ON THE NEWTON STRATIFICATION

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INTRODUCTION

This is a report on algebraic geometry in characteristic p. Let A/S be a family of abelian varieties over a base scheme of characteristic p. For any prime number $\ell \neq p$ the family of Tate modules $T_\ell(A_{\bar{s}})$ ($\bar{s}$ ranging over the geometric points of S) defines a local system of $\mathbb{Z}_\ell$-modules on S. The replacement for $\ell = p$ of the Tate module $T_\ell(A_{\bar{s}})$ is the *Dieudonné module $M(A_{\bar{s}})$* which is an *F-crystal.* However, in contrast to the ℓ-adic case, the Dieudonné module is not locally constant as $\bar{s}$ varies over the base. This leads to the *Newton stratification* of S into locally closed subsets where the isomorphism classes of the *rational Dieudonné modules $M(A_{\bar{s}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$* are constant. Recently de Jong and Oort proved some general qualitative facts on this stratification. These were applied by Oort to the universal family of abelian varieties over the Siegel moduli space. We formulate this result in imprecise terms as follows.

THEOREM 0.1. — *The Newton stratification of the moduli space of principally polarized abelian varieties of a fixed dimension g in characteristic p has the strong stratification property (the closure of a stratum is a union of strata). Furthermore, the jumps in this stratification all occur in codimension one.*

The main tools in the proof of this theorem are the purity theorem on families of F-isocrystals and deformation theory of p-divisible groups. The latter is based on the theory of displays for formal p-divisible groups, which has recently been completed by Zink.

The layout of the report is as follows. In section 1 we introduce the notion of an F-isocrystal over a base scheme S and of the corresponding Newton stratification. Section 2 is devoted to the general theorems on Newton stratifications, and section 3 to the particular case of the Siegel moduli space. In section 4 we give the main theorem of display theory. In the final section 5 we comment on other moduli spaces of abelian varieties.

The subject matter of this report has deep historical roots, with contributions by many mathematicians. I apologize in advance for any oversights and misrepresentations, which are not intentional but rather due to my ignorance.

1. F-CRYSTALS

DEFINITION 1.1. — *Let k be a perfect field of characteristic p, with ring of Witt vectors $W(k)$. Let L be the fraction field of $W(k)$ and denote by σ the Frobenius automorphisms on k, $W(k)$ and L.*

a) A (non-degenerate) F-crystal over Spec k is a free $W(k)$-module M of finite rank with a σ-linear endomorphism $F : M \to M$ such that $M/F(M)$ has finite length.

b) An F-isocrystal over Spec k is a finite-dimensional L-vector space N with a σ-linear bijective endomorphism $F : N \to N$.

Recall that $W(k)$ is the unique complete discrete valuation ring with residue field k and with p as uniformizer. An F-crystal (M, F) defines an F-isocrystal via $(N, F) := (M, F) \otimes_{W(k)} L$. Conversely, given an F-isocrystal (N, F), the corresponding set of F-crystals is the set of $W(k)$-lattices M in N such that $F(M) \subset M$ (such lattices need not exist).

The F-isocrystals over Spec k form a category in the obvious way which is abelian $\mathbb{Q}_p$-linear and noetherian and artinian. If k' is a perfect field extension of k, then an F-crystal over Spec k defines an F-crystal over Spec k' via base extension $\otimes_{W(k)} W(k')$.

THEOREM 1.2 (Dieudonné). — *Let k be algebraically closed. Then the category of F-isocrystals is semi-simple. The simple objects are parametrized by the set of rational numbers. To $\lambda \in \mathbb{Q}$ corresponds the simple object E_λ defined as follows. If $\lambda = \frac{r}{s}$, with $s, r \in \mathbb{Z}$, $s > 0$, $(r, s) = 1$, then*

$$E_\lambda = \left(L^s, F = \begin{bmatrix} 0 & & & p^r \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix} \cdot \sigma \right) \quad .$$

Furthermore

$$\operatorname{End}(E_\lambda) = D_\lambda \quad ,$$

where D_λ is the division algebra with center $\mathbb{Q}_p$ and invariant equal to the image of λ in $\mathbb{Q}/\mathbb{Z}$. $\qquad \square$

We may parametrize the F-isocrystals of rank n over the algebraically closed field k by their Newton polygons, or preferably but equivalently, by their *Newton vectors*.

COROLLARY 1.3. — *Let k be algebraically closed. Then there is an injection (the Newton map)*

$$\{\text{isomorphism classes of F-isocrystals of rank } n\} \to (\mathbb{Q}^n)_+, \ (N, F) \mapsto \nu(N, F).$$

Here $(\mathbb{Q}^n)_+ = \{(\nu_1, \dots, \nu_n) \in \mathbb{Q}^n, \ \nu_1 \geq \dots \geq \nu_n\}$. The Newton map sends (N, F) to $\nu(N, F) \in (\mathbb{Q}^n)_+$, where $\lambda \in \mathbb{Q}$ occurs in $\nu(N, F)$ with multiplicity equal to the dimension of the isotypical component of type λ. The image of the Newton map may be described as follows. Write $\nu \in (\mathbb{Q}^n)_+$ as

$$\nu = (\nu(1)^{m_1}, \dots, \nu(r)^{m_r}) \quad \text{with } \nu(1) > \dots > \nu(r) \ .$$

Then ν lies in the image if and only if ν satisfies the integrality condition

$$\nu(i)m_i \in \mathbb{Z}, \ \forall i = 1, \dots, r \ . \quad \square$$

The components of the *Newton vector $\nu(N, F)$* (i.e. the types of the isotypical components occurring in (N, F)) are called the *slopes* of the F-isocrystal.

Let (N, F) be an F-isocrystal over a perfect field k. Then the Newton vector of the F-isocrystal $(N, F) \otimes_{W(k)} W(\bar{k})$ over $\operatorname{Spec} \bar{k}$ is independent of the algebraically closed field $\bar{k}$ containing k. We may therefore speak of the Newton vector of (N, F).

