathcal{A}_0$ such that $\phi_0(y_0) = y_0$ and which satisfies $\phi_0(n \cdot y) = \phi(n) \cdot \phi_0(y)$ for each $y \in \mathcal{A}_0$ and n in the normalizer $N(T) \subset G(\mathbb{F}((u)))$. Now recall that the building $\mathcal{B}$ can be described as the quotient of $G(\mathbb{F}((u)) \times \mathcal{A}_0$ via the equivalence relation $(g, x) \sim (x', g')$ if there is $n \in N(T)$ such that $x' = n \cdot x$, $g' = ngn^{-1}$. Using the above we immediately see that $\phi(x, g) = (\phi_0(x), \phi(g))$ respects the equivalence relation and gives $\phi : \mathcal{B} \to \mathcal{B}$; since each apartment of $\mathcal{B}$ is of the form $g \cdot \mathcal{A}_0$ we see that the image of an apartment by ϕ is also an apartment. The desired properties now follow easily by using the above and the fact that any two points x, $y \in \mathcal{B}$ are contained in some apartment and that the geodesic $[x, y]$ is the straight line segment connecting x and y in that apartment. Note that the equality $d(\phi(x), \phi(y)) = p \cdot d(x, y)$ implies that there is at most one fixed point which then has to be the vertex y_0 given above.

Note that, by construction, the group $L^+G(\mathbb{F}) = G(\mathbb{F}[[u]])$ is the stabilizer of y_0 in $G(\mathbb{F}((u)))$. The map

$$\iota : \mathcal{F}_G(\mathbb{F}) = G(\mathbb{F}((u)))/G(\mathbb{F}[[u]]) \hookrightarrow \mathcal{B}, \quad g \cdot G(\mathbb{F}[[u]]) \mapsto g \cdot y_0 \tag{6.40}$$

allows us to identify the $\mathbb{F}$-valued points of the affine Grassmannian with a subset of the vertices in the building.

6.b.3. Suppose now that A is in $G(\mathbb{F}((u)))$ and gives an object in $\mathcal{R}_G(\mathbb{F})$. Then we have a map $\Phi = A \cdot \phi : \mathcal{B} \to \mathcal{B}$ which also satisfies

$$d(\Phi_A(x), \Phi_A(y)) = p \cdot d(x, y), \quad \delta(\Phi_A(x), \Phi_A(y)) = p \cdot \delta(x, y). \tag{6.41}$$

Then the $\mathbb{F}$-valued points of $\mathcal{C}_\nu(A) \subset \mathcal{F}_G(\mathbb{F})$ correspond to the following subset of vertices of the building,

$$\mathcal{C}_\nu(A) = \{x \text{ vertex in } \mathcal{B} \mid x \in \text{Im } \iota,\ 0 \leq \delta(x, \Phi_A(x)) \leq \nu\ \}.$$

If $N/\mathbb{F}((u))$ is a finite separable extension, we have an isometric embedding

$$\mathcal{B} \hookrightarrow \mathcal{B}(N).$$

We will use this to identify $\mathcal{B}$ with a subspace of $\mathcal{B}(N)$. The map Φ_A extends to a map $\mathcal{B}(N) \to \mathcal{B}(N)$.

Proposition 6.1. *There is a finite separable extension $M/\mathbb{F}((u))$ such that the above map $\Phi_A : \mathcal{B}(M) \to \mathcal{B}(M)$ has a fixed point. This fixed point is unique in $\cup_{N/\mathbb{F}((u))}\mathcal{B}(N)$.*

Proof. The uniqueness follows easily from (6.41). For simplicity, we will write Φ instead of Φ_A. Consider the "Lang isogeny"

$$G_{\mathbb{F}((u))} \to G_{\mathbb{F}((u))}\ ; \quad g \mapsto g^{-1}\phi(g).$$

