Derivatives of Eisenstein series and Faltings heights[1]

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In a series of papers, [**25**], [**30**], [**28**], [**29**], [**31**], [**26**], we showed that certain quantities from the arithmetic geometry of Shimura varieties associated to orthogonal groups occur in the Fourier coefficients of the derivative of suitable Siegel-Eisenstein series. It was essential in these examples that this derivative was the second term in the Laurent expansion of a Siegel-Eisenstein series at the center of symmetry, and that the first term in this Laurent expansion vanished (*incoherent case*). In the present paper we prove a relation between a generating function for the heights of Heegner cycles on the arithmetic surface associated to a Shimura curve and the second term in the Laurent expansion at $s=\frac{1}{2}$ of an Eisenstein series of weight $\frac{3}{2}$ for SL_2. It is remarkable that $s=\frac{1}{2}$ is not the center of symmetry and that the first term of the Laurent expansion is non-zero. In fact, this nonzero value has a geometric interpretation in terms of the Shimura curve over the field of complex numbers. Considering the fact that the Eisenstein series is a rather familiar classical object, it is surprising that this interpretation of its Laurent expansion at $s=\frac{1}{2}$ has not been noticed before. As we will argue below in this introduction, we believe that our result is part of a general pattern involving the heights of divisors on arithmetic models of Shimura varieties associated to orthogonal groups.

We now describe our results in more detail.

Let B be an indefinite division quaternion algebra over $\mathbb{Q}$ and let O_B be a maximal order in B. Let $D(B)$ be the product of all primes p at which B is division. Let $\mathcal{M}$ be the moduli space of abelian varieties of dimension 2 with a (special) action of O_B. Then $\mathcal{M}$ is an integral model of the Shimura curve attached to B; it is proper of relative dimension 1 over $\mathrm{Spec}\,(\mathbb{Z})$, with semi-stable reduction at all primes and is smooth at all primes p at which B splits, i.e., for $p \nmid D(B)$. Ignoring, for the moment, the fact that $\mathcal{M}$ is only a stack, we may consider $\mathcal{M}$ as an arithmetic surface in the sense of Arakelov theory, [**14**], [**3**], ...

For each $m\in\mathbb{Z}$ and for $v\in\mathbb{R}_+^\times$, we define a class in the arithmetic Chow group

$$\hat{\mathcal{Z}}(m,v) = \big(\,\mathcal{Z}(m), \Xi(m,v)\,\big) \in \widehat{CH}^1(\mathcal{M}). \tag{0.1}$$

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Here, for $m > 0$, $\mathcal{Z}(m)$ is the divisor on $\mathcal{M}$ corresponding to those O_B–abelian varieties which admit a special endomorphism x with $x^2 = -m$. These cycles can be viewed as the Shimura curve analogues of the cycles on the modular curve defined by elliptic curves with CM by the order $\mathbb{Z}[\sqrt{-m}]$. For $m < 0$, $\mathcal{Z}(m) = \emptyset$. For all $m \neq 0$, $\Xi(m, v)$ is the (non-standard) Green's function introduced in [**25**]. The class $\hat{\mathcal{Z}}(0, v)$ will be defined presently.

The moduli stack $\mathcal{M}$ carries a universal abelian variety $\mathcal{A}/\mathcal{M}$, and the Hodge bundle ω on $\mathcal{M}$ is defined by

$$\omega = \wedge^2(\,Lie(\mathcal{A}))^*. \tag{0.2}$$

We equip ω with the metric $||\ ||$ which, for $z \in \mathcal{M}(\mathbb{C})$ is given by

$$||\alpha||_z^2 = e^{-2C} \cdot \frac{1}{4\pi^2} \left| \int_{A_z(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|, \tag{0.3}$$

where

$$C = \frac{1}{2}\big(\,\log(4\pi) + \gamma\,\big), \tag{0.4}$$

for Euler's constant γ. The reason for this normalization will be explained below. We thus obtain a class $\hat{\omega} = (\omega, ||\ ||) \in \widehat{\mathrm{Pic}}(\mathcal{M})$, and we set

$$\hat{\mathcal{Z}}(0, v) = -\Big(\,\hat{\omega} + (0, \log(v))\Big) \in \widehat{CH}^1(\mathcal{M}). \tag{0.5}$$

Using the Gillet–Soulé height pairing $\langle\ ,\ \rangle$ between $\widehat{CH}^1(\mathcal{M})$ and $\widehat{\mathrm{Pic}}(\mathcal{M})$, [**18**], we form the height generating series

$$\phi_{\text{height}}(\tau) = \sum_m \langle\ \hat{\mathcal{Z}}(m, v), \hat{\omega}\ \rangle\ q^m, \tag{0.6}$$

where, for $\tau = u + iv$ in the upper half plane $\mathfrak{H}$, we have set $q = e(\tau) = e^{2\pi i\tau}$. The quantities $\langle\ \hat{\mathcal{Z}}(m, v), \hat{\omega}\ \rangle$ can be thought of as arithmetic degrees [**5**], [**2**]. At the same time, we can define the more elementary generating series

$$\phi_{\text{degree}}(\tau) = \sum_m \deg(\hat{\mathcal{Z}}(m, v))\, q^m \quad , \tag{0.7}$$

where $\deg(\hat{\mathcal{Z}}(m, v)) = \deg(\mathcal{Z}(m))$ is simply the usual (geometric) degree of the 0–cycle $\mathcal{Z}(m)_{\mathbb{C}}$ on the complex Shimura curve $\mathcal{M}_{\mathbb{C}}$.

To see that these generating series are the q–expansions of modular forms, we consider a family of Eisenstein series. In 1975, Zagier [**45**], [**8**], introduced a (non-holomorphic) Eisenstein series of weight $\frac{3}{2}$, whose Fourier expansion is given by

$$\mathcal{F}(\tau) = -\frac{1}{12} + \sum_{m>0} H(m)\, q^m + \sum_{n} \frac{1}{16\pi}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi n^2 v r}\, r^{-\frac{3}{2}}\, dr\, q^{-n^2}, \tag{0.8}$$

where $H(m)$ is the number of classes of positive definite integral binary quadratic forms of discriminant $-m$. This series, which played a key role in the work of Hirzebruch and Zagier [**21**] on generating functions for intersection numbers of curves on Hilbert modular surfaces, can be viewed as the value at $s = \frac{1}{2}$ of an Eisenstein series $\mathcal{F}(\tau, s)$, defined for $s \in \mathbb{C}$ and satisfying a functional equation $\mathcal{F}(\tau, -s) = \mathcal{F}(\tau, s)$. In fact, there is a whole family of such series, $\mathcal{E}(\tau, s; D)$, where D is a square free positive integer, whose values at $s = \frac{1}{2}$, for $D > 1$, are given by[4]

$$\mathcal{E}(\tau, \frac{1}{2}; D) = -(-1)^{\mathrm{ord}(D)} \frac{1}{12} \prod_{p|D} (p-1) + \sum_{m>0} 2\,\delta(d; D)\, H_0(m; D)\, q^m. \tag{0.9}$$

Here $H_0(m; D)$ is a variant of the class number $H(m)$, and $\delta(d; D)$ is either 0 or a power of 2, cf. (8.19) and (8.20) respectively, and $\mathrm{ord}(D)$ is the number of prime factors of D. In the case $D = D(B) > 1$, a simple calculation of $\deg(\mathcal{Z}(m))$ proves the (known) relation

$$\phi_{\mathrm{degree}}(\tau) = \mathcal{E}(\tau, \frac{1}{2}; D(B)), \tag{0.10}$$

so that the value of $\mathcal{E}(\tau, s; D(B))$ at $s = \frac{1}{2}$ is the degree generating function. The main result of this paper asserts that the second term in the Laurent expansion of the Eisenstein series $\mathcal{E}(\tau, s; D(B))$ at the point $s = \frac{1}{2}$ contains information about the arithmetic surface $\mathcal{M}$:

Theorem A. *For $D(B) > 1$,*

$$\phi_{\mathrm{height}}(\tau) = \mathcal{E}'(\tau, \frac{1}{2}; D(B)) + \mathbf{c}$$

for some constant $\mathbf{c}$.

This identity is proved by a direct computation of the two sides. The resulting formulas, cf. Theorem 8.8, are quite complicated. For example, for $m > 0$ in the case in which

[4] Our normalization of these series differs slightly at 2 from that used by Zagier, so that our $\mathcal{E}(\tau, \frac{1}{2}; 1)$ is not quite Zagier's function, cf (8.24) below.

$\deg(\mathcal{Z}(m)_{\mathbb{C}}) \neq 0$, the coefficient of q^m in the derivative of the Eisenstein series is given by

(0.11)
$$2\,\delta(d;D(B))\,H_0(m;D(B))\cdot\bigg[\,\frac{1}{2}\,\log(d)+\frac{L'(1,\chi_d)}{L(1,\chi_d)}-\frac{1}{2}\log(\pi)-\frac{1}{2}\gamma$$
$$+\frac{1}{2}J(4\pi mv)+\sum_{\substack{p\\ p\nmid D(B)}}\bigg(\ \log|n|_p-\frac{b_p'(n,0;D)}{b_p(n,0;D)}\bigg)+\sum_{\substack{p\\ p\mid D(B)}}K_p\,\log(p)\,\bigg].$$

Here we write the discriminant of the order $\mathbb{Z}[\sqrt{-m}]$ as $4m = n^2d$ for a fundamental discriminant $-d$, and the other notation is explained in Theorem 8.8. Theorem A asserts that this expression coincides with the height pairing $\langle\ \hat{\mathcal{Z}}(m,v),\hat{\omega}\ \rangle$! A point over $\bar{\mathbb{Q}}$ of $\mathcal{Z}(m)$ corresponds to an O_B–abelian surface A over $\bar{\mathbb{Q}}$, equipped with an action of $\mathbb{Z}[\sqrt{-m}]$ commuting with that of O_B. Such a surface is *isogenous* to a product $E_d\times E_d$, where E_d is an elliptic curve with complex multiplication by the maximal order $O_{\boldsymbol{k}}$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. With our normalization of the metric on the Hodge bundle, the Faltings height of $E_d\times E_d$ is given by

$$h^*_{\text{Fal}}(E_d\times E_d)=2\,h^*_{\text{Fal}}(E_d)=\frac{1}{2}\log(d)+\frac{L'(1,\chi_d)}{L(1,\chi_d)}-\frac{1}{2}\log(\pi)-\frac{1}{2}\gamma, \tag{0.12}$$

so that the geometric meaning of the first terms of (0.11), and of our title, emerges. The change in the Faltings height due to the isogeny is accounted for by the term involving the sum over $p\nmid D(B)$, where the logarithmic derivatives occurring there are given explicitly in Lemma 8.10. The sum over $p\mid D(B)$ has the following geometric meaning. The arithmetic surface $\mathcal{M}$ has bad reduction at such primes and the cycle $\mathcal{Z}(m)$, defined as a moduli space, can include components of the special fiber $\mathcal{M}_p$, i.e., vertical components [**29**]. Their contribution to the height pairing coincides with the term $K_p\cdot\log(p)$ in (0.11), where K_p is given explicitly in Theorem 8.8. Finally, there is an additional 'archimedean' term in the height pairing, which arises from the fact that the Green's function $\Xi(m,v)$ is not orthogonal to the Chern form $\mu = c_1(\hat{\omega})$ of the Hodge bundle. This contribution coincides with the term involving $\frac{1}{2}J(4\pi mv)$.

The ambiguous constant $\mathbf{c}$ occurs in Theorem A because we do not know the exact value of the quantity $\langle\ \hat{\omega},\hat{\omega}\ \rangle$. More precisely, for $m=0$, the constant term of the derivative of the Eisenstein series at $s=\frac{1}{2}$ with $D=D(B)>1$ is given by

$$\mathcal{E}_0'(\tau,\frac{1}{2};D(B))=\zeta_D(-1)\left[\frac{1}{2}\log(v)-2\frac{\zeta'(-1)}{\zeta(-1)}-1+2C+\sum_{p\mid D}\frac{p\log(p)}{p-1}\right], \tag{0.13}$$

where $\zeta_D(s)=\zeta(s)\prod_{p\mid D}(1-p^{-s})$. On the other hand, by (0.5), the constant term of the generating function for heights is given by

$$\phi_{\text{height},0}(\tau)=\langle\ \hat{\mathcal{Z}}(0,v),\hat{\omega}\ \rangle=-\langle\ \hat{\omega},\hat{\omega}\ \rangle-\frac{1}{2}\log(v)\,\deg(\omega). \tag{0.14}$$

Noting that $\deg(\omega) = -\zeta_D(-1)$, we see that the constant terms would coincide as well, i.e., the constant $\mathbf{c}$ in Theorem A would vanish, if

$$\langle\, \hat{\omega}, \hat{\omega}\,\rangle \overset{??}{=} \zeta_D(-1)\left[2\frac{\zeta'(-1)}{\zeta(-1)} + 1 - 2C - \sum_{p|D} \frac{p\log(p)}{p-1}\right]. \tag{0.15}$$

If we write $\hat{\omega}_o = (\omega, ||\ ||_{\mathrm{nat}})$ for the Hodge bundle with the more standard choice of metric, cf. (10.15) below and [**3**], and if we delete the 'extra' factor of $\frac{1}{2}$ which occurs due to the fact that $\mathcal{M}$ is a stack, cf. section 4, then (0.15) amounts to

$$\langle\, \hat{\omega}_o, \hat{\omega}_o\,\rangle^{\mathrm{nat}} \overset{??}{=} 4\,\zeta_D(-1)\left[\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} - \frac{1}{2}\sum_{p|D} \frac{p\log(p)}{p-1}\right]. \tag{0.16}$$

If we take formally $D = D(B) = 1$, so that $\mathcal{M}$ would be the modular curve and $\hat{\omega}_o$ the bundle of modular forms of weight 2 with the Petersson metric, then, indeed, by the result of Bost and Kühn, [**4**],[**34**],

$$\langle\, \hat{\omega}_o, \hat{\omega}_o\,\rangle^{\mathrm{nat}} = 4\,\zeta(-1)\left[\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right]. \tag{0.17}$$

These considerations are one of the motivations for our choice of metric on the Hodge bundle and our definition of $\hat{\mathcal{Z}}(0,v)$.

We expect that Theorem A will continue to hold when $D(B) = 1$, i.e., in the case of the modular curve $\mathcal{M}$, where (0.17) will allow us to eliminate the constant $\mathbf{c}$. There are, however, extra complications. The first is that the metric on $\hat{\omega}$ becomes singular at the cusp. This difficulty was overcome by Bost [**3**], [**4**] and Kühn [**34**] by extending the definition of $\widehat{\mathrm{Pic}}(\mathcal{M}) \simeq \widehat{CH}^1(\mathcal{M})$ to allow more general Green's functions. On these more general Chow groups, the geometric degree map should be defined as

$$\deg : \widehat{CH}^1(\mathcal{M}) \longrightarrow \mathbb{R}, \qquad (Z, g_Z) \mapsto \int_{\mathcal{M}(\mathbb{C})} \omega_Z, \tag{0.18}$$

where ω_Z is the (now not necessarily smooth) form occuring on the right hand side of the Green's equation

$$dd^c\, g_Z + \delta_Z = \omega_Z. \tag{0.19}$$

Note that this definition agrees with the previous one in the case $D(B) > 1$. With our previous definition of $\hat{\mathcal{Z}}(m,v)$ for $m \neq 0$ and $m = -n^2$ and with a slight modification when $m = 0$ or $m = -n^2$, the result of Funke [**15**], [**16**] shows that, indeed,

$$\begin{aligned}\phi_{\deg}(\tau) &= \mathcal{E}(\tau, \tfrac{1}{2}; 1)\\ &= -\frac{1}{12} + \sum_{m>0} 2\,H_0(m;1)\,q^m + \sum_{n\in\mathbb{Z}} \frac{1}{8\pi}\,v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi n^2 vr}\, r^{-\frac{3}{2}}\,dr \cdot q^{-n^2}\ .\end{aligned} \tag{0.20}$$

In fact, if we had defined the cycles $\mathcal{Z}(m)$ by imposing the action of an order of discriminant m (rather than $4m$) on our O_B–abelian surface, then the degree generating function would coincide exactly with Zagier's function (0.8)! We defer the calculations of the additional terms which occur in the derivative $\mathcal{E}'(\tau, \frac{1}{2}; 1)$ and in the height generating function for $D(B) = 1$ to a sequel to this paper [**32**].

Relations like the ones proved here between the first (resp. second) term of the Laurent expansion of an Eisenstein series and the generating function for degrees (resp. heights) should hold in much greater generality. More precisely, suppose that V is a rational vector space with nondegenerate inner product of signature $(n, 2)$. Let $H = \mathrm{GSpin}(V)$ and let D be the space of oriented negative 2–planes in $V(\mathbb{R})$. Then, for each compact open subgroup $K \subset G(\mathbb{A}_f)$, there is a quasiprojective variety X_K, defined over $\mathbb{Q}$, with

$$X_K(\mathbb{C}) \simeq H(\mathbb{Q})\backslash\bigg(\, D \times H(\mathbb{A}_f)/K\,\bigg). \tag{0.21}$$

For each integer $m > 0$, and each K–invariant 'weight function' $\varphi \in S(V(\mathbb{A}_f))^K$ in the Schwartz space of $V(\mathbb{A}_f)$, there is a divisor $Z(m, \varphi)_K$ on X_K, rational over $\mathbb{Q}$, [**24**]. The variety X_K comes equipped with a metrized line bundle $\hat{\mathcal{L}} = (\mathcal{L}, ||\ ||)$, and it is proved in [**27**] that, with the exception of the cases $n = 1$, V isotropic and $n = 2$, V split, the degree generating function

$$\phi_{\deg}(\tau; \varphi) := \mathrm{vol}(X_K)\, \varphi(0) + \sum_{m>0} \deg(Z(m, \varphi))\, q^m \tag{0.22}$$

coincides with the value $E(\tau, \frac{n}{2}; \varphi)$ of an Eisenstein series $E(\tau, s; \varphi)$ of weight $\frac{n}{2}+1$ associated to φ. Here $\mathrm{vol}(X_K)$ (resp. $\deg(Z(m, \varphi))$) is the volume of $X_K(\mathbb{C})$, (resp. $Z(m, \varphi)_K$) with respect to Ω^n (resp. Ω^{n-1}), where Ω is the negative of the first Chern form of $\hat{\mathcal{L}}$. We believe that there should be an analogue of Theorem A in this situation. To obtain such a result, one needs, first of all, suitable extensions $\mathcal{Z}(m; \varphi)$ of the cycles $Z(m, \varphi)$ to suitable integral models $\mathfrak{X}_K$ of the X_K's. Next, since the varieties X_K are, in general, not projective, one needs nice compactifications $\bar{\mathfrak{X}}_K$ and, more importantly, an extension of the Gillet–Soulé theory, general enough to allow the singularities of the metric on the extension $\hat{\omega}$ of $\hat{\mathcal{L}}^\vee$ to the compactification, etc. Assuming all of this, one would have cycles

$$\widehat{\mathcal{Z}}(m, v; \varphi) = (\mathcal{Z}(m; \varphi), \Xi(m, v)) \in \widehat{CH}^1(\bar{\mathfrak{X}})_K \tag{0.23}$$

and a class $\hat{\omega} \in \widehat{CH}^1(\bar{\mathfrak{X}})_K$. The analogue of Theorem A would identify the height generating series

$$\phi_{\mathrm{height}}(\tau; \varphi) := \sum_m \langle\ \widehat{\mathcal{Z}}(m, \varphi; v), \hat{\omega}^n\ \rangle\ q^m \tag{0.24}$$

with the derivative $\mathcal{E}'(\tau,\frac{n}{2};\varphi)$ at $s=\frac{n}{2}$ of a normalized (and possibly slightly modified, cf. section 6 below) version $\mathcal{E}(\tau,s;\varphi)$ of the Eisenstein series $E(\tau,s;\varphi)$. However, it seems a challenge to go beyond the case considered in the present paper and to obtain such results for more general level structures (even for $n=1$) and for higher values of n, e.g. for $n=2$ (Hilbert-Blumenthal surfaces) or $n=3$ (Siegel threefolds). Nonetheless, the results of [**27**] provide some additional evidence in favor of this picture for general n.

With hindsight, it may be said that the results in [**31**] support our picture in the case $n=0$. This is the one case which is common to the general picture developed here and the general picture of our papers [**25**], [**30**], [**28**], [**29**], [**31**], [**26**]. In this rather degenerate case, the variety X_K is zero dimensional, so that the cycles $Z(m;\varphi)$ are actually empty, and the degree generating function $\phi_{\text{deg}}(\tau;\varphi)$ is identically zero. On the other hand, the associated Eisenstein series $\mathcal{E}(\tau,s;\varphi)$ of weight 1 is incoherent, in the sense of [**25**], [**26**], so that $\mathcal{E}(\tau,0;\varphi)=0$ as well. The main result of [**31**], Theorem 3, may be interpreted as the identity

$$\phi_{\text{height}}(\tau,\varphi_0)=\mathcal{E}'(\tau,0;\varphi_0)\quad . \tag{0.26}$$

Here, as in the present paper, $\varphi=\varphi_0$ is the characteristic function of a certain standard lattice and K is the maximal open compact subgroup. To make the transition to the result in [**31**] one has to take into account the following two remarks. First, the arithmetic degree $\widehat{\deg}(\cdot)$ on $\widehat{CH}^1(\mathfrak{X}_K)$ used in [**31**] may be viewed as

$$\widehat{\deg}(Z)=\langle Z,\hat{\omega}^0\rangle\quad . \tag{0.27}$$

Second, in defining the degree generating function in this case, we set

$$\hat{Z}(0,v)=\hat{\omega}+(0,\log\, v)\in\widehat{CH}^1(\mathfrak{X}_K)\quad , \tag{0.28}$$

where $\hat{\omega}$ is the Hodge bundle on $\mathfrak{X}$ (the moduli stack of elliptic curves with complex multiplication by $\mathcal{O}_q$ for a prime $q\equiv 3\mod(4)$), with metric *normalized as in (10.16)* of the present paper. Indeed, this particular choice of normalization, i.e., the choice of the constant C in (0.3), was motivated by the requirement that no ambiguous constant like $\mathbf{c}$ in Theorem A should arise in (0.26). Specifically, the constant term which occurs in [**31**] is given by

$$\begin{aligned} a_0(\phi,v)&=-2\,h(\boldsymbol{k})\,\big(\frac{1}{2}\,\log(v)+2\,h_{\text{Fal}}(E)-\log(2\pi)-\frac{1}{2}\log(\pi)+\frac{1}{2}\gamma+2\log(2\pi)\,\big)\\ &=-2\,h(\boldsymbol{k})\,\big(\frac{1}{2}\,\log(v)+2\,h_{\text{Fal}}(E)+\frac{1}{2}\log(\pi)+\frac{1}{2}\gamma+\log(2)\,\big)\\ &=-2\,h(\boldsymbol{k})\,\big(\frac{1}{2}\,\log(v)+2\,h^*_{\text{Fal}}(E)\,\big). \end{aligned} \tag{0.29}$$

Here the quantity $h_{\mathrm{Fal}}(E) - \frac{1}{2}\log(2\pi)$ is the Faltings height in the normalization of Colmez [**9**], which was used in (0.16) of [**31**], cf. Proposition 10.10 below. We found it particularly striking that the normalization of the metric on the Hodge bundle which eliminates any garbage constant in the case $n = 0$ also gives a precise match in the positive Fourier coefficients in our Shimura curve case ($n = 1$). Of course, this is perhaps not so surprising, given the fact that the cycles in the case of signature $(n, 2)$ are themselves (weighted) combinations of Shimura varieties of the same type for signature $(n - 1, 2)$. Thus a main term in the arithmetic degrees which occur in the positive Fourier coefficients of the height generating function for the signature $(n, 2)$ case is the 'arithmetic volume' occuring in the constant term for the $(n - 1, 2)$ case. This 'explains' the relation between the present paper and the results of [**31**]. It should not be difficult to verify that the positive Fourier coefficients of the derivatives of Eisenstein series of weight $\frac{n}{2} + 1$ at $s = \frac{n}{2}$ are related to the constant terms of the derivatives of those of weight $\frac{n-1}{2} + 1$ at $s = \frac{n-1}{2}$ in a similar way.

In a similar vein, we remark that the height generating function which, according to our picture above, is related to the derivative of Eisenstein series on $\mathrm{SL}_2 = \mathrm{Sp}_1$ of weight $\frac{n}{2} + 1$ at $s = \frac{n}{2}$, is connected with the singular Fourier coefficients of the derivative of Eisenstein series of genus 2, i.e., on Sp_2, of weight $\frac{n}{2} + 1$ at $s = \frac{n-1}{2}$. In fact, this is how we arrived at the height generating function considered in this paper. We hope to elaborate on this point in a future paper.

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Part I. Arithmetic geometry.

1. The moduli stack $\mathcal{M}$.

Let B be an indefinite quaternion algebra over $\mathbb{Q}$. We fix a maximal order O_B in B, and we let $D(B)$ be the product of the primes p at which B_p is a division algebra. For the moment, we allow the case $B = M_2(\mathbb{Q})$ and $O_B = M_2(\mathbb{Z})$, where $D(B) = 1$.

We denote by $\mathcal{M}$ the stack over $\operatorname{Spec}\mathbb{Z}$ representing the following moduli problem. The moduli problem associates to a scheme S the category $\mathcal{M}(S)$ whose objects are pairs (A, ι), where A is an abelian scheme over S and

$$\iota : O_B \longrightarrow \operatorname{End}_S(A)$$

is a homomorphism such that, for $a \in O_B$,

$$\det(\iota(a); \operatorname{Lie}(A)) = Nm^o(a). \tag{1.1}$$

Here Nm^o is the reduced norm on B and, as usual, [**22**], [**38**], the identity (1.1) is meant as an identity of polynomial functions on S. All morphisms in this category are isomorphisms.

Proposition 1.1. *$\mathcal{M}$ is an algebraic stack in the sense of Deligne-Mumford. Furthermore, $\mathcal{M}$ is proper over $\operatorname{Spec}\mathbb{Z}$ if B is a division algebra. The restriction of $\mathcal{M}$ to $\operatorname{Spec}\mathbb{Z}[D(B)^{-1}]$ is smooth of relative dimension 1. Finally, if $p \mid D(B)$, then $\mathcal{M} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_p$ has semi-stable reduction.* $\square$

2. Uniformization.

Let $H = B^{\times}$, considered as an algebraic group over $\mathbb{Q}$. Let

$$D = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, B_{\mathbb{R}}), \tag{2.1}$$

be the set of homomorphisms of $\mathbb{R}$–algebras, taking 1 to 1, with the natural conjugation action of $H(\mathbb{R})$. This action is transitive. We fix an isomorphism $B_{\mathbb{R}} \simeq M_2(\mathbb{R})$, so that $H(\mathbb{R}) \simeq GL_2(\mathbb{R})$, and a compatible isomorphism $D \simeq \mathbb{C} \setminus \mathbb{R}$, the union of the upper and lower half planes. Also let $K = \hat{O}_B^{\times} \subset H(\mathbb{A}_f)$, be the compact open subgroup determined by O_B, where $\hat{O}_B = O_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then we have, as usual, an isomorphism of Deligne-Mumford stacks over $\mathbb{C}$,

$$\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C} = [\, H(\mathbb{Q}) \backslash D \times H(\mathbb{A}_f)/K \,], \tag{2.2}$$

where the right is to be understood in the sense of stacks, [**10**], p.99. The stack on the right hand side may be written in a simpler way using the fact that $H(\mathbb{A}_f) = H(\mathbb{Q})K$. Let

$$\Gamma = H(\mathbb{Q}) \cap K = O_B^{\times}. \tag{2.3}$$

Then

$$\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C} = [\Gamma \backslash D]. \tag{2.4}$$

Note that Γ acts on D through its image $\bar{\Gamma} = \Gamma/\{\pm 1\}$ in $\mathrm{PGL}_2(\mathbb{R})$, with finite stabilizer groups. Instead of considering $[\Gamma \backslash D]$ as an algebraic stack, it is more traditional to view this quotient as an orbifold [**20**]. Intuitively speaking, this means that the quotient of D by the action of Γ is not carried out, but rather, all information obtained from the action of Γ on D is stored. As a particular instance, consider the hyperbolic volume form μ on D, normalized as

$$\mu = \frac{1}{2\pi}\, y^{-2}\, dx \wedge dy, \tag{2.5}$$

in standard coordinates on $\mathbb{C} \setminus \mathbb{R}$. Since this volume form is Γ–invariant, it induces a volume form on the orbifold $[\Gamma \backslash D] = \mathcal{M}(\mathbb{C})$. The volume of the orbifold $\mathcal{M}(\mathbb{C})$ is given by

$$\mathrm{vol}(\mathcal{M}(\mathbb{C})) = \int_{[\Gamma \backslash D]} \mu = \frac{1}{2} \int_{\Gamma \backslash D} \mu, \tag{2.6}$$

where the extra factor of $\frac{1}{2}$ in the second expression is due to the fact that the stablizer in Γ of a generic point of D, has order 2. Explicitly, we have, [**13**],

$$\mathrm{vol}(\mathcal{M}(\mathbb{C})) = \frac{1}{12} \prod_{p | D(B)} (p-1) = \,-\, \zeta_D(-1). \tag{2.7}$$

where

$$\zeta_D(s) = \zeta(s) \prod_{p|D} (1 - p^{-s}).$$

We now turn to p–adic uniformization, [**12**], [**6**], [**29**]. We fix a prime $p \mid D(B)$. Let B' be the definite quaternion algebra over $\mathbb{Q}$ whose invariants agree with those of B at all primes $\ell \neq p,\ \infty$. Let $H' = B'^{\times}$ considered as an algebraic group over $\mathbb{Q}$. We fix identifications $H'(\mathbb{A}_f^p) \simeq H(\mathbb{A}_f^p)$ and $H'(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p)$. Let $\hat{\Omega}^2$ be the Deligne-Drinfeld formal scheme relative to $GL_2(\mathbb{Q}_p)$. Then

$$\text{(2.8)}\quad \mathcal{M} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} W(\bar{\mathbb{F}}_p) \simeq \Big[\, H'(\mathbb{Q})\backslash \Big(\, \hat{\Omega}^2 \times_{\operatorname{Spf}\mathbb{Z}_p} \operatorname{Spf} W(\bar{\mathbb{F}}_p) \Big) \times \mathbb{Z} \times H'(\mathbb{A}_f^p)/K'^{\,p} \,\Big].$$

Here $K'^{\,p}$ corresponds to $(O_B \otimes \hat{\mathbb{Z}}^p)^{\times}$ under the identification of $H(\mathbb{A}_f^p)$ with $H'(\mathbb{A}_f^p)$, and $g \in H'(\mathbb{Q})$ acts on the $\mathbb{Z}$ factor by shifting by $\operatorname{ord}_p(\det(g))$. Again, this formula can be simplified since $H'(\mathbb{Q})K_f^p$ maps surjectively onto $\mathbb{Z} \times H'(\mathbb{A}_f^p)$. Let

$$\text{(2.9)}\qquad H'(\mathbb{Q})^1 = \{\ g \in H'(\mathbb{Q}) \mid \operatorname{ord}_p(\det(g)) = 0\ \}.$$

Put $\Gamma' = H'(\mathbb{Q})^1 \cap K_f^p$. Then

$$\text{(2.10)}\qquad \mathcal{M} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} W(\bar{\mathbb{F}}_p) \simeq \Big[\ \Gamma'\backslash\hat{\Omega}^2 \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} W(\bar{\mathbb{F}}_p)\ \Big],$$

where, again the right hand side is considered as (the algebraization of) a formal Deligne-Mumford stack. The group Γ' acts through $\bar{\Gamma}' = \Gamma'/\{\pm 1\} \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ with finite stabilizer groups.

3. The Hodge bundle.

We denote by $(\mathcal{A}, \iota)$ the universal abelian scheme over $\mathcal{M}$ and by $\epsilon : \mathcal{M} \to \mathcal{A}$, its zero section. The *Hodge line bundle* on $\mathcal{M}$ is the following line bundle:

$$\text{(3.1)}\qquad \omega = \epsilon^*(\Omega^2_{\mathcal{A}/\mathcal{M}}) = \wedge^2 \mathrm{Lie}(\mathcal{A}/\mathcal{M})^*.$$

For convenience, we will refer to ω as the Hodge bundle.