Let X be a p-divisible group over a perfect field k of characteristic p. Then one may associate to X its (contravariant) Dieudonné module $(M(X), F)$, which is an F-crystal over $\operatorname{Spec} k$, compare [D]. *In this way one obtains*

a) an *anti-equivalence* of the category of p-divisible groups over $\operatorname{Spec} k$ and the full subcategory of the category of F-crystals over $\operatorname{Spec} k$ consisting of those F-crystals (M, F) such that $pM \subset FM$,

b) an *anti-equivalence* of the category of p-divisible groups over $\operatorname{Spec} k$ up to isogeny and the full subcategory of all F-isocrystals over $\operatorname{Spec} k$ such that all slopes lie between 0 and 1.

Let S be a scheme of characteristic p. An *F-crystal over S* is a crystal $\mathcal{E}$ of finite locally free $\mathcal{O}_{S_{\mathrm{cris}}}$-modules, with a morphism $F : \mathcal{E}^{(\sigma)} \to \mathcal{E}$ such that the kernel and cokernel of F are annihilated by a power of p. Here $\mathcal{O}_{S_{\mathrm{cris}}}$ denotes the structure sheaf on the big crystalline site of S over $\mathbb{Z}_p$. We often write $\mathcal{E}$ for the F-crystal $(\mathcal{E}, F)$. This notion makes precise the intuitive concept of a family of F-crystals parametrized by the (perfect closures of the residue fields of) points of S. The F-crystals over S form a $\mathbb{Z}_p$-linear category. A morphism of F-crystals $f : \mathcal{E} \to \mathcal{E}'$ is an *isogeny* if there exists locally on S a morphism $g : \mathcal{E}' \to \mathcal{E}$ such that $gf = p^n$ and $fg = p^n$ for some n. The category of F-isocrystals over S is obtained by formally inverting isogenies of F-crystals.

EXAMPLE 1.4. — Let X be a p-divisible group over S. Then crystalline Dieudonné theory [Me] associates to X an F-crystal over S. More precisely, the Lie algebra of the universal extension of X is a crystal, and its dual is an F-crystal where F is induced by the Frobenius $Fr : X \to X^{(\sigma)}$, compare [Me], IV.2.5.

For the sequel it is not essential to have mastered the notion of an F-crystal over a scheme, in order to understand the resulting statements for p-divisible groups (although some proofs in this special case are based on general F-crystals for which one can perform the usual linear algebra operations like tensor products etc.).

The most basic statement about families of F-crystals is the following semi-continuity theorem. Recall the usual dominance order on $(\mathbb{Q}^n)_+$, for which $(\nu_1, \dots, \nu_n) \leq (\nu'_1, \dots, \nu'_n)$ if and only if

$$\sum_{j=1}^{r} \nu_i \leq \sum_{j=1}^{r} \nu'_i, \quad \forall r = 1, \dots, n-1, \quad \text{and} \quad \sum_{j=1}^{n} \nu_j = \sum_{j=1}^{n} \nu'_j \ .$$

For $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Q}^n$ we set $\|\nu\| = \sum_{j=1}^{n} \nu_j$.

THEOREM 1.5 (Grothendieck [G]). — *Let $(\mathcal{E}, F)$ be an F-isocrystal over a scheme S of characteristic p. Then the Newton vector of $(\mathcal{E}_s, F_s)$, for s ranging over the points of S, goes down under specialization. More precisely, let $(\mathcal{E}, F)$ be of constant rank n. Then the function $s \mapsto \|\nu(\mathcal{E}_s, F_s)\|$ is locally constant on S and for any $\nu_0 \in (\mathbb{Q}^n)_+$ the set*

$$\{s \in S; \ \nu(\mathcal{E}_s, F_s) \leq \nu_0\}$$

is Zariski closed in S.

We note here that the Newton vector of the fiber of $\mathcal{E}$ at a geometric point $\bar{s}$ of S only depends on the underlying point $s \in S$.

The proof by Katz in [Ka] relies on the relation between the Newton vector of an F-isocrystal and the divisibility by p of F with respect to an underlying F-crystal. For (the F-isocrystal associated to) a p-divisible group over S a simple proof is contained in [D]. □

REMARK 1.6. — This theorem is reminiscent of a theorem on vector bundles on a compact Riemann surface. In this theory one associates to a vector bundle of rank n its *Harder-Narasimhan vector* in $(\mathbb{Q}^n)_+$, and it is a basic fact that the HN-vector *goes up* (!) under specialization [AB].

COROLLARY 1.7. — *Let $(\mathcal{E}, F)$ be an F-isocrystal over a noetherian scheme S of characteristic p. Then the set of points of S where the Newton vector is constant is locally closed in S and this defines a finite decomposition of S.*

Proof. — We may assume S connected. Let us only consider the case when $(\mathcal{E}, F)$ comes from a p-divisible group X over S. Then the height and the dimension of X are constant. The assertion then follows from the preceding theorem by the following two observations.

For any $\nu_0 \in (\mathbb{Q}^n)_+$ the set

$$\{\nu \in (\mathbb{Q}^n)_+ \ ; \ \nu \leq \nu_0 \text{ and } \nu \text{ satisfies the integrality condition in Cor. 1.3}\}$$

is finite.

If $\nu = \nu(M(X), F)$ for a p-divisible group X of dimension d and height n, then $\nu \leq (1^d, 0^{n-d})$. This is a consequence of Mazur's inequality between the Hodge vector of an F-crystal over a perfect field and the Newton vector of its underlying F-isocrystal, [Ka]. □

Let $(\mathcal{E}, F)$ be an F-isocrystal of rank n over a noetherian scheme S of characteristic p. Associating to a geometric point $\bar{s}$ of S the Newton vector of $(\mathcal{E}_{\bar{s}}, F_{\bar{s}})$, we obtain a map

$$S \longrightarrow (\mathbb{Q}^n)_+ \ .$$

Let S_ν be the fiber of this map over $\nu \in (\mathbb{Q}^n)_+$ (with its reduced scheme structure). The corresponding disjoint decomposition of S, finite according to Corollary 1.7, is called the *Newton stratification of S associated to the F-isocrystal* $(\mathcal{E}, F)$. The subschemes S_ν are called the *Newton strata*.

