The Langlands lemma and the Betti numbers of stacks of G–bundles on a curve

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Atiyah and Bott [AB] and Harder and Narasimhan [HN] have established a formula for the Poincaré series (the generating series formed using the Betti numbers) of the stack $\mathcal{M}(G,\nu'_G)$ of G–bundles with slope ν'_G on a Riemann surface, which expresses it in terms of the Poincaré series of the open substack of semi–stable G–bundles and the similar Poincaré series for all standard Levi subgroups of G. This relation is a consequence of the Harder–Narasimhan stratification of $\mathcal{M}(G,\nu'_G)$. A similar relation arises in the context of period domains over a finite or p–adic field, where the Euler–Poincaré characteristic of a generalized flag variety of a reductive group is expressed in terms of the Euler–Poincaré characteristics of the period domains associated with the various standard Levi subgroups of G (cf. [Rap]). These relations can be considered as recursion relations expressing the Poincaré series (resp. Euler–Poincaré characteristic) of the semi–stable sublocus in terms of the corresponding quantities for the ambiant spaces for G and its Levi subgroups.

In this paper we show that the Langlands lemma from the theory of Eisenstein series, which has become a standard tool in the development of the Arthur–Selberg trace formula, can be used to invert the recursion relation for the Poincaré series of the open substack of semi–stable G–bundles. This note is therefore of a purely combinatorial nature.

This application of the Langlands lemma has been noticed by Kottwitz (in the context of p–adic period domains). Our only contribution has been to formalize this suggestion in a different context. In the case of vector bundles on a curve the inversion of the recursion formula had been obtained earlier by Zagier [Za] using different techniques.

The paper is organized as follows. In Section 1 we fix our notations and recall the Langlands lemma. In section 2 we use the lemma to prove a general inversion formula. In section 3 we explain how to apply this inversion formula to the theory of G–bundles on a curve. The special case of vector bundles is discussed in section 4.

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1. The Langlands lemma.

Let G be a reductive algebraic group over a perfect field k. We fix a minimal parabolic subgroup P_0 of G and a Levi subgroup M_0. We denote by $\mathcal{P}$ the set of standard parabolic subgroups of G, i.e. parabolic subgroups of G containing P_0.

If $P \in \mathcal{P}$, we denote by N_P its unipotent radical and by M_P the unique Levi subgroup of P containing M_0. Moreover, we denote by

$$A_P = \mathrm{Hom}_{k-\mathrm{gr}}(\mathbb{G}_{\mathrm{m},k}, Z_P) \otimes \mathbb{G}_{\mathrm{m},k}$$

the maximal split torus in the center Z_P of M_P and by

$$A'_P = \mathrm{Hom}(\mathrm{Hom}_{k-\mathrm{gr}}(M_{P,\mathrm{ab}}, \mathbb{G}_{\mathrm{m},k}), \mathbb{G}_{\mathrm{m},k})$$

the maximal quotient split torus of $M_{P,\mathrm{ab}}$. The composite map

$$A_P \hookrightarrow Z_P \hookrightarrow M_P \twoheadrightarrow M_{P,\mathrm{ab}} \twoheadrightarrow A'_P$$

is an isogeny. In particular, we have an injective map of free abelian groups of the same finite rank

$$X_*(A_P) = \mathrm{Hom}_{k-\mathrm{gr}}(\mathbb{G}_{\mathrm{m},k}, Z_P) \hookrightarrow \mathrm{Hom}(\mathrm{Hom}_{k-\mathrm{gr}}(M_{P,\mathrm{ab}}, \mathbb{G}_{\mathrm{m},k}), \mathbb{Z}) = X_*(A'_P) \ .$$

Following Arthur [Ar1], for each $P \in \mathcal{P}$, we set

$$\mathfrak{a}_P = \mathbb{R} \otimes X_*(A_P) = \mathbb{R} \otimes X_*(A'_P) \ .$$