This is a finite étale surjective morphism; therefore, if A is in $G(\mathbb{F}((u)))$, then there is $g \in G(M)$ for some finite Galois extension $M/\mathbb{F}((u))$ such that $A = g^{-1}\phi(g)$. Consider $x_0 = g^{-1} \cdot y_0$ which is a special vertex in $\mathcal{B}(M)$. We have

$$\Phi(x_0) = A \cdot \phi(g^{-1} \cdot y_0) = g^{-1} \cdot \phi(g) \cdot \phi(g)^{-1} \cdot y_0 = g^{-1} \cdot y_0 = x_0$$

and so x_0 is a fixed point. □

In fact, if σ is an element of $\mathrm{Gal}(M/\mathbb{F}((u)))$, since $\sigma\cdot\phi=\phi\cdot\sigma$, we can see that $\sigma(A)=\sigma(g^{-1}\phi(g))=\sigma(g)^{-1}\phi(\sigma(g))=A=g^{-1}\phi(g)$; therefore $\sigma(g)g^{-1}\in G(\mathbb{F})$. Since $g_0\cdot y_0=y_0$ for $g_0\in G(\mathbb{F})$, the point $x_0=g^{-1}\cdot y_0$ depends only on A and is $\mathrm{Gal}(M/\mathbb{F}((u)))$-fixed. Therefore, if $M/\mathbb{F}((u))$ is tamely ramified, which implies $\mathcal{B}(M)^{\mathrm{Gal}(M/\mathbb{F}((u)))}=\mathcal{B}$, we conclude that x_0 belongs to $\mathcal{B}$.

6.b.4. We continue to write $\Phi=\Phi_A$. If x is in $\mathcal{B}$, we can apply the triangle inequality above to x, $\Phi(x)$ and $x_0=\Phi(x_0)$, in two different ways. We obtain:

$$\delta(\Phi(x),x)\le\delta(\Phi(x),x_0)+\delta(x_0,x)=p\cdot\delta(x,x_0)+\delta(x,x_0)^*,$$

$$\delta(\Phi(x),x_0)=p\cdot\delta(x,x_0)\le\delta(\Phi(x),x)+\delta(x,x_0).$$

Combining these we get

$$(p-1)\cdot\delta(x,x_0)\le\delta(\Phi(x),x)\le p\cdot\delta(x,x_0)+\delta(x,x_0)^*. \tag{6.42}$$

This implies that if $h\in G(\mathbb{F}((u)))$ is such that

$$p\cdot\delta(h\cdot y_0,x_0)+\delta(h\cdot y_0,x_0)^*\le\nu, \tag{6.43}$$

then the corresponding point $h\cdot G(\mathbb{F}[[u]])$ in $\mathcal{F}_G(\mathbb{F})$ belongs to $\mathcal{C}_{\nu,G}(A)$, which is then non-empty and is contained in the ball of radius $\nu/(p-1)$ around x_0.

6.b.5. Suppose that $A'=h^{-1}\cdot A\cdot\phi(h)$ with $h\in G(\mathbb{F}((u)))$. Then $A'=(gh)^{-1}\phi(gh)$ and the corresponding $\Phi_{A'}$-fixed vertex is $x_0'=(gh)^{-1}\cdot y_0=h^{-1}\cdot x_0$. We conclude that the orbit $G(\mathbb{F}((u)))\cdot x_0$ only depends on the ϕ-conjugacy class of A in $G(\mathbb{F}((u)))$. By the above, if $M/\mathbb{F}((u))$ is tamely ramified, x_0 belongs to $\mathcal{B}$.

6.c. We continue to assume that $k=\mathbb{F}_p$ and now take $G=H=GL_d$. Take T the standard maximal torus of GL_d. Then the finite Weyl group is the symmetric group S_d, $X_*(T)_{\mathbf{R}}=X_*(T)\otimes_{\mathbb{Z}}\mathbb{R}$, and the standard choice of a positive closed Weyl chamber is

$$C=\{(v_1,\dots,v_d)\in\mathbb{R}^d\mid v_1\ge v_2\ge\cdots\ge v_d\}.$$

The partial order on C is given by: $(v_1,\dots,v_d)\le(v_1',\dots,v_d')$ iff

$$\sum_{i=1}^r v_i\le\sum_{i=1}^r v_i',\ \text{for } r=1,\dots,d-1, \text{ and } v_1+\cdots+v_d=v_1'+\cdots+v_d'.$$