We speak of a stratification *in the strong sense* if the closure of a stratum is a union of strata. In general the Newton stratification associated to an F-isocrystal is *not* a stratification in the strong sense.

2. PURITY OF THE NEWTON STRATIFICATION

Let $(\mathcal{E}, F)$ be an F-isocrystal of rank n over a scheme S of characteristic p, with associated Newton stratification $(S_\nu)_{\nu \in (\mathbb{Q}^n)_+}$ of S. The purity theorem states that the jumps in this stratification all occur in codimension one. The corresponding statement for families of vector bundles on a Riemann surface (cf. Remark 1.6) is false.

THEOREM 2.1 (de Jong, Oort [JO]). — *Let $(\mathcal{E}, F)$ be an F-isocrystal of rank n over a locally noetherian scheme S of characteristic p, with associated Newton stratification $(S_\nu)_{\nu \in (\mathbb{Q}^n)_+}$. Let η be a generic point of the scheme $\bar{S}_\nu \setminus S_\nu$. Then*

$$\dim \mathcal{O}_{\bar{S}_\nu, \eta} = 1 \ .$$

An equivalent statement is the following.

THEOREM 2.2. — *Let $(\mathcal{E}, F)$ be an F-isocrystal over a locally noetherian scheme S of characteristic p. Let U be an open subset of S such that $\mathrm{codim}(S \setminus U) \geq 2$. If the Newton vector of $(\mathcal{E}, F)$ is constant at all points of U, then it is constant on all of S.*

These theorems are referred to as purity theorems since they are reminiscent of the purity theorem of Nagata-Zariski on étale coverings. We shall be mainly interested in this statement when $(\mathcal{E}, F)$ comes from a p-divisible group over S. The structure of a p-divisible group with constant Newton vector is addressed in the following two results.

THEOREM 2.3 (de Jong, Oort [JO]). — *Let $S = \operatorname{Spec} A$, where A is a complete noetherian local ring of characteristic p with algebraically closed residue field k. Let X be a p-divisible group over S with constant* scalar *Newton vector, i.e., there is only one slope at all points of S (isoclinic case). Then X is isogenous to a* constant *p-divisible group, i.e., one of the form $X_0 \times_{\operatorname{Spec} k} S$ for a p-divisible group X_0 over $\operatorname{Spec} k$.*

The motivation for this theorem is the heuristic idea that a p-divisible group with constant scalar Newton vector is analogous to a local system over S. The hypotheses on S in Theorem 2.3 ensure that it behaves like a simply connected space. One may expect a similar result for general F-isocrystals with constant scalar Newton vector, compare [JO], Remark 2.18. The case where A is a complete discrete valuation ring with algebraically closed residue field is due to Katz, [Ka], Thm. 2.7.1. □

The constancy up to isogeny becomes false when the constant Newton vector has more than one slope, for there can be highly nontrivial extensions of constant F-isocrystals over $\operatorname{Spec} k[[t]]$ (e.g. the p-divisible group of the universal deformation of an ordinary elliptic curve is a nontrivial extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\hat{\mathbb{G}}_m$). When there is more than one slope, there is the following analogue of the Harder-Narasimhan filtration of vector bundles, cf. Remark 1.6.

THEOREM 2.4 (Zink [Z3]). — *Let S be a regular scheme of characteristic p. Let X be a p-divisible group over S with constant Newton vector $\nu \in (\mathbb{Q}^n)_+$. Then X is isogenous to a p-divisible group Y which admits a filtration by closed embeddings of p-divisible groups*

$$(0) = Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y \quad ,$$

such that the following condition is satisfied. Let $\nu = (\nu(1)^{m_1}, \ldots, \nu(r)^{m_r})$ with $\nu(1) > \ldots > \nu(r)$, cf. integrality condition in Cor. 1.3. Then there are natural numbers $r_i \geq 0$, $s_i > 0$ $(i = 1, \ldots, r)$ such that $\nu(i) = r_i/s_i$ and such that

$$p^{-r_i} \operatorname{Fr}^{s_i} : \; Y_i \longrightarrow Y_i^{(\sigma^{s_i})} \quad \text{is an isogeny}$$

and

$$p^{-r_i} \operatorname{Fr}^{s_i} : \; Y_i/Y_{i-1} \longrightarrow (Y_i/Y_{i-1})^{(\sigma^{s_i})} \quad \text{is an isomorphism} ,$$

$\forall i = 1, \ldots, r.$

The degree of the isogeny between X and Y may be bounded in terms of the height of X. The heuristic idea behind this theorem is that the isotypic *direct sum decomposition* of an F-isocrystal over an algebraically closed field is replaced in the case of a more general base scheme by a filtration. In ongoing work of Oort and Zink, the regularity

hypothesis on S is weakened. The case where $S = \operatorname{Spec} k$ for an arbitrary field k is due to Grothendieck [G]. $\qquad\square$

The proof of Theorem 2.2 is based on the following result which is of independent interest.

THEOREM 2.5 (de Jong, Oort [JO]). — *Let $S = \operatorname{Spec} A$, where A is a normal complete noetherian local ring of dimension 2 with algebraically closed residue field k. Let $U = S \setminus \{s\}$, where s denotes the closed point. Let $\pi : \tilde{S} \to S$ be a resolution of singularities, i.e. a proper morphism from a regular scheme which induces an isomorphism over U and such that $E = \pi^{-1}(s) = \bigcup_{i=1}^{m} E_i$ is a union of smooth divisors crossing each other normally. Identifying $\pi^{-1}(U)$ with U we have the restriction map*

$$H^1_{et}(\tilde{S}, \mathbb{Z}_p) \longrightarrow H^1_{et}(U, \mathbb{Z}_p) \quad .$$

This map is an isomorphism.