If $P \subset Q$ are two standard parabolic subgroups of G, we have canonical maps

$$A_Q \hookrightarrow A_P \hookrightarrow A'_P \twoheadrightarrow A'_Q \ .$$

The canonical maps $A_Q \hookrightarrow A_P$ and $A'_P \twoheadrightarrow A'_Q$ induce a canonical embedding $\mathfrak{a}_Q \hookrightarrow \mathfrak{a}_P$ and a canonical retraction $\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$. Hence, we have a canonical splitting

$$\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q \ ,$$

where $\mathfrak{a}_P^Q$ is the kernel of the retraction. Taking the dual real vector spaces, we get a splitting

$$\mathfrak{a}_P^* = \mathfrak{a}_P^{Q*} \oplus \mathfrak{a}_Q^* \ .$$

More generally, if $P \subset Q \subset R$ are three standard parabolic subgroups of G, we have canonical splittings

$$\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q^R \oplus \mathfrak{a}_R$$

and

$$\mathfrak{a}_P^* = \mathfrak{a}_P^{Q*} \oplus \mathfrak{a}_Q^{R*} \oplus \mathfrak{a}_R^* \ .$$

We shall denote by $[\cdot]^Q$, $[\cdot]_Q^R$ and $[\cdot]_R$ the canonical projections of $\mathfrak{a}_P$ onto $\mathfrak{a}_P^Q$, $\mathfrak{a}_Q^R$ and $\mathfrak{a}_R$ respectively.

For each $P \in \mathcal{P}$, let $\Phi_P \subset \mathfrak{a}_P^{G*} \subset \mathfrak{a}_P^*$ be the set of the non trivial characters of A_P which occur in the Lie algebra $\mathfrak{g}$ of G and let $\Phi_P^+ \subset \Phi_P$ be the set of the non trivial characters of A_P which occur in the Lie algebra $\mathfrak{n}_P$ of the unipotent radical of P. It is well known that $\Phi_0 = \Phi_{P_0}$ is a root system and that $\Phi_0^+ = \Phi_{P_0}^+$ is an order on Φ_0. Let $\Delta_0 = \Delta_{P_0} \subset \Phi_0^+$ be the set of simple roots; Δ_0 is a basis of the real vector space $\mathfrak{a}_{P_0}^{G*}$. For each $\alpha \in \Phi_0$, there is a corresponding coroot $\alpha^\vee$ and $(\alpha^\vee)_{\alpha\in\Delta_0}$ is a basis of the real vector space $\mathfrak{a}_0^G = \mathfrak{a}_{P_0}^G \subset \mathfrak{a}_{P_0} = \mathfrak{a}_0$. For the other P's in $\mathcal{P}$, Φ_P is not a root system in general. Nevertheless, following Arthur, we define $\Delta_P \subset \Phi_P^+$ as the set of non trivial restrictions to A_P (or $\mathfrak{a}_P$) of the simple roots in Δ_0. Then, Δ_P is a basis of the real vector space $\mathfrak{a}_P^{G*}$ and, for each $\alpha \in \Delta_P$, there is a corresponding "coroot" $\alpha^\vee \in \mathfrak{a}_P^G$ with the property that $(\alpha^\vee)_{\alpha\in\Delta_P}$ is a basis of the real vector space $\mathfrak{a}_P^G$: α is the restriction to A_P of a unique $\beta \in \Delta_0$ and $\alpha^\vee$ is the projection of $\beta^\vee$ onto $\mathfrak{a}_P^G$.

If $P \subset Q$ are two standard parabolic subgroups of G, let $\Phi_P^Q = \Phi_{P\cap M_Q}$ (resp. $\Phi_P^{Q+} = \Phi_{P\cap M_Q}^+$, resp. $\Delta_P^Q = \Delta_{P\cap M_Q}$) be the set of α in Φ_P (resp. Φ_P^+, resp. Δ_P) which occur in the Lie algebra $\mathfrak{m}_Q$ of M_Q. Then, on the one hand, Δ_P^Q is contained in $\mathfrak{a}_P^{Q*} \subset \mathfrak{a}_P^{G*}$ and is a basis of the real vector space $\mathfrak{a}_P^{Q*}$. On the other hand, the projection of $(\alpha^\vee)_{\alpha\in\Delta_P^Q}$ onto $\mathfrak{a}_P^Q$ is a basis of the real vector space $\mathfrak{a}_P^Q$ and we may consider its dual basis $(\varpi_\alpha^Q)_{\alpha\in\Delta_P^Q} \subset \mathfrak{a}_P^{Q*}$.