In this case, we will explain the construction of the fixed point in a slightly different way. Start with $M_A=(k((u))^d,A\cdot\phi)$ and set

$$U=(k((u))^{\mathrm{sep}}\otimes_{k((u))}M_A)^{\phi\otimes\Phi_A=\mathrm{Id}}\subset k((u))^{\mathrm{sep}}\otimes_{k((u))}M_A$$

for the k-vector space of the corresponding $\mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$-representation ρ. (Here $\phi:k((u))^{\mathrm{sep}}\to k((u))^{\mathrm{sep}}$ denotes again the Frobenius of the separable closure.) In fact, one can see from the construction of ρ that there is a finite separable extension $L/k((u))$ such that

$$U=(L\otimes_{k((u))}M_A)^{\phi\otimes\Phi=\mathrm{Id}}\subset L\otimes_{k((u))}M_A=L^d$$

as $\mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$-modules. (Note that L^d also supports a Φ-module structure for the extension $\phi_{|L}$ of ϕ to L; this is given by $A\cdot\phi_{|L}$) Now set $\mathfrak{M}_0$ for the $\mathcal{O}_L$-submodule in L^d generated by the elements in U. Then $\mathfrak{M}_0/u_L\mathfrak{M}_0\simeq U$ and so $\mathfrak{M}_0$ is an $\mathcal{O}_L$-lattice in L^d. Since $\phi\otimes\Phi=A\cdot\phi_{|L}$ acts as identity on U, we can see that

$$(A\cdot\phi_{|L})^*(\mathfrak{M}_0)=\mathfrak{M}_0.$$

The lattice $\mathfrak{M}_0$ gives a point x_0 of $\mathcal{B}(L)$ which is fixed under the map Φ, i.e $\Phi(x_0)=x_0$.

6.d. In this paragraph, we will explain the picture in the building for $\mathbb{F}=\mathbb{F}_p$ and $G=H=GL_2$. Our main objective is the following. Given a dominant coweight $\nu=(a,b)$ with $a\geq b\geq 0$ and a matrix $A\in GL_2(\mathbb{F}((u)))$, describe the set of vertices in the building $\mathcal{B}$ which correspond to $\mathbb{F}$-valued points in $\mathcal{C}_\nu(A)$, i.e, to lattices $\mathfrak{M}\subset\mathbb{F}((u))^2$ for which $\Phi_A(\phi^*(\mathfrak{M}))\subset\mathfrak{M}$ and such that $\mathfrak{M}/\Phi_A(\phi^*(\mathfrak{M}))=\mathfrak{M}/\langle A\cdot\phi(\mathfrak{M})\rangle$ has elementary divisors (a',b'), $a'\geq b'\geq 0$ which are smaller than $\nu=(a,b)$, i.e $a'\leq a$, $a'+b'=a+b$. The corresponding set in the building is the set of vertices x such that $0\leq\delta(x,\Phi_A(x))\leq\nu$. To simplify our discussion, we will consider the projection $\mathcal{B}\to\mathcal{T}$ where $\mathcal{T}$ is the tree of homothety classes of lattices in $\mathbb{F}((u))^2$ (i.e the building for $PGL_2(\mathbb{F}((u)))$). Note that the Weyl chamber distance δ on the tree $\mathcal{T}$ coincides (up to sign) with the usual distance $d:\mathcal{T}\times\mathcal{T}\to\mathbb{R}_{\geq 0}$.