In terms of the fundamental groups (w.r.t. some geometric point of U) the assertion is that

$$(1) \qquad \operatorname{Hom}(\pi_1(\tilde{S}), \mathbb{Z}_p) \xrightarrow{\sim} \operatorname{Hom}(\pi_1(U), \mathbb{Z}_p) \quad .$$

Since both $\pi_1(\tilde{S})$ and $\pi(U)$ are factor groups of the Galois group of the fraction field of S, the homomorphism $\pi_1(U) \to \pi_1(\tilde{S})$ is surjective. Therefore we have the injectivity of the map (1), and to prove the surjectivity we may replace $\mathbb{Z}_p$ by $\mathbb{Q}_p$ in (1). Topologically or when $p \neq \operatorname{char} k$, this surjectivity is easy to see. Indeed, we need the injectivity of

$$(2) \qquad H^2_E(\tilde{S}, \mathbb{Q}_p) \longrightarrow H^2(\tilde{S}, \mathbb{Q}_p) \quad .$$

But $H^2_E(\tilde{S}, \mathbb{Q}_p)$ has the classes $\operatorname{cls}(E_i)$, $i = 1, \ldots, m$, as basis (purity). The image of $\operatorname{cls}(E_i)$ under the composition

$$H^2_E(\tilde{S}, \mathbb{Q}_p) \longrightarrow H^2(\tilde{S}, \mathbb{Q}_p) \xrightarrow{\operatorname{Res}} H^2(E_j, \mathbb{Q}_p)$$

is the intersection product $(E_i.E_j)$. The injectivity of (2) follows therefore from the negative-definiteness of the intersection matrix $(E_i.E_j)_{i,j=1,\ldots,m}$.

When $p = \operatorname{char} k$, the proof of Theorem 2.5 is much more difficult. (That the situation in this case is radically different is already apparent from the fact that $H^2_{et}(\tilde{S}, \mathbb{Z}_p) = (0)$ when $p = \operatorname{char} k$. This is easily checked using Artin-Schreier theory.) Suppose that A has characteristic p, i.e. $k \subset A$. In this case, using de Jong's technique of alterations there is a reduction to the following situation. Let $\mathcal{C} \to \operatorname{Spec} k[[t]]$ be a flat projective family of curves with smooth generic fiber and strict semistable reduction. Let $\mathcal{C}'$ be the scheme obtained from $\mathcal{C}$ by collapsing a proper union E of irreducible components of the special fiber to a point P. Then A is the complete local ring of P.

One now starts with an element $\alpha \in H^1_{et}(U, \mathbb{Z}_p)$ and first globalizes it into an element $\alpha_1 \in H^1_{et}(\mathcal{C}' \setminus \{P\}, \mathbb{Z}_p) = H^1_{et}(\mathcal{C} \setminus E, \mathbb{Z}_p)$. This element α_1 is then extended to

$\alpha_2 \in H^1_{et}(\mathcal{C}, \mathbb{Z}_p)$ by using de Jong's extension theorem on homomorphisms of p-divisible groups [J1], [J2]. According to this theorem, any homomorphism between the generic fibers of p-divisible groups over Spec $k[[t]]$ extends. $\qquad \square$

Let X be a p-divisible group of height h over a scheme S of characteristic p, for which there exists $r \geq 0$, $s > 0$ such that

$$(3) \qquad\qquad p^{-r} Fr^s : X \longrightarrow X^{(\sigma^s)} \quad \text{is an isomorphism.}$$

To X we associate the lisse p-adic sheaf of $W(\mathbb{F}_{p^s})$-modules $C_X = \varprojlim C_{X,n}$ for the étale topology on S, such that for any affine S-scheme Spec R

$$(4) \qquad\qquad C_{X,n}(\text{Spec } R) = \{x \in M/p^n M; \ p^{-r} F^s(x) = x\} \ .$$

Here M denotes the $W(R)$-module defined by the Dieudonné crystal of X. The fibers of C_X are free $W(\mathbb{F}_{p^s})$-modules of rank h. The formation of C_X is compatible with base change and defines a functor from the category of p-divisible groups over S with (3) to the category of lisse p-adic sheaves of $W(\mathbb{F}_{p^s})$-modules on S. The corresponding $W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-adic sheaf only depends on the isogeny class of X and corresponds to a representation of the fundamental group,

$$(5) \qquad\qquad \varrho_X : \pi_1(S) \longrightarrow GL_h(W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \ .$$

Let R be a discrete valuation ring of characteristic p, with uniformizer π, residue field k and fraction field K. Let X be a p-divisible group over R. After replacing X_K by an isogenous p-divisible group Y over K we have integers $r_i \geq 0$, $s > 0$ and a filtration $(0) = Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y$ as in Theorem 2.4. Applying the preceding considerations to $S = \text{Spec } K$ and Y_i/Y_{i-1} we therefore obtain p-adic Galois representations

$$(6) \qquad\qquad \varrho_i = \varrho_{Y_i/Y_{i-1}} : \text{Gal } (\bar{K}/K) \longrightarrow GL_{h_i}(W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \ ,$$

where $h_i = \text{height}(Y_i/Y_{i-1})$. The proof of Theorem 2.2 is based on the following lemma.

LEMMA 2.6. — With the previous notation, the following conditions are equivalent.

(i) $\wedge^{h_i} \varrho_i$ is unramified, $\forall i = 1, \ldots, r$.

(ii) The Newton vector of X is constant.

Under these conditions the representations ϱ_i are also unramified.