If $P \subset Q \subset R$ are three standard parabolic subgroups of G and if $H \in \mathfrak{a}_P^R$, we have

$$\langle \alpha, [H]^Q \rangle = \langle \alpha, H \rangle \,, \qquad \forall \alpha \in \Delta_P^Q \subset \Delta_P^R$$

and

$$\langle \varpi_\alpha^R, [H]_Q \rangle = \langle \varpi_\beta^R, H \rangle \,, \qquad \forall \alpha \in \Delta_Q^R \,,$$

where β is the unique element in Δ_P^R such that $\alpha = \beta | A_Q$.

Lemma 1.1. — *Let $P \subset R$ be two standard parabolic subgroups of G and let $H \in \mathfrak{a}_P^R$.*

(i) *Let us assume that $\langle \alpha, H \rangle > 0$ or $\langle \varpi_\alpha, H \rangle > 0$ for each $\alpha \in \Delta_P^R$. Then, we have $\langle \varpi_\alpha, H \rangle > 0$ for all $\alpha \in \Delta_P^R$.*

(ii) *Let us assume that $\langle \alpha, H \rangle \leq 0$ or $\langle \varpi_\alpha, H \rangle \leq 0$ for each $\alpha \in \Delta_P^R$. Then, we have $\langle \varpi_\alpha, H \rangle \leq 0$ for all $\alpha \in \Delta_P^R$.*

Proof : See [La] 3.1.

∎

If $P \subset Q$ are two standard parabolic subgroups of G, Arthur has introduced two characteristic functions on the real vector space $\mathfrak{a}_P^Q$: the characteristic function τ_P^Q of the acute Weyl chamber

$$\mathfrak{a}_P^{Q+} = \{H \in \mathfrak{a}_P^Q \mid \langle \alpha, H \rangle > 0 \,,\ \forall \alpha \in \Delta_P^Q\}$$

and the characteristic function $\widehat{\tau}_P^Q$ of the obtuse Weyl chamber

$$^+\mathfrak{a}_P^Q = \{H \in \mathfrak{a}_P^Q \mid \langle \varpi_\alpha^Q, H \rangle > 0 \,,\ \forall \alpha \in \Delta_P^Q\} \ .$$

It follows from lemma 1.1 (i) that $\mathfrak{a}_P^{Q+} \subset {}^+\mathfrak{a}_P^Q$.

Lemma 1.2 (Langlands). — *For any standard parabolic subgroups $P \subset R$ of G and any $H \in \mathfrak{a}_P^R$, we have*

$$\sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_Q^R)} \tau_P^Q([H]^Q)\widehat{\tau}_Q^R([H]_Q) = \delta_P^R \ .$$

and

$$\sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_P^Q)} \widehat{\tau}_P^Q([H]^Q)\tau_Q^R([H]_Q) = \delta_P^R \ .$$

Proof : See [Ar1] §6 or [La] 3.2.

□

Following Arthur (see [Ar2] §2), we set

$$\Gamma_P^R(H,T) = \sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_Q^R)} \tau_P^Q([H]^Q)\widehat{\tau}_Q^R([H-T]_Q)$$

and

$$\widehat{\Gamma}_P^R(H,T) = \sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_P^Q)} \tau_P^Q([H-T]^Q)\widehat{\tau}_Q^R([H]_Q) = (-1)^{\dim(\mathfrak{a}_P^R)}\Gamma_P^R(H-T,-T)\,,$$

for any standard parabolic subgroups $P \subset R$ of G and for any $H, T \in \mathfrak{a}_P^R$. As an immediate consequence of the Langlands lemma, we obtain

$$\sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_Q^R)} \Gamma_P^Q(H,T)\widehat{\Gamma}_Q^R(H,T) = \delta_P^R$$

and

$$\sum_{P \subset Q \subset R} (-1)^{\dim(\mathfrak{a}_P^Q)} \widehat{\Gamma}_P^Q(H,T)\Gamma_Q^R(H,T) = \delta_P^R \ .$$