We consider the sets $\mathrm{Vert}(\mathcal{T})_{\nu,A}$ of vertices x in the tree $\mathcal{T}$ for which

$$d(x,\Phi_A(x))\leq r=|a-b|.$$

Let x_0 be the fixed point of $\Phi=\Phi_A$ on $\mathcal{T}(M)$ and $\tilde{x}_0$ its projection to $\mathcal{T}$. Note that the inequalities (6.42) imply that

$$B(x_0,\frac{r}{p+1})\cap\mathrm{Vert}(\mathcal{T})\subset\mathrm{Vert}(\mathcal{T})_{\nu,A}\subset B(x_0,\frac{r}{p-1})\cap\mathrm{Vert}(\mathcal{T})$$

where $B(x_0,R)$ is the "ball"

$$d(x,x_0)\leq R \tag{6.44}$$

of radius R centered at the point x_0.

To refine this, we will consider several possible cases:

A) $\tilde{x}_0$ is not a vertex in $\mathcal{T}$. Then $\tilde{x}_0$ lies on a segment $[\eta,\eta']$ with η, η' the closest vertices to $\tilde{x}_0$. Now consider the images $\Phi(\eta)$, $\Phi(\eta')$. Since the geodesic $[x_0,\Phi(\eta)]$ passes through the projection $\tilde{x}_0$, it also has to pass through either η or η' (but not both). Similarly for $[x_0,\Phi(\eta')]$. There are several subcases:

1) $\eta\in[x_0,\Phi(\eta')]$, $\eta'\in[x_0,\Phi(\eta)]$. Apply Φ to conclude that $\Phi(\eta)$ lies in the geodesic from x_0 to $\Phi^2(\eta')$ and $\Phi(\eta')$ in the geodesic from x_0 to $\Phi^2(\eta)$. Note that if $\Phi(\tilde{x}_0)\neq\tilde{x}_0$ and is, for example, between $\tilde{x}_0$ and η', then $\Phi([x_0,\eta'])=[x_0,\Phi(\eta')]$ would pass first through $\tilde{x}_0$, then through $\Phi(\tilde{x}_0)$, and then through η. This contradicts the fact that this is a geodesic. A

similar contradiction is obtained if $\Phi(\tilde{x}_0)$ is between $\tilde{x}_0$ and η. We conclude that $\Phi(\tilde{x}_0) = \tilde{x}_0$ and hence $x_0 = \tilde{x}_0$.

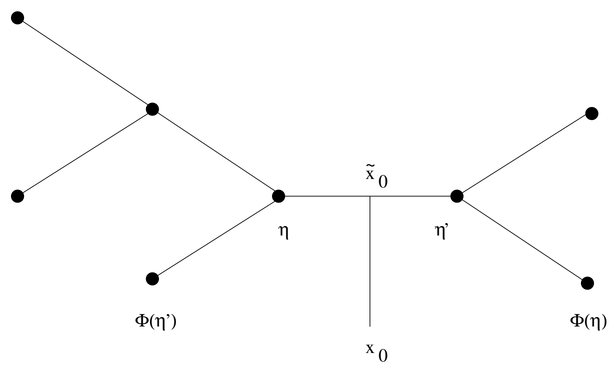

FIGURE 1. The case A1

By similar arguments, we deduce that the limit

$$\lim_{n\to\infty}[\Phi^{2n}(\eta), \Phi^{2n}(\eta')] = \lim_{n\to\infty}[\Phi^{2n+1}(\eta'), \Phi^{2n+1}(\eta)]$$

gives an apartment which is preserved (but flipped) by Φ. Indeed, Φ takes the half-apartment $\lim\limits_{n\to\infty}[x_0, \Phi^{2n}(\eta)]$ to $\lim\limits_{n\to\infty}[x_0, \Phi^{2n+1}(\eta)]$.

Note that in this case, there are no half-apartments in the tree $\mathcal{T}$ that are preserved by Φ. Indeed, consider a vertex y in such a half-apartment and connect this to x_0; the geodesic has to pass through either η or η'; in either case, since the geodesic from x_0 to $\Phi(y)$ has to pass through the opposite point η', resp. η, we obtain a contradiction. Recall that the set of half apartments in the tree can be naturally identified with the set of one-dimensional subspaces of the corresponding vector space $\mathbb{F}((u))^2$. Hence, we see that in case (A1) the Φ-module given by the matrix A is simple.