Remark 3.1. Assume that $(B, O_B) = (M_2(\mathbb{Q}), M_2(\mathbb{Z}))$. In this case, $\mathcal{M}$ may be identified with the moduli stack of elliptic curves, and the universal object $\mathcal{A}$ with $(\mathcal{E}^2, \iota_0)$ where $\mathcal{E}$ is the universal elliptic curve and $\iota_0 : M_2(\mathbb{Z}) \to \mathrm{End}(\mathcal{E}^2) = M_2(\mathrm{End}(\mathcal{E}))$ is the natural embedding. In this case, $\mathrm{Lie}(\mathcal{A}/\mathcal{M}) = \mathrm{Lie}(\mathcal{E}/\mathcal{M})^{\oplus 2}$ and hence

$$\text{(3.2)}\qquad \omega = \omega_{\mathcal{E}/\mathcal{M}}^{\otimes 2}.$$

Here $\omega_{\mathcal{E}/\mathcal{M}} = \mathrm{Lie}(\mathcal{E}/\mathcal{M})^*$. Recall [**11**], VI.4.5, that $\omega_{\mathcal{E}/\mathcal{M}}^{\otimes 2}$ can be identified with the module of relative differentials of $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$. The following proposition generalizes this fact. $\square$

Proposition 3.2. *The Hodge bundle ω is isomorphic to the relative dualizing sheaf $\omega_{\mathcal{M}/\mathbb{Z}}$.*

Proof. Since the fibers of $\mathcal{M}$ over Spec $\mathbb{Z}$ are Gorenstein, $\omega_{\mathcal{M}/\mathbb{Z}}$ is an invertible sheaf. Since $\mathcal{M}$ is regular of dimension 2, it suffices to show that the restrictions of ω to the smooth locus $\mathcal{M}^{\text{smooth}}$ is isomorphic to the restriction of $\omega_{\mathcal{M}/\mathbb{Z}}$ to $\mathcal{M}^{\text{smooth}}$, i.e., to the sheaf of relative differentials $\Omega^1_{\mathcal{M}^{\text{smooth}}/\mathbb{Z}}$.

By deformation theory we have a canonical identification

$$\Omega^1_{\mathcal{M}^{\text{smooth}}/\mathbb{Z}} = Hom_{O_B}(\operatorname{Lie}\mathcal{A}, (\operatorname{Lie}(\hat{\mathcal{A}})^*) \quad , \tag{3.3}$$

where $(\mathcal{A}, \iota)$ is, as before, the universal object over $\mathcal{M}$ and where $\hat{\mathcal{A}}$ denotes the dual abelian variety.

This formula shows that it suffices to check the claimed equality after passing to the completion $\mathbb{Z}_p$ for each prime p. For any prime $p \nmid D(B)$ our identification problem reduces to the situation considered in Remark 3.1. Hence, all we need to do is to extend to $\mathcal{M}^{\text{smooth}}$ the isomorphism between ω and $\Omega^1_{\mathcal{M}/\text{Spec}}$ over $\mathcal{M}[D(B)^{-1}]$. Fix a prime number $p \mid D(B)$. Denote by R the involution on O_B

$$\alpha \longmapsto \alpha^R = \delta\, \alpha^\iota\, \delta^{-1}, \tag{3.4}$$

where $\delta \in O_B$ satisfies $\delta^2 = D(B)$, and where $\alpha \mapsto \alpha^\iota$ is the main involution on B. After choosing a p–principal polarization on $\mathcal{A}$ whose Rosati involution induces the involution R on O_B, we may identify $(\operatorname{Lie}\hat{\mathcal{A}})^*$ with $(\operatorname{Lie}\mathcal{A})^*$, in such a way that $\alpha \in O_B$ acts on $(\operatorname{Lie}\mathcal{A})^*$ as $(\alpha^R)^*$. We write

$$O_{B_p} = \mathbb{Z}_{p^2}[\Pi]/(\Pi^2 = p, \ \Pi a = a^\sigma \Pi, \quad \forall a \in \mathbb{Z}_{p^2}) \quad . \tag{3.5}$$

We may assume that the restriction of R to $\mathbb{Z}_{p^2}$ is trivial and that $\Pi^R = -\Pi$. After extending scalars from $\mathbb{Z}$ to $\mathbb{Z}_{p^2}$, we have the eigenspace decomposition of $\operatorname{Lie}\mathcal{A}$ as

$$\operatorname{Lie}\mathcal{A} = \mathcal{L}_0 \oplus \mathcal{L}_1 \quad , \tag{3.6}$$

such that the action of Π on $\operatorname{Lie}\mathcal{A}$ is of degree 1 with respect to this $\mathbb{Z}/2$-grading. The condition (1.1) ensures that $\mathcal{L}_0$ and $\mathcal{L}_1$ are both line bundles. It now follows, via (3.3) and (3.6), that the local sections of $\Omega^1_{\mathcal{M}/\mathbb{Z}}$ are given by local homomorphisms $\varphi_i : \mathcal{L}_i \to \mathcal{L}_i^*$ $(i = 0, 1)$ forming a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\varphi_0} & \mathcal{L}_0^* \\ {\scriptstyle \Pi_0}\downarrow & & \downarrow{\scriptstyle -\Pi_1^*} \\ \mathcal{L}_1 & \xrightarrow{\varphi_1} & \mathcal{L}_1^* \\ {\scriptstyle \Pi_1}\downarrow & & \downarrow{\scriptstyle -\Pi_0^*} \\ \mathcal{L}_0 & \xrightarrow{\varphi_0} & \mathcal{L}_0^* \end{array} \tag{3.7}$$

The pair (φ_0, φ_1) therefore defines an injective homomorphism

$$\alpha : \Omega^1_{\mathcal{M}/\mathbb{Z}} \longrightarrow \mathcal{L}_0^{\otimes(-2)} \oplus \mathcal{L}_1^{\otimes(-2)} \;\subset\; (\mathrm{Lie}\,\mathcal{A})^* \otimes (\mathrm{Lie}\,\mathcal{A}). \tag{3.8}$$

On the generic fiber, $\mathcal{L}_0^{\otimes(-2)}$ and $\mathcal{L}_1^{\otimes(-2)}$ both coincide with ω and α induces an isomorphism of $\Omega^1_{\mathcal{M}/\mathbb{Z}}$ with the diagonal. On the smooth locus either Π_0 or Π_1 is an isomorphism locally around any given point. Assume for instance that Π_0 is an isomorphism. Then φ_0 determines φ_1 by the commutativity of the upper square in (3.7),

$$\varphi_1 = (-\Pi_1^*) \circ \varphi_0 \circ \Pi_0^{-1} \quad . \tag{3.9}$$

But then also the lower square commutes. Since Π_0 is an isomorphism, it suffices to check this after premultiplying φ_1 with Π_0. But

$$(-\Pi_0^*) \circ \varphi_1 \circ \Pi_0 = (-\Pi_0^*) \circ (-\Pi_1^*) \circ \varphi_0 = \varphi_0 \circ \Pi_1 \circ \Pi_0 \quad . \tag{3.10}$$

It follows that on the open sublocus of $\mathcal{M}^{\mathrm{smooth}}$ where Π_0 is an isomorphism, the first projection applied to (3.8) induces an isomorphism between $\Omega^1_{\mathcal{M}/\mathbb{Z}}$ and $\mathcal{L}_0^{\otimes(-2)}$. On the other hand on this open sublocus, ω can be identified with $\mathcal{L}_0^{\otimes(-2)}$, which proves the claim. □

By base change to $\mathbb{C}$, the Hodge bundle induces a line bundle $\omega_{\mathbb{C}}$ on $\mathcal{M}_{\mathbb{C}} = [\Gamma\backslash D]$. In the orbifold picture, we may view $\omega_{\mathbb{C}}$ as being given by a descent datum with respect to the action of Γ on the pullback of $\omega_{\mathbb{C}}$ to D. At a point z of $\mathcal{M}_{\mathbb{C}}$, a section α of $\omega_{\mathbb{C}}$ corresponds a holomorphic 2 form on $\mathcal{A}_z$, and so there is a natural norm [**3**] on $\omega_{\mathbb{C}}$ given by:

$$||\alpha_z||^2_{\mathrm{nat}} = \left| \left(\frac{i}{2\pi}\right)^2 \int_{\mathcal{A}_z(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|. \tag{3.11}$$

Equivalently, $\omega_{\mathbb{C}}$ is given by the automorphy factor $(cz+d)^2$, i.e., by the action of Γ on $D \times \mathbb{C}$ defined by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, \zeta) \mapsto (\gamma(z), (cz+d)^2 \zeta\,). \tag{3.12}$$

More precisely, for $z \in D \simeq \mathbb{C} \setminus \mathbb{R}$, we have an isomorphism

$$B_{\mathbb{R}} \simeq M_2(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^2, \qquad u \mapsto u \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.13}$$

and the corresponding abelian variety $\mathcal{A}_z$ has $\mathcal{A}_z(\mathbb{C}) = \mathbb{C}^2/\Lambda_z$, where Λ_z is the image of O_B in $\mathbb{C}^2$. The pullback of $\omega_{\mathbb{C}}$ to D is trivialized via the section $\alpha = dw_1 \wedge dw_2$. The Petersson norm, $||\;||_{\mathrm{Pet}}$ on the bundle of modular forms of weight 2, is defined by the Γ-invariant norm on the trivial line bundle $D \times \mathbb{C}$ given by

$$||(z,\zeta)||^2_{\mathrm{Pet}} = |\zeta|^2\,(4\pi \mathrm{Im}(z))^2. \tag{3.14}$$

If f is such a modular form, then we identify f with the section, [**7**], pp141–2,

$$\alpha(f) = f(z)\,(2\pi i\, dw_1 \wedge 2\pi i\, dw_2) = -4\pi^2 f(z)\,\alpha.$$

Lemma 3.3. *The two norms on $\omega_{\mathbb{C}}$ are related by*

$$|| \, ||^2_{\rm nat} = D(B)^2 \, || \, ||^2_{\rm Pet}.$$

Proof. The pullback to $M_2(\mathbb{R})$ of the form $\alpha \wedge \bar{\alpha} = dw_1 \wedge dw_2 \wedge d\bar{w}_1 \wedge d\bar{w}_2$ above is $4\,\mathrm{Im}(z)^2$ times the standard volume form, and so, via (3.11),

$$||\alpha||^2_{\rm nat} = \frac{1}{4\pi^2} \cdot 4\,\mathrm{Im}(z)^2 \,\mathrm{vol}(M_2(\mathbb{R})/O_B). \tag{3.15}$$

Then

$$||\alpha(f)||^2_{\rm nat} = 16\pi^4 \, |f(z)|^2 \cdot \frac{D(B)^2}{\pi^2} \, Im(z)^2. \quad \square$$

Definition 3.4. *The metrized Hodge bundle $\hat{\omega}$ is ω equipped with the metric*

$$|| \, || = e^{-C} \, || \, ||_{\rm nat},$$

where

$$C = \frac{1}{2}\big(\log(4\pi) + \gamma \big).$$

Here γ is Euler's constant.

The motivation for this normalization is explained in the introduction.

The Chern form $c_1(\hat{\omega})$ for this metric is then

$$c_1(\hat{\omega}) = -dd^c \log ||\alpha||^2 = \mu \tag{3.16}$$

with μ as in (2.5), and so

$$\deg(\hat{\omega}) = \int_{[\Gamma\backslash D]} c_1(\hat{\omega}) = \mathrm{vol}(\mathcal{M}(\mathbb{C})). \tag{3.17}$$

4. The arithmetic Picard group and the arithmetic Chow group.

From now on, we assume that $D(B) > 1$, so that B is a division algebra and $\mathcal{M}$ is proper over $\mathrm{Spec}\,\mathbb{Z}$. If we had imposed a sufficient level structure, then $\mathcal{M}$ would be an arithmetic surface over $\mathrm{Spec}\,\mathbb{Z}$, [**18**], [**3**], etc. Then the Chow groups (tensored with $\mathbb{Q}$) $CH^r(\mathcal{M})$ and arithmetic Chow groups $\widehat{CH}^r(\mathcal{M})$ would be defined, with $\widehat{CH}^1(\mathcal{M}) \simeq \widehat{\mathrm{Pic}}(\mathcal{M})$, the group of isomorphism classes of metrized line bundles, and these would be equipped with a height pairing

$$\langle\ ,\ \rangle : \widehat{CH}^1(\mathcal{M}) \times \widehat{CH}^1(\mathcal{M}) \longrightarrow \widehat{CH}^2(\mathcal{M}) \tag{4.1}$$

and the arithmetic degree

$$\widehat{\deg}\ : \widehat{CH}^2(\mathcal{M}) \longrightarrow \mathbb{C}. \tag{4.2}$$

In this section, we explain how to carry over (parts of) this formalism to our DM–stack $\mathcal{M}$.

We begin with $\widehat{\mathrm{Pic}}(\mathcal{M})$. There are two ways to define the concept of a metrized line bundle on $\mathcal{M}$. First, one can define such an object to be a rule which associates, functorially to any S–valued point $S \to \mathcal{M}$ of $\mathcal{M}$, a line bundle $\mathcal{L}_S$ on S equipped with a C^∞–metric on the line bundle $\mathcal{L}_{S,\mathbb{C}}$ on $S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$. Second, one can define a metrized line bundle on $\mathcal{M}$ to be an invertible sheaf on $\mathcal{M}$ together with a Γ–invariant metric on the pullback of $\mathcal{L}_\mathbb{C}$ to D under the identification of $\mathcal{M}_\mathbb{C}$ with the orbifold $[\Gamma \backslash D]$. These definitions are equivalent. As usual, we denote the set of isomorphism classes of metrized line bundles on $\mathcal{M}$ by $\widehat{\mathrm{Pic}}(\mathcal{M})$. This is an abelian group under the tensor product operation.

Let $\hat{\mathcal{L}} = (\mathcal{L}, ||\cdot||)$ be a metrized line bundle on $\mathcal{M}$. Then $\hat{\mathcal{L}}$ determines a *height* of a one dimensional irreducible reduced proper DM–stack $\mathcal{Z}$ mapping to $\mathcal{M}$. To define it , we are guided by the heuristic principle that, in a numerical formula, a geometric point x of a stack counts with fractional multiplicity $1/|\mathrm{Aut}(x)|$. Let $\tilde{\mathcal{Z}}$ be the normalization and $\nu : \tilde{\mathcal{Z}} \to \mathcal{M}$ be the natural morphism.

If $\Gamma(\tilde{\mathcal{Z}}, \mathcal{O}_{\tilde{\mathcal{Z}}}) = O_K$ for a number field K, then

$$h_{\hat{\mathcal{L}}}(\mathcal{Z}) = \widehat{\deg}\, \nu^*(\hat{\mathcal{L}}) \tag{4.3}$$

Here the right hand side is defined by setting, for a meromorphic section s of $\nu^*(\mathcal{L})$,

$$\widehat{\deg}\, \nu^*(\hat{\mathcal{L}}) = \sum_p \Bigg(\sum_{x \in \tilde{\mathcal{Z}}(\bar{\mathbb{F}}_p)} \frac{\mathrm{ord}_x(s)}{|\mathrm{Aut}(x)|} \Bigg) \cdot \log p - \frac{1}{2} \int_{\tilde{\mathcal{Z}}(\mathbb{C})} \log \|s\|^2. \tag{4.4}$$

Here the integral is defined as

$$\int_{\tilde{\mathcal{Z}}(\mathbb{C})} \log \|s\|^2 = \sum_{z \in \tilde{\mathcal{Z}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(z)|} \cdot \log \|s(z)\|^2. \tag{4.5}$$

Also $\mathrm{ord}_x(s)$ is defined by noting that the strict henselization $\tilde{\mathcal{O}}_{\tilde{\mathcal{Z}},x}$ of the local ring $\mathcal{O}_{\tilde{\mathcal{Z}},x}$ is a discrete valuation ring. Let us check that the expression (4.4) is independent of the choice of s. This comes down to checking for a function $f \in \mathbb{Q}(\tilde{\mathcal{Z}})^\times = K^\times$ that

$$0 = \sum_p \Bigg(\sum_{x \in \tilde{\mathcal{Z}}(\bar{\mathbb{F}}_p)} \frac{\mathrm{ord}_x(f)}{|\mathrm{Aut}(x)|} \Bigg) \cdot \log p - \frac{1}{2} \sum_{\sigma : K \to \mathbb{C}} \frac{1}{|\mathrm{Aut}(\sigma)|} \cdot \log |\sigma(f)|^2 \tag{4.6}$$

For $x \in \tilde{\mathcal{Z}}(\bar{\mathbb{F}}_p)$, let $\underline{x}$ be the corresponding geometric point of the coarse moduli scheme $Z = \operatorname{Spec} O_K$ of $\tilde{\mathcal{Z}}$. Then

$$\tilde{\mathcal{O}}_{Z,\underline{x}} = (\tilde{\mathcal{O}}_{\tilde{\mathcal{Z}},x})^{\mathrm{Aut}(x)/\mathrm{Aut}(\bar{\eta})} \tag{4.7}$$

where $\bar{\eta}$ is any generic geometric point of $\tilde{\mathcal{Z}}$, and $\tilde{\mathcal{O}}_{\tilde{\mathcal{Z}},x}$ is a totally ramified extension of degree $|\mathrm{Aut}(x)|/|\mathrm{Aut}(\bar{\eta})|$ of $\tilde{\mathcal{O}}_{Z,\underline{x}}$. Inserting this into (4.6), we obtain for the right hand side the expression

$$\frac{1}{|\mathrm{Aut}(\bar{\eta})|}\Big(\sum_p \sum_{x \in (\operatorname{Spec} O_K)(\bar{\mathbb{F}}_p)} \mathrm{ord}_x(f) \cdot \log p - \sum_\sigma \log|\sigma(f)|\Big) \tag{4.8}$$

which is zero by the product formula for $f \in K^\times$.

If $\Gamma(\tilde{\mathcal{Z}}, \mathcal{O}_{\tilde{\mathcal{Z}}}) = \mathbb{F}_q$, we put

$$h_{\hat{\mathcal{L}}}(\mathcal{Z}) = \deg \nu^*(\mathcal{L}) \cdot \log q = \Big(\sum_{x \in \tilde{\mathcal{Z}}(\bar{\mathbb{F}}_p)} \frac{\mathrm{ord}_x(s)}{|\mathrm{Aut}(x)|}\Big) \cdot \log p, \tag{4.9}$$

where s is a meromorphic section of $\nu^*(\mathcal{L})$. Here $\deg \nu^*(\mathcal{L})$ coincides with the definition given in [**11**], V.4.3.

Next we need to define the (arithmetic) Chow group of $\mathcal{M}$. By a prime divisor on $\mathcal{M}$ we mean a closed substack $\mathcal{Z}$ of $\mathcal{M}$ which is locally for the étale topology a Cartier divisor defined by an irreducible equation. Let $Z^1(\mathcal{M})$ be the free abelian group generated by the prime divisors on $\mathcal{M}$. Any rational function $f \in \mathbb{Q}(\mathcal{M})^\times$ (i.e. a morphism $\mathcal{U} \to \mathbb{A}^1$ defined on a non-empty open substack $\mathcal{U}$ of $\mathcal{M}$) defines a principal divisor

$$\mathrm{div}(f) = \sum_{\mathcal{Z}} \mathrm{ord}_{\mathcal{Z}}(f) \cdot \mathcal{Z} \quad , \tag{4.10}$$

where the sum is over the prime divisors $\mathcal{Z}$ of $\mathcal{M}$, and where we note that the strict henselization of the local ring at $\mathcal{Z}$, $\tilde{\mathcal{O}}_{\mathcal{M},\mathcal{Z}}$, is a discrete valuation ring. The factor group of $Z^1(\mathcal{M})$ by the group of principal divisors is the Chow group $CH^1(\mathcal{M})$, comp. [**41**].

Let $\mathcal{Z} \in Z^1(\mathcal{M})$. Then the divisor $\mathcal{Z}_{\mathbb{C}}$ of $\mathcal{M}_{\mathbb{C}} = [\Gamma \backslash D]$ is of the form $\mathcal{Z}_{\mathbb{C}} = [\Gamma \backslash D_{\mathcal{Z}}]$ for a unique Γ-invariant divisor $D_{\mathcal{Z}}$ of D. By a Green's function for $\mathcal{Z}$ we mean a real current g of degree 0 on D which is Γ-invariant and such that

$$\omega = dd^c g + \delta_{D_{\mathcal{Z}}} \tag{4.11}$$

is C^∞. We denote by $\hat{Z}^1(\mathcal{M})$ the group of Arakelov divisors, i.e., of pairs $(\mathcal{Z}, g)$ consisting of a divisor $\mathcal{Z}$ on $\mathcal{M}$ and a Green's function for $\mathcal{Z}$, with componentwise addition. If $f \in$

$\mathbb{Q}(\mathcal{M})^\times$, then $f|\mathcal{M}_\mathbb{C}$ corresponds to a Γ-invariant meromorphic function $\tilde{f}_\mathbb{C}$ on D and we define the associated principal Arakelov divisor

$$\widehat{\mathrm{div}}(f) = (\,\mathrm{div}(f), -\log|\tilde{f}_\mathbb{C}|^2\,) \quad . \tag{4.12}$$

The factor group of $\hat{Z}^1(\mathcal{M})$ by the group of principal Arakelov divisors is the arithmetic Chow group $\widehat{CH}^1(\mathcal{M})$. The groups $\widehat{CH}^1(\mathcal{M})$ and $\widehat{\mathrm{Pic}}(\mathcal{M})$ are isomorphic. Under this isomorphism, an element $\hat{\mathcal{L}}$ goes to the class of

$$(\,\sum_{\mathcal{Z}} \mathrm{ord}_\mathcal{Z}(s)\,\mathcal{Z},\ -\log\|s\|^2\,) \quad , \tag{4.13}$$

where s is a meromorphic section of $\mathcal{L}$. Conversely, if $(\mathcal{Z}, g) \in Z^1(\mathcal{M})$, then its preimage under this isomorphism is

$$(\mathcal{O}(\mathcal{Z}), \|\ \|) \quad , \tag{4.14}$$

where $-\log\|\mathbf{1}\|^2 = g$, with $\mathbf{1}$ the canonical Γ-invariant section of the pullback of $\hat{\mathcal{O}}(\mathcal{Z})$ to D.

We define a pairing

$$\langle\ ,\ \rangle : \hat{Z}^1(\mathcal{M}) \times \widehat{\mathrm{Pic}}(\mathcal{M}) \longrightarrow \mathbb{C} \tag{4.15}$$

by formula (5.11) in Bost [**3**],

$$\langle\,(\mathcal{Z}, g), \hat{\mathcal{L}}\,\rangle = h_{\hat{\mathcal{L}}}(\mathcal{Z}) + \frac{1}{2}\int_{[\Gamma\backslash D]} g \cdot c_1(\hat{\mathcal{L}}). \tag{4.16}$$

Here $c_1(\hat{\mathcal{L}})$ is the Γ-invariant form on D defined by the pullback to D of $\hat{\mathcal{L}}$ (analogously to $c_1(\hat{\omega})$ in section 3 above). The integral is defined as

$$\int_{[\Gamma\backslash D]} g \cdot c_1(\hat{\mathcal{L}}) = [\Gamma : \Gamma']^{-1} \cdot \int_{\Gamma\backslash D} g \cdot c_1(\hat{\mathcal{L}}), \tag{4.17}$$

where $\Gamma' = \ker(\Gamma \to \mathrm{Aut}(D))$.

It seems very likely that under the identification $\widehat{CH}^1(\mathcal{M}) \simeq \widehat{\mathrm{Pic}}(\mathcal{M})$, the pairing (4.15) descends to a symmetric bilinear pairing

$$\langle\ ,\ \rangle : \widehat{CH}^1(\mathcal{M}) \times \widehat{CH}^1(\mathcal{M}) \longrightarrow \mathbb{C} \quad , \tag{4.18}$$

as is the case for arithmetic surfaces. For ease of expression we will proceed as if this were the case, although we have not checked it. Since all we will actually use is the paring (4.15), this will cause no harm.

5. Special cycles and the generating function.

In this section, we will define for each $m \in \mathbb{Z}$ and each $v \in \mathbb{R}_+^\times$ a class

$$\hat{\mathcal{Z}}(m,v) = (\mathcal{Z}(m), \Xi(m,v)) \in \widehat{CH}^1(\mathcal{M}). \tag{5.1}$$

We first assume that $m > 0$. Then we consider the DM–stack $\mathcal{Z}(m)$ classifying triples (A, ι, x) where (A, ι) is an object of $\mathcal{M}$ and where x is a *special endomorphism*, [**25**],[**29**], with $x^2 = -m$, i.e.,

$$x \in \mathrm{End}(A, \iota), \qquad \mathrm{tr}^o(x) = 0, \qquad x^2 = -m. \tag{5.2}$$

Then $\mathcal{Z}(m)$ maps to $\mathcal{M}$ by a finite unramified morphism. Furthermore, $\mathcal{Z}(m)$ is purely one dimensional, except in the following cases, [**29**] and the Appendix to section 11,

$$\exists p \mid D(B),\ p \neq 2, \qquad \text{such that } m \in \mathbb{Z}_p^{\times,2}. \tag{5.3}$$

In the cases covered by (5.3), we set $\hat{\mathcal{Z}}(m,v) = 0$. In all other cases, we define a Green's function for the unramified morphism $\mathcal{Z}(m) \to \mathcal{M}$, in the sense of section 4, as follows ([**25**]). Let

$$V = \{ x \in B \mid \mathrm{tr}^o(x) = 0 \ \} \tag{5.4}$$

with quadratic form $Q(x) = -x^2 = N^o(x)$ given by the restriction of the reduced norm and with associated inner product $(x,y) = \mathrm{tr}^o(xy^\iota)$. Note that the signature of $V(\mathbb{R})$ is $(1,2)$. As in [**25**], we can identify D with the space of oriented negative 2-planes in $V(\mathbb{R})$. For $x \in V(\mathbb{R})$ and $z \in D$, let $\mathrm{pr}_z(x)$ be the projection of x to z and let

$$R(x,z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x)) \geq 0. \tag{5.5}$$

This quantity vanishes precisely when $\mathrm{pr}_z(x) = 0$, i.e., when $z \in D_x$ where

$$D_x = \{ \ z \in D \mid (x,z) = 0 \ \}. \tag{5.6}$$

Let $L = V(\mathbb{Q}) \cap O_B$, and let

$$L(m) = \{ \ x \in L \mid Q(x) = m \ \}. \tag{5.7}$$

Then, for $m \in \mathbb{Z}_{\neq 0}$, and $v \in \mathbb{R}_+^\times$, let

$$\Xi(m,v) = \sum_{x \in L(m)} \xi(v^{\frac{1}{2}}x, z) \tag{5.8}$$

where

$$\xi(x,z) = -\mathrm{Ei}(-2\pi R(x,z)) \tag{5.9}$$

for the exponential integral

$$-\mathrm{Ei}(-t) = \int_1^\infty e^{-tr}\, r^{-1}\, dr. \tag{5.10}$$

The properties of this function are described in [**25**], section 11. For $m > 0$, $\Xi(m,v)$ is a Γ–invariant Green's function for the divisor

$$D_{\mathcal{Z}(m)} := \coprod_{x\in L(m)} D_x \tag{5.11}$$

in D.

When $m < 0$, $\Xi(m,v)$ is a smooth Γ–invariant function on D. Therefore,

$$\hat{\mathcal{Z}}(m,v) = (0,\Xi(m,v)), \qquad m < 0 \tag{5.12}$$

again defines an element of $\widehat{CH}^1(\mathcal{M})$.

For $m = 0$, the definition of $\hat{\mathcal{Z}}(0,v)$ is more speculative. Using the canonical map from $\widehat{\mathrm{Pic}}(\mathcal{M})$ to $\widehat{CH}^1(\mathcal{M})$, we let

$$\hat{\mathcal{Z}}(0,v) = -\Big(\,\hat{\omega} + (0,\log v)\,\Big). \tag{5.13}$$

We now have defined elements $\hat{\mathcal{Z}}(m,v) \in \widehat{CH}^1(\mathcal{M})$ for all $m \in \mathbb{Z}$ and $v \in \mathbb{R}_+^\times$. We define the following two generating series, which are formal Laurent series in a parameter q. Later we will take $q = e(\tau) = e^{2\pi i\tau}$, where $\tau = u + iv \in \mathfrak{H}$.

The first generating function involves only the orbifold $\mathcal{M}(\mathbb{C}) = [\Gamma\backslash D]$. Let

$$\omega(m,v) = dd^c\Xi(m,v) + \delta_{D_{\mathcal{Z}(m)}} \tag{5.14}$$

be the right hand side of the Green's equation for $\Xi(m,v)$. Then let

$$\deg(\hat{\mathcal{Z}}(m,v)) = \int_{[\Gamma\backslash D]} \omega(m,v). \tag{5.15}$$

If $m > 0$, then $\deg(\hat{\mathcal{Z}}(m, v))$ is just the usual degree of the 0–cycle $\mathcal{Z}(m)_{\mathbb{C}}$ (in the stack sense). If $m < 0$, then $\deg(\hat{\mathcal{Z}}(m, v)) = 0$, since $\Xi(m, v)$ is smooth is this case so that $\omega(m, v)$ is exact. For $m = 0$, we take $\omega(0, v)$ to be the Chern form of $-\hat{\omega}$, i.e., $-\mu$, and hence

$$\deg(\hat{\mathcal{Z}}(0, v)) := \int_{[\Gamma \backslash D]} \omega(0, v) = -\operatorname{vol}(\mathcal{M}(\mathbb{C})). \tag{5.16}$$

The generating function for degrees is then

$$\begin{aligned} \phi_{\deg}(\tau) : &= \sum_{m} \deg(\hat{\mathcal{Z}}(m, v))\, q^m \\ &= -\operatorname{vol}(\mathcal{M}(\mathbb{C})) + \sum_{m>0} \deg(\mathcal{Z}(m)_{\mathbb{C}})\, q^m. \end{aligned} \tag{5.17}$$

For the second generating function, we use the height pairing (4.15) of our cycles with the class $\hat{\omega} \in \widehat{\operatorname{Pic}}(\mathcal{M})$, and let

$$\phi_{\text{height}}(\tau) = \sum_{m} \langle\, \hat{\mathcal{Z}}(m, v), \hat{\omega}\, \rangle\, q^m. \tag{5.18}$$

At the moment, we regard $\phi_{\deg}(\tau)$ (resp. $\phi_{\text{height}}(\tau)$) as a formal generating series, but our main theorem will identify it as a bona fide holomorphic (resp. non-holomorphic) function of the variable τ by identifying it with the Fourier expansion of a special value of an Eisenstein series (resp. of the derivative of an Eisenstein series).

Part II. Eisenstein series.

6. Eisenstein series of weight 3/2.

In this section, we introduce the Eisenstein series of half–integral weight which will be connected with the arithmetic geometry discussed in Part I. A more general discussion of such series from an adelic point of view can be found in [**25**]. The series we consider are, of course, rather familiar from a classical point of view, and an expression for them in this language will emerge in section 8 and 16 below. Thus, one purpose of the present section is to explain how such classical series are associated to indefinite quaternion algebras in a natural way, via the Weil representation. A second advantage of the adelic viewpoint is that it allows one to assemble the Fourier coefficients out of local quantities. This construction shows in a very clear way the dependence of these coefficients, and more importantly, of their derivatives on the choice of local data.