Lemma 1.3 (Arthur). — *If $T \in \mathfrak{a}_P^{R+} \subset {}^+\mathfrak{a}_P^R$, the function $H \mapsto \Gamma_P^R(H,T)$ (resp. $H \mapsto \widehat{\Gamma}_P^R(H,T)$) is the characteristic function of the bounded subset*

$$\{H \in \mathfrak{a}_P^R \mid \langle \alpha, H\rangle > 0\,,\ \langle \varpi_\alpha, H\rangle \leq \langle \varpi_\alpha, T\rangle\,,\ \forall \alpha \in \Delta_P^R\} \subset \mathfrak{a}_P^{R+}$$

(*resp.*

$$\{H \in \mathfrak{a}_P^R \mid \langle \varpi_\alpha^R, H\rangle > 0\,,\ \langle \alpha, H\rangle \leq \langle \alpha, T\rangle\,,\ \forall \alpha \in \Delta_P^R\} \subset {}^+\mathfrak{a}_P^R\,)$$

of $\mathfrak{a}_P^R$.

In particular, when T goes to infinity, the supports of the functions $H \mapsto \Gamma_P^R(H,T)$ (resp. $H \mapsto \widehat{\Gamma}_P^R(H,T)$) cover $\mathfrak{a}_P^{R+}$ (resp. ${}^+\mathfrak{a}_P^R$) (by definition, $T \in \mathfrak{a}_P^{R+}$ *goes to infinity* if $\langle \alpha, T\rangle > 0$ goes to infinity, for each $\alpha \in \Delta_P^R$).

Proof : Let us prove the statement about $\Gamma_P^R(H,T)$. Let us fix H and let us set

$$I = \{\alpha \in \Delta_P^R \mid \langle \varpi_\alpha^R, H - T\rangle \leq 0\}$$

and

$$J = \{\alpha \in \Delta_P^R \mid \langle \alpha, H\rangle > 0\}\ .$$

We clearly have

$$\Gamma_P^R(H,T) = (-1)^{|\Delta_P^R - I|} \sum_{\substack{P\subset Q\subset R \\ I\subset \Delta_P^Q \subset J}} (-1)^{|\Delta_P^Q - I|} = (-1)^{|\Delta_P^R - I|}\delta_I^J\ .$$

Therefore, $\Gamma_P^R(H,T) \neq 0$ if and only if $I = J$. Now, if $I = J$, we have

$$\langle \alpha, H - T\rangle \leq -\langle \alpha, T\rangle < 0\,, \qquad \forall \alpha \in \Delta_P^R - I$$

and

$$\langle \varpi_\alpha^R, H - T\rangle \leq 0\,, \qquad \forall \alpha \in I\,,$$

so that

$$\langle \varpi_\alpha^R, H - T\rangle \leq 0\,, \qquad \forall \alpha \in \Delta_P^R$$

by the lemma 1.1 (ii). Therefore, if $I = J$, we have $I = J = \Delta_P^R$ and H satifies the relations

$$\langle \alpha, H\rangle > 0$$

and

$$\langle \varpi_\alpha, H\rangle \leq \langle \varpi_\alpha, T\rangle$$

for all $\alpha \in \Delta_P^R$. Conversely, if H satisfies these relations, it is obvious that $I = J = \Delta_P^R$.

The proof of the statement about $\widehat{\Gamma}_P^R(H,T)$ is similar.