Now suppose x is a vertex in $\mathcal{T}$. The geodesic $[x_0, x]$ has to pass through either η or η'. Suppose that $\eta' \in [x_0, x]$ (the other case is similar) and consider $\Phi([x_0, x]) = [\Phi(x_0), \Phi(x)] = [x_0, \Phi(x)]$. This contains $\Phi(\eta')$ and therefore has to pass through η (since $\eta \in [x_0, \Phi(\eta')]$). Therefore, the geodesic $[x, \Phi(x)]$ passes through both η and η' (and also x_0) and we have

$$\begin{aligned} d(x, \Phi(x)) &= d(x, x_0) + d(x_0, \Phi(x)) \\ &= (p+1)d(x, x_0). \end{aligned}$$

Hence, in this case, $d(x, \Phi_A(x)) \le r$ amounts to $d(x, x_0) \le r/(p+1)$ and we have

$$\mathrm{Vert}(\mathcal{T})_{\nu,A} = B(x_0, \frac{r}{p+1}) \cap \mathrm{Vert}(\mathcal{T}).$$

2) $\eta \in [x_0, \Phi(\eta)]$, $\eta' \in [x_0, \Phi(\eta')]$. Then, we can see that the limits $\lim_{\to n}[x_0, \Phi^n(\eta)]$, $\lim_{\to n}[x_0, \Phi^n(\eta')]$ give two half-apartments that are both preserved by Φ. As above, this implies that the Φ-module given by A contains two 1-dimensional Φ-submodules; we can easily see that these are distinct. Hence the Φ-module given by A is decomposable.

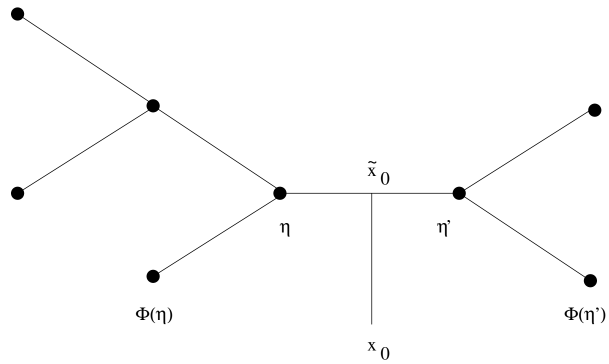

FIGURE 2. The case A2

Now suppose x is a vertex in $\mathcal{T}$. The geodesic $[x_0, x]$ has to pass through either η or η'. Suppose that $\eta \in [x_0, x]$ and in fact suppose that $a \geq 0$ is the largest integer for which $\Phi^a(\eta)$ is contained in $[x_0, x]$. Consider $\Phi([x_0, x]) = [x_0, \Phi(x)]$ which has to contain $\Phi^{a+1}(\eta)$. Therefore, the geodesic $[x, \Phi(x)]$ has to pass through $\Phi^{a+1}(\eta)$. We obtain

$$\begin{aligned} d(x, \Phi(x)) &= d(x, \Phi^{a+1}(\eta)) + d(\Phi^{a+1}(\eta), \Phi(x)) \\ &= d(x, \Phi^{a+1}(\eta)) + p \cdot d(\Phi^a(\eta), x). \end{aligned}$$

If x' is the projection of x to the half-apartment $\lim_{\to n}[x_0, \Phi^n(\eta)]$, then we can rewrite this distance as

$$\begin{aligned} d(x, \Phi(x)) &= d(\Phi(x), \Phi(x')) + d(\Phi(x'), \Phi^{a+1}(\eta)) + \\ &\quad + d(\Phi^{a+1}(\eta), \Phi^a(\eta)) - d(x', \Phi^a(\eta)) + d(x, x') \\ &= (p+1)d(x, x') + (p-1)d(x', \Phi^a(\eta)) + p^a d(\Phi(\eta), \eta). \end{aligned}$$

There is a similar expression if η' is in $[x_0, x]$. Hence, $d(x, \Phi_A(x)) \leq r$ can be described as the union of two "thinning tubes" around the two half-apartments that are preserved by Φ. Note that, in the above, when $d(x, \Phi(x))$ is bounded, the possible values of a are bounded too.