Let $G'_{\mathbb{A}}$ be the metaplectic extension of $Sp_1(\mathbb{A}) = SL_2(\mathbb{A})$ by $\mathbb{C}^1$, and let $P'_{\mathbb{A}}$ be the preimage of the subgroup $N(\mathbb{A})M(\mathbb{A})$ of $\mathrm{SL}_2(\mathbb{A})$ where

$$N(\mathbb{A}) = \{n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{A}\} \tag{6.1}$$

and

$$M(\mathbb{A}) = \{m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{A}^\times\}. \tag{6.2}$$

As in [**42**] we have an identification $G'_{\mathbb{A}} \simeq \mathrm{SL}_2(\mathbb{A}) \times \mathbb{C}^1$ where the multiplication on the right is given by $[g_1, z_1][g_2, z_2] = [g_1 g_2, c(g_1, g_2) z_1 z_2]$ with cocycle $c(g_1, g_2)$ as in [**42**] or [**17**]. Let $G'_{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Q})$, identified with a subgroup of $G'_{\mathbb{A}}$ via the canonical splitting homomorphism $G'_{\mathbb{Q}} \to G'_{\mathbb{A}}$, and let $P'_{\mathbb{Q}} = P'_{\mathbb{A}} \cap G'_{\mathbb{Q}}$. An idèle character χ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, determines a character χ^ψ of $P'_{\mathbb{Q}} \backslash P'_{\mathbb{A}}$ via

$$\chi^\psi([n(b)m(a), z]) = z\, \chi(a)\, \gamma(a, \psi)^{-1}, \tag{6.3}$$

where ψ is our fixed additive character of $\mathbb{Q}\backslash\mathbb{A}$ and $\gamma(a, \psi)$ is the Weil index ([**43**] or [**37**], appendix). For $s \in \mathbb{C}$, let

$$I(s, \chi) = \mathrm{Ind}_{P'_{\mathbb{A}}}^{G'_{\mathbb{A}}} \chi^\psi |\,|^s \tag{6.4}$$

be the principal series representation of $G'_{\mathbb{A}}$ determined by χ^ψ. A section $\Phi(s) \in I(s, \chi)$ is thus a smooth function on $G'_{\mathbb{A}}$ such that

$$\Phi(p'g', s) = \chi^\psi(p')\, |a|^{s+1} \Phi(g', s). \tag{6.5}$$

where $p' = [n(b)m(a), z]$. Such a section is called *standard* if its restriction to the maximal compact subgroup $K' \subset G'_{\mathbb{A}}$ is independent of s and *factorizable* if $\Phi(s) = \otimes_p \Phi_p(s)$ for the decomposition of the induced representation $I(s, \chi) = \otimes'_p I_p(s, \chi_p)$. Here, for each prime p, $I_p(s, \chi_p)$ is the corresponding induced representation of G'_p, the metaplectic extension of $\mathrm{SL}_2(\mathbb{Q}_p)$. The Eisenstein series associated to a standard section $\Phi(s) \in I(s, \chi)$ is given by

$$E(g, s, \Phi) = \sum_{\gamma \in P'_{\mathbb{Q}} \backslash G'_{\mathbb{Q}}} \Phi(\gamma g, s). \tag{6.6}$$

This series is absolutely convergent for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the whole complex s-plane. Note that this series is normalized so that it has a functional equation

$$E(g', -s, M(s)\Phi) = E(g', s, \Phi), \tag{6.7}$$

where $M(s) : I(s, \chi) \to I(-s, \chi^{-1})$ is the intertwining operator. It has a Fourier expansion

$$E(g', s, \Phi) = \sum_{m \in \mathbb{Q}} E_m(g', s, \Phi) \tag{6.8}$$

where, in the half-plane of absolute convergence,

$$E_m(g', s, \Phi) = \int_{\mathbb{Q}\backslash\mathbb{A}} E(n(b)g', s, \Phi)\, \psi(-mb)\, db, \tag{6.9}$$

for db the self-dual measure on $\mathbb{A}$ with respect to ψ. When $m \neq 0$ and $\Phi(s) = \otimes_p \Phi_p(s)$ is factorizable, the mth Fourier coefficient has a product expansion

$$E_m(g', s, \Phi) = \prod_{p \le \infty} W_{m,p}(g'_p, s, \Phi_p), \tag{6.10}$$

where

$$W_{m,p}(g'_p, s, \Phi_p) = \int_{\mathbb{Q}_p} \Phi_p(wn(b)g'_p, s)\, \psi(-mb)\, db \tag{6.11}$$

is the local Whittaker function, and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G'_{\mathbb{Q}}$. Here db is the self dual measure on $\mathbb{Q}_p$ for ψ_p. On the other hand, the constant term is

$$E_0(g', s, \Phi) = \Phi(g', s) + \prod_{p \le \infty} W_{0,p}(g'_p, s, \Phi). \tag{6.12}$$

Recall that the poles of the Eisenstein series are precisely those of its constant term.

In this paper, we will only be concerned with the case of a quadratic character χ given by $\chi(x) = (x, \kappa)_{\mathbb{A}}$ for $\kappa \in \mathbb{Q}^\times$, where $(\ ,\ )_{\mathbb{A}}$ denotes the global quadratic Hilbert symbol, so we will omit χ from the notation and write $I(s) = \otimes'_p I_p(s)$ for the induced representation, etc. We now begin to make specific choices of the local sections $\Phi_p(s)$.

As before, let B be an indefinite quaternion algebra over $\mathbb{Q}$ with a fixed maximal order O_B. Once again, the case $B = M_2(\mathbb{Q})$ and $O_B = M_2(\mathbb{Z})$ will be allowed. Let

$$V = \{\ x \in B \mid \mathrm{tr}^o(x) = 0\ \} \tag{6.13}$$

with quadratic form defined by $Q(x) = -x^2$, and let $L = O_B \cap V$. Note that the determinant of the quadratic space (V, Q), i.e., $\det(S)$ where S is the matrix for the quadratic form, is a square. Therefore, the discriminant $-\det(S)$ is -1 and the quadratic character χ_V associated to V is given by $\chi_V(x) = (x, -1)_{\mathbb{A}}$. We therefore take $\chi = \chi_V$ and $\kappa = -1$ in this case.

The group $G'_{\mathbb{A}}$ (resp. G'_p) acts on the Schwartz space $S(V(\mathbb{A}))$ (resp. $S(V_p)$) via the Weil representation ω (resp. ω_p) determined by ψ (resp. ψ_p).

For a finite prime p, let $\Phi_p(s) \in I_p(s)$ be the standard section extending $\lambda_p(\varphi_p)$, where

$$\lambda_p : S(V_p) \to I_p(\frac{1}{2}), \qquad \lambda_p(\varphi_p)(g') = \big(\omega(g')\varphi_p\big)(0) \tag{6.14}$$

is the usual map and $\varphi_p \in S(V_p)$ is the characteristic function of $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let K'_∞ be the inverse image in $G'_{\mathbb{A}}$ of $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$. For $\ell \in \frac{1}{2}\mathbb{Z}$, there is a character ν_ℓ of K'_∞ such that

$$\nu_\ell([k_\theta, 1])^2 = e^{2i\theta\ell}. \tag{6.15}$$

For $\ell \in \frac{3}{2} + 2\mathbb{Z}$, there is a unique standard section $\Phi^\ell_\infty(s) \in I_\infty(s)$ with

$$\Phi^\ell_\infty(k, s) = \nu_\ell(k), \tag{6.16}$$

for $k \in K'_\infty$.

Let

$$\Phi^{\ell, D(B)}(s) = \Phi^\ell_\infty(s) \otimes \Big(\otimes_p \Phi_p(s) \Big) \tag{6.17}$$

be the associated global standard section. A little more generally, for a finite prime p, let $\Phi^+_p(s)$ be the standard section arising from the maximal order $M_2(\mathbb{Z}_p)$ in $M_2(\mathbb{Q}_p)$ and let $\Phi^-_p(s)$ be the standard section arising from the maximal order in the division quaternion algebra over $\mathbb{Q}_p$. Then for any square free positive integer D, we have a global section

$$\Phi^{\ell, D}(s) = \Phi^\ell_\infty(s) \otimes \Big(\otimes_p \Phi^{\epsilon(D)}_p(s) \Big), \tag{6.18}$$

where $\epsilon(D) = (-1)^{\mathrm{ord}_p(D)}$.

Since, by strong approximation, $G'_{\mathbb{A}} = G'_{\mathbb{Q}} G'_{\mathbb{R}} K^0$ for any open subgroup K^0 of $G'_{\mathbb{A}_f}$, we loose no information by restricting automorphic forms to $G'_{\mathbb{R}}$, the inverse image of $\mathrm{SL}_2(\mathbb{R})$ in $G'_{\mathbb{A}}$. For $\tau = u + iv \in \mathfrak{H}$, let

$$g'_\tau = [\, n(u) m(v^{\frac{1}{2}}), 1\,] \in G'_\infty \subset G'_{\mathbb{A}}. \tag{6.19}$$

Then, if $\Phi(s)$ is a standard factorizable section with $\Phi_\infty(s) = \Phi^\ell_\infty(s)$, we set

$$E(\tau, s, \Phi) = v^{-\frac{\ell}{2}}\, E(g'_\tau, s, \Phi), \tag{6.20}$$

and, by (6.10), we have

$$E_m(\tau,s,\Phi)=v^{-\frac{\ell}{2}}\,W_{m,\infty}(g'_\tau,s,\Phi^\ell_\infty)\cdot\prod_p W_{m,p}(s,\Phi_p), \tag{6.21}$$

for $m\neq 0$, and

$$E_0(\tau,s,\Phi)=v^{\frac{1}{2}(s+1-\ell)}\cdot\Phi_f(e)+v^{-\frac{\ell}{2}}\,W_{0,\infty}(g'_\tau,s,\Phi^\ell_\infty)\prod_p W_{0,p}(s,\Phi_p). \tag{6.22}$$

The main series of interest to us will be $E(\tau,s,\Phi^{\ell,D})$, associated to the standard section $\Phi^{\ell,D}(s)$ of (6.18). This series has weight ℓ, where $\ell=\frac{3}{2},\ \frac{7}{2},\ \frac{11}{2},\dots$. Note that the character χ is given by $\chi(x)=(x,-1)_{\mathbb{A}}$ in this case. A second family $E(\tau,s,\Phi^{\ell,D})$, with $\ell=\frac{1}{2},\ \frac{5}{2},\ \frac{9}{2},\dots$ etc. is obtained by the same construction applied to the quadratic space (V,Q_-) where $Q_-(x)=x^2$. In this case, $\kappa=1$ and χ is trivial. These cases will be discussed in more detail in [**33**]. In the present paper, we will only be concerned with the case $\ell=\frac{3}{2}$, and so, from now on, we take $\kappa=-1$.

In the next section, we will give a geometric interpretation of the first two terms of the Laurent expansion of the series $E(\tau,s,\Phi^{\frac{3}{2},D(B)})$ at the point $s=\frac{1}{2}$. For this it will be convenient to normalize the series as follows. For any square free positive integer D, let

$$\mathbb{E}(\tau,s;D):=(s+\frac{1}{2})\,c(D)\,\Lambda_D(2s+1)\,E(\tau,s,\Phi^{\frac{3}{2},D}). \tag{6.23}$$

where

$$\Lambda_D(2s+1)=\left(\frac{D}{\pi}\right)^{s+\frac{1}{2}}\Gamma(s+\frac{1}{2})\,\zeta(2s+1)\cdot\prod_{p|D}(1-p^{-2s-1}), \tag{6.24}$$

and

$$c(D)=-(-1)^{\mathrm{ord}(D)}\,\frac{1}{2\pi}\,D\prod_{p|D}(p+1)^{-1}, \tag{6.25}$$

where $\mathrm{ord}(D)=\sum_p\mathrm{ord}_p(D)$. Note that at the point $s=\frac{1}{2}$, of interest to us, the normalizing factor has value

$$c(D)\,\Lambda_D(2)=-(-1)^{\mathrm{ord}(D)}\,\frac{1}{12}\prod_{p|D}(p-1). \tag{6.26}$$

Then, in the case $D=D(B)$, and recalling (2.7),

$$c(D)\,\Lambda_D(2)=-\,\mathrm{vol}(\mathcal{M}(\mathbb{C})). \tag{6.27}$$

This expression explains the choice of $c(D)$. In addition, the normalized Eisenstein series satisfies the simple functional equation, c.f. section 16,

$$\mathbb{E}(\tau, s; D) = \mathbb{E}(\tau, -s; D). \tag{6.28}$$

Finally, we restrict to the case $D = D(B) > 1$ and introduce the *modified* Eisenstein series

$$\mathcal{E}(\tau, s; D(B)) := \mathbb{E}(\tau, s; D(B)) + \sum_{p|D} c_p(s)\, \mathbb{E}(\tau, s; D(B)/p), \tag{6.29}$$

where $c_p(s)$ is any rational function of p^{-s} satisfying

$$c_p(\frac{1}{2}) = 0, \qquad \text{and} \qquad c_p'(\frac{1}{2}) = -\frac{p-1}{p+1}\log(p). \tag{6.30}$$

To retain the functional equation (6.28) one should also require that $c_p(s) = c_p(-s)$, although we will not use this. The motivation for the definition of $\mathcal{E}(\tau, s, D(B))$ comes from geometric considerations which will emerge below. Note that

$$\mathcal{E}(\tau, \frac{1}{2}; D(B)) = \mathbb{E}(\tau, \frac{1}{2}; D(B)), \tag{6.31}$$

and

$$\mathcal{E}'(\tau, \frac{1}{2}; D(B)) = \mathbb{E}'(\tau, \frac{1}{2}; D(B)) + \sum_{p|D} c_p'(\frac{1}{2}) \cdot \mathbb{E}(\tau, \frac{1}{2}; D(B)/p). \tag{6.32}$$

7. The main identities.

In this section, we state our main results on the generating functions

$$\phi_{\text{deg}}(\tau) = -\text{vol}(\mathcal{M}(\mathbb{C})) + \sum_{m>0} \deg(\mathcal{Z}(m)_{\mathbb{C}})\, q^m \tag{7.1}$$

and

$$\phi_{\text{height}}(\tau) = \sum_m \langle\ \hat{\mathcal{Z}}(m, v), \hat{\omega}\ \rangle\, q^m. \tag{7.2}$$

introduced in section 5.

The following result is actually well known, cf., for example, [**16**]. We state it here to bring out the analogy with Theorem 7.2.

Proposition 7.1. *For any indefinite quaternion division algebra B over $\mathbb{Q}$ with associated moduli stack $\mathcal{M}$, as in section 1–5 above, the generating function for the degrees of the special cycles coincides with the value at $s = \frac{1}{2}$ of the Eisenstein series $\mathcal{E}(\tau, s; D(B))$ of weight $\frac{3}{2}$:*

$$\phi_{\mathrm{deg}}(\tau) = \mathcal{E}(\tau, \frac{1}{2}; D(B)).$$

Theorem 7.2. *Under the same assumptions, the generating series for heights of the special cycles coincides, up to an additive constant, with the derivative at $s = \frac{1}{2}$ of the Eisenstein series $\mathcal{E}(\tau, s; D(B))$ of weight $\frac{3}{2}$:*

$$\phi_{\mathrm{height}}(\tau) = \mathcal{E}'(\tau, \frac{1}{2}; D(B)) + \mathbf{c}.$$

for some constant $\mathbf{c}$.

These identities are to be understood as follows. We write the Fourier expansion of the modified Eisenstein series as

$$\mathcal{E}(\tau, s; D(B)) = \sum_m A_m(s, v)\, q^m, \tag{7.3}$$

so that the Fourier expansions of the value and derivative at $s = \frac{1}{2}$ are

$$\mathcal{E}(\tau, \frac{1}{2}; D(B)) = \sum_m A_m(\frac{1}{2}, v)\, q^m, \tag{7.4}$$

and

$$\mathcal{E}'(\tau, \frac{1}{2}; D(B)) = \sum_m A'_m(\frac{1}{2}, v)\, q^m. \tag{7.5}$$

Proposition 7.1 then says that

$$A_m(\frac{1}{2}, v) = \begin{cases} \deg(\mathcal{Z}(m)_{\mathbb{C}}) & \text{if } m > 0, \\ -\operatorname{vol}(\mathcal{M}(\mathbb{C})) & \text{if } m = 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{7.6}$$

Analogously, Theorem 7.2 says that, for $m \neq 0$,

$$A'_m(\frac{1}{2}, v) = \langle\, \hat{\mathcal{Z}}(m, v), \hat{\omega}\, \rangle\,. \tag{7.7}$$

The ambiguity $\mathbf{c}$ in Theorem 7.2 thus arises from the fact that we do not have an explicit expression for the quantity $\langle\, \hat{\omega}, \hat{\omega}\, \rangle$. If we knew, a priori, that $\phi_{\mathrm{height}}(\tau)$ was (the Fourier expansion of) a modular form of weight $\frac{3}{2}$, then we could conclude that $\mathbf{c} = 0$ and, by formula (5.11) of Bost [**3**], that

$$A_0'(\frac{1}{2}, v) \overset{??}{=} \langle\, \hat{\mathcal{Z}}(0,v), \hat{\omega}\, \rangle = -\langle\, \hat{\omega}, \hat{\omega}\, \rangle - \frac{1}{2}\log(v)\deg(\hat{\omega}). \tag{7.8}$$

For further discussion of this point, see section 13.

As already explained in the introduction, Theorem 7.2 is proved by an explicit computation of both sides of (7.7). For the left hand side, this will be done in the next section. The right hand side will be computed in sections 9–12.

8. Fourier expansions and derivatives.

In this section, we describe the first two terms of the Laurent expansion of the Eisenstein series $\mathcal{E}(\tau, s; D(B))$ at the point $s = \frac{1}{2}$. By (6.21) and (6.22), the essential point is to describe the behaviour of the local Whittaker functions $W_{m,p}(s, \Phi_p^D)$ and

$$W_{m,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}}) := v^{-\frac{3}{4}}\, W_{m,\infty}(g_\tau', s, \Phi_\infty^{\frac{3}{2}}). \tag{8.1}$$

The calculations of this section will be elementary manipulations based on results about these Whittaker functions quoted from Part IV below.

In what follows, for a nonzero integer m, we write

$$4m = n^2 d \tag{8.2}$$

where $-d$ is a fundamental discriminant, i.e., discriminant of the field $\boldsymbol{k} = \boldsymbol{k}_d = \mathbb{Q}(\sqrt{-m})$. Note that if $4m = -n^2$, then $\boldsymbol{k} = \mathbb{Q} \oplus \mathbb{Q}$. Let χ_d be the corresponding Dirichlet character, so that

$$\chi_d(p) = \begin{cases} 1 & \text{if } p \text{ is split in } \boldsymbol{k}_d, \\ -1 & \text{if } p \text{ is inert in } \boldsymbol{k}_d, \\ 0 & \text{if } p \text{ is ramified in } \boldsymbol{k}_d. \end{cases} \tag{8.3}$$

For a given m and for a square free positive integer D, define a modification of the standard Dirichlet L-series $L(s, \chi_d)$ by

$$L(s, \chi_m; D) := L(s, \chi_d) \prod_{p \mid nD} b_p(n, s; D) \tag{8.4}$$

where $b_p(n, s; D)$ is defined as follows. Set

$$k = k_p(n) = \mathrm{ord}_p(n) \tag{8.5}$$

and $X = p^{-s}$. Then for $p \nmid D$

$$b_p(n,s;D) = \frac{1-\chi_d(p)\,X + \chi_d(p)\,p^k\,X^{(1+2k)} - (pX^2)^{k+1}}{1-pX^2}, \tag{8.6}$$

and, for $p \mid D$,

(8.7)

$$b_p(n,s;D)$$
$$= \frac{(1-\chi\,X)(1-p^2X^2) - \chi\,p^{k+1}X^{2k+1} + p^{k+2}X^{2k+2} + \chi\,p^{k+1}X^{2k+3} - p^{2k+2}X^{2k+4}}{1-pX^2}.$$

Here, for a moment, we write χ for $\chi_d(p)$. Depending on whether or not $p \mid d$ we can rewrite (8.7) as

$$b_p(n,s;D) = \frac{1-p^2X^2 + p^{k+2}X^{2k+2}(1-X^2)}{1-pX^2} \qquad \text{if } p \mid d \text{ and } p \mid D, \tag{8.8}$$

and

(8.9)

$$b_p(n,s;D) = \frac{(1-\chi\,X)(1-p^2X^2) - \chi\,p^{k+1}X^{2k+1}(1-\chi\,pX)(1-X^2)}{1-pX^2} \qquad \text{if } p \nmid d \text{ and } p \mid D.$$

In all cases the local factor $b_p(n,s;D)$ is, in fact, a polynomial in $X = p^{-s}$ and is, hence, entire in s. It satisfies the functional equation

$$|nD|_p^{-s}\,b_p(n,s;D) = |nD|_p^{s-1}\,b_p(n,1-s;D). \tag{8.10}$$

One of the main results of section 14 is the following.

Proposition 8.1. *For a fixed prime p,*
(i) if $m \neq 0$, then

$$W_{m,p}(s+\frac{1}{2},\Phi_p^D) = L_p(s+1,\chi_d)\,b_p(n,s+1;D)\cdot\begin{cases} C_p^+\cdot\frac{1}{\zeta_p(2s+2)} & \text{for } p \nmid D,\\ C_p^- & \text{if } p \mid D.\end{cases}$$

where the constants $C_p^{\pm}$ are given by

$$C_p^+ = \begin{cases} 1 & \text{if } p \neq 2,\\ \frac{1}{\sqrt{2}}\zeta_8^{-1} & \text{if } p = 2,\end{cases}$$

and $C_p^- = -p^{-1}C_p^+$. Here $\zeta_8 = e(\frac{1}{8})$.
(ii) If $m = 0$, then

$$W_{0,p}(s+\frac{1}{2},\Phi_p^D) = \zeta_p(2s)\cdot\begin{cases} C_p^+\cdot\frac{1}{\zeta_p(2s+1)} & \text{for } p \nmid D,\\ C_p^-\cdot\frac{1}{\zeta_p(2s-1)} & \text{if } p \mid D.\end{cases}$$

From (6.21), (6.22), and these formulas, we obtain a nice description of the Fourier expansion of $\mathbb{E}(\tau,s;D)$.

Corollary 8.2. *Let* $C_f(D) = \prod_p C_p^{\epsilon(D)}$.
(i) For $m \neq 0$,

$$E_m(\tau, s; D) = C_f(D) \cdot W_{m,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}}) \cdot \frac{L(s+\frac{1}{2}, \chi_d)}{\zeta_D(2s+1)} \cdot (nD)^{-2s} \prod_p b_p(n, \frac{1}{2} - s; D),$$

and

$$\mathbb{E}_m(\tau, s; D) = c(D)\, C_f(D) \left(\frac{D}{\pi}\right)^{s+\frac{1}{2}} \Gamma(s+\frac{3}{2}) \cdot W_{m,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}})$$
$$\times L(s+\frac{1}{2}, \chi_d) \cdot (nD)^{-2s} \prod_p b_p(n, \frac{1}{2} - s; D).$$

(ii) For $m = 0$,

$$E_0(\tau, s; D) = v^{\frac{1}{2}(s-\frac{1}{2})} + W_{0,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}})\, C_f(D) \cdot \frac{\zeta(2s)}{\zeta_D(2s+1)} \cdot \prod_{p|D} \frac{1}{\zeta_p(2s-1)},$$

and

$$\mathbb{E}_0(\tau, s; D) = v^{\frac{1}{2}(s-\frac{1}{2})}\, (s+\frac{1}{2})\, c(D)\, \Lambda_D(2s+1)$$
$$+ W_{0,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}})\, c(D)\, C_f(D) \left(\frac{D}{\pi}\right)^{s+\frac{1}{2}} \Gamma(s+\frac{3}{2}) \cdot \zeta(2s) \cdot \prod_{p|D} \frac{1}{\zeta_p(2s-1)},$$

Here

$$c(D)\, C_f(D) = -\frac{1}{\sqrt{2}}\, \zeta_8^{-1}\, \frac{1}{2\pi} \prod_{p|D} (p+1)^{-1}. \tag{8.11}$$

Using Corollary 8.2, we now compute the value of $\mathbb{E}(\tau, s; D)$ at $s = \frac{1}{2}$. We start with the constant term.

The following result is a special case of (iii) of Proposition 15.1 below.

Lemma 8.3.

$$W_{0,\infty}(\tau, s, \Phi_\infty^{\frac{3}{2}}) = 2\pi\, (-i)^{\frac{3}{2}}\, v^{-\frac{1}{2}(s+\frac{1}{2})}\, 2^{-s}\, \frac{\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta)},$$

for $\alpha = \frac{1}{2}(s+\frac{5}{2})$ *and* $\beta = \frac{1}{2}(s-\frac{1}{2})$. *Here* $(-i)^{\frac{3}{2}} = e(-\frac{3}{8})$.

Since the zero of $\Gamma(\beta)^{-1}$ at $s = \frac{1}{2}$ cancels the pole of $\zeta(2s)$ there, the second term in $\mathbb{E}_0(\tau, s; D)$ has a zero of order equal to the number of primes dividing D, and we obtain

Corollary 8.4. *For $D>1$, the constant term at $s=\frac{1}{2}$ is*

$$\mathbb{E}_0(\tau,\frac{1}{2};D) = c(D)\,\Lambda_D(2) = -(-1)^{\mathrm{ord}(D)}\,\frac{1}{12}\prod_{p|D}(p-1) = \zeta_D(-1).$$

Next we consider the coefficients of $\mathbb{E}(\tau,\frac{1}{2};D)$ for $m\neq 0$.

If $p\nmid D$, then by (8.6),

$$b_p(n,0;D) = \frac{1-\chi_d(p)\ +\chi_d(p)\,p^k - p^{k+1}}{1-p}. \tag{8.13}$$

Note that, when $\chi_d(p)=1$, this simplifies to $p^k=|n|_p^{-1}$.

If $p\mid D$, then by (8.7)–(8.9),

$$b_p(n,0;D) = (1-\chi_d(p))(1+p). \tag{8.14}$$

Note that this quantity is actually independent of n, and that $b_p(n,0;D)=0$ if and only if $p\mid D$ and $\chi_d(p)=1$.

The proof of the following identity is a simple combinatorial exercise, which we omit.

Lemma 8.5. *(i) For $p\nmid D$, and $k=\mathrm{ord}_p(n)$,*

$$b_p(n,0;D) = \sum_{c|p^k} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1}),$$

where ℓ runs over the prime factors of c and the product is taken to be 1 when $c=1$.
(ii)

$$\prod_{p\nmid D} b_p(n,0;D) = \sum_{\substack{c|n\\(c,D)=1}} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1}).$$

Here, again, ℓ runs over the prime factors of c and the product is taken to be 1 when $c=1$. □

On the other hand, the following fact is a special case of (iv) of Proposition 15.1 below.

Lemma 8.6.

$$W_{m,\infty}(\tau,\frac{1}{2},\Phi_\infty^{\frac{3}{2}}) = \begin{cases} 2\,C_\infty\cdot m^{\frac{1}{2}}\,q^m & \text{if } m>0, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

where $C_\infty = (-2i)^{\frac{3}{2}}\,\pi$.

Combining these facts, we obtain the following results.
For $m<0$, the vanishing of the archimedean factor yields:

$$\mathbb{E}_m(\tau, \frac{1}{2}; D) = 0, \qquad \text{when } \chi_d \neq 1, \text{ or } D > 1. \tag{8.16}$$

For $m>0$, (i) of Corollary 8.2 gives

(8.17)

$$\begin{aligned}
\mathbb{E}_m(\tau, \frac{1}{2}; D) &= c(D)\,C_f(D)\,C_\infty \cdot \frac{D}{\pi} \cdot 2\,m^{\frac{1}{2}}\,q^m \cdot L(1,\chi_d)\,(nD)^{-1}\prod_p b_p(n,0;D)\\
&= c(D)\,C_f(D)\,C_\infty \cdot q^m \cdot 2\,\frac{h(d)}{w(d)} \cdot \Big(\sum_{\substack{c|n\\(c,D)=1}} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1})\Big)\\
&\qquad \times \Big(\prod_{p|D}(1-\chi_d(p))(1+p)\Big)\\
&= q^m \cdot 2\,\frac{h(d)}{w(d)} \cdot \Big(\sum_{\substack{c|n\\(c,D)=1}} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1})\Big) \cdot \Big(\prod_{p|D}(1-\chi_d(p))\Big).
\end{aligned}$$

Here, $w(d) = |O_{\boldsymbol{k}}^\times|$ is the number of roots of unity in the maximal order $O_{\boldsymbol{k}}$ of $\boldsymbol{k}_d$, $h(d)$ is the class number, and

$$c(D)\,C_f(D)\,C_\infty = \prod_{p|D}(p+1)^{-1}. \tag{8.18}$$

For $m>0$, let

$$H_0(m;D) = \frac{h(d)}{w(d)} \cdot \Big(\sum_{\substack{c|n\\(c,D)=1}} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1})\Big), \tag{8.19}$$

where, as before, in the product, ℓ runs over the prime factors of c and the product is taken to be 1 when $c=1$, and

$$\delta(d;D) = \prod_{p|D}(1-\chi_d(p)). \tag{8.20}$$

Thus, we obtain the Fourier expansion of $\mathcal{E}(\tau, \frac{1}{2}; D)$.

Proposition 8.7. *For $D > 1$,*

$$\mathcal{E}(\tau, \frac{1}{2}; D) = c(D)\,\Lambda_D(2) + \sum_{m>0} 2\,\delta(d; D)\,H_0(m; D)\,q^m.$$

Here

$$c(D)\,\Lambda_D(2) = -(-1)^{\mathrm{ord}(D)}\,\frac{1}{12}\prod_{p|D}(p-1).$$

This Eisenstein series of weight $\frac{3}{2}$ is a familiar object. Recall that, if O_{c^2d} is the order in $O_{\boldsymbol{k}}$ of conductor c, with class number $h(c^2d)$ and with $w(c^2d) = |O_{c^2d}^\times|$, then [**1**], p.250[5],

$$\frac{h(c^2d)}{w(c^2d)} = \frac{h(d)}{w(d)} \cdot c \prod_{\ell|c}(1 - \chi_d(\ell)\ell^{-1}). \tag{8.21}$$

Thus,

$$H_0(m; D) = \sum_{\substack{c|n \\ (c,D)=1}} \frac{h(c^2d)}{w(c^2d)}. \tag{8.22}$$

For example, if $D = 1$, i.e., in the case of $B = M_2(\mathbb{Q})$,

$$H_0(m; 1) = \sum_{c|n} \frac{h(c^2d)}{w(c^2d)} \tag{8.23}$$

is quite close to[6] the 'class number' $H(m)$ which appears in the Fourier expansion

$$\mathcal{F}(\tau) = -\frac{1}{12} + \sum_{m>0} H(m)\,q^m + \sum_{n\in\mathbb{Z}} \frac{1}{16\pi}\,v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi n^2 vr}\,r^{-\frac{3}{2}}\,dr\,q^{-n^2}, \tag{8.24}$$

of Zagier's nonholomorphic Eisenstein series of weight $\frac{3}{2}$, [**8**], [**45**]. In fact, when $D = 1$, we have

$$\mathcal{E}(\tau, \frac{1}{2}; 1) = -\frac{1}{12} + \sum_{m>0} 2\,H_0(m; 1)\,q^m + \sum_{n\in\mathbb{Z}} \frac{1}{8\pi}\,v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi n^2 vr}\,r^{-\frac{3}{2}}\,dr \cdot q^{-n^2}. \tag{8.25}$$

This case will be discussed in detail in the sequel [**32**].

Next we consider the derivative $\mathcal{E}'(\tau, \frac{1}{2}; D)$ in the case $D = D(B) > 1$. In this case, the only terms which contribute are the following:

(i) $m > 0$ and $\delta(d; D) \neq 0$,
(ii) $m > 0$ and there is a unique $p \mid D(B)$ such that $\chi_d(p) = 1$,
(iii) $m < 0$ and $\delta(d; D) \neq 0$, and
(iv) $m = 0$.

[5] The quantity e_c there is $|\mathcal{O}_d^\times : \mathcal{O}_{c^2d}^\times| = w(d)/w(c^2d)$

[6] Precisely, $2\,H_0(m; 1) = H(4m)$.

In cases (i) and (iv), $\mathcal{E}_m(\tau, \frac{1}{2}; D) \neq 0$. In cases (ii) and (iii), $\mathcal{E}_m(\tau, s; D)$ has a simple zero at $s = \frac{1}{2}$ due to the vanishing of the local factor $b_p(n, 0; D)$ in case (ii) and the archimedean factor $W_{m,\infty}(\tau, \frac{1}{2}; \Phi_\infty^{\frac{3}{2}})$ in case (iii). In all other cases, $\mathcal{E}_m(\tau, \frac{1}{2}; D)$ has a zero of order at least 2 at $s = \frac{1}{2}$.

Theorem 8.8. *Assume that $D = D(B) > 1$.*
(i) If $m > 0$ and there is no prime $p \mid D$ for which $\chi_d(p) = 1$, then

$$\begin{aligned}
&\mathcal{E}_m'(\tau, \frac{1}{2}; D) \\
&= 2\,\delta(d; D)\, H_0(m; D) \cdot q^m \cdot \Bigg[\frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma \\
&\qquad + \frac{1}{2}J(4\pi m v) + \sum_{\substack{p \\ p\nmid D}} \Big(\log|n|_p - \frac{b_p'(n,0;D)}{b_p(n,0;D)} \Big) + \sum_{\substack{p \\ p\mid D}} K_p \log(p) \Bigg].
\end{aligned}$$

Here

$$K_p = \begin{cases} -k + \frac{(p+1)(p^k-1)}{2(p-1)} & \text{if } \chi_d(p) = -1, \text{ and} \\ -1 - k + \frac{p^{k+1}-1}{p-1} & \text{if } \chi_d(p) = 0, \end{cases}$$

with $k = \mathrm{ord}_p(n)$, and

$$J(t) = \int_0^\infty e^{-tr}\left[(1+r)^{\frac{1}{2}} - 1 \right] r^{-1}\, dr.$$

An explicit expression for the logarithmic derivative of $b_p(n, s; D)$ is given by (i) of Lemma 8.10, and $H_0(m; D)$ and $\delta(d; D)$ are given by (8.19) and (8.20) respectively.
(ii) If there is a unique prime $p \mid D$ such that $\chi_d(p) = 1$, then

$$\mathcal{E}_m'(\tau, \frac{1}{2}; D) = 2\,\delta(d; D/p)\, H_0(m; D) \cdot (p^k - 1)\log(p) \cdot q^m.$$

(iii) If $m < 0$, then

$$\mathcal{E}_m'(\tau, \frac{1}{2}; D) = 2\,\delta(d; D)\, H_0(m; D) \cdot q^m \cdot \frac{1}{4\pi}\,|m|^{-\frac{1}{2}}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vr} r^{-\frac{3}{2}}\, dr,$$

where, for $m < 0$, $H_0(m; D)$ is defined by (8.32) below.
(iv)

$$\mathcal{E}_0'(\tau, \frac{1}{2}; D) = c(D)\,\Lambda_D(2) \left[\frac{1}{2}\log(v) - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + 2C + \sum_{p\mid D} \frac{p\log(p)}{p-1} \right].$$

(v) All other Fourier coefficients of $\mathcal{E}'(\tau, \frac{1}{2}; D)$ vanish.