In the proof of this lemma, again, as in the proof of Theorem 2.5, de Jong's theorem on extension of homomorphisms of p-divisible groups (or rather the techniques entering into the proof) plays a key role. $\qquad \square$

Proof of Theorem 2.2. — We limit ourselves to the case where $(\mathcal{E}, F)$ comes from a p-divisible group X on S. An easy reduction allows us to assume that $S = \text{Spec } A$, where A is a complete normal noetherian local ring of dimension 2 and where $U = S \setminus \{s\}$, with s denoting the special point. In proving Theorem 2.2, we may replace A by an A-algebra A' of the same kind such that the special point s' of Spec A' is the unique point mapping

to s. Since the Newton vector of X is constant on the regular scheme U, we obtain via Theorem 2.4 p-adic Galois representations,

$$(7) \qquad \varrho_i : \pi_1(U) \longrightarrow GL_{h_i}(W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \ ,$$

$i = 1, \dots, r$. The determinant representation of each ϱ_i is a character of $\pi_1(U)$ with values in $(W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$. Since $(W(\mathbb{F}_{p^s}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$ contains an open subgroup of the form $\mathbb{Z}_p^s$, we may assume by the initial remark that $\wedge^{h_i} \varrho_i$ is an s-tuple of homomorphisms from $\pi_1(U)$ to $\mathbb{Z}_p$. Let $\pi : \tilde{S} \to S$ be a resolution of singularities. Then by Theorem 2.5 this s-tuple of homomorphisms factors through $\pi_1(\tilde{S})$. Let E be an irreducible component of the exceptional fiber $\pi^{-1}(s)$. Applying Lemma 2.6 to the discrete valuation ring $\mathcal{O}_{\tilde{S},E}$, the pullback of X to Spec $\mathcal{O}_{\tilde{S},E}$ has constant Newton vector, as had to be shown. $\qquad \square$

3. THE SIEGEL MODULI SPACE

We fix a positive integer g. For an auxiliary integer $m \geq 3$ prime to p, we denote by $\mathcal{M} = \mathcal{M}_g = \mathcal{M}_{g,m}$ the *Siegel moduli space of genus g* over Spec $\mathbb{F}_p$. It represents the functor which to a locally noetherian scheme S in characteristic p associates the set of isomorphism classes of triples (A, λ, η), where A is an abelian scheme of relative dimension g over S, and $\lambda : A \to \hat{A}$ is a principal polarization, and η is a (full) level-m-structure on (A, λ).

The universal abelian scheme over $\mathcal{M}$ defines a p-divisible group X on $\mathcal{M}$. The existence of a polarization implies that the Newton vectors of the fibers of X lie in the subset $(\mathbb{Q}^{2g})_+^1$ of $(\mathbb{Q}^{2g})_+$,

$$(\mathbb{Q}^{2g})_+^1 = \{(\nu_1, \dots, \nu_{2g}) \in (\mathbb{Q}^{2g})_+; \nu_i + \nu_{2g-i+1} = 1, \ \forall i = 1, \dots, g,$$
$$0 \leq \nu_i \leq 1, \ \forall i = 1, \dots, 2g\}.$$

Let B_g be the set of elements in $(\mathbb{Q}^{2g})_+^1$ which satisfy the integrality condition in Cor. 1.3. Then B_g is a finite partially ordered set (*poset*), which has a unique maximal element and a unique minimal element,
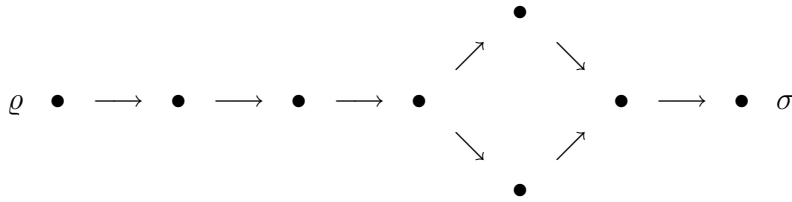
$$\varrho = (1^g, 0^g) \text{ the ordinary Newton vector (maximal)}$$
$$\sigma = ((\tfrac{1}{2})^{2g}) \text{ the supersingular Newton vector (minimal)}.$$

For $\nu \in B_g$ we denote by $\mathcal{S}_\nu$ the corresponding Newton stratum in $\mathcal{M}$. By Grothendieck's semi-continuity theorem, Theorem 1.5, we have

$$(8) \qquad \bar{\mathcal{S}}_\nu \subset \bigcup_{\nu' \leq \nu} \mathcal{S}_{\nu'} \ .$$

In particular $\mathcal{S}_\sigma$ is a closed subset and $\mathcal{S}_\varrho$ is an open subset.

Picture of B_4: —

For $\nu \in B_g$, with $\nu = (\nu_1, \ldots, \nu_{2g})$, let

$$(9) \qquad \Delta(\nu) \;=\; \{(i,j) \in \mathbb{Z}^2;\; 0 \le i \le g,\; \textstyle\sum_{\ell=1}^{i} \nu_{2g-\ell+1} \le j < i\}$$

$$(10) \qquad d(\nu) \;=\; \#\Delta(\nu)$$

Example 3.1. —

$$d(\varrho) \;=\; \frac{g(g+1)}{2}$$

$$d(\sigma) \;=\; \left[\frac{g^2}{4}\right] \;.$$

Lemma 3.2. — Let $\nu, \nu' \in B_g$. Then

(i) $\nu \le \nu'$ if and only if $\Delta(\nu) \subseteq \Delta(\nu')$.