3) $\eta \in [x_0, \Phi(\eta)]$, $\eta \in [x_0, \Phi(\eta')]$ (the case $\eta' \in [x_0, \Phi(\eta')]$, $\eta' \in [x_0, \Phi(\eta)]$ is symmetric). Then $\lim_{\to n}[x_0, \Phi^n(\eta)]$ gives a half-apartment which is preserved by Φ. As above, this implies that the Φ-module given by A contains a 1-dimensional Φ-submodule and, therefore, it is not simple. In this case, we can see that this is the unique half-apartment preserved by Φ. Indeed, consider a vertex y in such a half-apartment $\mathcal{A}'$ and connect this with a geodesic to x_0; the geodesic has to pass through either η or η' and we can easily rule out η'. Now the geodesic $[x_0, \Phi(y)]$ has to pass through $\Phi(\eta)$. Since $\Phi(y)$ is also in $\mathcal{A}'$, we can conclude that $\mathcal{A}'$ also contain $\Phi(\eta)$. Inductively, $\mathcal{A}'$ contains $\Phi^n(\eta)$ for all n. We conclude that the Φ-module given by A is not simple and not decomposable.

Now suppose x is a vertex in $\mathcal{T}$. The geodesic $[x_0, x]$ has to pass through either η or η'.

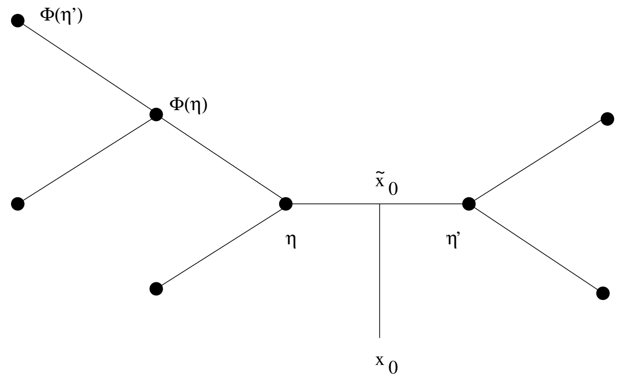

FIGURE 3. The case A3

• Suppose first that the geodesic $[x_0, x]$ passes through η. In fact, suppose that $a \geq 0$ is the largest integer for which $\Phi^a(\eta)$ is contained in $[x_0, x]$. Then as before, we obtain

$$\begin{aligned} d(x, \Phi(x)) &= d(x, \Phi^{a+1}(\eta)) + pd(\Phi^a(\eta), x) \\ &= (p+1)d(x, x') + (p-1)d(x', \Phi^a(\eta)) + p^a d(\Phi(\eta), \eta) \end{aligned}$$

with x' the projection of x to the apartment $\lim_{\to n}[x_0, \Phi^n(\eta)]$. Hence again the set $d(x, \Phi_A(x)) \leq r$ can be described for these vertices as a union of thinning tubes.

• Now suppose that the geodesic $[x_0, x]$ passes through η'. Then an argument as in case (A1) gives

$$d(x, \Phi(x)) = (p+1)d(x, x_0) - 2d(\tilde{x}_0, x_0).$$

Hence for this kind of vertices this set $d(x, \Phi_A(x)) \leq r$ is a ball around x_0.

B) Suppose now that $\tilde{x}_0 = \eta$ is a vertex of $\mathcal{T}$. There are two subcases:

1) $\tilde{x}_0 = \eta$ is not fixed by Φ. Then, $\lim_{\to n}[x_0, \Phi^n(\eta)]$ gives again a half-apartment that is preserved by Φ. We can, in fact, see as before, that this is the unique such half-apartment. Hence, in this case, the Φ-module given by A is not simple and not decomposable. Note that after replacing $\mathbb{F}$ by a finite extension, x_0 becomes of type B2 below.