Proof. We begin with the Eisenstein series $\mathbb{E}(\tau, s; D)$ for any $D > 1$.

First consider case (i), so that $m > 0$ and that there are no primes $p \mid D$ with $\chi_d(p) = 1$. Then, using (i) of Corollary 8.2, we have

$$\mathbb{E}'_m(\tau, \frac{1}{2}; D) = \mathbb{E}_m(\tau, \frac{1}{2}; D) \Bigg[\log(D) - \log(\pi) + 1 - \gamma + \frac{W'_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}})}{W_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}})} + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - 2\log(nD) - \sum_p \frac{b'_p(n,0;D)}{b_p(n,0;D)} \Bigg]$$

The following fact is proved in section 15.

Lemma 8.9. *For $m > 0$,*

$$\frac{W'_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}})}{W_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}})} = \frac{1}{2}\Big[\log(\pi m) - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + J(4\pi m v)\Big].$$

Using this result and the fact that

$$\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} = 2 - \gamma - 2\log(2),$$

and recalling that $4m = n^2 d$, we obtain

$$\begin{aligned}\mathbb{E}'_m(\tau, \tfrac{1}{2}; D) &\\ = \mathbb{E}_m(\tau, \tfrac{1}{2}; D) \Bigg[&\frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma + \frac{1}{2}J(4\pi m v) \\ &+ \sum_{\substack{p \\ p\nmid D}} \Big(\log|n|_p - \frac{b'_p(n,0;D)}{b_p(n,0;D)} \Big) \\ &- \log(D) + \sum_{\substack{p \\ p\mid D}} \Big(\log|n|_p - \frac{b'_p(n,0;D)}{b_p(n,0;D)} \Big) \Bigg].\end{aligned} \tag{8.27}$$

Next we note the explicit expressions for the logarithmic derivatives of the $b_p(n, s; D)$'s which will be useful later.

Lemma 8.10. *(i) For a prime $p \nmid D$,*

$$\begin{aligned}\frac{1}{\log p}\cdot\frac{b'_p(n,0;D)}{b_p(n,0;D)} &= \frac{\chi_d(p) - \chi_d(p)\,(2k+1)p^k + (2k+2)p^{k+1}}{1 - \chi_d(p) + \chi_d(p)\,p^k - p^{k+1}} - \frac{2p}{1-p} \\ &= \begin{cases} \frac{p^k-1}{p^k(p-1)} - 2k & \text{if } \chi_d(p) = 1, \\ -\frac{2p(1-(k+1)p^k + kp^{k+1})}{(p-1)(p^{k+1}-1)} & \text{if } \chi_d(p) = 0, \\ -\frac{1+3p-(2k+1)p^k - 3p^{k+1} + 2kp^{k+2}}{(p-1)(p^{k+1}+p^k-2)} & \text{if } \chi_d(p) = -1. \end{cases}\end{aligned}$$

(ii) For a prime $p \mid D$ with $\chi_d(p) \neq 1$,

$$\frac{1}{\log p} \cdot \frac{b'_p(n,0;D)}{b_p(n,0;D)} = \begin{cases} -\frac{2p(p^{k+1}-1)}{p^2-1} & \text{if } \chi_d(p) = 0, \\ -\frac{2(1+p)p^{k+1}+p^2-4p-1}{2(p^2-1)} & \text{if } \chi_d(p) = -1. \end{cases}$$

Here $k = \mathrm{ord}_p(n)$.

In case (ii), $m > 0$ and there is a unique prime $p \mid D$ such that $\chi_d(p) = 1$. In this case, it is easy to verify

(8.28) $$b'_p(n,0;D) = (1+p-2p^{k+1})\log p.$$

Then, with the notation introduced above and using (8.14), we have

(8.29) $$\mathbb{E}'_m(\tau, \frac{1}{2}; D) = -2\,\delta(d; D/p)\, H_0(m;D) \cdot q^m \cdot (p+1)^{-1} \cdot (1+p-2p^{k+1}) \log p.$$

Recall that $k = \mathrm{ord}_p(n)$.

Finally, in case (iii), we need another result to be proved in section 15.

Lemma 8.11. *For $m < 0$:*

$$\begin{aligned} W'_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}}_\infty) &= C_\infty\, |m|^{\frac{1}{2}}\, q^m\, e^{-4\pi|m|v} \int_0^\infty e^{-4\pi|m|vr} (r+1)^{-1}\, r^{\frac{1}{2}}\, dr \\ &= C_\infty \cdot \frac{1}{4}\, q^m\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vr}\, r^{-\frac{3}{2}}\, dr. \end{aligned}$$

Using the second expression of this Lemma and (i) of Corollary 8.2, we have, for $m < 0$,

(8.30)
$$\begin{aligned} \mathbb{E}'_m(\tau, \frac{1}{2}; D) &= c(D)\, C_f(D) \cdot \frac{D}{\pi} \cdot W'_{m,\infty}(\tau, \frac{1}{2}, \Phi^{\frac{3}{2}}) \cdot L(1,\chi_d) \cdot (nD)^{-1} \prod_p b_p(n,0;D) \\ &= c(D)\, C_f(D)\, C_\infty \cdot \frac{4\, h(d)\, \log|\epsilon(d)|}{w(d)\, |d|^{\frac{1}{2}}} \cdot n^{-1} \Bigg(\sum_{\substack{c \mid n \\ (c,D)=1}} c \prod_{\ell \mid c} (1 - \chi_d(\ell)\ell^{-1}) \Bigg) \\ &\qquad \times \Bigg(\prod_{p \mid D} (1-\chi_d(p))(1+p) \Bigg) \\ &\qquad\qquad \times q^m\, \frac{1}{4\pi}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vr}\, r^{-\frac{3}{2}}\, dr. \end{aligned}$$

where $\epsilon(d)$ is the fundamental unit of the real quadratic field $\boldsymbol{k}_d = \mathbb{Q}(\sqrt{|d|})$. Using the value (8.18), this can rewritten as

$$\mathbb{E}'_m(\tau, \frac{1}{2}; D) = 2\, H_0(m; D)\, \delta(d; D) \cdot q^m \cdot \frac{1}{4\pi}\, |m|^{-\frac{1}{2}}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vr} r^{-\frac{3}{2}}\, dr \tag{8.31}$$

where

$$\begin{aligned} H_0(m; D) &= \frac{h(d)\, \log|\epsilon(d)|}{w(d)} \cdot \Big(\sum_{\substack{c|n \\ (c,D)=1}} c \prod_{\ell | c} (1 - \chi_d(\ell)\ell^{-1}) \Big) \\ &= \sum_{\substack{c|n \\ (c,D)=1}} h(c^2 d) \cdot \frac{\log|\epsilon(c^2 d)|}{w(c^2 d)}. \end{aligned} \tag{8.32}$$

is the analogue of (8.19) and (8.22) in the case of a real quadratic field, i.e., for $m < 0$.

Finally, we consider the constant term using (ii) of Corollary 8.2 and noting that for $D = D(B)$, the second term there has a zero of order at least 2. This gives

(8.33)
$$\begin{aligned} \mathbb{E}'_0(\tau, \frac{1}{2}; D) &= c(D)\, \Lambda_D(2) \left[\frac{1}{2}\, \log(v) + 1 + 2\frac{\Lambda'_D(2)}{\Lambda_D(2)} \right] \\ &= c(D)\, \Lambda_D(2) \left[\frac{1}{2}\, \log(v) + 1 + \log(D) - \log(\pi) - \gamma + 2\frac{\zeta'(2)}{\zeta(2)} + 2\sum_{p|D} \frac{\log(p)}{p^2 - 1} \right]. \end{aligned}$$

Now we return to the modified Eisenstein series

$$\mathcal{E}(\tau, s; D) = \mathbb{E}(\tau, s; D) + \sum_{p|D} c_p(s)\, \mathbb{E}(\tau, s; D/p)$$

of (6.29) for $D = D(B) > 1$. By (6.32), the Fourier coefficients of $\mathcal{E}'(\tau, \frac{1}{2}; D)$ for $m < 0$ agree with those of $\mathbb{E}'(\tau, \frac{1}{2}; D)$, so that (8.31) gives part (iii) of Theorem 8.8.

If $m > 0$ and for all $p \mid D$, $\chi_d(p) \neq 1$, note that by (8.13) and Lemma 8.5,

(8.34)
$$\begin{aligned} \mathbb{E}_m(\tau, \frac{1}{2}; D/p) &= 2\, \delta(d; D/p)\, H_0(m; D/p)\, q^m \\ &= (1 - \chi_d(p))^{-1} \cdot \frac{1 - \chi_d(p) + \chi_d(p)p^k - p^{k+1}}{1 - p} \cdot 2\, \delta(d; D)\, H_0(m; D) \cdot q^m \\ &= (1 - \chi_d(p))^{-1} \cdot \frac{1 - \chi_d(p) + \chi_d(p)p^k - p^{k+1}}{1 - p} \cdot \mathbb{E}_m(\tau, \frac{1}{2}; D). \end{aligned}$$

Therefore,

(8.35)
$$\mathcal{E}'_m(\tau,\frac{1}{2};D)=\mathbb{E}_m(\tau,\frac{1}{2};D)\Big[\cdots+\sum_{p|D}c'_p(\frac{1}{2})\cdot(1-\chi_d(p))^{-1}\cdot\frac{1-\chi_d(p)+\chi_d(p)p^k-p^{k+1}}{1-p}\ \Big]$$

where the dots indicate the expression in (8.27) for $\mathbb{E}'_m(\tau,\frac{1}{2};D)$. For a prime $p\mid D$, we write c' for $c'_p(\frac{1}{2})/\log(p)$ and $k=\mathrm{ord}_p(n)$. Then, the coefficient of $\log(p)$ inside the bracket is

$$-1-k-\frac{1}{\log(p)}\cdot\frac{b'_p(n,0;D)}{b_p(n,0;D)}+c'\cdot(1-\chi_d(p))^{-1}\cdot\frac{1-\chi_d(p)+\chi_d(p)p^k-p^{k+1}}{1-p}. \tag{8.36}$$

We now use (ii) of Lemma 8.10. If $\chi_d(p)=-1$, (8.36) gives

$$\begin{aligned}K_p:=-1-k+&\frac{2(p+1)p^{k+1}+p^2-4p-1}{2(p^2-1)}+c'\cdot\frac{1}{2}\cdot\frac{p^{k+1}+p^k-2}{p-1}\\ &=-1-k+\frac{1}{2}+\frac{p^{k+1}+p^k-2}{2(p-1)}\\ &=-k+\frac{(p+1)(p^k-1)}{2(p-1)}.\end{aligned}\tag{8.37}$$

If $\chi_d(p)=0$, (8.36) gives

$$\begin{aligned}K_p:=-1-k+&\frac{2p(p^{k+1}-1)}{p^2-1}+c'\cdot\frac{p^{k+1}-1}{p-1}\\ &=-1-k+\frac{p^{k+1}-1}{p-1}.\end{aligned}\tag{8.38}$$

Thus, (8.27), (8.35), and these expressions for the coefficients K_p of $\log(p)$ for $p\mid D$ yield (i) of Theorem 8.8.

To prove (ii), suppose that $m>0$ and that there is a unique prime $p\mid D$ for which $\chi_d(p)=1$. Then, using (8.29) and (6.32), we have

(8.39)
$$\begin{aligned}\mathcal{E}'_m(\tau,\frac{1}{2};D)&=\mathbb{E}'_m(\tau,\frac{1}{2};D)+\sum_{\ell|D}c'_\ell(\frac{1}{2})\cdot\mathbb{E}_m(\tau,\frac{1}{2};D/\ell)\\ &=-2\,\delta(d;D/p)\,H_0(m;D)\cdot q^m\cdot(p+1)^{-1}\cdot(1+p-2p^{k+1})\log p\\ &\qquad+c'_p(\frac{1}{2})\cdot 2\,\delta(d;D/p)\,H_0(m;D/p)\,q^m\\ &=2\,\delta(d;D/p)\,H_0(m;D)\cdot q^m\cdot\Big[-(p+1)^{-1}\cdot(1+p-2p^{k+1})+c'\cdot p^k\Big]\log(p)\\ &=2\,\delta(d;D/p)\,H_0(m;D)\cdot q^m\cdot(p^k-1)\,\log(p),\end{aligned}$$

as claimed.

Finally, we consider the constant term. By (8.33), Corollary 8.4 and (6.32), we have

(8.40)

$$\mathcal{E}'_0(\tau,\frac{1}{2};D)$$

$$= \mathbb{E}'_0(\tau,\frac{1}{2};D) - \sum_{p|D} \frac{p-1}{p+1}\,\log(p)\cdot c(D/p)\,\Lambda_{D/p}(2)$$

$$= c(D)\Lambda_D(2)\Big[\,\frac{1}{2}\,\log(v) + 1 + \log(D) - \log(\pi) - \gamma + 2\frac{\zeta'(2)}{\zeta(2)} + \sum_{p|D}\Big(\frac{2}{p^2-1} + \frac{1}{p+1}\Big)\log(p)\Big]$$

$$= c(D)\Lambda_D(2)\Big[\,\frac{1}{2}\,\log(v) + 1 - \log(\pi) - \gamma + 2\frac{\zeta'(2)}{\zeta(2)} + \sum_{p|D}\frac{p\log(p)}{p-1}\,\Big]$$

(13.1)

$$= c(D)\,\Lambda_D(2)\,\Big[\frac{1}{2}\,\log(v) - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + 2\log(2) + \log(\pi) + \gamma + \sum_{p|D}\frac{p\log(p)}{p-1}\,\Big]$$

$$= c(D)\,\Lambda_D(2)\,\Big[\frac{1}{2}\,\log(v) - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + 2C + \sum_{p|D}\frac{p\log(p)}{p-1}\,\Big],$$

where C is as in Definition 3.4. Here we use the fact that

$$c(D/p)\,\Lambda_{D/p}(2) = -c(D)\,\Lambda_D(2)\,\frac{p+1}{p^2-1}. \tag{8.41}$$

□

For later comparison, we note that the coefficient $A'_m(\frac{1}{2},v)$ in the term

$$\mathcal{E}'_m(\tau,\frac{1}{2};D) = A'_m(\frac{1}{2},v)\,q^m \tag{8.42}$$

in (i) of Theorem 8.8 can be written as a sum of four quantities:

$$2\,\delta(d;D)\,H_0(m;D)\cdot\Big[\,\frac{1}{2}\,\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma\,\Big], \tag{8.43}$$

$$2\,\delta(d;D)\,H_0(m;D)\cdot\frac{1}{2}J(4\pi mv), \tag{8.44}$$

$$2\,\delta(d;D)\,H_0(m;D)\cdot\sum_{\substack{p\\ p\nmid D}}\Big(\;\log|n|_p - \frac{b'_p(n,0;D)}{b_p(n,0;D)}\;\Big), \tag{8.45}$$

and

$$2\,\delta(d;D)\,H_0(m;D)\cdot\sum_{\substack{p\\ p|D}} K_p\,\log(p). \tag{8.46}$$

Part III. Computations: geometric.

§9. The geometry of $\mathcal{Z}(m)$'s.

In this section we will prepare the calculation of the coefficients of the generating series $\phi_{\text{deg}}(\tau)$ and $\phi_{\text{height}}(\tau)$ by describing some of the geometry of the special cycles $\mathcal{Z}(m)$.

It turns out that the primes of bad reduction (i.e. $p \mid D(B)$) play a very special role. Namely,

(9.1) $\mathcal{Z}(m) \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}[D(B)^{-1}]$ is reduced and is finite and flat over Spec $\mathbb{Z}[D(B)^{-1}]$.

We denote by $\mathcal{Z}(m)^{\text{horiz}}$ the closure of $\mathcal{Z}(m) \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}\,\mathbb{Z}[D(B)^{-1}]$ in $\mathcal{Z}(m)$ and call it the *horizontal part of the special cycle.*

We first describe the generic fiber of $\mathcal{Z}(m)$. As in section 5, let $L = V \cap O_B$, let $L(m)$ be as in (5.7), and let

$$D_{\mathcal{Z}(m)} = \coprod_{x \in L(m)} D_x \quad , \tag{9.2}$$

as in (5.11). Then Γ acts on $D_{\mathcal{Z}(m)}$ compatibly with its action on D, and we may represent $\mathcal{Z}(m)_{\mathbb{C}}$ as an orbifold mapping to $[\Gamma \setminus D]$

$$\mathcal{Z}(m)_{\mathbb{C}} = [\Gamma \setminus D_{\mathcal{Z}(m)}] \quad . \tag{9.3}$$

We next give the degree of this orbifold. We recall the following notation from (8.2). For $x \in L$ with $Q(x) = m > 0$, let

$$\boldsymbol{k} = \mathbb{Q}[x] \simeq \mathbb{Q}[X]/(X^2 + m) = \mathbb{Q}(\sqrt{-m}) \quad . \tag{9.4}$$

Let $O_{\boldsymbol{k}}$ be its ring of integers and let $-d = \text{disc}(O_{\boldsymbol{k}})$ be its discriminant. Then the discriminant of the order $\mathbb{Z}[x] = \mathbb{Z}[X]/(X^2 + m)$ is equal to $-4m$. Write $4m = n^2 d$, as in (8.2). We note that there is a map of discrete orbifolds

$$[\Gamma\backslash D_{\mathcal{Z}(m)}] \longrightarrow [\Gamma\backslash L(m)], \tag{9.5}$$

which is 2 to 1.

Proposition 9.1. *For $m > 0$ and $\boldsymbol{k}$ as above,*
(i) if $\boldsymbol{k}$ cannot be embedded into B, then $\mathcal{Z}(m)_{\mathbb{Q}} = \emptyset$,
(ii) otherwise,

$$\deg\, \mathcal{Z}(m)_{\mathbb{Q}} = 2\,\delta(m, D)\, H_0(m, D).$$

Here the degree of $\mathcal{Z}(m)_{\mathbb{Q}}$ is taken in the stack sense, i.e. each geometric point η of $\mathcal{Z}(m)_{\mathbb{Q}}$ counts with multiplicity $1/|\mathrm{Aut}(\eta)|$.

Proof. (i) For any $\mathbb{C}$-valued point (A,ι) of $\mathcal{M}$ we have an injection

$$\mathrm{End}(A,\iota) \hookrightarrow B \quad .$$

Hence, if (A,ι,x) is a $\mathbb{C}$-valued point of $\mathcal{Z}(m)$ we obtain an injection $\boldsymbol{k} = \mathbb{Q}[x] \hookrightarrow B$.

(ii) Using (9.8), we obtain

$$\deg\, \mathcal{Z}(m)_{\mathbb{Q}} = 2 \sum_{\substack{x\in L(m)\\ \mathrm{mod}\ \Gamma}} \frac{1}{|\Gamma_x|} \quad . \tag{9.6}$$

The following result finishes the proof of the Proposition.

Lemma 9.2. *If $m>0$, then*

$$\sum_{\substack{x\in L(m)\\ \mathrm{mod}\ \Gamma}} \frac{1}{|\Gamma_x|} = \delta(d;D)\cdot H_0(m;D),$$

where $H_0(m;D)$ and $\delta(d;D)$ are given by (8.19) and (8.20).

Proof. For any $x \in V(\mathbb{Q}) \cap O_B$ with $Q(x) = m$, there is an associated embedding $i_x : \mathbb{Q}(\sqrt{-m}) \to B$, taking $\sqrt{-m}$ to x. The order $O_{c^2d} = i_x^{-1}(O_B)$ is an invariant of the $\Gamma = O_B^\times$ conjugacy class of i_x and i_x is an optimal embedding of O_{c^2d}, in the terminology of Eichler, [**13**]. Recall that the order $\mathbb{Z}[\sqrt{-m}]$ has discriminant $-4m$ and hence, conductor n. Let

$$\mathrm{Opt}(O_{c^2d},O_B) = \{\, i : \mathbb{Q}(\sqrt{-m}) \to B \mid i^{-1}(O_B) = O_{c^2d}\,\}/\Gamma \tag{9.7}$$

be the set of Γ orbits of optimal embeddings. Recall that the order $i^{-1}(O_B)$ is maximal at all primes $p|D(B)$. The following fact is classical, [**13**]:

$$|\mathrm{Opt}(O_{c^2d},O_B)| = \delta(d;D)\cdot h(c^2d). \tag{9.8}$$

Using (8.21), we have

$$\begin{aligned}
\Big(\sum_{\substack{x\in L(m)\\ Q(x)=m\\ \mathrm{mod}\ \Gamma}} |\Gamma_x|^{-1}\Big) &= \sum_{\substack{c|n\\ (c,D)=1}} |\mathrm{Opt}(O_{c^2d},O_B)|\cdot|O_{c^2d}^\times|^{-1}\\
&= \delta(d;D)\,\frac{h(d)}{w(d)} \sum_{\substack{c|n\\ (c,D)=1}} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1}) \qquad (9.9)\\
&= \delta(d;D)\,H_0(m;D).
\end{aligned}$$

□

Proof of Proposition 7.1. Comparing the expression just found for $\deg \mathcal{Z}(m)_{\mathbb{Q}}$ together with (2.7), we have

$$\phi_{\deg}(\tau) = \zeta_D(-1) + \sum_{m>0} 2\,\delta(d; D(B))\, H_0(m; D(B))\, q^m\,,$$

which coincides with $\mathcal{E}(\tau, \frac{1}{2}; D(B))$, via Proposition 8.5, so that Proposition 7.1 is proved. □

Remark 9.3. The map $\mathcal{Z}(m)_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}}$ is not a closed immersion, hence $\mathcal{Z}(m)_{\mathbb{Q}}$ is not a divisor on $\mathcal{M}_{\mathbb{Q}}$. In fact, the morphism is of degree 2 over its image. To see this, note that if $(A, \iota, x) \in \mathcal{Z}(m)(\mathbb{C})$, then $\mathrm{End}(A, \iota)_{\mathbb{Q}} = \boldsymbol{k} = \mathbb{Q}(\sqrt{-m})$. Hence the only other point of $\mathcal{Z}(m)(\mathbb{C})$ mapping to $(A, \iota) \in \mathcal{M}(\mathbb{C})$ is $(A, \iota, -x)$. That the degree is 2 even in the stack sense follows from

$$\mathrm{Aut}(A, \iota) = \mathrm{Aut}(A, \iota, x).$$

□

The stack $\mathcal{Z}(m)$ can have some pathological features in characteristic p for $p \mid D(B)$. Namely, as already mentioned in section 5, it can happen that $\mathcal{Z}(m)$ has dimension 0 (only if $\boldsymbol{k} = \mathbb{Q}(\sqrt{-m})$ does not embed into B, cf. (i) of Proposition 9.1) and also that $\mathcal{Z}(m)$ has embedded components. This leads us to introduce the Cohen-Macauleyfication $\mathcal{Z}(m)^{\mathrm{pure}}$, [**29**]. In the case that $\mathcal{Z}(m)$ has dimension 0, this is empty. In all other cases $\mathcal{Z}(m)$ may be considered a divisor on $\mathcal{M}$ (but note that since $\mathcal{Z}(m)$ is not a closed substack of $\mathcal{M}$, the degree of $\mathcal{Z}(m)$ over its image must be taken into account). But even after $\mathcal{Z}(m)$ is replaced by $\mathcal{Z}(m)^{\mathrm{pure}}$, one interesting feature remains, namely the existence of vertical components in characteristic $p \mid D(B)$. We write

$$\mathcal{Z}(m)^{\mathrm{pure}} = \mathcal{Z}(m)^{\mathrm{horiz}} + \mathcal{Z}(m)^{\mathrm{vert}} \tag{9.10}$$

(equality of "divisors" on $\mathcal{M}$), where $\mathcal{Z}(m)^{\mathrm{vert}}$ is the sum with multiplicities of the irreducible vertical components in characteristic p as p runs over primes dividing $D(B)$. We note that if we redefine

$$\hat{\mathcal{Z}}(m, v) = (\, \mathcal{Z}(m)^{\mathrm{pure}}, \Xi(m, v)\,) \tag{9.11}$$

then the expression $\langle \hat{\mathcal{Z}}(m, v), \hat{\omega} \rangle$ appearing in the definition of $\phi_{\mathrm{height}}(\tau)$ remains unchanged, cf. [**29**], section 4. We remark in passing that if $\mathcal{Z}(m)_{\mathbb{Q}} = \emptyset$, then $\Xi(m, v) = 0$, cf. (5.8)

Taking (9.10) into account and using formula (5.11) of Bost [**3**], we may write

$$\langle\, \hat{\mathcal{Z}}(m, v), \hat{\omega}\,\rangle = h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) + h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{vert}}) + \frac{1}{2} \int_{[\Gamma \backslash D]} \Xi(m, v) c_1(\hat{\omega})\ . \tag{9.12}$$

Also note that the second summand on the right hand side may be written as the sum over contributions of the bad fibers ,

$$h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{vert}}) = \sum_{p|D(B)} h_{\hat{\omega}}(\mathcal{Z}(m)_p^{\mathrm{vert}}) \quad , \tag{9.13}$$

where, since $\mathcal{Z}(m)^{\mathrm{vert}}$ has empty generic fiber,

$$h_{\hat{\omega}}(\mathcal{Z}(m)_p^{\mathrm{vert}}) = \deg(\omega|\mathcal{Z}(m)_p^{\mathrm{vert}}) \log(p) \ . \tag{9.14}$$

Here $\mathcal{Z}(m)_p^{\mathrm{vert}} = \mathcal{Z}(m)^{\mathrm{vert}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}_p$.

In the next three sections we will evaluate explicitly each summand on the right hand side of (9.12).

10. Contributions of horizontal components.

In this section, we compute the quantity $h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}})$. This will be done in two steps. We first express this quantity in terms of the Faltings heights of certain abelian surfaces which are isogenous to products of CM–elliptic curves. We then determine the effect of the isogeny on the Faltings height.

Observe that $\mathcal{Z}(m)^{\mathrm{horiz}}$ is a union of horizontal integral substacks

$$\mathcal{Z}(m)^{\mathrm{horiz}} = \sum_{\xi \in \mathcal{Z}(m)_{\mathbb{Q}}^{\mathrm{horiz}}} \mathcal{Z}_\xi, \tag{10.1}$$

where ξ is the generic point of $\mathcal{Z}_\xi$. Let $\tilde{\mathcal{Z}}_\xi$ be the normalization of $\mathcal{Z}_\xi$ and let $j_\xi : \tilde{\mathcal{Z}}_\xi \to \mathcal{M}$ be the composition of the normalization map $\tilde{\mathcal{Z}}_\xi \to \mathcal{Z}_\xi$ with the morphism $\mathcal{Z}_\xi \to \mathcal{M}$. By linearity and the definition of $h_{\hat{\omega}}$ for horizontal cycles, cf. (4.4) section 4 and [**3**], we have

$$\begin{aligned} h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) &= \sum_{\xi \in \mathcal{Z}(m)_{\mathbb{Q}}^{\mathrm{horiz}}} h_{\hat{\omega}}(\mathcal{Z}_\xi) \\ &= \sum_{\xi \in \mathcal{Z}(m)_{\mathbb{Q}}^{\mathrm{horiz}}} \widehat{\deg}\ j_\xi^* \hat{\omega} \cdot \frac{1}{|\mathrm{Aut}(\bar{\xi})|} \\ &= \frac{1}{|L:\mathbb{Q}|} \cdot \sum_{\eta \in \mathcal{Z}(m)^{\mathrm{horiz}}(L)} \widehat{\deg}\ j_\eta^* \hat{\omega} \cdot \frac{1}{|\mathrm{Aut}(\eta)|}, \end{aligned} \tag{10.2}$$

for any sufficiently large number field $L \subset \bar{\mathbb{Q}}$, where η runs over the L points of $\mathcal{Z}(m)$ and where the factors involving $\mathrm{Aut}(\bar{\xi}) = \mathrm{Aut}((A,\iota,x)_{\bar{\xi}})$ and $\mathrm{Aut}(\eta) = \mathrm{Aut}((A,\iota,x)_\eta)$ come in due to the stack. Here we also write η for the extension to $\eta : \mathrm{Spec}\,(O_L) \to \mathcal{M}$. We may assume that A_η, the abelian variety over L determined by η, has semistable reduction over L. Then, by definition, the Faltings height $h^*_{\mathrm{Fal}}(A_\eta)$ is given by

$$h^*_{\mathrm{Fal}}(A_\eta) = |L:\mathbb{Q}|^{-1}\widehat{\deg}\; j_\eta^*\hat{\omega}. \tag{10.3}$$

Here the notation h^*_{Fal} indicates that we have used the metric $||\;||$ given in Definition 3.4, rather than the standard metric of (3.11). This point is discussed further below. Then:

$$\begin{aligned} h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) &= \sum_{\eta \in \mathcal{Z}(m)^{\mathrm{horiz}}(L)} h^*_{\mathrm{Fal}}(A_\eta)\cdot \frac{1}{|\mathrm{Aut}(\eta)|} \\ &= 2 \sum_{\substack{x \in L(m) \\ \mathrm{mod}\ \Gamma}} h^*_{\mathrm{Fal}}(A_x)\cdot \frac{1}{|\Gamma_x|}. \end{aligned} \tag{10.4}$$

In the last expression, we have used the description of $\mathcal{Z}(m)(\mathbb{C}) = \mathcal{Z}(m)^{\mathrm{horiz}}(\mathbb{C})$ as the orbifold $[\Gamma\backslash D_{\mathcal{Z}(m)}]$, as in section 9, where the map $[\Gamma\backslash D_{\mathcal{Z}(m)}] \to [\Gamma\backslash L(m)]$ is 2 to 1, together with the fact that the abelian varieties associated to the two points in D_x (i.e., having opposite complex structures) have the same Faltings height.

We next turn to the computation of the Faltings height $h^*_{\mathrm{Fal}}(A)$ of an abelian surface A occuring in a triple (A,ι,x) where $\iota : O_B \to \mathrm{End}(A)$ and $x \in \mathrm{End}(A,\iota)$ is a special endomorphism with $Q(x) = m$, all defined over a number field L, where they all have good reduction. Let $\phi_x : \boldsymbol{k} \to \mathrm{End}^0(A)$ be the embedding determined by $\phi_x(\sqrt{-m}) = x$ and let

$$O_{c^2d} = \phi_x^{-1}(\mathbb{Q}[x] \cap \mathrm{End}(A)), \tag{10.5}$$

where O_{c^2d} is the order in $O_{\boldsymbol{k}}$ of conductor c. In this case, we will say that the triple (A,ι,x) is of type c. Recall that the order $\mathbb{Z}[\sqrt{-m}\,] \subset O_{c^2d} \subset O_{\boldsymbol{k}}$ has discriminant $4m$, and that we have written $4m = n^2\,d$, where $-d$ is the discriminant of $O_{\boldsymbol{k}}$. Then the order $\mathbb{Z}[\sqrt{-m}\,]$ has conductor n and $c \mid n$. Note, in addition, that the order O_{c^2d} must be maximal at all primes $p \mid D(B)$, and hence $(c, D(B)) = 1$.

Here the key point is that the special endomorphism x with $x^2 = -Q(x)\cdot 1_A$ forces A to be isogenous to a product of elliptic curves with CM by $\boldsymbol{k} = \mathbb{Q}(\sqrt{-m})$. The isogeny of interest is constructed as follows. Since $\boldsymbol{k}$ splits B, we can choose an embedding of $\psi : \boldsymbol{k} \hookrightarrow B$ such that

$$O_{\boldsymbol{k}} = \psi^{-1}(\mathcal{O}_B), \tag{10.6}$$

i.e., an optimal embedding, in the sense of Eichler. Then the endomorphisms

$$\alpha_x^{\pm} := x \pm \iota\circ\psi(\sqrt{-m}) \in \mathrm{End}(A) \tag{10.7}$$

satisfy

$$\alpha_x^+ + \alpha_x^- = 2x \qquad \text{and} \qquad \alpha_x^+ \cdot \alpha_x^- = 0. \tag{10.8}$$

Let

$$E^{\pm} = (\ker(\alpha_x^{\pm}))^0 \tag{10.9}$$

be the identity component of the kernel of the endomorphism $\alpha_x^{\pm}$. Choose an element $\eta \in O_B$ with $\mathrm{tr}(\eta) = 0$ and such that conjugation by η induces the Galois automorphism on $\psi(\boldsymbol{k})$. Note that

$$\iota(\eta)\, \alpha_x^+ = \alpha_x^-\, \iota(\eta), \tag{10.10}$$

and so

$$\iota(\eta) E^{\pm} = E^{\mp}. \tag{10.11}$$

Thus $E^{\pm}$ are both elliptic curves, and we obtain an isogeny

$$u_L : E^+ \times E^- = (\ker(\alpha_x^+))^0 \times (\ker(\alpha_z^-))^0 \longrightarrow A, \tag{10.12}$$

rational over L. Moreover, the elliptic curves $E^{\pm} = (\ker(\alpha_x^{\pm}))^0$ are stable under $\psi(O_{\boldsymbol{k}})$, i.e., have complex multiplication by the full ring of integers of $\boldsymbol{k}$. Note that the kernel of u_L is the subgroup

$$M_L = (\ker(\alpha_x^+))^0 \cap (\ker(\alpha_x^-))^0 \tag{10.13}$$

embedded antidiagonally in $E^+ \times E^-$.