(ii) If $\nu \le \nu'$, then any shortest chain in the poset B_g starting at ν and ending at ν' has length $d(\nu') - d(\nu)$. In particular, B_g is a catenary poset. $\qquad \square$

Theorem 3.3 (Oort [O2]). — *Each Newton stratum $\mathcal{S}_\nu$ is equidimensional of dimension $d(\nu)$. The Newton stratification of $\mathcal{M}$ has the strong stratification property,*

$$\bar{\mathcal{S}}_\nu = \bigcup_{\nu' \le \nu} \mathcal{S}_{\nu'} \;.$$

Remark 3.4. — The analogous statement for moduli spaces of vector bundles on a Riemann surface is false. More precisely, consider the stack $\mathcal{M} = \mathcal{M}_X(r,d)$ of holomorphic vector bundles of rank r and degree d over a Riemann surface X. Then $\mathcal{M}$ is the disjoint union of its Harder-Narasimhan strata $\mathcal{M}_\nu$, and we have

$$\overline{\mathcal{M}}_\nu \subset \bigcup_{\nu' \ge \nu} \mathcal{M}_{\nu'} \;,$$

cf. Remark 1.6. If $g(X) = 0$ we have equality here and the same holds for $g(X) = 1$, according to a recent paper of Friedman and Morgan [FM]. This fails for $g(X) \ge 2$ [FM]. It is conceivable that

$$\nu' \ge \nu \Longleftrightarrow \overline{\mathcal{M}}_\nu \cap \mathcal{M}_{\nu'} \ne \emptyset$$

and in loc. cit. this is proved, provided ν' and ν are adjacent. $\qquad \square$

The proof of Theorem 3.3 is based on the following theorem.

THEOREM 3.5 (Oort [O2]). — *Let (X_0, λ_0) be a p-divisible group with principal polarization of dimension g and height $2g$ over an algebraically closed field k of characteristic p. Let $\nu_0 \in B_g$ be the Newton vector of X_0, and let $\nu \in B_g$ with $\nu_0 \leq \nu$. Then there exists a principally polarized p-divisible group (X, λ) over $k[[t]]$ with special fiber (X_0, λ_0) and with Newton vector of the generic fiber of X equal to ν.*

Theorem 3.5 implies via the Serre-Tate theorem the second statement in Theorem 3.3. Since, as is easily seen, $\mathcal{S}_\sigma \neq \emptyset$ (the supersingular locus), Theorem 3.5 implies $\mathcal{S}_\nu \neq \emptyset$ for all $\nu \in B_g$. The first statement in Theorem 3.3 now follows from Lemma 3.2 and the purity theorem. $\qquad \square$

The two extreme cases in the Newton stratification deserve a separate discussion.

THE SUPERSINGULAR STRATUM $\mathcal{S}_\sigma$: The fact that $\mathcal{S}_\sigma$ is equidimensional of dimension $d(\sigma) = \left[\frac{g^2}{4}\right]$ (a special case of Theorem 3.3) was proved earlier by Li and Oort [LO], among other things. The supersingular Newton stratum is exceptional in several aspects. Every abelian variety occurring as a fiber of the universal abelian scheme at a point of $\mathcal{S}_\sigma$ is isogenous to E^g, where E is a supersingular elliptic curve. Moreover, let $x \in \mathcal{S}_\sigma$ and let (A, λ) be the fiber of x of the universal object on $\mathcal{M}$. Then one can represent (A, λ) in an almost canonical way as the quotient of E^g by a finite group scheme. This leads to the dimension formula for this stratum and in fact much more. There is the hope for an *explicit synthetic* description of $\mathcal{S}_\sigma$, like the one of Kaiser [K] for $g = 2$ (compare also [KR], and [R] for $g = 3$). In [LO] the number of irreducible components of $\mathcal{S}_\sigma$ is given.

THE ORDINARY STRATUM $\mathcal{S}_\varrho$: In contrast to the supersingular stratum, the ordinary stratum is quite amorphous and nonlinear, and there is no hope of an explicit description of it. According to Theorem 3.3, $\mathcal{S}_\varrho$ is open and dense in $\mathcal{M}$. This fact has been known for a long time, by more direct and easier proofs: 1) There is the proof by Mumford [M], and Norman and Oort [NoO] (compare also Chai and Faltings [FC]) using Cartier theory to construct deformations. 2) There is the proof by Koblitz [Kob], compare also [Ill], App. 2, who investigated by deformation-theoretic arguments the stratification of $\mathcal{M}$ by the p-rank of X. 3) There is the global proof using toroidal compactifications [FC]. 4) There is the proof of Ngo and Genestier [NG] who deduce the density result from a corresponding density result (of a combinatorial nature) in *bad reduction.* Furthermore, Chai [C1] has proved the much stronger assertion that the orbit of an arbitrary ordinary point under the Hecke correspondences of degree prime to p is dense in $\mathcal{M}$. $\qquad \square$

The proof of Theorem 3.5 is rather round-about. For a p-divisible group X over an algebraically closed field k, let

$$(11) \qquad a(X) = \mathrm{Hom}_k(\alpha_p, X) = \dim_k M/(F(M) + V(M)) \ .$$

Here (M, F) is the Dieudonné module of X and $V = pF^{-1}$. Hence $a(X) = 0$ if and only if $X \simeq \hat{\mathbb{G}}_m^d \times (\mathbb{Q}_p/\mathbb{Z}_p)^{d'}$. In a first step one proves Theorem 3.5 under the additional assumption $a(X_0) = 1$, cf. Oort [O1] (the case where $a(X_0) = 0$, where $\nu_0 = \varrho$, is trivial). The technical tool for this is the theory of displays, compare Section 4, which allows one to write down explicitly by display equations deformations of a (polarized) formal p-divisible group. The hypothesis $a(X_0) = 1$ is then needed to read off from these display equations the Newton vectors of the p-divisible groups occurring in the deformation.

In a second step, one shows that (X_0, λ_0) can be deformed into a principally polarized p-divisible group (X, λ) with the same Newton vector and with $a(X) = 1$. This in turn is reduced to the following statement which is of independent interest.