If x is a vertex of $\mathcal{T}$ then $\eta \in [x_0, x]$. If $a \geq 0$ is the largest integer for which $\Phi^a(\eta)$ is contained in $[x_0, x]$ we obtain for $d(x, \Phi(x))$ the same formula as in cases A2 or A3a. Hence in this case $d(x, \Phi_A(x)) \leq r$ is a union of thinning tubes.

2) $\tilde{x}_0 = \eta$ is fixed by Φ, in other words the fixed point x_0 is a vertex of $\mathcal{T}$. This corresponds to the homothety class of a lattice $\mathfrak{M}_0$; we have $\Phi(\phi^*\mathfrak{M}_0) = u^s\mathfrak{M}_0$ for some s. Denote by $\{\eta_i\}_{i=0,\dots,p}$ its neighborhood vertices. The link of η is identified with the projective space of lines in $\mathfrak{M}_0/u\mathfrak{M}_0$ and the action of Φ on the link then corresponds to the action on the projective space given the linear action of $u^{-s} \cdot \Phi$ on $\mathfrak{M}_0/u\mathfrak{M}_0$. Now observe that the geodesic $[x_0, \Phi(\eta_i)]$ passes through η_i if and only if the action of Φ on the link leaves the point of the link that is given by η_i fixed. Depending on whether the number of fixed points in the link is ≥ 2, resp. 1, resp. 0, the Φ-module is decomposable, resp. not simple and

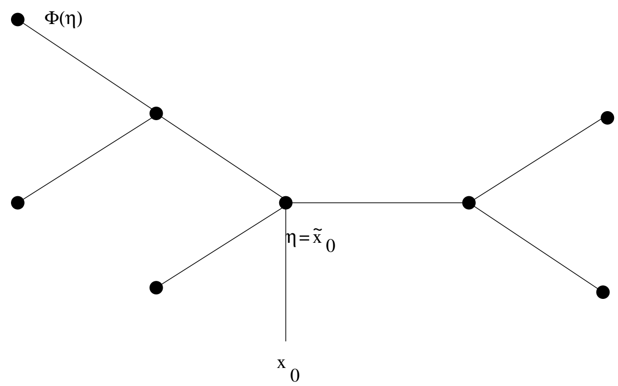

FIGURE 4. The case B1

not decomposable, resp. simple. Note that in the last case it is not absolutely simple, it disappears if $\mathbb{F}$ is replaced by a finite extension.

Now let x be a vertex of $\mathcal{T}$ and consider the geodesic $[x_0, x]$ which has to pass through one of the vertices η_j. We distinguish cases according as η_j gives a fixed point of the link, or not.

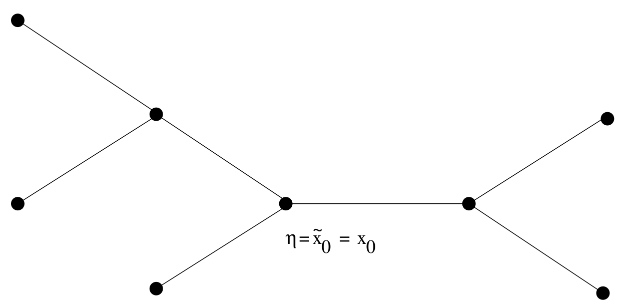

FIGURE 5. The case B2

• Suppose first that $[x_0, x]$ passes through a vertex η_j with the corresponding point of the link fixed by Φ. Then the argument of case A2 applies to obtain $d(x, \Phi(x))$. (It involves the largest integer $a \geq 0$ such that $\Phi^a(\eta_j)$ is in $[x_0, x]$.) For this kind of vertices $d(x, \Phi_A(x)) \leq r$ is a union of thinning tubes.

• Now suppose that $[x_0, x]$ passes through a vertex η_j with the corresponding point of the link not fixed by Φ. Then the argument of case A1 applies to give

$$d(x, \Phi(x)) = (p + 1)d(x, x_0).$$

For this kind of vertices $d(x, \Phi_A(x)) \leq r$ is a ball around x_0.

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