The behavior of the Faltings height under isogeny is nicely described in the article of Raynaud, [**39**]. Our normalization of the Faltings height is the following. For an abelian variety B of dimension g over a number field L, let $N(B)$ be the Néron model of B over $S = \mathrm{Spec}\,(O_L)$, and let $\omega_B = \epsilon^*(\wedge^g \Omega_{N(B)/S})$ be the pullback by the zero section ϵ of the top power of the sheaf of relative differentials on $N(B)$. This invertible sheaf on $\mathrm{Spec}\,(O_L)$ has natural metrics; if $\sigma : L \hookrightarrow \mathbb{C}$ is an embedding of L, then a section β of ω_B determines a holomorphic g–form on $B_\sigma(\mathbb{C})$, and[7]

$$||\beta||^2_{\sigma,\mathrm{nat}} = \left| \left(\frac{i}{2\pi}\right)^g \int_{B_\sigma(\mathbb{C})} \beta \wedge \bar{\beta} \right|. \tag{10.14}$$

[7] Note that we use the factor $\left(\frac{i}{2\pi}\right)^g$ rather than $\left(\frac{i}{2}\right)^g$. This is the normalization used in Bost, [**3**], for example.

If B has semi–stable reduction, then the Faltings height of B is given by

$$h_{\mathrm{Fal}}(B) = |L:\mathbb{Q}|^{-1}\,\widehat{\deg}\,(\omega_B) \tag{10.15}$$

where $\widehat{\deg}(\omega_B)$ denotes that Arakelov degree of ω_B. The quantity $h_{\mathrm{Fal}}(B)$ does not depend on the choice of L over which B has semi–stable reduction.

In view of the normalization used in Definition 3.4, we introduce the metrics

$$||\beta||_\sigma^2 = \left|\,\left(e^{-C}\frac{i}{2\pi}\right)^g \int_{B_\sigma(\mathbb{C})} \beta\wedge\bar\beta\,\right|, \tag{10.16}$$

where, as before, $C = \frac{1}{2}\big(\log(4\pi)+\gamma\big)$. We denote the resulting height by $h^*_{\mathrm{Fal}}(B)$. The two heights are related by

$$h^*_{\mathrm{Fal}}(B) = h_{\mathrm{Fal}}(B) + \frac{1}{2}gC. \tag{10.17}$$

Assume that A has good reduction over L and let $u_L : A \to B$ be an isogeny defined over L. Let $u : N(A) \to N(B)$ be the resulting homomorphism of Néron models with $M := \ker(u)$. Then, as a special case of [**39**], p.205,

$$h_{\mathrm{Fal}}(B) = h_{\mathrm{Fal}}(A) + \frac{1}{2}\log(\deg(u_L)) - |L:\mathbb{Q}|^{-1}\,\log|\epsilon^*(\Omega^1_{M/R})|. \tag{10.18}$$

The quantity $\delta(u) := \log|\epsilon^*(\Omega^1_{M/S})|$ is a sum of local contributions as follows. For each prime v of L with $v \mid \deg(u_L)$, let R_v be the completion of O_L at v and let $M_v = M\otimes_{O_L} R_v$. Then

$$\delta(u) = \log|\epsilon^*(\Omega^1_{M/R})| = \sum_{v|\deg(u_L)} \log|\epsilon^*(\Omega^1_{M_v/R_v})|. \tag{10.19}$$

For convenience, we set

$$\delta_v(u) = \log|\epsilon^*(\Omega^1_{M_v/R_v})|. \tag{10.20}$$

This quantity is invariant under base change in the sense that if L' is a finite extension of L and if u' is the base change of u to Spec $(O_{L'})$, then

$$\delta_v(u) = \sum_{w|v} \delta_w(u'), \tag{10.21}$$

where w runs over the primes of L' dividing v.

We now return to our isogeny u_L, noting that A and $E^{\pm}$ all have good reduction over L. Let

$$u : N(E^+) \times N(E^-) \longrightarrow N(A) \tag{10.22}$$

be the homomorphism induced by u_L and let M be its kernel. To calculate $\delta_v(u)$ for a prime $v \mid \deg(u_L)$, we pass to the p-divisible groups, where p is the residue characteristic for v.

Let $G = G^+ \times G^-$ (resp. $A(p)$) be the p-divisible group over R_v associated to $E^+ \times E^-$ (resp. A) so that we have an exact sequence

$$0 \longrightarrow C \longrightarrow G \longrightarrow A(p) \longrightarrow 0 \tag{10.23}$$

determined by the isogeny u. Since the prime to p part of the kernel of u is automatically étale over R_v, the invariant $\delta_v(u)$ depends only on the p–divisible groups and hence on C. The isogeny (10.23) corresponds to a submodule T' of $V(G)$, the rational Tate module of G,

$$T(G) \subset T' \subset V(G) = V(G^+) \oplus V(G^-). \tag{10.24}$$

The fact that $E^{\pm}$ maps injectively into A implies that $G^{\pm} \hookrightarrow A(p)$ and hence

$$T' \cap V(G^{\pm}) = T(G^{\pm}). \tag{10.25}$$

Thus there are isomorphisms

$$\mathrm{pr}^+(T')/T(G^+) \overset{\sim}{\longleftarrow} T'/T(G) \overset{\sim}{\longrightarrow} \mathrm{pr}^-(T')/T(G^-). \tag{10.26}$$

Proposition 10.1. *Suppose that p splits in $\boldsymbol{k}$. Then $\mathrm{ord}_p(\deg(u_L)) = 2\,\mathrm{ord}_p(c)$. Moreover, for any place v of L with $v \mid p$, the group C is étale over R_v, and hence*

$$\delta_v(u) = \log|\epsilon^*(\Omega^1_{M_v/R_v})| = \log|\epsilon^*(\Omega^1_{C/R_v})| = 0.$$

Proof. We write

$$\boldsymbol{k}_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p, \qquad \alpha \mapsto (\alpha_1, \alpha_2), \tag{10.27}$$

and let λ_1 and λ_2 be the corresponding algebra homomorphisms from $\boldsymbol{k}_p$ to $\mathbb{Q}_p$. Let G_0 (resp. $G_{\text{ét}}$) be the connected (resp. étale) part of G, and note that, for example

$$G_0 = G_0^+ \times G_0^-. \tag{10.28}$$

Since the action of $\mathcal{O} := O_{\boldsymbol{k}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ preserves $G_0^{\pm}$, the action of $\mathcal{O}$ on the Tate module $T(G^{\pm})$ must have the form

$$T(G^+) = T(G_0^+) \oplus T(G_{\text{ét}}^+) \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p, \qquad \lambda_1 \oplus \lambda_2 \tag{10.29}$$
$$T(G^-) = T(G_0^-) \oplus T(G_{\text{ét}}^-) \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p, \qquad \lambda_2 \oplus \lambda_1,$$

where the switch of characters is due to the fact that the isogeny induced by η is $\mathcal{O}$–antilinear but must preserve the connected–étale decomposition. Thus there is a canonical decomposition

$$T(G) = T(G_0^+) \oplus T(G_0^-) \oplus T(G_{\text{ét}}^+) \oplus T(G_{\text{ét}}^-), \qquad \lambda_1 \oplus \lambda_2 \oplus \lambda_2 \oplus \lambda_1. \tag{10.30}$$

Since the $\mathbb{Z}_p$–lattice $T' \subset V(G)$ is stable under $\mathcal{O}$, it must be generated by coset representatives of the form $(x, 0, 0, w)$ and $(0, y, z, 0)$. Condition (10.24) implies, for example, that if $x = 0$ for such a representative, then $w = 0$ (i.e., lies in $\mathbb{Z}_p$). It follows that

$$T' = \mathbb{Z}_p \cdot (p^{-r}, 0, 0, \epsilon_1 p^{-r}) + \mathbb{Z}_p \cdot (0, p^{-s}, \epsilon_2 p^{-s}, 0) + T(G). \tag{10.31}$$

for units ϵ_1 and ϵ_2 and non–negative integers r and s. In addition, we can choose the element $\eta \in O_B$ above so that η^2 is prime to p and so η induces an automorphism on $A(p)$. Since T' is stable under this automorphism, we must have $r = s$ and $\epsilon_1 \equiv \epsilon_2 \mod p^r$. On the other hand, the ϕ_x–action of $\mathcal{O}$ on $V(G)$ is given by

$$\lambda_2 \oplus \lambda_2 \oplus \lambda_1 \oplus \lambda_1. \tag{10.32}$$

The ϕ_x action of $\alpha \in \mathcal{O}$ preserves T' if and only if

$$\alpha \in O_{c^2 d} \otimes \mathbb{Z}_p = \{(a_1, a_2)\mathcal{O} \mid a_1 \equiv a_2 \mod p^{\operatorname{ord}_p(c)}\}. \tag{10.33}$$

Thus, we conclude that $r = \operatorname{ord}_p(c)$. This proves that $\operatorname{ord}_p(\deg(u_L)) = 2 \operatorname{ord}_p(c)$.

To finish the proof of the Proposition, it suffices to show that

$$T' \cap V(G_0) = T(G_0), \tag{10.34}$$

so that the projection to the étale part $G \to G_{\text{ét}}$ induces an isomorphism on C. But this is clear from our description of the coset representatives. $\square$

Proposition 10.2. *Suppose that p is inert or ramified in $\boldsymbol{k}$.*
(i) If $p \nmid D(B)$, then for any place v of L with $v \mid p$, there is a factorization $u = u^{\dagger} \circ u^{o}$ with isogenies $u^{\dagger}$ and u^{o} such that

$$\delta_v(u^o) = \frac{1}{2} \cdot |L_v : \mathbb{Q}_p| \cdot \big(\operatorname{ord}_p(\deg(u_L)) - 2 \operatorname{ord}_p(c)\big),$$

and $\mathrm{ord}_p(\deg(u^\dagger)) = 2\mathrm{ord}_p(c) = 2s$. *Moreover,*

$$\delta_v(u^\dagger) = |L_v : \mathbb{Q}_p| \frac{(1-p^{-s})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))} \log(p).$$

Here $\chi(p) = -1$ *if* p *is inert and* $\chi(p) = 0$ *if* p *is ramified in* $\boldsymbol{k}$.
(ii) If $p \mid D(B)$, *then* $\mathrm{ord}_p(\deg(u_L)) = 0$ *and* $\delta_v(u) = 0$.

Proof. Let $\mathbb{F}_q$ be the residue field $\mathcal{O}/\pi\mathcal{O}$, where π is a fixed prime element of $\mathcal{O}$. Also write $\mathcal{O}_s = O_{c^2 d} \otimes \mathbb{Z}_p$, where $s = \mathrm{ord}_p(c)$. For convenience, we temporarily write L in place of L_v and $\mathcal{O}_L$ in place of O_{L_v}.

Now $G^\pm = E^\pm(p)$ is a formal group of dimension 1 and height 2 over $R_v = \mathcal{O}_{L_v}$ with an action of $\mathcal{O}$, i.e., a special formal $\mathcal{O}$–module in the sense of Drinfeld. We consider the sequence

$$C \longrightarrow G^+ \times G^- \xrightarrow{\ u\ } A(p) \tag{10.35}$$

and note that $O_B \otimes \mathcal{O}_s$ acts on $A(p)$. After replacing L by a finite extension and using the invariance property (10.21), we may assume that $G_0 := G^+ \simeq G^-$.

First suppose that $p \nmid D(B)$. Then, fixing an isomorphism

$$O_B \otimes_{\mathbb{Z}} \mathcal{O}_s \simeq M_2(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_s \simeq M_2(\mathcal{O}_s), \tag{10.36}$$

we may write $A(p) \simeq G_s \times G_s$, where G_s is a 1–dimensional formal group of height 2. Since (A, ι, x) was supposed to be of type c, we have $\mathrm{End}(G_s) = \mathcal{O}_s$, hence the notation for G_s, consistent with that for G_0. The isogeny u corresponds to an inclusion of Tate modules

$$T(G_0) \oplus T(G_0) = T(G_0 \times G_0) \subset T(A(p)) = T(G_s \times G_s) = T(G_s) \oplus T(G_s). \tag{10.37}$$

Note that the two direct sums here are not necessarily compatible. Let $T(G_s)^\dagger$ be the largest $\mathcal{O}$–module contained in the $\mathcal{O}_s$–module $T(G_s)$. Note that

$$T(G_s)^\dagger \oplus T(G_s)^\dagger \tag{10.38}$$

is the largest $\mathcal{O}$–module contained in $T(G_s) \oplus T(G_s)$. Hence the inclusion (10.37) gives rise to a chain of inclusions

$$T(G_0) \oplus T(G_0) \subset T(G_s)^\dagger \oplus T(G_s)^\dagger \subset T(G_s) \oplus T(G_s). \tag{10.39}$$

Hence the isogeny u factors as

$$G_0 \times G_0 \xrightarrow{\ u^o\ } G_s^\dagger \times G_s^\dagger \xrightarrow{\ u^\dagger\ } G_s \times G_s. \tag{10.40}$$

By the elementary divisor theorem, we can then find automorphisms α of $G_0 \times G_0$ and β of $G_s^\dagger \times G_s^\dagger$ such that $\beta \circ u^o \circ \alpha$ is of the form $u_1^o \times u_2^o$ for isogenies $u_i^o : G_0 \to G_s^\dagger$, $i = 1, 2$. Each of the isogenies u_i^o corresponds to an inclusion of Tate modules $T(G_0) \subset T(G_s^\dagger) \subset V(G_0)$. Since both Tate modules are free $\mathcal{O}$–modules of rank 1, there exists an isomorphism $G_0 \simeq G_s^\dagger$ such that u_i^o is given by multiplication by an element $\nu_i \in \mathcal{O}$. On the other hand, the isogeny $u^\dagger$ is of the form $u^\dagger = u_1^\dagger \times u_1^\dagger$, where the isogeny $u_1^\dagger$ is determined by the inclusion $T(G_s)^\dagger \subset T(G_s)$. If we choose an isomorphism $T(G_s)^\dagger \simeq \mathcal{O}$, then

$$T(G_s) \simeq p^{-s}\epsilon\,\mathbb{Z}_p + \mathcal{O}, \tag{10.41}$$

where ϵ is a unit in $\mathcal{O}$. This implies that the degree of the isogeny $u_1^\dagger$ is p^s, as claimed.

The contribution $\delta_v(u^\dagger) = 2\delta_v(u_1^\dagger)$ of the isogeny $u^\dagger$ to the invariant $\delta_v(u) = \delta_v(u^0) + \delta_v(u^\dagger)$ can now be obtained from the following result, whose proof we include, for the sake of completeness.

Proposition 10.3. (Nakkajima–Taguchi, [**36**]) *Let $\boldsymbol{k}_p/\mathbb{Q}_p$ be a quadratic extension and let $L/\boldsymbol{k}_p$ be a finite extension. Let G_0 be a one-dimensional formal $\mathcal{O}$-module over $\mathcal{O}_L$. For $s \geq 0$ suppose that $\lambda : G_0 \to G_s$ is an isogeny of degree p^s over $\mathcal{O}_L$ such that* $\mathrm{End}(G_s) = \mathcal{O}_s$. *Let $D = \mathrm{Ker}\,\lambda$. Then*

$$\log|\varepsilon^*\Omega^1_{D/\mathcal{O}_L}| = \frac{1}{2}\,|L : \mathbb{Q}_p|\,\frac{(1-p^{-s})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))}\log p \quad .$$

Here $\chi(p)$ is -1 (resp. 0) depending on whether $\boldsymbol{k}_p/\mathbb{Q}_p$ is unramified or ramified.

Proof. (Sketch) Let G_0 be defined by the formal group law $g(X,Y) \in \mathcal{O}_L[[X,Y]]$. Then by Serre's isogeny formula, [**19**],

$$D = \mathrm{Spec}\ \mathcal{O}_L[[X]]/\prod_{d\in D} g(X,d). \tag{10.42}$$

It follows that

$$\begin{aligned} \varepsilon^*\Omega^1_{D/\mathcal{O}_L} &= \mathcal{O}_L/\Big(\prod_{d\in D} g(X,d)\Big)'_{X=0} \\ &= \mathcal{O}_L/(\prod_{d\in D\setminus\{0\}} d) \quad . \end{aligned} \tag{10.43}$$

A consideration of the Newton polygon of $[\pi^r]_{G_0} = \pi^r X + \ldots$ shows that if $d \in D \setminus \{0\}$ is of precise π-order r then

$$\mathrm{ord}(d) = \frac{1}{q^r - q^{r-1}}. \tag{10.44}$$

Here, as before, π denotes a prime element of $\boldsymbol{k}_p$ and the ord function is normalized by $\mathrm{ord}(\pi) = 1$. It follows that

$$\begin{aligned} \mathrm{lg}_{\mathcal{O}_L}(\varepsilon^*\Omega^1_{D/\mathcal{O}_L}) &= e_{L/\boldsymbol{k}_p} \cdot \sum_{d \in D\setminus\{0\}} \mathrm{ord}(d) \\ &= e_{L/\boldsymbol{k}_p} \cdot \sum_{r=1}^{\infty} \cdot \frac{\ell(r)}{q^r - q^{r-1}} \quad . \end{aligned} \tag{10.45}$$

Here $\ell(r)$ denotes the number of elements of $D(\bar{L})$ of exact π-order $\ell(r)$. But the conditions $\deg(\lambda) = p^s$ and $\mathrm{End}(G_s) = \mathcal{O}_s$ imply that

$$D(\bar{L}) \;\cong\; T(G_s)/T(G_0) \;\cong\; \big(p^{-s}\varepsilon\,\mathbb{Z}_p + \mathcal{O}\big)/\mathcal{O} \;\cong\; p^{-s}\,\mathbb{Z}_p/\mathbb{Z}_p, \tag{10.46}$$

where ε is a unit in $\mathcal{O}$. Hence, for $r \geq 1$.

$$\ell(r) = \begin{cases} p^r - p^{r-1} & 1 \leq r \leq s, \quad \text{if } \boldsymbol{k}_p/\mathbb{Q}_p \text{ is unramified} \\ p^{\frac{r}{2}} - p^{\frac{r}{2}-1} & 1 \leq r \leq 2s,\ 2 \mid r, \text{ if } \boldsymbol{k}_p/\mathbb{Q}_p \text{ is ramified} \end{cases} \tag{10.47}$$

and is zero in all other cases. It follows that

$$\begin{aligned} \log|\varepsilon^*\Omega^1_{D/\mathcal{O}_L}| &= f_{L/\mathbb{Q}_p} \cdot \mathrm{lg}_{\mathcal{O}_L}(\varepsilon^*\Omega^1_{D/\mathcal{O}_L}) \cdot \log p \\ &= f_{L/\mathbb{Q}_p} \cdot e_{L/\boldsymbol{k}_p} \cdot \sum_{r=1}^{s} \frac{\ell(r)}{q^r - q^{r-1}} \log p \quad , \end{aligned} \tag{10.48}$$

which yields the assertion. $\square$

Finally, the contribution $\delta_v(u^o) = \delta_v(u_1^o) + \delta_v(u_2^o)$ is given by the following result.

Lemma 10.4. *For extensions $L/\boldsymbol{k}_p/\mathbb{Q}_p$ as in the previous Proposition, suppose that G_0 is a one-dimensional formal $\mathcal{O}$-module over $\mathcal{O}_L$. Let $\lambda : G_0 \to G_0$ be the isogeny given by multiplication by a non-zero element $\nu \in \mathcal{O}$. Let $D = \ker(\lambda)$. Then*

$$\log|\varepsilon^*\Omega^1_{D/\mathcal{O}_L}| = \frac{1}{2}\,|L : \mathbb{Q}_p|\log(\deg\,\lambda) \quad .$$

Proof. Let $s = \mathrm{ord}(\nu)$. Then, there are $q^r - q^{r-1}$ elements in $D(\bar{L}) \simeq \pi^{-s}\mathcal{O}/\mathcal{O}$ of exact order π^r, for $1 \leq r \leq s$, and none for all other r. Hence

$$\mathrm{lg}_{\mathcal{O}_L}(\varepsilon^*\Omega^1_{D/\mathcal{O}_L}) = e_{L/k_p} \cdot \sum_{r=1}^{s} \frac{q^r - q^{r-1}}{q^r - q^{r-1}} = e_{L/k_p} \cdot s. \tag{10.49}$$

It follows that

$$\log|\varepsilon^*\Omega^1_{D/O_L}| = e_{L/k_p} \cdot f_{L/\mathbb{Q}_p} \cdot s \cdot \log p = \frac{1}{2}\,|L:\mathbb{Q}_p| \cdot s \cdot \log(q). \tag{10.50}$$

The assertion follows since the isogeny λ has degree q^s. $\square$

Finally, we consider the case $p \mid D(B)$, and we recall that $\mathrm{ord}_p(c) = 0$, so that $\mathcal{O}_s = \mathcal{O}_0 = \mathcal{O}$. Once again, we consider the action of $O_B \otimes \mathcal{O}$ on $A(p)$. We fix an isomorphism

$$B \otimes \boldsymbol{k}_p \simeq M_2(\boldsymbol{k}_p), \qquad \text{with} \qquad O_B \otimes \mathcal{O} \hookrightarrow M_2(\mathcal{O}). \tag{10.51}$$

and such that, for $\alpha \in \boldsymbol{k}_p$,

$$1 \otimes \alpha \mapsto \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \qquad \text{and} \qquad \psi(\alpha) \otimes 1 \mapsto \begin{pmatrix} \alpha & \\ & \alpha^\sigma \end{pmatrix}, \tag{10.52}$$

where ψ is the embedding of $\boldsymbol{k}$ into B chosen above.

Lemma 10.5. *(i) If $\boldsymbol{k}_p/\mathbb{Q}_p$ is unramified, then the image of $O_B \otimes \mathcal{O}$ in $M_2(\mathcal{O})$ is the order*

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p\mathcal{O} & \mathcal{O} \end{pmatrix}.$$

(ii) If $\boldsymbol{k}_p/\mathbb{Q}_p$ is ramified, then there is an element $\lambda \in (O_B \otimes \mathcal{O})^\times$ whose image in $GL_2(\mathbb{F}_p)$ under the composition of maps

$$O_B \otimes \mathcal{O} \longrightarrow M_2(\mathcal{O}) \longrightarrow M_2(\mathbb{F}_p)$$

has eigenvalues which are not rational over $\mathbb{F}_p$.
(iii) In the ramified case, suppose that Λ is an $\mathcal{O}$–lattice contained in $\boldsymbol{k}^2$ which is stable under $O_B \otimes \mathcal{O}$. Then Λ is homothetic to $\Lambda_0 = \mathcal{O}^2$.

Remark 10.6. In fact, if $p \neq 2$ and $\boldsymbol{k}_p/\mathbb{Q}_p$ is ramified, then the image of $O_B \otimes \mathcal{O}$ in $M_2(\mathcal{O})$ is conjugate to $\mathrm{red}^{-1}(\mathbb{F}_{p^2})$, where

$$\mathrm{red} : M_2(\mathcal{O}) \longrightarrow M_2(\mathbb{F}_p)$$

is the reduction modulo π, and $\mathbb{F}_{p^2}$ is the nontrivial quadratic extension of $\mathbb{F}_p$, viewed as a subalgebra of $M_2(\mathbb{F}_p)$.

Proof. In the unramified case, we may write

$$O_B \otimes \mathbb{Z}_p = \mathcal{O}\,\langle\, \Pi \,\rangle, \tag{10.53}$$

where $\Pi \in O_B$ is an element with $\Pi^2 = p$ which normalizes $\psi(\boldsymbol{k}_p)$ and acts on it by the Galois automorphism σ, i.e., $\Pi\alpha = \alpha^\sigma \Pi$. The image of $O_B \otimes \mathcal{O}$ in $M_2(\mathcal{O})$ is then generated by the elements of the form

$$\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \\ & \alpha^\sigma \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} & 1 \\ p & \end{pmatrix}, \tag{10.54}$$

for $\alpha \in \mathcal{O}$. Since $\boldsymbol{k}_p$ is unramified, there is an $\alpha \in \mathcal{O}$ such that $\alpha - \alpha^\sigma$ is a unit, and hence such elements generate the Eichler order as claimed in (i).

Next suppose that $\boldsymbol{k}_p$ is ramified. Let $\boldsymbol{k}_o$ be the unramified quadratic extention of $\mathbb{Q}_p$ and let $\mathcal{O}_o$ be its ring of integers, with generator λ having unit norm. Again, we can write $O_B \otimes \mathbb{Z}_p = \mathcal{O}_o \langle\, \Pi \,\rangle$. But now, the image of $\lambda \otimes 1 \in (O_B \otimes \mathcal{O})^\times$ gives the element required in (ii).

Finally, to prove (iii), observe that the lattice $\Lambda_0 = \mathcal{O}^2$ is preserved by $O_B \otimes \mathcal{O}$, and hence is fixed by the element λ of part (ii). If Λ is another $\mathcal{O}$–lattice, preserved by $O_B \otimes \mathcal{O}$, then Λ must also be fixed by λ. The whole geodesic joining the vertices $[\Lambda_0]$ and $[\Lambda]$ in the building of $PGL_2(\boldsymbol{k}_p)$ is then fixed by λ. In particular, λ must then fix a vertex at distance 1 from $[\Lambda_0]$. But this implies that the image of λ in $PGL_2(\mathbb{F}_p)$ has a fixed point on $\mathbb{P}^1(\mathbb{F}_p) = \mathbb{P}(\Lambda_0/\pi\Lambda_0)$, and hence an $\mathbb{F}_p$–rational eigenvector/eigenvalue, which has been excluded. $\square$

Returning to $A(p)$ and our isogeny, the isomorphism (10.51) determines an isomorphism

$$V(A(p)) \simeq \boldsymbol{k}^2, \tag{10.55}$$

under which

$$V(G^+) = \boldsymbol{k} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad V(G^-) = \boldsymbol{k} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.56}$$

The image of the Tate module $T(A(p))$ in $\boldsymbol{k}^2$ is an $\mathcal{O}$–lattice which is stable under the action of $O_B \otimes \mathcal{O}$.

If $\boldsymbol{k}_p$ is unramified, then $T(A(p)) \subset \boldsymbol{k}^2$ is an $\mathcal{O}$–lattice stable under the Eichler order $O' \subset M_2(\mathcal{O})$ in (i) of Lemma. Let $1_2 = e_+ + e_- \in O'$ where

$$e_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad e_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in O'. \tag{10.57}$$

If $y = y_+ + y_- \in T(A(p))$ with $y_\pm \in V(G^\pm)$, then $y_\pm = e_\pm y \in V(G^\pm) \cap T(A(p)) = T(G^\pm)$, and hence $T(A(p)) = T(G^+ \times G^-)$. Thus, our isogeny has degree 1 and $\delta_v(u) = 0$.

If $\boldsymbol{k}_p/\mathbb{Q}_p$ is ramified, then, by (iii) of Lemma 10.5, $T(A(p))$ is homothetic to $\Lambda_0 = \mathcal{O}^2$. But then, since $T(G^\pm) = T(A(p)) \cap V(G^\pm)$, we have simply $T(G^+ \times G^-) = T(A(p))$, so again our isogeny has degree 1 and $\delta_v(u) = 0$. This finishes the proof of (ii) of Proposition 10.2. $\square$

We can now compute the Faltings height.

Theorem 10.7. *Suppose that the triple (A, ι, x), defined over a number field L, is of type c. Write $4m = n^2 d$, with $-d$ a fundamental discriminant, and let $E = E_d$ be an elliptic curve over L with complex multiplication by $O_{\boldsymbol{k}}$, the ring of integers in $\boldsymbol{k} = \mathbb{Q}(\sqrt{-d})$. Then*

$$h^*_{\text{Fal}}(A) = 2\, h^*_{\text{Fal}}(E) + \log(c) - \sum_p \frac{(1 - p^{-\text{ord}_p(c)}) \cdot (1 - \chi(p))}{(1 - p^{-1}) \cdot (p - \chi(p))} \cdot \log(p).$$

Here $\chi = \chi_d$ is as in (8.3).

Proof. We apply formula (10.18) to the isogeny $E^+ \times E^- \to A$ defined above. The change in the Faltings height due to the isogeny has the form

$$\frac{1}{2}\log(\deg(u_L)) - \frac{1}{|L:\mathbb{Q}|}\sum_v \delta_v(u). \tag{10.58}$$

We write

$$\frac{1}{2}\log(\deg(u_L)) = \sum_p \frac{1}{2}\text{ord}_p(\deg(u_L)) \cdot \log(p), \tag{10.59}$$

so that (10.58) can be written as a sum of local contributions $(10.58)_p$ which we now describe case by case.

If p is split in $\boldsymbol{k}$, then by Proposition 10.1,

$$(10.58)_p = \text{ord}_p(c) \cdot \log(p). \tag{10.60}$$

If $p \nmid D(B)$ is inert or ramified, by Proposition 10.2 , $(10.58)_p$ is equal to:

(10.61)
$$\begin{aligned}
&\frac{1}{2}\text{ord}_p(\deg(u_L)) \cdot \log(p)\\
&\quad - \frac{1}{|L:\mathbb{Q}|}\sum_{v|p} |L_v : \mathbb{Q}_p| \left(\frac{1}{2}\big[\text{ord}_p(\deg(u_L)) - 2r\big] + \frac{(1-p^{-r})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))}\right)\log(p)\\
&= \left(r - \frac{(1-p^{-r})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))}\right)\log(p)
\end{aligned}$$

where we have set $r = \text{ord}_p(c)$.

If $p \mid D(B)$, then by the considerations after Lemma 10.5, $(10.58)_p = 0$. $\square$

Theorem 10.8. *The contribution of the 'horizontal' part to the pairing $\langle \hat{\mathcal{Z}}(m,v), \hat{\omega} \rangle$ is*

$$h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) = 2\,\delta(d,D)\,H_0(m,D(B))\,2\,h^*_{\mathrm{Fal}}(E) + 2\delta(d,D)\,\frac{h(d)}{w(d)} \sum_{\substack{c|n \\ (c,D(B))=1}} c\prod_{\ell|c}(1-\chi(\ell)\ell^{-1})\cdot\sum_p \eta_p(\mathrm{ord}_p(c))\log(p),$$

where, for $r \in \mathbb{Z}_{\geq 0}$,

$$\eta_p(r) = r - \frac{(1-p^{-r})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))}.$$

Proof. Continuing (10.4) above, we have

$$\begin{aligned}
&h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) \\
&= 2\sum_{c|n} \sum_{\substack{x\in L(m) \\ \mathrm{mod}\ \Gamma \\ \textit{type}\ c}} h^*_{\mathrm{Fal}}(A_x)\cdot\frac{1}{|\Gamma_x|} \\
&= 2\sum_{c|n}\Bigg(\sum_{\substack{x\in L(m) \\ \mathrm{mod}\ \Gamma \\ \textit{type}\ c}} \frac{1}{|\Gamma_x|}\Bigg)\Bigg(2\,h^*_{\mathrm{Fal}}(E) + \log(c) - \sum_p \frac{(1-p^{-\mathrm{ord}_p(c)})\cdot(1-\chi(p))}{(1-p^{-1})\cdot(p-\chi(p))}\log(p)\Bigg) \\
&= 2\,\delta(d,D)\,H_0(m,D(B))\,2\,h^*_{\mathrm{Fal}}(E) \\
&\qquad + 2\delta(d,D)\,\frac{h(d)}{w(d)} \sum_{\substack{c|n \\ (c,D(B))=1}} c\prod_{\ell|c}(1-\chi(\ell)\ell^{-1})\cdot\sum_p \eta_p(\mathrm{ord}_p(c))\log(p),
\end{aligned}$$

as claimed. □

We now make the comparison of this expression with terms arising in the the positive Fourier coefficients of the derivative of the modified Eisenstein series. To do this, we need a better expression for the sum on c in the second term in Theorem 10.8. For convenience in the calculations, we let

$$\beta_p(k) = -2k + \begin{cases} \frac{p^k-1}{p^k(p-1)}, & \text{if } \chi_d(p)=1, \\ \frac{(3p+1)(p^k-1)-4k(p-1)}{(p-1)(p^{k+1}+p^k-2)}, & \text{if } \chi_d(p)=-1, \\ \frac{2}{p-1} - \frac{2k+2}{p^{k+1}-1} & \text{if } \chi_d(p)=0. \end{cases} \tag{10.62}$$

Note that when $k = \mathrm{ord}_p(n)$, then, by (i) of Lemma 8.7,

$$\beta_p(k) = \frac{1}{\log(p)} \cdot \frac{b_p'(n,0;D)}{b_p(n,0;D)}. \tag{10.63}$$

Lemma 10.9. *Let $4m = n^2 d$, as before. Then the following identity holds for any square-free $D > 0$:*

$$\begin{aligned}
&\frac{h(d)}{w(d)} \sum_{\substack{c|n\\(c,D)=1}} c \prod_{p|c} (1 - \chi_d(p)\, p^{-1}) \sum_{p|c} \eta_p(\mathrm{ord}_p(c)) \log(p) \\
&= H_0(m;D) \cdot \sum_{\substack{p\\(p,D)=1}} \big(-\mathrm{ord}_p(n) - \beta_p(\mathrm{ord}_p(n)) \big) \log(p) \\
&= H_0(m;D) \cdot \sum_{\substack{p\\(p,D)=1}} \left(\log|n|_p - \frac{b_p'(n,0;D)}{b_p(n,0;D)} \right).
\end{aligned}$$

Proof. We note that the sum on the last expression of the Lemma (right hand side) is finite since only summands for p with $p|n$ are non-zero. We proceed by induction on the number of prime factors of n. To start the induction, let $n = p^t$. Then the first expression of the Lemma (left hand side) is equal to

$$\frac{h(d)}{w(d)} \cdot \sum_{r=1}^{t} p^r \,(1 - \chi_d(p) p^{-1})\, \eta_p(r) \cdot \log p \tag{10.64}$$

(note that the contribution of $c = 1$ is trivial). By (8.19) and (10.62), the right hand side is equal to

$$\frac{h(d)}{w(d)} \cdot \left(\sum_{r=1}^{t} p^r (1 - \chi_d(p) p^{-1}) + 1 \right) \cdot (-t - \beta_p(t)) \cdot \log p. \tag{10.65}$$

Case by case, one can check that these two expressions coincide.