THEOREM 3.6 (de Jong, Oort [JO]). — *Let X_0 be a p-divisible group over an algebraically closed field k of characteristic p such that its F-isocrystal is irreducible. Then there exists an irreducible scheme T over k and a p-divisible group X over T together with an isogeny $X_0 \times_{\mathrm{Spec}\, k} T \to X$ over T, such that any p-divisible group over k isogenous to X_0 occurs as a fiber of X at a k-rational point of T.*

An equivalent formulation of the previous theorem is the following. Let X_0 be as in the previous theorem. By [RZ] the following functor on $(Sch/\mathrm{Spec}\, k)$ is representable by a formal scheme $\mathcal{S}$ locally formally of finite type over $\mathrm{Spec}\, k$,

$$S \longmapsto \quad \{\text{isomorphism classes of pairs } (X, \varrho), \text{ where } X \text{ is a } p\text{-divisible group}$$
$$\text{over } S \text{ and } \varrho : X_0 \times_{\mathrm{Spec}\, k} S \longrightarrow X \text{ is a quasi-isogeny of height } 0\} \ .$$

Then $\mathcal{S}$ is irreducible. $\square$

COROLLARY 3.7. — *Let X_0 be as in the previous theorem. Then there exists a deformation of X_0 into a p-divisible group X isogenous to X_0 with $a(X) = 1$.*

Indeed, the locus in T where $a(X) = 1$ is open. It therefore suffices to produce one point $t \in T(k)$ where $a(X_t) = 1$. This is easy. $\square$

CONJECTURE 3.8. — *Let $\nu, \nu' \in B_g$ with $\nu' \leq \nu$. The closure of each irreducible component of $\mathcal{S}_\nu$ meets $\mathcal{S}_{\nu'}$.*

Conjecture 3.8 would certainly hold if Oort's conjecture [O2] was true, according to which for $\nu \neq \sigma$ the intersection of $\mathcal{S}_\nu$ with any connected component of $\mathcal{M}$ is irreducible.

4. DISPLAYS

Display equations for formal p-divisible groups were introduced by Mumford [M]. These techniques were applied to moduli problems of abelian varieties by Norman [No] and Norman and Oort [NoO]. We will follow here the recent formulation of the theory due to Zink [Z1].

Let R be a ring of characteristic p. We denote by $W(R)$ its ring of Witt vectors and by $x \mapsto {}^F x$ resp. $x \mapsto {}^V x$ its Frobenius resp. Verschiebung endomorphisms. Let $I_R \subset W(R)$ be the ideal of Witt vectors with trivial 0-component.

DEFINITION 4.1. — *A not necessarily nilpotent display (= 3n-display) over R is a quadruple (P, Q, F, V^{-1}) consisting of a finitely generated projective $W(R)$-module P, a submodule $Q \subset P$ and ^{F}-linear maps $F : P \to P$ and $V^{-1} : Q \to P$. The following conditions are required:*
(i) *$I_R P \subset Q \subset P$ and the quotient P/Q is a projective R-module.*
(ii) *$V^{-1} : Q \to P$ is a ^{F}-linear epimorphism.*
(iii) *For $x \in P$ and $w \in W(R)$ we have ${}^V w \cdot x \in Q$ and we require that*

$$V^{-1}({}^V w \cdot x) = w \cdot F(x) \quad .$$

We note that F is determined by the remaining data. There is no operator V. The reason for the notation comes from the following example.

EXAMPLE 4.2. — Let $R = k$ be a perfect field. Then an F-crystal (M, F) over k such that $pM \subset FM$ defines a 3n-display (M, VM, F, V^{-1}). Here as usual $V = pF^{-1}$. This defines an equivalence of categories.

The notion of a display is obtained by imposing a nilpotency condition as follows. After localization in R there exists a $W(R)$-basis $e_1, \dots, e_n$ of P such that

$$Q = I_R e_1 \oplus \dots \oplus I_R e_d \oplus W(R) e_{d+1} \oplus \dots \oplus W(R) e_n \quad ,$$

for some d with $0 \leq d \leq n$. Then there exists an invertible matrix $(\alpha_{ij}) \in GL_n(W(R))$ such that

$$\begin{aligned} F e_j &= \sum_{i=1}^n \alpha_{ij} e_i \quad \text{for } j = 1, \dots, d \\ V^{-1} e_j &= \sum_{i=1}^n \alpha_{ij} e_i \quad \text{for } j = d+1, \dots, n \ . \end{aligned}$$

Conversely, any $(\alpha_{ij}) \in GL_n(W(R))$ defines a 3n-display. Let $(\beta_{k\ell})$ be the inverse of (α_{ij}). Let $B \in M_{n-d}(R)$ be the image of $(\beta_{k\ell})_{k,\ell=d+1,\dots,n}$ under the 0-component map

$$M_{n-d}(W(R)) \longrightarrow M_{n-d}(R) \quad .$$

Let $B^{(p)}$ be the matrix obtained from B by raising its coefficients to the power p. The nilpotency condition can now be formulated: there exists N such that

$$B^{(p^N)} \cdot \dots \cdot B^{(p)} \cdot B = 0 \quad .$$

In the context of Example 4.2 the F-crystal (M, F) defines a display if and only if $pM \subset FM$ and if $V = pF^{-1}$ is topologically nilpotent on M.

THEOREM 4.3 (Zink [Z1]). — *We assume that the nilideal of R is nilpotent. Then there is a fully faithful functor BT from the category of displays over R to the category of formal*

p-divisible groups over R. This is an equivalence of categories if either R is an excellent local ring or an algebra of finite type over a field k.

It is quite likely that this equivalence of categories holds for any noetherian ring R of characteristic p. The functor BT has the following properties: 1) It commutes with arbitrary base change. 2) Lie $BT(P, Q, F, V^{-1}) = P/Q$. 3) P can be identified with the value at $W(R)$ of the crystal defined by the universal extension of $BT(P, Q, F, V^{-1})$, cf. Example 1.4. 4) The passage from a formal p-divisible group to its dual p-divisible group can be expressed in terms of displays, provided that the dual p-divisible group is a formal group, i.e. has trivial étale part.

The theory also works if p is only supposed to be nilpotent in R. For an extension of the theory to p-divisible groups with an étale part, compare [Z2].