□

2. A general inversion formula.

We denote by $\mathfrak{P}$ the set of pairs (P, ν'_P), where $P \in \mathcal{P}$ and $\nu'_P \in X_*(A'_P)$. We fix a topological abelian group A. A function

$$a : \mathfrak{P} \to A$$

is said to be $\widehat{\Gamma}$*–converging* if it has the following property :

For each standard parabolic subgroup $P \subset Q$ of G and each $\nu'_Q \in X_*(A'_Q)$, the finite sum

$$\sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \widehat{\Gamma}_P^Q([\nu'_P]^Q, T)a(P, \nu'_P)$$

admits a limit as $T \in \mathfrak{a}_P^{Q+}$ *goes to infinity.*

If this is the case, we shall denote this limit by

$$\sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \widehat{\tau}_P^Q([\nu'_P]^Q) a(P, \nu'_P) \ .$$

A function

$$b : \mathfrak{P} \to A$$

is said to be Γ–*converging* if it has the following property :

For each standard parabolic subgroup $P \subset Q$ *of* G *and each* $\nu'_Q \in X_*(A'_Q)$*, the finite sum*

$$\sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \Gamma_P^Q([\nu'_P]^Q, T) b(P, \nu'_P)$$

admits a limit as $T \in \mathfrak{a}_P^{Q+}$ *goes to infinity.*

If this is the case, we shall denote this limit by

$$\sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \tau_P^Q([\nu'_P]^Q) b(P, \nu'_P) \ .$$

Theorem 2.1. — *For each* $\widehat{\Gamma}$*–converging function* $a : \mathfrak{P} \to A$*, there exists a unique* Γ*–converging function* $b : \mathfrak{P} \to A$ *such that, for each* $(Q, \nu'_Q) \in \mathfrak{P}$*, we have*

$$a(Q, \nu'_Q) = \sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} \sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \tau_P^Q([\nu'_P]^Q) b(P, \nu'_P) \ .$$

The function b *is given by the following formula : for each* $(Q, \nu'_Q) \in \mathfrak{P}$*, we have*

$$b(Q, \nu'_Q) = \sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} (-1)^{\dim(\mathfrak{a}_P^Q)} \sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_Q = \nu'_Q}} \widehat{\tau}_P^Q([\nu'_P]^Q) a(P, \nu'_P) \ .$$

Proof : This is an easy consequence of lemmas 1.2 and 1.3.

$\square$

Let us now consider a particular case of this theorem which is relevant for the computation of the Poincaré series of the stack of semi–stable G–bundles on a curve.

For any standard parabolic subgroup P of G, we fix $n_P \in \mathbb{Z}_{\geq 0}$ and $\delta_0^P \in \mathfrak{a}_0^{P*} \subset \mathfrak{a}_0^*$. We assume that, for any standard parabolic subgroups $P \subset Q$ of G, we have

$$n_P \geq n_Q \ ,$$

$$(\delta_0^Q - \delta_0^P)|\mathfrak{a}_0^P = 0$$

and

$$\langle \delta_P^Q, \alpha^\vee \rangle \in \mathbb{Z}_{>0} \qquad (\forall \alpha \in \Delta_P^Q)$$

where we have set

$$\delta_P^Q = (\delta_0^Q - \delta_0^P)|\mathfrak{a}_P^Q \ .$$

We have

$$\langle \delta_P^Q, [H]^Q \rangle = \langle \delta_P^G, H \rangle - \langle \delta_Q^G, [H]_Q \rangle$$

for every $H \in \mathfrak{a}_P$.

We set

$$m(P, \nu'_P) = n_P + \langle \delta_P^G, \nu'_P \rangle$$

for each $(P, \nu'_P) \in \mathfrak{P}$.