Case $\chi_d(p) = 1$: Then (10.64) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\sum_{r=1}^{t} p^r \,(1 - p^{-1})\, r = (p-1) \sum_{r=1}^{t} r\, p^{r-1} = t\, p^t - \frac{p^t - 1}{p-1}. \tag{10.66}$$

On the other hand, (10.65) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\left(\sum_{r=1}^{t} p^r (1 - p^{-1}) + 1 \right) \cdot \left(t - p^{-t} \frac{p^t - 1}{p-1} \right) = p^t \,\left(t - p^{-t} \frac{p^t - 1}{p-1} \right). \tag{10.67}$$

Case $\chi_d(p) = -1$: Then (10.64) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\begin{aligned}
&\sum_{r=1}^{t} p^r (1+p^{-1}) \left(r - 2\, p^{-r+1} \frac{p^r - 1}{p^2 - 1} \right) \\
&= (1+p^{-1}) \sum_{r=1}^{t} \left(rp^r - 2p \frac{p^r - 1}{p^2 - 1} \right) \qquad (10.68) \\
&= \frac{(p-1)\, t\, (p^t(p+1)+2) - (3p+1)\,(p^t-1)}{(p-1)^2} \quad .
\end{aligned}$$

On the other hand, (10.65) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\begin{aligned}
&\left(\sum_{r=1}^{t} p^r (1+p^{-1}) + 1 \right) \left(t - \frac{(3p+1)\,(p^t-1) - 4t(p-1)}{(p-1)\,(p^{t+1}+p^t-2)} \right) \\
&= \frac{p^{t+1}+p^t-2}{p-1} \left(t - \frac{(3p+1)(p^t-1) - 4t(p-1)}{(p-1)\,(p^{t+1}+p^t-2)} \right) \qquad (10.69) \\
&= \frac{(p-1)\, t\, (p^t(p+1)+2) - (3p+1)\,(p^t-1)}{(p-1)^2} \quad .
\end{aligned}$$

Case $\chi_d(p) = 0$: Then (10.64) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\begin{aligned}
&\sum_{r=1}^{t} p^r \left(r - p^{-r} \frac{p^r - 1}{p - 1} \right) \\
&= \sum_{r=1}^{t} r\, p^r - \sum_{r=1}^{t} \frac{p^r - 1}{p - 1} \qquad (10.70) \\
&= \frac{p^{t+1} - 1}{p - 1} \left(t - \frac{2}{p-1} + \frac{2t+2}{p^{t+1}-1} \right) \quad .
\end{aligned}$$

On the other hand, (10.65) without the factor $\frac{h(d)}{w(d)} \cdot \log(p)$ is equal to

$$\left(\sum_{r=0}^{t} p^r \right) \left(t - \frac{2}{p-1} + \frac{2t+2}{p^{t+1}-1} \right) = \frac{p^{t+1}-1}{p-1} \left(t - \frac{2}{p-1} + \frac{2t+2}{p^{t+1}-1} \right) \quad . \qquad (10.71)$$

We therefore have checked the beginning of the induction. Let us now perform the induction step. Let $n = p^t \cdot n_0$ where $p \nmid n_0$. Let us put $m_0 = m/p^{2t}$, so that $4m_0 = n_0^2 d$. We may assume that $p \nmid D$ because otherwise both sides of the identity for n coincide with the corresponding sides of the identity for n_0, so that we may apply the induction hypothesis.

We write $\mathcal{L}(m)$ resp. $\mathcal{R}(m)$ for the left hand side resp. right hand side of our identity corresponding to m. Then $\mathcal{L}(m)$ is equal to

(10.72)

$$\frac{h(d)}{w(d)}\sum_{r=1}^{t} p^r(1-\chi_d(p)p^{-1}) \sum_{\substack{c_0|n_0\\(c_0,D)=1}} c_0\cdot\Big(\prod_{\ell|c_0}(1-\chi_d(\ell)\ell^{-1})\Big)\Big[\eta_p(r)\log(p)+\sum_{\ell|c_0}\eta_\ell(r_\ell(c_0))\log(\ell)\Big]$$

$$+\frac{h(d)}{w(d)}\sum_{\substack{c_0|n_0\\(c_0,D)=1}} c_0\cdot\Big(\prod_{\ell|c_0}(1-\chi_d(\ell)\ell^{-1})\Big)\cdot\sum_{\ell|c_0}\eta_\ell(r_\ell(c_0))\,\log(\ell)$$

We recall that

$$\frac{h(d)}{w(d)}\cdot\sum_{\substack{c_0|n_0\\(c_0,D)=1}} c_0\prod_{\ell|c_0}(1-\chi_d(\ell)\ell^{-1}) = H_0(m_0;D). \tag{10.73}$$

Hence we can write the above expression as a sum of three terms, the first one being

(10.74)

$$\frac{h(d)}{w(d)}\sum_{\substack{c_0|n_0\\(c_0,D)=1}} c_0\Big(\prod_{\ell|c_0}(1-\chi_d(\ell)\ell^{-1})\Big)\cdot(1-\chi_d(p)p^{-1})\sum_{r=1}^{t} p^r\,\eta_p(r)\log(p)$$

$$=H_0(m_0,D)\cdot(1-\chi_d(p)p^{-1})\sum_{r=1}^{t} p^r\,\eta_p(r)\,\log(p)\quad .$$

The second and the third term are respectively equal to

$$\mathcal{L}(m_0)\cdot(1-\chi_d(p)\cdot p^{-1})\sum_{r=1}^{t} p^r = \mathcal{L}\,(m_0)\cdot(1-\chi_d(p)p^{-1})\,p\,\frac{p^t-1}{p-1} \tag{10.75}$$

and $\mathcal{L}(m_0)$.

We thus obtain

(10.76)

$$\mathcal{L}(m)=\frac{p^{t+1}-\chi_d(p)p^t+\chi_d(p)-1}{p-1}\cdot\mathcal{L}(m_0)+H_0(m_0,D)\cdot(1-\chi_d(p)p^{-1})\sum_{r=1}^{t} p^r\,\eta_p(r)\log(p)\quad .$$

By induction we have for the last summand

$$H_0(m_0;D)\cdot(1-\chi_d(p)p^{-1})\sum_{r=1}^{t} p^r\eta_p(r)\,\log(p)$$

(10.77)

$$=H_0(m_0;D)\cdot\Big(\sum_{r=1}^{t} p^r\,(1-\chi_d(p)p^{-1})+1\Big)\big(-t-\beta_p(t)\big)\,\log(p)$$

$$=H_0(m;D)\cdot\big(-t-\beta_p(t)\big)\,\log(p)\quad .$$

Hence

$$\mathcal{L}(m) = \frac{p^{t+1} - \chi_d(p)p^t + \chi_d(p) - 1}{p-1} \cdot \mathcal{L}(m_0) + H_0(m;D) \cdot (-t - \beta_p(t)) \log(p). \tag{10.78}$$

Now recall from (8.13) and Lemma 8.5 that

$$H_0(m;D) = \frac{h(d)}{w(d)} \prod_{q \nmid D} \frac{q^{t+1} - \chi_d(q)q^t + \chi_d(q) - 1}{q-1} \quad . \tag{10.79}$$

It follows that

$$\frac{H_0(m;D)}{H_0(m_0;D)} = \frac{p^{t+1} - \chi_d(p)p^t + \chi_d(p) - 1}{p-1} \quad . \tag{10.80}$$

From the definition of $\mathcal{R}(m)$ we have

(10.81)

$$\begin{aligned} \mathcal{R}(m) &= \frac{H_0(m,D)}{H_0(m_0;D)} \cdot \mathcal{R}(m_0) + H_0(m,D) \cdot (-t - \beta_p(t)) \log(p) \\ &= \frac{p^{t+1} - \chi_d(p)p^t + \chi_d(p) - 1}{p-1} \cdot \mathcal{R}(m_0) + H_0(m;D) \cdot (-t - \beta_p(t)) \log(p) \quad . \end{aligned}$$

Comparing (10.78) with (10.81), the induction hypothesis $\mathcal{L}(m_0) = \mathcal{R}(m_0)$ implies the assertion. $\square$

The following result is well known, cf., for example, [**9**].

Proposition 10.10. *With the normalization given by (10.14) above, the Faltings height $h_{\mathrm{Fal}}(E)$ of an elliptic curve E with CM by $O_{\boldsymbol{k}}$ is given by*

$$\begin{aligned} 2\, h_{\mathrm{Fal}}(E) &= -\frac{1}{2}\log(d) - \frac{L'(0,\chi_d)}{L(0,\chi_d)} \\ &= \frac{1}{2}\log(d) - \frac{w(d)}{2h(d)} \sum_{a=1}^{d-1} \chi_d(a) \log\Gamma\left(\frac{a}{d}\right) \\ &= \frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \log(2\pi) - \gamma. \end{aligned}$$

Remark 10.11. The value for $2h_{\mathrm{Fal}}(E)$ in Colmez [**9**], p.633 is our $2h_{\mathrm{Fal}}(E) - \log(2\pi)$ due to a difference in the normalization of the metric on the Hodge bundle.

Our *renormalized* Faltings height is then given by

$$
\begin{aligned}
2\,h^*_{\mathrm{Fal}}(E) &= 2\,h_{\mathrm{Fal}}(E) + \frac{1}{2}\log(\pi) + \frac{1}{2}\gamma + \log(2) \\
&= \frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma
\end{aligned} \tag{10.82}
$$

Combining these facts, we have

Corollary 10.12. *The contribution of the 'horizontal' part to the pairing* $\langle\, \hat{\mathcal{Z}}(m,v), \hat{\omega}\,\rangle$ *is*

$$
h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}}) = 2\,\delta(d;D)\,H_0(m;D)\,\Bigg[\,\frac{1}{2}\log(d) + \frac{L'(1,\chi_d)}{L(1,\chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma \\
+ \sum_{\substack{p \\ p\nmid D=1}} \left(\log|n|_p - \frac{b'_p(n,0;D)}{b_p(n,0;D)} \right) \Bigg].
$$

Proof of Theorem 7.2. Looking back to the end of section 8, we see that the expression of Corollary 10.12 for $h_{\hat{\omega}}(\mathcal{Z}(m)^{\mathrm{horiz}})$ coincides exactly with the sum of (8.43) and (8.45). The remaining terms will be considered in the next two sections.

§11. Contributions of vertical components.

In this section we fix a prime number p with $p \mid D(B)$. We wish to determine the quantity $\deg(\omega|\mathcal{Z}(m)_p^{\mathrm{vert}})$, cf. (9.11), using the results of [**29**].

We describe $\mathcal{Z}(m) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, W(\bar{\mathbb{F}}_p)$ in terms of the p-adic uniformization of $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,(\mathbb{F}_p)$, comp. section 2. To this end we fix $x \in O_{B'}$ with $\mathrm{tr}^\circ(x) = 0$ and $x^2 = -m$. As in section 2 we identify $B' \otimes \mathbb{A}_f^p$ with $B \otimes \mathbb{A}_f^p$ and $H(\mathbb{A}_f^p)$ with $H'(\mathbb{A}_f^p)$ and K^p with K'^p. Put

$$
I(x) = \{ gK^p \in H'(\mathbb{A}_f^p)/K'^{,p} \mid \; g^{-1}xg \in \hat{O}^p_{B'} \} \quad . \tag{11.1}
$$

We also use the abbreviation $\hat{\Omega}_{W(\bar{\mathbb{F}}_p)}$ for $\hat{\Omega}^2 \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\, W(\bar{\mathbb{F}}_p)$. Let $\tilde{x} = x$, if $\mathrm{ord}_p(m) = 0$ (resp. $\tilde{x} = 1 + x$, if $\mathrm{ord}_p(m) > 0$). Let

$$
Z(x) = (\hat{\Omega}_{W(\bar{\mathbb{F}}_p)} \times \mathbb{Z})^{\tilde{x}} \tag{11.2}
$$

be the fixed point set of $\tilde{x} \in H'(\mathbb{Q}_p)$. Denoting by H'_x the stabilizer of x in H', we have, [**29**],

$$
\mathcal{Z}(m) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, W(\bar{\mathbb{F}}_p) = [H'_x(\mathbb{Q}) \setminus I(x) \times Z(x)] \tag{11.3}
$$

(quotient in the sense of stacks). Since $\mathrm{ord}_p \det(\tilde{x}) = 0$, we have

$$Z(x) = \hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)} \times \mathbb{Z} \quad . \tag{11.4}$$

Since the set

$$H'_x(\mathbb{A}^p_f) \setminus \{g \in H'(\mathbb{A}^p_f) \mid \ g^{-1}xg \in \hat{O}^p_{B'}\}$$

is compact, the group $H'_x(\mathbb{A}^p_f)$ has only finitely many orbits on $I(x)$. Let $g_1, \dots, g_r \in H'(\mathbb{A}^p_f)$ such that

$$I(x) = \coprod_{i=1}^{r} H'_x(\mathbb{A}^p_f)\, g_i\, K'^{,p} \quad . \tag{11.5}$$

Then we may rewrite (11.3) as

$$\coprod_{i=1}^{r} \Big[H'_x(\mathbb{Q}) \setminus \Big(H'_x(\mathbb{A}^p_f)/(K'^{,p}_i \cap H'_x(\mathbb{A}^p_f)) \times \mathbb{Z} \times \hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)} \Big) \Big] \quad ,$$

where $K'^{p}_i = g_i K'^{p} g_i$.

Note that $H'_x(\mathbb{Q}) \cong \boldsymbol{k}^\times$, where $\boldsymbol{k} = \mathbb{Q}(\sqrt{-m})$ is the imaginary quadratic field associated to m. Let us first consider the case where p does not split in $\boldsymbol{k}$. Then

$$\mathrm{ord}_p \det(\boldsymbol{k}^\times_p) = \delta_p \cdot \mathbb{Z} \quad , \tag{11.6}$$

where $\delta_p = 2$ if p is unramified in $\boldsymbol{k}$ and $\delta_p = 1$ if p is ramified in $\boldsymbol{k}$. Let

$$H'_x(\mathbb{Q})^1 = \{g \in H'_x(\mathbb{Q}) \mid \ \mathrm{ord}_p(\det(g)) = 0\} \quad . \tag{11.7}$$

Then $H'_x(\mathbb{Q})^1$ acts with finite stabilizer groups on $H'_x(\mathbb{A}^p_f)/(K'^{,p}_i \cap H'_x(\mathbb{A}^p_f))$. Hence we may rewrite (11.3) as δ_p copies of

$$\coprod_{i=1}^{r} \Big[H'_x(\mathbb{Q})^1 \setminus \Big(H'(\mathbb{A}^p_f)/(K'^{,p}_i \cap H'_x(\mathbb{A}^p_f)) \Big) \Big] \times \hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)} \tag{11.8}$$

(here the first factor is taken in the sense of stacks).

Appealing now to [**29**], Proposition 3.2, we obtain the following expression for the vertical components of $\mathcal{Z}(m)$:

(11.9)

$$\begin{aligned} &\mathcal{Z}(m)^{\mathrm{vert}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathrm{Spec}\ W(\bar{\mathbb{F}}_p) \\ &= \mathbb{Z}/\delta_p\mathbb{Z} \times \coprod_{i=1}^{r} \Big[H'_x(\mathbb{Q})^1 \setminus \Big(H'(\mathbb{A}^p_f)/(K'^{,p}_i \cap H'_x(\mathbb{A}^p_f)) \Big) \Big] \times \Big(\sum_{[\Lambda] \in \mathcal{B}} \mathrm{mult}_{[\Lambda]}(x) \cdot \mathbb{P}_{[\Lambda]} \Big). \end{aligned}$$

Here $[\Lambda]$ ranges over the vertices of the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$ and the multiplicity with which the prime divisor $\mathbb{P}_{[\Lambda]}$ occurs is given by loc.cit., (3.9) for $p \neq 2$ and by Proposition A.1 in the Appendix below for $p = 2$.

Proposition 11.1. *For any* $[\Lambda] \in \mathcal{B}$ *we have*

$$\deg(\omega|\mathbb{P}_{[\Lambda]}) = p - 1 \quad .$$

Proof. Of course, here $\deg(\omega|\mathbb{P}_{[\Lambda]})$ is shorthand for

$$\deg\, i^*_{[\Lambda]}(\omega \otimes_{\mathbb{Z}} W(\bar{\mathbb{F}}_p)) \quad ,$$

where $i_{[\Lambda]} : \mathbb{P}_{[\Lambda]} \to \mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, W(\bar{\mathbb{F}}_p)$ is the natural morphism. We write O_{B_p} as

$$O_{B_p} = \mathbb{Z}_{p^2}[\Pi]/(\Pi^2 = p,\ \Pi a = a^\sigma \Pi, \quad \forall a \in \mathbb{Z}_{p^2}) \quad .$$

For the inverse image of the universal abelian scheme $(\mathcal{A}, \iota)$ on $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, W(\bar{\mathbb{F}}_p)$ we have

$$\mathrm{Lie}\,\mathcal{A} = \mathcal{L}_0 \oplus \mathcal{L}_1 \quad , \tag{11.10}$$

where $\mathcal{L}_i = \{x \in \mathrm{Lie}\,\mathcal{A} \mid \ \iota(a)x = a^{\sigma^{-i}}x, \quad \forall a \in \mathbb{Z}_{p^2}\}$.

Due to the determinant condition (1.1), both $\mathcal{L}_0$ and $\mathcal{L}_1$ are line bundles on $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, W(\bar{\mathbb{F}}_p)$ and

$$\omega \otimes_{\mathbb{Z}} W(\bar{\mathbb{F}}_p) = \mathcal{L}_0^{-1} \otimes \mathcal{L}_1^{-1} \quad . \tag{11.11}$$

The fiber of $\mathcal{L}_i$ at a $\bar{\mathbb{F}}_p$-valued point of $\mathcal{M}$ is expressed as follows in terms of the Dieudonné module M of the corresponding abelian variety,

$$\mathcal{L}_0 = M_0/VM_1 \quad , \quad \mathcal{L}_1 = M_1/VM_0 \quad . \tag{11.12}$$

Here $M = M_0 \oplus M_1$ is the eigenspace decomposition under the action of $\mathbb{Z}_{p^2}$ analogous to (11.10).

To fix ideas assume that Λ is even ([**29**]). Then for every $x \in \mathbb{P}_{[\Lambda]}(\bar{\mathbb{F}}_p)$ we have

$$M_0 = \Lambda \otimes_{\mathbb{Z}_p} W(\bar{\mathbb{F}}_p),\ VM_0 = \Pi M_0;\ \mathcal{L}_{0x} = M_0/\ell_x \quad , \tag{11.13}$$

where ℓ_x is the line in $\Lambda \otimes_{\mathbb{Z}_p} \bar{\mathbb{F}}_p$ corresponding to x. It follows that

$$i^*_{[\Lambda]}(\mathcal{L}_0) = \mathcal{O}_{\mathbb{P}_{[\Lambda]}}(1) \quad . \tag{11.14}$$

(It is $\mathcal{O}_{\mathbb{P}_{[\Lambda]}}(1)$ rather than $\mathcal{O}_{\mathbb{P}_{[\Lambda]}}(-1)$ since $\mathcal{L}_0$ obviously has global sections.) To calculate $i^*_{[\Lambda]}(\mathcal{L}_1)$, we use the exact sequence of coherent sheaves on $\hat{\Omega}_{W(\bar{\mathbb{F}}_p)}$

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{\Pi} \mathcal{L}_0 \longrightarrow \mathcal{O}_{\mathbb{P}^{\mathrm{odd}}} \longrightarrow 0 \quad . \tag{11.15}$$

Here $\mathcal{O}_{\mathbb{P}^{\rm odd}}$ denotes the structure sheaf of the closed subscheme of the special fiber ([**29**], section 2),

$$\bigcup_{[\Lambda]\ \mathrm{odd}} \mathbb{P}_{[\Lambda]} \subset \hat{\Omega}_{W(\bar{\mathbb{F}}_p)} \otimes_{W(\bar{\mathbb{F}}_p)} \bar{\mathbb{F}}_p \quad . \tag{11.16}$$

This sequence remains exact after pulling back and yields

$$0 \longrightarrow i^*_{[\Lambda]}(\mathcal{L}_1) \longrightarrow i^*_{[\Lambda]}(\mathcal{L}_0) \longrightarrow \mathcal{O}_{\mathbb{P}_{[\Lambda]}(\mathbb{F}_p)} \longrightarrow 0 \tag{11.17}$$

(note that $i^*_{[\Lambda]}(\mathcal{L}_1)$ is torsion-free). Here we used that $\mathbb{P}_{[\Lambda]}(\mathbb{F}_p) = \mathbb{P}_{[\Lambda]} \cap (\bigcup_{[\Lambda]\ \mathrm{odd}} \mathbb{P}_{[\Lambda]})$. From (11.14) we obtain

$$\begin{aligned} \deg\, i^*_{[\Lambda]}(\mathcal{L}_1) &= \deg\, i^*_{[\Lambda]}(\mathcal{L}_0) - (p+1) \\ &= -p \quad . \end{aligned} \tag{11.18}$$

Now the identity (11.11) yields the assertion. The case where $[\Lambda]$ is odd is similar. □

Remark 11.2. Another proof of Proposition 11.1 may be obtained by using Proposition 3.2. Indeed, by that proposition we may identify ω and the relative dualizing sheaf $\omega_{\mathcal{M}/\mathbb{Z}}$. It follows that

$$\deg(\omega|\mathbb{P}_{[\Lambda]}) = \deg(\omega_{\mathcal{M}/\mathbb{Z}}|\,\mathbb{P}_{[\Lambda]}) = \deg(\omega_{\mathcal{M}\otimes_{\mathbb{Z}}\bar{\mathbb{F}}_p/\bar{\mathbb{F}}_p}|\,\mathbb{P}_{[\Lambda]}) \quad .$$

By expressing the dualizing sheaf $\omega_{\mathcal{M}\otimes_{\mathbb{Z}}\bar{\mathbb{F}}_p/\bar{\mathbb{F}}_p}$ explicitly, it is easy to calculate the last term. □

Corollary 11.3. *Let $k = \mathrm{ord}_p n$, where, as usual, $4m = n^2 d$. Then*

$$\sum_{[\Lambda]\in\mathcal{B}} \mathrm{mult}_{[\Lambda]}(x)\cdot\deg(\omega|\mathbb{P}_{[\Lambda]}) = \begin{cases} -2k + (p+1)\cdot\frac{p^k-1}{p-1} & \textit{if } p \textit{ is unramified in } \boldsymbol{k} \\ -2k-2+2\cdot\frac{p^{k+1}-1}{p-1} & \textit{if } p \textit{ is ramified in } \boldsymbol{k}. \end{cases}$$

Proof. It remains to calculate $\sum_{[\Lambda]} \mathrm{mult}_{[\Lambda]}(x)$. If p is odd, this is an easy exercise using the results of [**29**], section 6. For instance, let p be odd and unramified and put $\alpha = \mathrm{ord}_p(m)$ so that $\alpha = 2k$. Then

$$\sum_{[\Lambda]} \mathrm{mult}_{[\Lambda]}(x) = \frac{\alpha}{2} + (p+1)\sum_{r=1}^{\frac{\alpha}{2}-1} p^{r-1}\left(\frac{\alpha}{2} - r\right) = \frac{-\alpha}{p-1} + \frac{p+1}{p-1}\cdot\frac{p^{\frac{\alpha}{2}}-1}{p-1} \ .$$

The case $p = 2$ is handled in the Appendix to this section. $\square$

It now remains to determine the degree of the discrete stack

$$\coprod_{i=1}^{r} \left[H'_x(\mathbb{Q})^1 \setminus \left(H'(\mathbb{A}_f^p)/(K_i^{\prime,p} \cap H'_x(\mathbb{A}_f^p)) \right) \right].$$

Now the elements $g_1, \ldots, g_r$ are in one-to-one correspondence with the $\hat{O}_B^{p,\times}$–conjugacy classes of embeddings of rings

$$j^p : \boldsymbol{k} \otimes \mathbb{A}_f^p \longrightarrow B \otimes \mathbb{A}_f^p,$$

with $x \in (j^p)^{-1}(\hat{\mathcal{O}}_B^p)$. To each embedding j^p there is associated an order of $\boldsymbol{k}$,

$$O(j^p) = ((j^p)^{-1}(\hat{O}_B^p).O_{\boldsymbol{k}_p}) \cap \boldsymbol{k}.$$

The conductor $c = c(j^p)$ of this order satisfies $(c, D(B)) = 1$ and $c \mid n$, because $x \in (j^p)^{-1}(O_B^p)$. Conversely, given c with those two properties, there are precisely

$$\prod_{\ell | D(B), \ell \neq p} (1 - \chi_d(\ell))$$

classes of embeddings j^p yielding the order of conductor c. Finally, if g_i yields the order of conductor c, the stack $[H'_x(\mathbb{Q})^1 \setminus (H'(\mathbb{A}_f^p)/K'_i \cap H'_x(\mathbb{A}_f^p))]$ may be identified with

$$[\boldsymbol{k}^{\times,1} \setminus (\boldsymbol{k} \otimes \mathbb{A}_f^p / \hat{O}_{c^2 d}^{p,\times})]$$

which has degree $h(c^2 d)/w(c^2 d)$. Summarizing these arguments, we therefore obtain

Lemma 11.4.

$$\begin{aligned} &\delta_p \cdot \deg \left(\coprod_{i=1} [H'_x(\mathbb{Q})^1 \setminus H'(\mathbb{A}_f^p)/K_i^{\prime,p} \cap H'_x(\mathbb{A}_f^p)] \right) \\ &= \prod_{\ell | D(B)} (1 - \chi_d(\ell)) \cdot \sum_{\substack{c | n \\ (c, D(B))=1}} h(c^2 d)/w(c^2 d) \\ &= \delta(d; D(B)) \cdot H_0(m; D(B)) \quad . \end{aligned}$$

$\square$

Now let us consider the case when p splits in $\boldsymbol{k}$. In this case $\mathrm{ord}_p(\det(\boldsymbol{k}_p^\times)) = \mathbb{Z}$, but $H'_x(\mathbb{Q})^1$ does not act with finite stabilizer groups on $H'_x(\mathbb{A}_f^p)/(K_i^{\prime p} \cap H'_x(\mathbb{A}_f^p))$. Let $\epsilon(x) \in H'_x(\mathbb{Q}) = \boldsymbol{k}^\times$

be an element whose localization in $\boldsymbol{k}_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$ has valuation $(1, -1)$. Let $H'_x(\mathbb{Q})^{1,1}$ be the subgroup of elements of $H'_x(\mathbb{Q})$ which are units at p. Then $H'_x(\mathbb{Q})^1 = H'_x(\mathbb{Q})^{1,1} \times \langle \epsilon(x)^{\mathbb{Z}} \rangle$, and $H'_x(\mathbb{Q})^{1,1}$ acts with finite stabilizer groups on $H'_x(A_f^p)/(K_i'^p \cap H'_x(\mathbb{A}_f^p))$, whereas $\epsilon(x)$ acts freely on $\hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)}$ by translations by 2 on the 'apartment of central components' ([**29**]). We obtain therefore the following expression for (11.3) in this case

$$\left(\coprod_{i=1}^{r} \left[H'_x(\mathbb{Q})^{1,1} \setminus \Big(H'_x(\mathbb{A}_f^p)/(K_i'^{,p} \cap H'_x(\mathbb{A}_f^p)) \Big) \right] \right) \times \Big(\langle \epsilon(x)^{\mathbb{Z}} \rangle \setminus \hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)} \Big) . \tag{11.19}$$

The same analysis as before yields

$$\begin{aligned} &\deg(\coprod_{i=1}^{r} \left[H'_x(\mathbb{Q})^{1,1} \setminus \Big(H'_x(\mathbb{A}_f^p)/(K_i'^{,p} \cap H'_x(\mathbb{A}_f^p)) \Big) \right]) \\ &= \prod_{\substack{\ell \mid D(B) \\ \ell \neq p}} (1 - \chi_d(\ell)) \cdot \sum_{\substack{c \mid n \\ (c, D(B))=1}} h(c^2 d)/w(c^2 d) \\ &= \delta(d; D(B)/p) \cdot H_0(m; D(B)) \quad . \end{aligned} \tag{11.20}$$

Using Proposition 11.1 we have

$$\deg(\omega \mid \big(\langle \epsilon(x)^{\mathbb{Z}} \rangle \setminus \hat{\Omega}^{\tilde{x}}_{W(\bar{\mathbb{F}}_p)} \big)) = 2\,(p-1) \sum_{[\Lambda]} \mathrm{mult}_{[\Lambda]}(x). \tag{11.21}$$

The sum on the right hand side runs over all vertices $[\Lambda]$ such that the closest vertex on the apartment corresponding to $\boldsymbol{k}_p^\times$ is a fixed vertex. This sum can again be evaluated using [**29**], 3.9, for p odd (resp. the appendix to this section for $p = 2$).

We summarize our findings in the following theorem.

Theorem 11.5. *Let $k = \mathrm{ord}_p(n)$, where, as usual, $4m = n^2 d$.*
(i) If p splits in $\boldsymbol{k}$, then

$$\deg(\,\omega \mid \mathcal{Z}(m)_p^{\mathrm{vert}}\,) = 2\, H_0(m; D(B))\, \delta(d; D(B)/p) \cdot (p^k - 1).$$

(ii)

$$\deg(\,\omega \mid \mathcal{Z}(m)_p^{\mathrm{vert}}\,) = 2\, H_0(m; D(B))\, \delta(d; D(B)) \cdot \begin{cases} -k + \frac{(p+1)(p^k-1)}{2(p-1)} & \text{if } \chi_d(p) = -1, \\ -1 - k + \frac{p^{k+1}-1}{p-1} & \text{if } \chi_d(p) = 0. \end{cases}$$

Proof of Theorem 7.2 (continued). In the case $\chi_d(p) = 1$, the quantity

$$\deg(\,\omega \mid \mathcal{Z}(m)_p^{\mathrm{vert}}\,)\ \log(p) \cdot q^m$$

coincides exactly with the term (ii) of Theorem 8.8. On the other hand, in the cases $\chi_d(p) = -1$ and $\chi_d(p) = 0$, we find that

$$\deg(\omega \mid \mathcal{Z}(m)_p^{\mathrm{vert}}) \log(p) \cdot q^m = 2\,\delta(d; D)\, H_0(m; D)\, K_p \log(p) \cdot q^m,$$

where K_p is as in Theorem 8.8. Thus, summing on $p \mid D$, we obtain (8.45).

Appendix to section 11: The case $p = 2$.