5. OTHER MODULI SPACES OF ABELIAN VARIETIES

Let F be a finite-dimensional semisimple $\mathbb{Q}$-algebra equipped with a positive involution $*$ and let V be a finite F-module equipped with an alternating non-degenerate $\mathbb{Q}$-valued skew-hermitian pairing $\langle \ , \ \rangle$. The F-linear similitudes of $(V, \langle \ , \ \rangle)$ form an algebraic group G over $\mathbb{Q}$. We assume that G is a *connected* reductive algebraic group. We also fix a conjugacy class of algebraic homomorphisms $h : \mathbb{C}^\times \to G(\mathbb{R})$ satisfying the usual Riemann conditions. Let E be the corresponding Shimura field, i.e. the field of definition of the corresponding conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\mathbb{C}} \to G_\mathbb{C}$. We assume that p is a prime of good reduction, in particular $G_{\mathbb{Q}_p}$ is unramified, and we choose a hyperspecial maximal compact subgroup K_p of $G(\mathbb{Q}_p)$. We choose a prime ideal of E over p with residue field κ. After a choice of some sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, Kottwitz [Kot1] has defined a moduli problem of abelian varieties which is representable by a smooth quasi-projective scheme $\mathcal{M} = \mathcal{M}(G, h)_K = \mathcal{M}(F, V, \langle \ , \ \rangle, h, K^p.K_p)$ over Spec κ.

Let L be the fraction field of $W(\overline{\mathbb{F}}_p)$ and let $B(G)$ be the set of σ-conjugacy classes in $G(L)$. By associating to a point $s \in \mathcal{M}$ the *F-isocrystal with G-structure* defined by the fiber at (a geometric point over) s of the universal abelian scheme over $\mathcal{M}$ with its auxiliary structure (endomorphisms and polarization), we obtain a map

$$\mathcal{M} \longrightarrow B(G) \ .$$

The conjugacy class μ defines a finite subset $B(G, \mu)$ of $B(G)$ ([Kot2], §6). It is defined by the group-theoretic version of Mazur's inequality, compare the proof of Corollary 1.7. The image of the map above is contained in $B(G, \mu)$ [RR]. (In the case of the Siegel moduli space $\mathcal{M}_g$ we have $G = GSp_{2g}$ and $B(G, \mu) = B_g$, cf. section 3.) Furthermore, $B(G, \mu)$ is partially ordered and the semicontinuity theorem 1.5 continues to hold in this

context [RR]. We therefore obtain the *generalized Newton stratification of* $\mathcal{M}$ (by the locally closed subsets arising as inverse images of elements of $B(G, \mu)$),

$$\mathcal{M} = \bigcup_{b \in B(G,\mu)} \mathcal{M}_b \ .$$

Just as B_g, also $B(G, \mu)$ is a catenary poset [C2] with a unique minimal element b_0 (the *μ-basic element*) and a unique maximal element b_1 (the *μ-ordinary element*).

THEOREM 5.1 (Wedhorn[W]). — *The μ-ordinary locus $\mathcal{M}_{b_1}$ is open and dense in $\mathcal{M}$.* $\square$

This is about the only known general statement in direction of the following conjecture.

CONJECTURE 5.2. — (i) *The generalized Newton stratification of $\mathcal{M} = \mathcal{M}(G, h)_K$ has the strong stratification property.*
(ii) *The generalized Newton stratum corresponding to $b \in B(G, \mu)$ is equidimensional of dimension $d(b) = \dim \mathcal{M} - c(b)$, where $c(b)$ is the length of a chain joining b to b_1.*
(iii) *Let $b, b' \in B(G, \mu)$ with $b' \le b$. The closure of each irreducible component of $\mathcal{M}_b$ meets $\mathcal{M}_{b'}$.*

We note that Chai [C2] has given a group theoretical formula for $d(b)$. When G is a group of unitary similitudes, there are results supporting (i) and (ii) of this conjecture:

THEOREM 5.3 (Oort). — *Let F be an imaginary quadratic field such that p splits in F. Then* (i) *and* (ii) *of Conjecture 5.2 hold true for $\mathcal{M} = \mathcal{M}(F, V, \langle \ , \ \rangle, h, K^p.K_p)$.*

The proof is analogous to the proof of Theorem 3.3 (which proves (i) and (ii) of Conjecture 5.2 for the Siegel moduli space). The analogue of Theorem 3.5 is the following statement which confirms a conjecture of Grothendieck [G]. Its proof is similar to that of Theorem 3.5, but simpler.

THEOREM 5.4 (Oort[O2]). — *Let X_0 be a p-divisible group of height n and dimension d over an algebraically closed field k of characteristic p, with Newton vector $\nu_0 \in (\mathbb{Q}^n)_+$. Let $\nu \in (\mathbb{Q}^n)_+$ such that ν satisfies the integrality condition of Corollary 1.3 and with $\nu_0 \le \nu \le (1^d, 0^{n-d})$. Then there exists a p-divisible group X over $k[[t]]$ with special fiber X_0 and with Newton vector of the generic fiber of X equal to ν.* $\square$

Using the Serre-Tate theorem, Theorem 5.4 implies that property (i) holds in Theorem 5.3. Using Honda-Tate theory one shows that the μ-basic locus of $\mathcal{M}$ is non-empty, compare [Z4]. Therefore as in the proof of Theorem 3.3, Theorem 5.4 implies that $\mathcal{M}_b$ is non-empty for all $b \in B(G, \mu)$, and the purity theorem allows one now to deduce also property (ii) in Theorem 5.3 from Theorem 5.4. $\square$

We mention that when F is an imaginary quadratic field such that $p \ne 2$ is inert, Bültel and Wedhorn [BW] have proved (i) and (ii) of Conjecture 5.2, provided that the signature of the skew-hermitian form $\langle \ , \ \rangle$ on V is of the form $(n-1, 1)$. On the other hand, the conjecture seems to be open even for such classical moduli spaces as the Hilbert-Blumenthal varieties.

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