Lemma 2.2. — *Let $(Q, \nu'_Q) \in \mathfrak{P}$.*

(i) *For each $(P, \nu'_P) \in \mathfrak{P}$ such that $P \subset Q$, $[\nu'_P]_Q = \nu'_Q$ and $\widehat{\tau}_P^Q([\nu'_P]^Q) \neq 0$, we have*

$$m(P, \nu'_P) \geq m(Q, \nu'_Q) \ .$$

(ii) *For each positive integer m, there are only finitely many $(P, \nu'_P) \in \mathfrak{P}$ such that $P \subset Q$, $[\nu'_P]_Q = \nu'_Q$, $\widehat{\tau}_P^Q([\nu'_P]^Q) \neq 0$ and $m(P, \nu'_P) \leq m$.*

$\square$

We take A to be a $\mathbb{Z}[[t]]$–module equipped with the t–adic topology and we assume that A is complete for this topology. We consider an arbitrary function

$$a_0 : \mathcal{P} \to A$$

and set

$$a(P, \nu'_P) = a_0(P) t^{m(P, \nu'_P)} \in A$$

for any $(P, \nu'_P) \in \mathfrak{P}$. It follows from part (ii) of lemma 2.2 that the function a is $\widehat{\Gamma}$–converging. Therefore, by theorem 2.1, there exists a unique Γ–converging function

$$b : \mathfrak{P} \to A$$

such that
$$a(Q,\nu'_Q)=\sum_{\substack{P\in\mathcal{P}\\ P\subset Q}}\ \sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\tau_P^Q([\nu'_P]^Q)b(P,\nu'_P)\ .$$
Moreover, the function b is given by
$$b(Q,\nu'_Q)=b_0(Q,\nu'_Q)t^{m(Q,\nu'_Q)}\qquad(\forall(Q,\nu'_Q)\in\mathfrak{P})\,,$$
where
$$b_0(Q,\nu'_Q)=\sum_{\substack{P\in\mathcal{P}\\ P\subset Q}}(-1)^{\dim(\mathfrak{a}_P^Q)}a_0(P)\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\widehat{\tau}_P^Q([\nu'_P]^Q)t^{m(P,\nu'_P)-m(Q,\nu'_Q)}\in A$$
for any $(Q,\nu'_Q)\in\mathfrak{P}$ (cf. Lemma 2.2 (i)).

Let us consider the lattices
$$\sum_{\alpha\in\Delta_P^Q}\mathbb{Z}\alpha^\vee\subset X_*(A'_P)$$
and let us set
$$\Lambda_P^Q=X_*(A'_P)/\sum_{\alpha\in\Delta_P^Q}\mathbb{Z}\alpha^\vee\ .$$
Clearly, the projection $[\cdot]_Q:X_*(A'_P)\to\mathfrak{a}_Q$ factors through Λ_P^Q and, for each $\alpha\in\Delta_P^Q$, $\varpi_\alpha^Q:X_*(A'_P)\to\mathbb{R}$ induces a homomorphism from Λ_P^Q to $\mathbb{R}/\mathbb{Z}$.

Lemma 2.3. — *For each $(Q,\nu'_Q)\in\mathfrak{P}$ and each standard parabolic subgroup $P\subset Q$ of G, we have*
$$\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\widehat{\tau}_P^Q([\nu'_P]^Q)t^{\langle\delta_P^Q,[\nu'_P]^Q\rangle}$$
$$=\Big(\prod_{\alpha\in\Delta_P^Q}\frac{1}{1-t^{\langle\delta_P^Q,\alpha^\vee\rangle}}\Big)\sum_{\substack{\lambda\in\Lambda_P^Q\\ [\lambda]_Q=\nu'_Q}}t^{\sum_{\alpha\in\Delta_P^Q}\langle\delta_P^Q,\alpha^\vee\rangle\langle\varpi_\alpha^Q(\lambda)\rangle}\,,$$
where, for each $\mu\in\mathbb{R}/\mathbb{Z}$, $\langle\mu\rangle\in\mathbb{R}$ is the unique representative of the class μ such that $0<\langle\mu\rangle\leq 1$.

As $\delta_P^Q=\sum_{\alpha\in\Delta_P^Q}\langle\delta_P^Q,\alpha^\vee\rangle\varpi_\alpha^Q$, we have
$$\sum_{\alpha\in\Delta_P^Q}\langle\delta_P^Q,\alpha^\vee\rangle\langle\varpi_\alpha^Q([\nu'_P]^Q+\mathbb{Z})\rangle\equiv\langle\delta_P^Q,[\nu'_P]^Q\rangle\equiv 0\qquad(\mathrm{mod}\ \mathbb{Z})$$
for any $\nu'_P\in X_*(A'_P)$.