In [**29**] we made the blanket assumption $p \neq 2$. In this appendix we indicate the modifications needed to arrive at the formulas given in Theorem 11.5 in the case $p = 2$.

We will use the same notation as in [**29**]. We fix a special endomorphism $j \in V$ with $q(j) = j^2 \in \mathbb{Z}_2 \setminus \{0\}$. We denote by $Z(j)$ the associated closed formal subscheme of the Drinfeld moduli space $\mathcal{M} \simeq \hat{\Omega} \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)$. We will content ourselves with giving the structure of the divisor $Z(j)^{\mathrm{pure}}$ associated to $Z(j)$, loc.cit., section 4. Our discussion will proceed by distinguishing cases. Let $\boldsymbol{k} = \mathbb{Q}_2(j)$ (hence in the global case $\boldsymbol{k}$ is the localization at 2 of the imaginary quadratic field). Let $\mathcal{O} = \mathcal{O}_{\boldsymbol{k}}$ be the ring of integers in $\boldsymbol{k}$. We write as usual

$$(A.1) \qquad q(j) = \varepsilon \cdot 2^\alpha \quad , \quad \varepsilon \in \mathbb{Z}_2^\times \quad , \quad \alpha \geq 0 \quad .$$

We define $k \geq 0$ by

$$(A.2) \qquad \alpha + 2 = 2k + \mathrm{ord}_2(d) \quad ,$$

where d denotes the discriminant of $\mathcal{O}/\mathbb{Z}_2$. Note that in the global context, when $\mathcal{Z}(m)$ is p-adically uniformized by $Z(j)$ (cf. (11.3) above), then $\alpha = \mathrm{ord}_p(m)$. If we write as usual $4m = n^2 d$, then $k = \mathrm{ord}_p(n)$.

We have then the following cases

Case	$q(j)$	2 in $\boldsymbol{k}$	value of k	$\mathcal{B}^{\mathcal{O}^\times}$
1	$2\mid\alpha,\ \varepsilon \equiv 1(8)$	split	$k = \frac{\alpha}{2} + 1$	$\mathcal{A}$
2	$2\mid\alpha,\ \varepsilon \equiv 5(8)$	unramified	$k = \frac{\alpha}{2} + 1$	$\{[\Lambda_0]\}$
3	$2\mid\alpha,\ \varepsilon \equiv -1(4)$	ramified	$k = \frac{\alpha}{2}$	$\{[\Lambda_0],\ [\Lambda_1]\}$
4	$2 \nmid \alpha$	ramified	$k = \frac{\alpha-1}{2}$	$\{[\Lambda_0],\ [\Lambda_1]\}$

We explain the last column in this table. In cases 1 and 2, writing $j = 2^{\alpha/2} \cdot \bar{j}$, the index of $\mathbb{Z}_2[\bar{j}]$ in $\mathcal{O}$ is 2. In case 1, the fixed point set of $\mathcal{O}^\times$ is the apartment $\mathcal{A}$ in $\mathcal{B}$ corresponding to the split Cartan subgroup $\boldsymbol{k}^\times$ of $GL_2(\mathbb{Q}_2)$. In case 2, the fixed point set of $\mathcal{O}^\times$ is the vertex corresponding to the lattice $\Lambda_0 = \mathcal{O}$ in $\mathbb{Q}_2^2$. Note that in case 1 the fixed point set of j is

$$\mathcal{B}^j = \{[\Lambda];\ d([\Lambda],\ \mathcal{A}) \leq 1\} \quad . \tag{$A.3$}$$

In case 2, denoting by $[\Lambda_1]$ the vertex corresponding to the lattice $\Lambda_1 = \mathbb{Z}_2[\bar{j}]$, we have

$$\mathcal{B}^j = \{[\Lambda_0], [\Lambda_1]\} \quad . \tag{$A.4$}$$

In cases 3 and 4 we write $j = 2^{[\alpha/2]}\bar{j}$. Then we have $\mathcal{O} = \mathbb{Z}_2[\bar{j}]$. The fixed point set of $\mathcal{O}^\times$ consists of the vertices corresponding to the lattices $\Lambda_0 = \mathcal{O}$ and $\Lambda_1 = \pi\mathcal{O}$, where π denotes a uniformizer in $\mathcal{O}$. In case 3 this coincides with the fixed point set of j, whereas, in case 4, j permutes the two vertices $[\Lambda_0]$ and $[\Lambda_1]$ so that $\mathcal{B}^j$ consists of the midpoint of the edge formed by $[\Lambda_0]$ and $[\Lambda_1]$.

To formulate the theorem we write the divisor as usual as a sum of a vertical part and a horizontal part,

$$Z(j)^{\text{pure}} = Z(j)^{\text{vert}} + Z(j)^{\text{horiz}} \quad .$$

Proposition A.1. *(i) Let*

$$Z(j)^{\text{vert}} = \sum_{[\Lambda]\in\mathcal{B}} \text{mult}_{[\Lambda]}(j) \cdot \mathbb{P}_{[\Lambda]} \quad .$$

Then the multiplicity $\text{mult}_{[\Lambda]}(j)$ *is given by*

$$\text{mult}_{[\Lambda]}(j) = \max(k - d([\Lambda], \mathcal{B}^{\mathcal{O}^\times}), 0) \quad .$$

(ii) In case 1, $Z(j)^{\text{horiz}} = 0$*. In case 2,* $Z(j)^{\text{horiz}}$ *is isomorphic to the disjoint union of two copies of* Spf $W(\bar{\mathbb{F}}_2)$ *and meets the special fiber in two ordinary special points of* $\mathbb{P}_{[\Lambda_0]}$*. In cases 3 and 4,* $Z(j)^{\text{horiz}}$ *is isomorphic to* Spf W'*, where* W' *is the ring of integers in a ramified quadratic extension of* $W(\bar{\mathbb{F}}_2)$*, and meets the special fiber in the superspecial point corresponding to the midpoint of the edge formed by* $[\Lambda_0]$ *and* $[\Lambda_1]$*.*

Proof. We first determine $Z(j) \cap (\hat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_2} \text{Spf } W(\bar{\mathbb{F}}_2))$ for a vertex $[\Lambda]$ where the intersection is non-empty. Let $m = \max\{r;\ j(\Lambda) \subset 2^r\Lambda\}$. Then

$$m = \alpha/2 - d([\Lambda],\ \mathcal{B}^j) \quad , \tag{$A.5$}$$

cf. loc.cit., Lemma 2.8. After choosing a basis of Λ we may write

$$j = 2^m \cdot \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} = 2^m \cdot \bar{j} \quad , \tag{$A.6$}$$

where $\bar{a}, \bar{b}, \bar{c}$ are not simultaneously divisible by p. The equation of $Z(j)$ on $\hat{\Omega}_{[\Lambda]} \times_{\mathrm{Spf}\ \mathbb{Z}_2}$ Spf $W(\bar{\mathbb{F}}_2) = $ Spf $W(\bar{\mathbb{F}}_2)[T, (T^2 - T)^{-1}]^{\widehat{}}$ is given by

$$(A.7) \qquad 2^m \cdot (\bar{b}T^2 - 2\bar{a}T - \bar{c}) = 0 \quad ,$$

cf. loc.cit., (3.5). We now distinguish 2 cases.

Case a: $2|\bar{b}$ and $2|\bar{c}$. Then $2 \not| \, \bar{a}$ and we may write (A.7) in the form

$$(A.8) \qquad 2^{m+1} \cdot (\bar{b}_0 T^2 - \bar{a}T - \bar{c}_0) = 0 \quad ,$$

where $\bar{b} = 2\bar{b}_0$ and $\bar{c} = 2\bar{c}_0$.

Hence in this case the multiplicity $\mathrm{mult}_{[\Lambda]}(j)$ equals $m+1$. However, in this case $\bar{j} \in GL_2(\mathbb{Z}_2)$ and hence $[\Lambda]$ is fixed by j. We check now case by case when alternative a) can occur. Case 4 can be excluded right away since in this case no vertex is fixed by j. In case 3 let $[\Lambda] = [\Lambda_0]$ with the notation introduced in this case, i.e. $\Lambda_0 = \mathcal{O} = \mathbb{Z}_2[\bar{j}]$. Choosing as basis $1, \bar{j}$ we see that j is given by the matrix

$$(A.9) \qquad j = 2^{\alpha/2} \cdot \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} = 2^m \, \bar{j} \quad ,$$

hence alternative a) does not occur for $[\Lambda_0]$. The case when $[\Lambda] = [\Lambda_1]$ with $\Lambda_1 = \pi\mathcal{O}$ is identical and hence alternative a) does not occur in case 3.

In case 2, the vertex $[\Lambda_1]$ with $\Lambda_1 = \mathbb{Z}_2[\bar{j}]$ is excluded for the same reason. Now let us consider the vertex $[\Lambda_0]$, with $\Lambda_0 = \mathcal{O}$. We may choose the basis $1, \frac{1+\bar{j}}{2}$ of Λ_0 and then j is given by the matrix

$$(A.10) \qquad j = 2^{\frac{\alpha}{2}} \cdot \begin{pmatrix} -1 & 2\lambda \\ 2 & 1 \end{pmatrix} \quad , \quad \text{where } \varepsilon - 1 = 4\lambda \quad .$$

Hence alternative a) applies here. Furthermore, in this case, the second factor in (A.8) is equal to

$$(A.11) \qquad \lambda T^2 + T - 1 \quad .$$

Since, in case 2, we have $\lambda \in \mathbb{Z}_2^\times$, the ring

$$\mathbb{Z}_2[T]/(\lambda T^2 + T - 1)$$

is the ring of integers in an unramified quadratic extension of $\mathbb{Q}_2$ and the zero's of the polynomial $T^2 + T - 1 \in \mathbb{F}_2[T]$ lie in $\mathbb{F}_4 \setminus \mathbb{F}_2$ and define 2 ordinary special points of $\mathbb{P}_{[\Lambda_0]}$.

In case 1 the analysis is similar to case 2. First one checks that if $[\Lambda] \notin \mathcal{B}^{\mathcal{O}^\times}$, then alternative a) does not occur. If $[\Lambda] \in \mathcal{B}^{\mathcal{O}^\times}$, then after replacing $[\Lambda]$ by $[g\Lambda]$ for some $g \in \boldsymbol{k}^\times$ we may assume that either $[\Lambda] = [\Lambda_0]$ with $\Lambda_0 = \mathcal{O} = \langle 1, \frac{1+j}{2} \rangle$ or $[\Lambda] = [\Lambda_0']$ with $\Lambda_0' = \langle 2, \frac{1+\bar{j}}{2} \rangle$. In the first case the matrix of j is given by (A.10) and hence we are in alternative a). The second factor in (A.8) is equal to

$$\lambda T^2 + T - 1 \equiv T - 1 \bmod 2 \quad ,$$

since $2|\lambda$ in case 1. It follows that $Z(j)^{\mathrm{horiz}} \cap (\hat{\Omega}_{[\Lambda_0]} \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)) = \emptyset$. In the second case the matrix of j is given by

$$(A.12) \qquad j = 2^{\frac{\alpha}{2}} \cdot \begin{pmatrix} -1 & \lambda \\ 4 & 1 \end{pmatrix} \quad .$$

Again we are in alternative a), since $2|\lambda$. The second factor in (A.8) is equal to

$$\lambda_0 T^2 + T - 2 \equiv \lambda_0 T^2 + T \bmod 2 \quad ,$$

where we have set $\lambda = 2\lambda_0$. Since $T^2 - T$ is invertible on $\hat{\Omega}_{[\Lambda_0']}$, we again have $Z(j)^{\mathrm{horiz}} \cap (\hat{\Omega}_{[\Lambda_0']} \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)) = \emptyset$. This concludes our analysis of the alternative a).

Case b: $2 \nmid \bar{b}$ or $2 \nmid \bar{c}$. In this case the second factor of (A.7) is not divisible by 2, and hence $\mathrm{mult}_{[\Lambda]}(j) = m$. At this point we have shown that for any $[\Lambda] \in \mathcal{B}$ the multiplicity $\mathrm{mult}_{[\Lambda]}(j)$ is given by the formula in (i). Indeed, this follows by listing case by case the fixed point sets of j, and the expressions for k and for m, cf. (A.5), and comparing them with the multiplicities calculated above.

Now let us analyze the second factor in (A.7) in the alternative b). Its image in $\mathbb{F}_2[T]$ is

$$\bar{b} T^2 - \bar{c} \quad ,$$

hence is equal to either $T^2 - 1$, 1, or T^2. In all cases $Z(j)^{\mathrm{horiz}} \cap (\hat{\Omega}_{[\Lambda]} \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)) = \emptyset$.

Now we determine $Z(j)^{\mathrm{horiz}} \cap (\hat{\Omega}_\Delta \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2))$ for an edge $\Delta = \{[\Lambda], [\Lambda']\}$, where the intersection is non-empty. As in the proof of Prop. 3.3 in [**29**] we see that this intersection is non-empty only when $d([\Lambda], \mathcal{B}^j) = d([\Lambda'], \mathcal{B}^j)$. In cases 3 and 4, we therefore must have

$$(A.13) \qquad [\Lambda] = [\Lambda_0] \quad , \quad [\Lambda'] = [\Lambda_1] \quad .$$

In case 3, we take as basis in standard form for Λ_0, Λ_1, and, noting that $1+\bar{j}$ is a uniformizer of $\mathcal{O}$,

$$(A.14) \qquad \Lambda_0 = \langle 1, 1+\bar{j} \rangle \quad , \quad \Lambda_1 = \langle 2, 1+\bar{j} \rangle \quad .$$

In terms of the basis $1, 1+\bar{j}$ of Λ_0 the matrix of j is

$$(A.15) \qquad j = p^{\frac{\alpha}{2}} \cdot \begin{pmatrix} -1 & -2\cdot(1-2\lambda) \\ 1 & 1 \end{pmatrix} \quad , \text{ where } \varepsilon = 4\lambda - 1 \quad .$$

By loc. cit. the equations for $Z(j) \cap (\hat{\Omega}_\Delta \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2))$ in

$$\hat{\Omega}_\Delta \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2) = \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)[T_0, T_1, (1-T_0)^{-1}, (1-T_1)^{-1}]\widehat{\ }/(T_0T_1 - 2)$$

are given by

$$\begin{aligned} p^{\alpha/2} \cdot T_0\big(-(1-2\lambda)T_0 + 2 - T_1\big) &= 0 \\ p^{\alpha/2} \cdot T_1\big(-(1-2\lambda)T_0 + 2 - T_1\big) &= 0 \ . \end{aligned}$$

Hence, $Z(j)^{\mathrm{horiz}}$ is defined by the second factor in these equations. Now putting $\mu = (-(1-2\lambda))^{-1} \in \mathbb{Z}_2^\times$, we obtain

$$(A.16) \qquad Z(j)^{\mathrm{horiz}} = \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)[T_0]/(T_0^2 + 2\mu T_0 - 2\mu) \quad .$$

Since $T_0^2+2\mu T_0-2\mu$ is an Eisenstein polynomial, we see that $Z(j)^{\mathrm{horiz}}$ is the formal spectrum of the ring of integers in a ramified quadratic extension of $W(\bar{\mathbb{F}}_2)$ and it meets the special fiber of $\hat{\Omega}_{\{[\Lambda_0],[\Lambda_1]\}} \times_{\mathrm{Spf}\ \mathbb{Z}_2} \mathrm{Spf}\ W(\bar{\mathbb{F}}_2)$ in pt_Δ, which finishes the proof in this case.

The case 4 is similar to the case of loc. cit., p. 180.[8] In this case we have

$$(A.17) \qquad j(\Lambda_0) = 2^{\frac{\alpha-1}{2}} \cdot \Lambda_1 \quad , \quad j(\Lambda_1) = 2^{\frac{\alpha+1}{2}} \cdot \Lambda_0 \quad .$$

Hence, as in loc. cit., we can write j in terms of standard coordinates for Λ_0, Λ_1

$$j = 2^{\frac{\alpha-1}{2}} \cdot \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} \quad \text{with} \quad \bar{b} = 2\cdot \bar{b}_0$$

and where $2 \mid \bar{a}$ and $\bar{b}_0$ and $\bar{c}$ are units. Hence $Z(j)^{\mathrm{horiz}}$ is isomorphic to

$$\mathrm{Spf}\ W(\bar{\mathbb{F}}_2)[T_0, T_1, (1-T_0)^{-1}, (1-T_1)^{-1}]\widehat{\ }/(T_0T_1 - 2, \bar{b}_0T_0 - 2\bar{a} - \bar{c}T_1) \quad .$$

[8]We note that at this point in loc. cit. there is a slight error. The equations (3.23) of loc. cit. do not define the same closed subscheme as equations (3.22). The correct expression for $Z(j)^h$, replacing (3.24) is

$$\begin{aligned} Z(j)^h &= \mathrm{Spf}\ W[T_0, T_1]\widehat{\ }/(\bar{b}_0T_0 - 2\bar{a} - \bar{c}T_1, T_0T_1 - p) \\ &= \mathrm{Spf}\ W[T_0]/(T_0^2 + \alpha T_0 + \beta) \ , \end{aligned}$$

where $\alpha \in p\mathbb{Z}_p$ and $\beta \in p\mathbb{Z}_p^\times$. The conclusions drawn from these corrected equations (pp. 181, 182/183 loc. cit.) are unchanged.

Putting $\mu = \left(\frac{b_0}{\bar{c}}\right)^{-1} \in \mathbb{Z}_2^\times$ and $\nu = \frac{\bar{a}}{\bar{c}}$, we see that

$$Z(j)^{\text{horiz}} = \text{Spf } W(\bar{\mathbb{F}}_2)[T_0]/(T_0^2 - 2\mu\nu T_0 - 2\mu) \tag{$A.18$}$$

which yields the assertion as in case 3.

In case 2, we must have $\Delta = \{[\Lambda_0], [\Lambda_1]\}$, where $\Lambda_0 = \mathcal{O}$ and $\Lambda_1 = \mathbb{Z}_2[\bar{j}]$. In this case we take as standard bases

$$\Lambda_0 = \left\langle \frac{1+j}{2}, 1 \right\rangle \quad , \quad \Lambda_1 = \left\langle 2 \cdot \frac{1+j}{2}, 1 \right\rangle \quad .$$

But then, by (A.10), the equation for $Z(j)^{\text{horiz}}$ is given by

$$T_0 - 1 - \lambda T_1 = 0 \quad , \quad \text{where } \varepsilon - 1 = 4\lambda \quad .$$

It follows that $\text{pt}_\Delta \notin Z(j)^{\text{horiz}}$, which finishes the proof in this case.

Finally there is case 1. In this case either $\Delta \subset \mathcal{A}$ or, after replacing Δ by $g\Delta$ where $g \in \boldsymbol{k}^\times$, we may assume that $\Delta = \{[\Lambda_0], [\Lambda_1]\}$ where Λ_0 and Λ_1 are as in case 3. The second alternative is treated as the case 3 above. If $\Delta \subset \mathcal{A}$, we may assume that

$$\Lambda = \left\langle 1, \frac{1+\bar{j}}{2} \right\rangle \quad , \quad \Lambda' = \left\langle 2, \frac{1+\bar{j}}{2} \right\rangle \quad .$$

In this case j is given by the matrix (A.10) which yields the following equations for $Z(j)$,

$$\begin{aligned} p^{\alpha/2} \cdot T_0\big(\lambda T_0 + 2 - 2T_1\big) &= 0 \ , \\ p^{\alpha/2} \cdot T_1\big(\lambda T_0 + 2 - 2T_1\big) &= 0 \ . \end{aligned}$$

Since we are in case 1, we may write $\lambda = 2\lambda_0$ in the defining equation $\varepsilon - 1 = 4\lambda$. It follows that, after pulling a 2 out of the last factor in both equations, $Z(j)^{\text{horiz}}$ is defined by the equation

$$\lambda_0 T_0 + 1 - T_1 = 0 \quad ,$$

and again $\text{pt}_\Delta \notin Z(j)^{\text{horiz}}$. $\square$

§12. Archimedean contributions.

In this section, we compute the additional contribution

$$\kappa(m, v) = \frac{1}{2} \int_{[\Gamma \backslash D]} \Xi(m, v)\, c_1(\hat{\omega}) \tag{12.1}$$

to the height pairing coming from the fact that we are using nonstandard Green functions defined in [**25**] for the cycles $\mathcal{Z}(m)$. Recall that, by (5.8), for $z \in D$, we have

$$\Xi(m,v)(z) = \sum_{x \in L(m)} \xi(v^{\frac{1}{2}}x, z), \tag{12.2}$$

where

$$\xi(x,z) = -\mathrm{Ei}(-2\pi R(x,z)).$$

Note that the quantity

$$R(x,z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x)),$$

and hence $\xi(x,z)$, is independent of the orientation of the plane z. Also recall that $c_1(\hat{\omega}) = \mu$.

Proposition 12.1. *(i) If $m > 0$, then*

$$\kappa(m,v) = 2\,\delta(d,D)\, H_0(m;D) \cdot \frac{1}{2}\, J(4\pi m v),$$

where

$$J(t) = \int_0^\infty e^{-tw}\,\big[\,(w+1)^{\frac{1}{2}} - 1\big]\, w^{-1}\, dw,$$

is as in Theorem 8.8, and $H_0(m;D)$ is given by (8.19).
(ii) If $m < 0$, then

$$\kappa(m,v) = 2\,\delta(d;D)\, H_0(m;D)\, \frac{1}{4\pi}\, |m|^{-\frac{1}{2}}\, v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi|m|vw}\, w^{-\frac{3}{2}}\, dw,$$

where $H_0(m;D)$ is given by (8.32).

Proof. We have

$$\begin{aligned}
\kappa(m,v) &= \frac{1}{4}\int_{\Gamma\backslash D} \Xi(m,v)\cdot \mu \\
&= \frac{1}{4}\int_{\Gamma\backslash D} \sum_{x\in L(m)} \xi(v^{\frac{1}{2}}x, z)\, d\mu(z) \\
&= \frac{1}{4} \sum_{\substack{x \in L(m) \\ \mathrm{mod}\ \Gamma}} \int_{\Gamma_x\backslash D} \xi(v^{\frac{1}{2}}x, z)\, d\mu(z).
\end{aligned}$$

First suppose that $m = Q(x) > 0$, so that Γ_x is finite. Then

$$\int_{\Gamma_x\backslash D} \xi(x,z)\, d\mu(z) = 2\,|\Gamma_x|^{-1}\cdot \int_D \xi(x,z)\, d\mu(z) = 4\,|\Gamma_x|^{-1}\cdot \int_{D^+} \xi(x,z)\, d\mu(z). \tag{12.3}$$

Here the factor of 2 occurs since Γ_x contains ± 1, but these elements act trivially on D, while in the second step, we use the fact that $\xi(x,z)$ does not depend on the orientation of z. Since

$$\xi(gx, gz) = \xi(x,z), \tag{12.4}$$

for $g \in GL_2(\mathbb{R})$, we may assume that

$$x = m^{\frac{1}{2}} \cdot x_0 = m^{\frac{1}{2}} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{12.5}$$

Then, writing $z = k_\theta(e^t i) \in \mathfrak{H} \simeq D^+$, [**25**], p.601 , we have

$$R(x,z) = 2m \sinh^2(t). \tag{12.6}$$

Now, noting that t runs from 0 to ∞ and θ runs from 0 to π,

$$\begin{aligned}
I &:= \int_{D^+} -\mathrm{Ei}(-2\pi R(v^{\frac{1}{2}}x, z))\, d\mu(z) \\
&= \frac{1}{2\pi} \int_0^\pi \int_0^\infty -\mathrm{Ei}(-4\pi m v \sinh^2(t))\ 2\sinh(t)\, dt\, d\theta \\
&= \frac{1}{2} \int_0^\infty \left(\int_1^\infty e^{-4\pi m v \sinh^2(t) r} r^{-1}\, dr \right) 2\sinh(t)\, dt \\
&= \frac{1}{2} \int_0^\infty \left(\int_1^\infty e^{-4\pi m v w r} r^{-1}\, dr \right) (w+1)^{-\frac{1}{2}}\, dw
\end{aligned}$$

But

$$\begin{aligned}
&\int_0^\infty e^{-4\pi m v w r}\, (w+1)^{-\frac{1}{2}}\, dw \\
&= e^{-4\pi m v w r}\, 2(w+1)^{\frac{1}{2}} \Big|_0^\infty + 4\pi m v r \int_0^\infty e^{-4\pi m v w r}\, 2(w+1)^{\frac{1}{2}}\, dw \\
&= -2 + 4\pi m v r \int_0^\infty e^{-4\pi m v w r}\, 2(w+1)^{\frac{1}{2}}\, dw \\
&= 8\pi m v r \int_0^\infty e^{-4\pi m v w r}\, \big[\, (w+1)^{\frac{1}{2}} - 1 \big]\, dw
\end{aligned}$$

so that

$$\begin{aligned}
I &= \frac{1}{2} \int_1^\infty 8\pi m v \int_0^\infty e^{-4\pi m v w r}\, \big[\, (w+1)^{\frac{1}{2}} - 1 \big]\, dw\, dr \\
&= \int_0^\infty e^{-4\pi m v w}\, \big[\, (w+1)^{\frac{1}{2}} - 1 \big]\, w^{-1}\, dw \\
&= J(4\pi m v).
\end{aligned}$$

By Lemma 9.2, we have

$$\sum_{\substack{x\in L(m)\\ \bmod \Gamma}} |\Gamma_x|^{-1} = \delta(d;D)\cdot H_0(m;D). \tag{12.7}$$

Collecting terms, we obtain (i).

Next suppose that $m<0$. Let $\Gamma^+ = \Gamma\cap \mathrm{GL}_2(\mathbb{R})^+$, and let $\delta_x = |\Gamma_x:\Gamma_x^+|$, where $\Gamma_x^+ = \Gamma_x\cap\Gamma^+$. Then

$$\int_{\Gamma_x\backslash D} \xi(x,z)\,d\mu(z) = \delta_x^{-1}\int_{\Gamma_x^+\backslash D} \xi(x,z)\,d\mu(z) = 2\delta_x^{-1}\int_{\Gamma_x^+\backslash D^+} \xi(x,z)\,d\mu(z). \tag{12.8}$$

By conjugating by a suitable $g\in GL_2(\mathbb{R})$, we can take

$$g\cdot x = |m|^{\frac12}\cdot x_0 = |m|^{\frac12}\cdot\begin{pmatrix}1&\\&-1\end{pmatrix}. \tag{12.9}$$

Let Γ' be the corresponding conjugate of Γ^+ in $SL_2(\mathbb{R})$, and note that Γ'_{gx} will then be generated by $\pm 1_2$ and a unique element

$$\begin{pmatrix}\epsilon(x)&\\&\epsilon(x)^{-1}\end{pmatrix} \tag{12.10}$$

for $\epsilon(x)>1$ the fundamental unit *of norm 1* in the order $i_x^{-1}(O_B)$. If we write $z=re^{i\theta}\in\mathfrak{H}\simeq D^+$, then Γ'_{gx} acts by multiplication by powers of $\epsilon(x)^2$. Note that

$$R(gx,z) = \frac{2|m|}{\sin^2(\theta)}. \tag{12.11}$$

Then

$$\begin{aligned}
\int_{\Gamma_x^+\backslash D^+} \xi(v^{\frac12}x,z)\,d\mu(z) &= \frac{1}{2\pi}\int_1^{\epsilon(x)^2}\int_0^{\pi} -\mathrm{Ei}\left(-\frac{4\pi|m|v}{\sin(\theta)^2}\right)\ r^{-1}(\sin(\theta))^{-2}\,d\theta\,dr\\
&= \frac{2}{\pi}\log|\epsilon(x)|\cdot\int_0^{\pi/2} -\mathrm{Ei}\left(-\frac{4\pi|m|v}{\sin(\theta)^2}\right)\ (\sin(\theta))^{-2}\,d\theta\\
&= \frac{1}{\pi}\log|\epsilon(x)|\cdot\int_1^\infty\left(\int_1^\infty e^{-4\pi|m|vtw}\,w^{-1}\,dw\right)(t-1)^{-\frac12}\,dt\\
&= \frac{1}{\pi}\log|\epsilon(x)|\cdot\int_1^\infty e^{-4\pi|m|vw}\left(\int_0^\infty e^{-4\pi|m|vtw}\,t^{-\frac12}\,dt\right)w^{-1}\,dw\\
&= \frac{1}{\pi}\log|\epsilon(x)|\cdot\Gamma(\frac12)\,(4\pi|m|v)^{-\frac12}\int_1^\infty e^{-4\pi|m|vw}\,w^{-\frac32}\,dw\\
&= \frac{1}{2\pi}\log|\epsilon(x)|\cdot(|m|v)^{-\frac12}\int_1^\infty e^{-4\pi|m|vw}\,w^{-\frac32}\,dw.
\end{aligned}$$

The analogue of Lemma 9.2 for $m<0$ is the following.

Lemma 12.3. *If $m < 0$, then*

$$\Big(\sum_{\substack{x\in L(m)\\ \bmod \Gamma}} 2\delta_x^{-1} \log|\epsilon(x)|\Big) = 4\,\delta(d;D)\,H_0(m;D),$$

where $H_0(m;D)$ is as in (8.32).

Proof. We proceed as in the proof of Lemma 9.2. Note that for x of type c, $\Gamma_x \simeq O_{c^2d}^\times$ and $\Gamma_x^+ \simeq O_{c^2d}^1$, the subgroup of norm 1 elements, so that $\delta_x = \delta_c = |O_{c^2d}^\times : O_{c^2d}^1|$. Let $\epsilon(c^2d)$ be a fundamental unit in O_{c^2d} with $\epsilon(c^2d) > 1$. Note that $\epsilon^+(c^2d) = \epsilon(c^2d)^{\delta_c}$ is a generator of $O_{c^2d}^1/\pm 1$. Then we have

$$\begin{aligned}
&\Big(\sum_{\substack{x\in L(m)\\ \bmod \Gamma}} 2\delta_x^{-1} \log|\epsilon(x)|\Big)\\
&= \sum_{\substack{c\mid n\\ (c,D)=1}} 2\delta_c^{-1}|\mathrm{Opt}(O_{c^2d}, O_B)| \cdot \log|\epsilon^+(c^2d)|\\
&= 2\sum_{\substack{c\mid n\\ (c,D)=1}} |\mathrm{Opt}(O_{c^2d}, O_B)| \cdot \log|\epsilon(c^2d)|\\
&= 2\,\delta(d;D) \sum_{\substack{c\mid n\\ (c,D)=1}} h(d)\cdot \frac{\log|\epsilon(d)|}{\log|\epsilon(c^2d)|}\cdot \Big(c\prod_{\ell\mid c}(1-\chi_d(\ell)\ell^{-1})\Big)\log|\epsilon(c^2d)|\\
&= 4\,\delta(d;D)\,\frac{h(d)\,\log|\epsilon(d)|}{w(d)} \sum_{\substack{c\mid n\\ (c,D)=1}} c\prod_{\ell\mid c}(1-\chi_d(\ell)\ell^{-1})\\
&= 4\,\delta(d;D)\,H_0(m;D).
\end{aligned}$$

□

Proof of Theorem 7.2 (concluded). We observe that the expression in Corollary 12.2 when $m > 0$ coincides with the term (8.44) in the Fourier coefficient $\mathcal{E}'_m(\tau, \frac{1}{2}; D) = A'_m(\frac{1}{2}, v)\, q^m$. On the other hand, when $m < 0$, the expression in Corollary 12.2 coincides with that in (iii) of Theorem 8.8. □

§13. Remarks about the constant term.

In this section we explain the motivation for our definition of $\hat{\mathcal{Z}}(0,v)$. The key point is the comparison of our expression for the constant term of $\mathcal{E}'(\tau,\frac{1}{2};D)$ with the result of Bost [**4**] and Kühn [**34**] concerning $\langle\,\hat{\omega},\hat{\omega}\,\rangle$ in the case of the modular curve.

From Theorem 8.8, we have

$$\mathcal{E}_0'(\tau,\frac{1}{2};D) = c(D)\,\Lambda_D(2)\left[\frac{1}{2}\,\log(v) - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + 2C + \sum_{p|D}\frac{p\log(p)}{p-1}\right], \tag{13.1}$$

where C is as in Definition 3.4.

Now suppose, for a moment, that $D=1$. Then, recalling (6.28), we have

$$c(D)\zeta_D(2) = -\frac{1}{12} = \zeta(-1) = -\mathrm{vol}(\mathcal{M}(\mathbb{C})) = -\,\deg(\hat{\omega})$$

and so we obtain:

$$\mathcal{E}_0'(\tau,\frac{1}{2};D)\big|_{D=1} = -2\big[\,\zeta'(-1)+\frac{1}{2}\,\zeta(-1)\,\big] - \deg(\hat{\omega})\left[\frac{1}{2}\,\log(v)+2C\right]. \tag{13.2}$$

Let $\hat{\omega}_o$ denote the Hodge bundle with the metric defined by (3.11) or (10.15), so that, as elements of $\widehat{CH}^1(\mathcal{M})$,

$$\hat{\omega} = \hat{\omega}_o + (0,2\,C). \tag{13.3}$$

Then, $\hat{\omega}_o$ is the bundle of modular forms of weight 2 with its Petersson metric and hence, by Kühn and Bost,

$$\langle\,\hat{\omega}_o,\hat{\omega}_o\,\rangle^{\natural} = 4\,\big[\,\zeta'(-1)+\frac{1}{2}\,\zeta(-1)\,\big]. \tag{13.4}$$

Here $\langle\ ,\ \rangle^{\natural}$ denotes the height pairing without the 'stack' aspect! Thus, for the renormalized metric, we have

$$\begin{aligned}\langle\,\hat{\omega},\hat{\omega}\,\rangle^{\natural} &= \langle\,\hat{\omega}_o,\hat{\omega}_o\,\rangle^{\natural} + 2\,C\,\deg^{\natural}(\omega)\\ &= 4\,\big[\,\zeta'(-1)+\frac{1}{2}\,\zeta(-1)\,\big] + \deg^{\natural}(\omega)\,2C\end{aligned} \tag{13.5}$$

Note that

$$\langle\ ,\ \rangle = \frac{1}{2}\,\langle\ ,\ \rangle^{\natural}. \tag{13.6}$$

Then, recalling that

$$\hat{\mathcal{Z}}(0,v) = -\Big(\,\hat{\omega} + (0,\log(v))\,\Big),$$

we have

$$\begin{aligned}
&\langle\, \hat{\mathcal{Z}}(0,v), \hat{\omega}\,\rangle \\
&= -\langle\, \hat{\omega},\hat{\omega}\,\rangle - \frac{1}{2}\deg(\omega)\,\log(v) \qquad (13.7)\\
&= -2\big[\,\zeta'(-1) + \frac{1}{2}\,\zeta(-1)\big] - \deg(\omega)\left[\frac{1}{2}\log(v) + 2C\right]
\end{aligned}$$

This agrees perfectly with our constant term $\mathcal{E}_0'(\tau,\frac{1}{2};D)|_{D=1}$.

Finally, for general $D = D(B) > 1$, this discussion suggests that

$$\begin{aligned}
\langle\, \hat{\omega}_o,\hat{\omega}_o\,\rangle &= -\langle\, \hat{\mathcal{Z}}(0,v),\hat{\omega}\,\rangle - \frac{1}{2}\deg(\omega)\,\log(v) - \deg(\omega)\,2C \\
&\overset{??}{=} -\mathcal{E}_0'(\tau,\frac{1}{2};D) - \frac{1}{2}\deg(\omega)\,\log(v) - \deg(\omega)\,2C \qquad (13.8)\\
&= -c(D)\,\Lambda_D(2)\left[\,-2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + \sum_{p|D}\frac{p\log(p)}{p-1}\,\right] \\
&= \zeta_D(-1)\left[2\frac{\zeta'(-1)}{\zeta(-1)} + 1 - \sum_{p|D}\frac{p\log(p)}{p-1}\,\right],
\end{aligned}$$

since $c(D)\,\Lambda_D(2) = \zeta_D(-1) = \zeta(-1)\,\prod_{p|D}(p-1)$.

Part IV. Computations: analytic.

§14. Local Whittaker functions: the non-archimedean case.

The main purpose of this section is to prove Proposition 8.1. We fix a prime p and frequently drop the subscript p to lighten the notation. Recall that $B = B_p$ is a quaternion algebra which is a matrix algebra or a division algebra depending on whether $p \nmid D$ or $p \mid D$. Here D is a fixed square-free positive integer. Let O_B be a maximal order of B and let

$$V = \{\; x \in B \mid \operatorname{tr}^0 x = 0\}$$

with the quadratic form $Q(x) = \kappa\, x^2$, where $\kappa = \pm 1$. Actually, only the case $\kappa = -1$ is needed in section 8, but we treat the slightly more general case for future reference. Let

$L = V \cap O_B$, and let $S \in \mathrm{Sym}_2(\mathbb{Q}_p)$ be the matrix associated to L in the following sense. With respect to a basis of L over $\mathbb{Z}_p$, identify L with $\mathbb{Z}_p^3$. Then, for any $x \in L = \mathbb{Z}_p^3$,

$$Q(x) = \frac{1}{2}(x, x) = {}^t x S x. \tag{14.1}$$

Let $S_r = S \perp \frac{1}{2}\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, and let $L_r = \mathbb{Z}_p^{2r+3}$ be the associated quadratic lattice, viewed as the direct sum of L and r hyperbolic planes. Let $dx = \prod dx_i$ be the standard Haar measure on L_r, where

$$\int_{\mathbb{Z}_p} dx_i = 1.$$

Let

$$W(m, S_r) = \int_{\mathbb{Q}_p} \int_{L_r} \psi(b^t x S_r x)\, \psi(-mb)\, dx\, db \tag{14.2}$$

be the integral defined in [**44**], (1.2). It is the same as the local quadratic density polynomial $\alpha_p(X, m, S)$ with $X = p^{-r}$ defined in [**44**], page 312. Here $\psi = \psi_p$ is the local component of our standard additive character of $\mathbb{A}/\mathbb{Q}$.

Lemma 14.1. *With the notation as above, for any $r \geq 0$ and any $m \in \mathbb{Q}_p$,*

$$W_{m,p}(1, r + \frac{1}{2}, \Phi_p) = \beta(V)\, |\det 2S\,|_p^{\frac{1}{2}}\, W(m, S_r).$$

Here

$$\beta(V) = \Big(\epsilon(V)\, \gamma(\frac{1}{2}\psi_p)^3\, \gamma(\det V, \frac{1}{2}\psi)\Big)^{-1}$$

is the local splitting index defined in [**23**]*, Theorem 3.1. Here $\epsilon(V)$ is the Hasse invariant of V and $\gamma(\psi)$ and $\gamma(a, \psi)$ are the local Weil indices as in* [**37**].

Proof. We remark that this proposition is true in general for any 3-dimensional quadratic space V over $\mathbb{Q}_p$. Let $\phi = \mathrm{char}(L)$ and $\phi_r = \mathrm{char}(L_r)$. Then [**25**], Appendix, asserts that

$$\omega(g)\phi_r(0) = \Phi_p(g, r + \frac{1}{2})$$

where $\omega = \omega_\psi$ is the Weil representation of G'_p (the metaplectic cover of $SL_2(\mathbb{Q}_p)$) on the space $S(V)$ of Schwartz functions on V. Thus

$$\begin{aligned} & W_{m,p}(1, r + \frac{1}{2}, \Phi_p) \\ & = \int_{\mathbb{Q}_p} \omega(wn(b))\phi_r(0)\, \psi(-bm)\, db \\ & = \beta(V) \int_{\mathbb{Q}_p} \int_V \phi_r(x)\, d_r x\, \psi(-bm)\, db, \end{aligned} \tag{14.3}$$

where $d_r x$ is the self-dual Haar measure with respect to the bi-character $(x, y) \mapsto \psi_p((x, y)_r)$ with $(x, y)_r$ the bilinear from on $V_r = L_r \otimes \mathbb{Q}_p$ associated to S_r. It is easy to check

$$d_r x = |\det 2S_r|_p^{\frac{1}{2}}\, dx = |\det 2S\,|_p^{\frac{1}{2}}\, dx \tag{14.4}$$

under the identification $L_r = \mathbb{Z}_p^{2r+3}$ as above. This proves the proposition. □

The following lemma is standard.

Lemma 14.2. *Let V and L be as above.*
(i) When $p \nmid D$, one has $L \cong \mathbb{Z}_p^3$ with the quadratic form and symmetric matrix as follows:

$$Q(x) = \kappa\,(x_1^2 + x_2 x_3), \qquad S = \kappa \begin{pmatrix} 1 & & \\ & & \frac{1}{2} \\ & \frac{1}{2} & \end{pmatrix}.$$

(ii) When $2 \neq p \mid D$, one has $L \cong \mathbb{Z}_p^3$ with the quadratic form and symmetric matrix as follows:

$$Q(x) = \kappa\,(\beta x_1^2 + p x_2^2 - \beta p x_3^2), \qquad S = \kappa \operatorname{diag}(\beta, p, -\beta p).$$

Here $\beta \in \mathbb{Z}_p^\times$ with $(\beta, p)_p = -1$.
(iii) When $p = 2 \mid D(B)$, one has $L \cong \mathbb{Z}_p^3$ with the quadratic form and symmetric matrix as follows:

$$q = \kappa\,(-3x_1^2 + 2x_2^2 + 2x_2 x_3 + 2x_3^2), \qquad S = \kappa \begin{pmatrix} -3 & & \\ & 2 & 1 \\ & 1 & 2 \end{pmatrix}.$$

(iv) Finally,

$$|\det 2S\,|_p = |2|_p \cdot \begin{cases} 1 & \text{if } p \nmid D, \\ p^{-2} & \text{if } p \mid D. \end{cases}$$

Notice that $\beta(V)$ in Lemma 14.1 depends only on V and is well-defined even when $p = \infty$.

Lemma 14.3. *Let V be as above. Let $\zeta_8 = e^{\frac{2\pi i}{8}}$. Then*

$$\beta(V) = \epsilon(B) \cdot \begin{cases} 1 & \text{if } p \nmid 2\infty, \\ \zeta_8^{-\kappa} & \text{if } p = \infty, \\ \zeta_8^{\kappa} & \text{if } p = 2. \end{cases}$$

Here $\epsilon(B) = \pm 1$ depending on whether $B = B_p$ is split or not.

Proof. It is easy to see from Lemma 14.2 that $\det V \in -\kappa\, \mathbb{Q}_p^{\times,2}$ in all cases, and so, by [**37**],

$$\gamma(\det V, \frac{1}{2}\psi) = \gamma(-\kappa, \frac{1}{2}\psi) = \gamma(\frac{1}{2}\psi)^{-2a(\kappa)}. \tag{14.5}$$

Here $a(\kappa) = (1+\kappa)/2$. By [**37**], A.10, A.11, one has

$$\gamma(\frac{1}{2}\psi_p) = \begin{cases} 1 & \text{if } p \nmid 2\infty, \\ \zeta_8 & \text{if } p = \infty. \end{cases} \tag{14.6}$$

When $p = 2$, following the principle in [**37**], page 370, one has

$$\gamma(\frac{1}{2}\psi) = \frac{1}{2} \sum_{x \in \mathbb{Z}/4} \psi(\frac{1}{8}x^2) = \zeta_8^{-1}. \tag{14.7}$$

As for the Hasse invariant $\epsilon(V)$, one has in the split case

$$\epsilon(V) = (\kappa, -1)_p = \gamma(\frac{1}{2}\psi_p)^{4(1-a(\kappa))}, \tag{14.8}$$

where $(\ ,\ )_p$ is the quadratic HIlbert symbol for $\mathbb{Q}_p$. In the ramified case, one has

$$\epsilon(V) = (\eta\kappa, -\kappa^2\eta p^2)_p(\kappa p, -\kappa\eta p)_p = (\kappa, -1)_p(p, \eta)_p = -\gamma(\frac{1}{2}\psi_p)^{4(1-a(\kappa))}. \tag{14.9}$$

Now the lemma follows from the formula

$$\beta(V) = \{\epsilon(V)\gamma(\frac{1}{2}\psi_p)^3\gamma(\det V, \frac{1}{2}\psi)\}^{-1}. \quad \Box \tag{14.10}$$

Proof of Proposition 8.1 By Lemmas 14.1 and 14.3, Proposition 8.1 is equivalent to the following proposition for $\kappa = -1$.

Proposition 14.4. *For a nonzero integer m, write $4m = n^2 d$ such that κd is a fundamental discriminant of a quadratic field.*
(i) If $p \nmid D$,

$$W_p(m, S_r) = \frac{L_p(r+1, \chi_{-\kappa d})\, b_p(n, r+1; D)}{\zeta_p(2r+2)}.$$

(ii) If $p \mid D$,

$$W_p(m, S_r) = L_p(r+1, \chi_{-\kappa d})\, b_p(n, r+1; D).$$

(iii) If $m = 0$,

$$W_p(0, S_r) = \begin{cases} \frac{\zeta_p(2r+1)}{\zeta_p(2r+2)} & \textit{if } p \nmid D, \\ \frac{\zeta_p(2r+1)}{\zeta_p(2r)} & \textit{if } p \mid D, \end{cases}$$

Proof. Part (i) is a better reformulation of [**44**], Propositions 8.3. Part (ii) follows from (i) and [**44**], Proposition 8.2. Part (iii) follows from (i) and (ii) when we let $a = \mathrm{ord}_p m$

tends to infinity. We verify the case $p = 2 \nmid D$ and leave the other (easier) cases to the reader. We recall again that $W_p(m, S_r)$ is just the local density polynomial $\alpha_p(X, m, S)$ in [**44**] with $X = p^{-r}$. Write $m = \alpha p^a$ with $a = \mathrm{ord}_p m = 2k + \mathrm{ord}_p \frac{d}{4}$ and $\alpha \in \mathbb{Z}_p^\times$ where $k = k_p(n) = \mathrm{ord}_p n$ as before. In the notation of [**44**], Proposition 8.3, one has $(\frac{\alpha\kappa}{p}) = \chi_{-\kappa d}(p)$ if $p \nmid d$. In our case $p = 2 \nmid D$, $a = \mathrm{ord}_2 m = 2k - 2 + \mathrm{ord}_2 d$ with $k = \mathrm{ord}_2 n$ as before.

Subcase 1. First we assume $8 \mid d$. Then $a = 2k + 1$ is odd, and [**44**], Proposition 8.3(1), implies

$$
\begin{aligned}
W_2(m, S_r) &= (1 - 2^{-2}X^2) \sum_{l=0}^{k} (2^{-1}X^2)^l \\
&= \frac{(1 - 2^{-2}X^2)(1 - (2^{-1}X^2)^{k+1})}{1 - 2^{-1}X^2} \\
&= \frac{L_2(r+1, \chi_{-\kappa d})\, b_2(n, r+1; D)}{\zeta_2(2s+2)}
\end{aligned}
\tag{14.11}
$$

as desired.

Subcase 2. Now assume that $\mathrm{ord}_2 d = 2$. Then $a = 2k$, $\alpha\kappa = \frac{\kappa d}{4}(n2^{-k})^2 \equiv -1 \mod 4$, and thus $(\frac{-1}{\alpha\kappa}) = -1$ and $\delta_8(\alpha - \kappa) = 0$ So [**44**], Proposition 8.3(3), ($\frac{a-1}{2}$ in the summation there should be $\frac{a}{2}$) implies

$$
\begin{aligned}
W_2(m, S_r) &= 1 + 2^{-1} \sum_{l=1}^{k} (2^{-1}X^2)^l - 2^{-k-2}X^{2k+2} \\
&= \frac{(1 - 2^{-2}X^2)(1 - (2^{-1}X^2)^{k+1})}{1 - 2^{-1}X^2} \\
&= \frac{L_2(r+1, \chi_{-\kappa d})}{\zeta_2(2r+2)}\, b_2(n, r+1; D).
\end{aligned}
\tag{14.12}
$$

Subcase 3. Finally if $2 \nmid d$, i.e., $\kappa d \equiv 1 \mod 4$. Then $a = 2k - 2$ and $\alpha = d(n2^{-k+1})^2 \equiv d \mod 8$. In this case, $(\frac{-1}{\alpha\kappa}) = 1$ and

$$
\delta_8(\alpha - \kappa) = \delta_8(d - \kappa) = \chi_{-\kappa d}(2).
\tag{14.13}
$$

Set $v_2 = \chi_{-\kappa d}(2)$. Then [**44**], Proposition 8.3(3), gives

(14.14)

$$
\begin{aligned}
W_2(m, S_r) &= 1 + 2^{-1} \sum_{l=1}^{k-1} (2^{-1}X^2)^l + 2^{-k-1}X^{2k} + v_2 2^{-k-1}X^{2k+1} \\
&= \frac{1 - 2^{-2}X^2 - 2^{-k-1}X^{2k+1}(-v_2 + 2^{-1}X + v_2 2^{-1}X^2)}{1 - 2^{-1}X^2}.
\end{aligned}
$$

On the other hand,

$$(14.15)\qquad \begin{aligned} &\frac{L_2(r+1,\chi_{-\kappa d})b_2(n,r+1;D)}{\zeta_2(2r+2)} \\ &= \frac{(1+v_2 2^{-1}X)(1-v_2X+v_2 2^{-k-2}X^{2k+1}-2^{-k-1}X^{2k+2})}{1-2^{-1}X^2} \\ &= \frac{1-2^{-2}X^2-2^{-k-1}X^{2k+1}(1+v_2 2^{-1}X)(-v_2+X)}{1-2^{-1}X^2} \\ &= \frac{1-2^{-2}X^2-2^{-k-1}X^{2k+1}(-v_2+2^{-1}X+v_2 2^{-1}X^2)}{1-2^{-1}X^2}. \end{aligned}$$

Therefore

$$(14.16)\qquad W_2(m,S_r) = \frac{L_2(r+1,\chi_{-\kappa d})}{\zeta_2(2r+2)}\, b_2(n,r+1;D). \quad \square$$

§15. Local Whittaker functions: the archimedean case.

In this section, we compute the local Whittaker function

$$(15.1)\qquad W_{m,\infty}(\tau,s,\Phi^\ell_\infty) = v^{-\frac{1}{2}\ell}\int_{\mathbb{R}} \Phi^\ell_\infty(wn(b)g'_\tau,s)\,\psi_\infty(-mb)\,db$$

and prove Lemmas 8.9, and 8.11. Here $\ell \in \frac{1}{2}\mathbb{Z}$ is such that $\ell \equiv \frac{3}{2} \mod 2\mathbb{Z}$. In this paper, we only need $\ell = \frac{3}{2}$.

Let

$$(15.2)\qquad \Psi(a,b;z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zr}(r+1)^{b-a-1}r^{a-1}dr$$

be the standard confluent hypergeometric function of the second kind, [**35**], where $a > 0$, $z > 0$ and b is any real number. It satisfies the functional equation, [**35**], p. 265

$$(15.3)\qquad \Psi(a,b;z) = z^{1-b}\Psi(1+a-b,2-b;z).$$

For convenience, we also define

$$(15.4)\qquad \Psi(0,b;z) = \lim_{a\to 0+}\Psi(a,b;z) = 1.$$

So $\Psi(a,b;z)$ is well-defined for $z > 0$, $a \geq \min\{0,b-1\}$. Finally, for any number n, we define

$$(15.5)\qquad \Psi_n(s,z) = \Psi(\frac{1}{2}(1+n+s),s+1;z).$$

Then (15.3) implies

$$(15.6)\qquad \Psi_n(s,z) = z^{-s}\Psi_n(-s,z).$$

Proposition 15.1. *Let $q = e(m\tau)$, $(-i)^\ell = e(-\ell/4)$, and*

$$\alpha = \frac{s+1+\ell}{2}, \qquad \beta = \frac{s+1-\ell}{2}.$$

(i) For $m > 0$,

$$W_{m,\infty}(\tau, s, \Phi^\ell_\infty) = 2\pi\, (-i)^\ell\, v^\beta\, (2\pi m)^s\, \frac{\Psi_{-\ell}(s, 4\pi m v)}{\Gamma(\alpha)} \cdot q^m.$$

(ii) For $m < 0$,

$$W_{m,\infty}(\tau, s, \Phi^\ell_\infty) = 2\pi\, (-i)^\ell\, v^\beta\, (2\pi |m|)^s\, \frac{\Psi_{\ell}(s, 4\pi |m| v)}{\Gamma(\beta)}\, e^{-4\pi|m|v} \cdot q^m.$$

(iii) For $m = 0$,

$$W_{0,\infty}(\tau, s, \Phi^\ell_\infty) = 2\pi\, (-i)^\ell\, v^{\frac{1}{2}(1-\ell-s)}\, \frac{2^{-s}\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta)}.$$

(iv) The special value at $s = \ell - 1$ is

$$W_{m,\infty}(\tau, \ell-1, \Phi^\ell_\infty) = \begin{cases} 0 & \text{if } m \le 0, \\ \frac{(-2\pi i)^\ell}{\Gamma(\ell)}\, m^{\ell-1}\, q^m & \text{if } m > 0. \end{cases}$$

Proof. A standard calculation, [**40**], (see also [**25**] pages 585-586, for this special case) gives

$$W_{m,\infty}(\tau, s, \Phi^\ell_\infty) = (-i)^\ell v^\beta e(m\bar\tau) \frac{(2\pi)^{s+1}}{\Gamma(\alpha)\Gamma(\beta)} \int_{\substack{r>0\\ r>m}} e^{-4\pi v r} (r-m)^{\beta-1} r^{\alpha-1}\, dr \tag{15.7}$$

When $m = 0$, this gives (iii) immediately.

When $m > 0$, the integral equals

$$\begin{aligned}\int_{r>m} e^{-4\pi v r}(r-m)^{\beta-1} r^{\alpha-1}\, dr &= m^s e^{-4\pi m v} \int_0^\infty e^{-4\pi m v r} r^{\beta-1}(r+1)^{\alpha-1} dr \\ &= m^s e^{-4\pi m v} \Gamma(\beta) \Psi_{-\ell}(s, 4\pi m v).\end{aligned}$$

This proves (i). The special value at $s = \ell - 1$ is

$$\begin{aligned}W_{m,\infty}(\tau, \ell-1, \Phi^\ell_\infty) &= 2\pi (-i)^\ell (2\pi m)^{\ell-1} q^m \frac{\Psi_{-\ell}(\ell-1, 4\pi m v)}{\Gamma(\ell)} \\ &= \frac{(-2\pi i m)^\ell}{m\Gamma(\ell)} q^m,\end{aligned}$$

as claimed in (iv).

When $m < 0$, the integral is

$$\int_{r>0} e^{-4\pi vr}(r-m)^{\beta-1}r^{\alpha-1}\,dr = |m|^s\int_0^\infty e^{-4\pi mvr}r^{\alpha-1}(r+1)^{\beta-1}dr$$
$$= |m|^s\Gamma(\alpha)\Psi_\ell(s,4\pi mv).$$

This proves (ii). The special value at $s=\ell-1$ is 0 since $\frac{1}{\Gamma(\beta)}=0$ at $s=\ell-1$ and $\Psi_\ell(\ell-1,4\pi|m|v)$ is finite. □

Proof of Lemma 8.9. Since $m>0$, (i) of Proposition 15.1 implies

$$\frac{W'_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty)}{W_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty)} = \frac{1}{2}\log v+\log(2\pi m)-\frac{1}{2}\frac{\Gamma'(\ell)}{\Gamma(\ell)}+\frac{\Psi'_{-\ell}(\ell-1,4\pi mv)}{\Psi_{-\ell}(\ell-1,4\pi mv)}.$$

Notice that, for any $z>0$,

$$\Psi_{-\ell}(\ell-1,z)=\Psi(0,\ell;z)=1,$$

by (15.4). Observe that

$$\Psi_{-\ell}(s,z)=z^{-\beta}+\frac{1}{\Gamma(\beta)}\int_0^\infty e^{-zr}((r+1)^{s-\beta}-1)r^{\beta-1}dr.$$

The integral here is well-defined at $s=\ell-1$ and is equal to

$$J(\ell-1,z):=\int_0^\infty e^{-zr}\,\frac{(r+1)^{\ell-1}-1}{r}\,dr. \tag{15.8}$$

Notice also that the function $\frac{1}{\Gamma(\beta)}$ vanishes at $s=\ell-1$ and has the first derivative $\frac{1}{2}$ at $s=\ell-1$. Thus,

$$\Psi'_{-\ell}(\ell-1,z)=-\frac{1}{2}\log z+\frac{1}{2}\,J(\ell-1,z), \tag{15.9}$$

and we have

$$\frac{W'_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty)}{W_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty)} = \frac{1}{2}\left[\log(\pi m)-\frac{\Gamma'(\ell)}{\Gamma(\ell)}+J(\ell-1,4\pi mv)\right]. \tag{15.10}$$

When $\ell=\frac{3}{2}$, $J(\frac{1}{2},4\pi mv)=J(4\pi mv)$ is the quantity defined in Theorem 8.8, so this gives Lemma 8.9. □

Proof of Lemma 8.11. (The case $m<0$, with derivative). We now assume $m<0$. Since the function $\frac{1}{\Gamma(\beta)}$ vanishes at $s=\ell-1$ and has the first derivative $\frac{1}{2}$ there, one has, by (ii) of Proposition 15.1,

$$W'_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty)=2\pi(-i)^\ell\,(2\pi|m|)^{\ell-1}\,\frac{1}{2}\,\Psi_\ell(\ell-1,4\pi|m|v)\cdot e^{-4\pi|m|v}\cdot q^m.$$

By (15.6), one has

$$\begin{aligned}\Psi_\ell(\ell-1,4\pi|m|v) &= (4\pi|m|v)^{1-\ell}\,\Psi_\ell(1-\ell,4\pi|m|v)\\ &= (4\pi|m|v)^{1-\ell}\,\Psi(1,2-\ell;4\pi|m|v)\\ &= (4\pi|m|v)^{1-\ell}\int_0^\infty e^{-4\pi|m|vr}(1+r)^{-\ell}\,dr\\ &= (4\pi|m|v)^{1-\ell}\,e^{4\pi|m|v}\int_1^\infty e^{-4\pi|m|vr}r^{-\ell}\,dr.\end{aligned}$$

Therefore,

$$W'_{m,\infty}(\tau,\ell-1,\Phi^\ell_\infty) = 2\pi\,(-i)^\ell\,2^{-\ell}\,v^{1-\ell}\,q^m\int_1^\infty e^{-4\pi|m|vr}r^{-\ell}\,dr. \tag{15.11}$$

When $\ell=\frac{3}{2}$, this gives Lemma 8.11. $\square$

§16. The functional equation.

Let D be a square-free positive integer, not necessarily the discriminant of an indefinite quaternion algebra, and let

$$\mathbb{E}(\tau,s,\Phi^{\frac{3}{2},D}) = c(D)\,(s+\frac{1}{2})\,\Lambda_D(2s+1)\,E(\tau,s,\Phi^{\frac{3}{2},D}) \tag{16.1}$$

be the renormalized Eisenstein series of (6.23). In this section, we prove that it is invariant when s goes to $-s$, i.e., that

$$\mathbb{E}(\tau,s,\Phi^{\frac{3}{2},D}) = \mathbb{E}(\tau,-s,\Phi^{\frac{3}{2},D}). \tag{16.2}$$

First we need

Proposition 16.1. *Set*

$$\Lambda(s,\chi_m;D) = \left(\frac{4|m|D^2}{\pi}\right)^{\frac{1}{2}s}\Gamma(\frac{s+a}{2})\,L(s,\chi_d)\prod_p b_p(n,s,D)$$

with $a=(1+\operatorname{sgn}(m))/2$. *Then* $\Lambda(s,\chi_m,D)$ *has a meromorphic continuation to the whole complex* s*-plane with possible poles at* $s=0$ *and* 1, *which occurs precisely when* $D=-d=1$. *Furthermore, it satisfies the following function equation*

$$\Lambda(s,\chi_m;D) = \Lambda(1-s,\chi_m;D),$$

and

$$ord_{s=0}\,\Lambda(s,\chi_m;D) = ord_{s=1}\,\Lambda(s,\chi_m;D) = ord_{s=1}\,L(s,\chi_d) + \#\{p|D : \chi_d(p) = 1\}.$$

Proof. The functional equation follows from that of $L(s,\chi_d)$ and (8.10). The vanishing order at $s = 0$ follows from (8.14). We remark that $b_p(n,s;D)$ is a polynomial of p^{-s} even though it was written as a rational function and thus is regular at $s = 1/2$. $\square$

Theorem 16.2. *Let*

$$\mathbb{E}(\tau,s,\Phi^{\frac{3}{2},D}) = \sum_m \mathcal{A}_m(v,s)\,q^m$$

be the Fourier expansion of $\mathbb{E}(\tau,s,\Phi^{\frac{3}{2},D})$.
(i) For $m > 0$, *one has*

$$\mathcal{A}_m(v,s) = \frac{\Lambda(\frac{1}{2}+s,\chi_m;D)\,(4\pi mv)^{\frac{s}{2}-\frac{1}{4}}\,\Psi_{-\frac{3}{2}}(s,4\pi mv)}{\sqrt{\pi}\,\prod_{p|D}(1+p)}.$$

(ii) For $m < 0$, *one has*

$$\mathcal{A}_m(v,s) = \frac{(s^2-\frac{1}{4})\Lambda(\frac{1}{2}+s,\chi_m;D)\,(4\pi|m|v)^{\frac{s}{2}-\frac{1}{4}}\,\Psi_{\frac{3}{2}}(s,4\pi|m|v)}{4\sqrt{\pi}\prod_{p|D}(1+p)}\cdot e^{-4\pi|m|v}.$$

(iii) The constant term is

$$\mathcal{A}_0(v,s) = -\frac{D}{2\pi\prod_{p|D}(p+1)}(G_D(s)+G_D(-s)),$$

where

$$G_D(s) = v^{-\frac{1}{4}+\frac{s}{2}}\,\Lambda(1+2s)\,(s+\frac{1}{2})\prod_{p|D}(p^{-\frac{1}{2}-s}-p^{\frac{1}{2}+s}).$$

Proof. When $m > 0$, one has, by Propositions 8.1 and 15.1 and formula (8.18),

$$\begin{aligned}\mathcal{A}_m(v,s) &= c(D)(\frac{D}{\pi})^{s-\frac{1}{2}}\Gamma(s+\frac{3}{2})\zeta_D(2s+1)\frac{C_\infty}{\sqrt{2}}v^{\frac{s}{2}-\frac{1}{4}}(2\pi m)^s\frac{\Psi_{-\frac{3}{2}}(s,4\pi mv)}{\Gamma(\frac{s}{2}+\frac{5}{4})}\\ &\qquad\times C_f(D)\frac{L(s+\frac{1}{2},\chi_d)\prod b_p(n,s+\frac{1}{2};D)}{\zeta_D(2s+1)}\\ &= \frac{\Lambda(s+\frac{1}{2},\chi_m;D)}{\prod_{p|D}(p+1)}\,\frac{\Gamma(s+\frac{3}{2})2^{-s-\frac{1}{2}}}{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{s}{2}+\frac{5}{4})}(4\pi mv)^{\frac{s}{2}-\frac{1}{4}}\Psi_{-\frac{3}{2}}(s,4\pi mv)\end{aligned}$$

Now the doubling formula of the gamma function gives

$$\frac{\Gamma(s+\frac{3}{2})}{\Gamma(\frac{s}{2}+\frac{3}{4})\Gamma(\frac{s}{2}+\frac{5}{4})} = 2^{s+\frac{3}{2}-1}\pi^{-\frac{1}{2}} = 2^{s+\frac{1}{2}}\pi^{-\frac{1}{2}}.$$

This proves (i). The case $m<0$ is the same and is left to the reader. When $m=0$, one has by Corollary 8.2 and (8.12)

$$\begin{aligned}\mathcal{A}_0(v,s) &= c(D)(\frac{D}{\pi})^{s-\frac{1}{2}}\Gamma(s+\frac{3}{2})\zeta_D(2s+1) \\ &\qquad \times \left[v^{-\frac{1}{4}+\frac{s}{2}} + \frac{C_\infty}{\sqrt{2}} \frac{2^{-s}v^{-\frac{1}{4}-\frac{s}{2}}\Gamma(s)}{\Gamma(\frac{s}{2}-\frac{1}{4})\Gamma(\frac{s}{2}+\frac{5}{4})} \frac{C_f(D)\zeta(2s)}{\zeta_D(2s+1)} \prod_{p|D}(1-p^{1-2s}) \right] \\ &= -\frac{v^{-\frac{1}{4}+\frac{s}{2}}\Lambda(1+2s)(\frac{1}{2}+s)}{2\pi\prod_{p|D}(p+1)}(-1)^{\mathrm{ord}(D)}D^{s+\frac{3}{2}}\prod_{p|D}(1-p^{-1-2s}) \\ &\qquad - \frac{v^{-\frac{1}{4}-\frac{s}{2}}\Lambda(2s)(\frac{1}{2}-s)}{\sqrt{\pi}\prod_{p|D}(p+1)} \frac{\Gamma(s-\frac{1}{2})2^{\frac{1}{2}-s}}{\Gamma(\frac{s}{2}-\frac{1}{4})\Gamma(\frac{s}{2}+\frac{1}{4})} D^{\frac{1}{2}+s}\prod_{p|D}(1-p^{1-2s}) \\ &= -\frac{D}{2\pi\prod_{p|D}(p+1)}(G_D(s)+G_D(-s)).\end{aligned}$$

Here we have used the doubling formula for the gamma functions again. □

Proof of the functional equation (16.2). Now the functional equation (16.2) follows immediately from Theorem 16.2, Proposition 16.1, and (15.6). □

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