Proof : We have

$$\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \widehat{\tau}_P^Q([\nu'_P]^Q)t^{\langle\delta_P^Q,[\nu'_P]^Q\rangle} = \sum_{\substack{\lambda\in\Lambda_P^Q\\ [\lambda]_Q=\nu'_Q}} t^{\langle\delta_P^Q,\dot{\lambda}\rangle}\prod_{\alpha\in\Delta_P^Q}\sum_{\substack{m_\alpha\in\mathbb{Z}\\ m_\alpha+\varpi_\alpha(\dot{\lambda})>0}} t^{\langle\delta_P^Q,\alpha^\vee\rangle m_\alpha}$$

where $\dot{\lambda}\in X_*(A'_P)$ is a representative of the class λ. But, for each $p\in\mathbb{Z}_{>0}$ and each $x\in\mathbb{R}$, we have

$$\sum_{\substack{m\in\mathbb{Z}\\ m+x>0}} t^{pm} = \frac{t^{p(\langle x+\mathbb{Z}\rangle-x)}}{1-t^p}\ .$$

Hence, we have

$$\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \widehat{\tau}_P^Q([\nu'_P]^Q)t^{\langle\delta_P^Q,[\nu'_P]^Q\rangle} = \sum_{\substack{\lambda\in\Lambda_P^Q\\ [\lambda]_Q=\nu'_Q}}\prod_{\alpha\in\Delta_P^Q}\frac{t^{\langle\delta_P^Q,\alpha^\vee\rangle\langle\varpi_\alpha^Q(\lambda)\rangle}}{1-t^{\langle\delta_P^Q,\alpha^\vee\rangle}}\ .$$

∎

To sum up, we can state :

Theorem 2.4. — *There exists a unique function $b_0:\mathfrak{P}\to A$ which satisfies the relation*

$$a_0(Q)=\sum_{\substack{P\in\mathcal{P}\\ P\subset Q}}\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}}\tau_P^Q([\nu'_P]^Q)b_0(P,\nu'_P)t^{m(P,\nu'_P)-m(Q,\nu'_Q)}\,,$$

for each $(Q,\nu'_Q)\in\mathfrak{P}$. This function is given by

$$b_0(Q,\nu'_Q)=\sum_{\substack{P\in\mathcal{P}\\ P\subset Q}}(-1)^{\dim(\mathfrak{a}_P^Q)}a_0(P)t^{n_P-n_Q}\Big(\prod_{\alpha\in\Delta_P^Q}\frac{1}{1-t^{\langle\delta_P^Q,\alpha^\vee\rangle}}\Big)\cdot$$
$$\cdot\sum_{\substack{\lambda\in\Lambda_P^Q\\ [\lambda]_Q=\nu'_Q}} t^{\sum_{\alpha\in\Delta_P^Q}\langle\delta_P^Q,\alpha^\vee\rangle\langle\varpi_\alpha^Q(\lambda)\rangle}\in A\,,$$

for each $(Q,\nu'_Q)\in\mathfrak{P}$. ∎

Remark 2.5. — It follows from the definition of the function b_0 that $b_0(P,\nu'_P)$ only depends on the class $\overline{\nu}'_P$ of ν'_P in $X_*(A'_P)/X_*(A_P)$.

∎

3. Application to G-bundles.

From now on, let us assume that k is algebraically closed, so that P_0 is a Borel subgroup of G, and let us fix a smooth, projective and connected curve X of genus $g \geq 2$ over k.

Let us recall that, for any $P \in \mathcal{P}$ and any P–bundle $\mathcal{T}_P$ on X, the *slope* of $\mathcal{T}_P$ is the element $\mu(\mathcal{T}_P) \in X_*(A'_P)$ defined by the condition

$$\langle \xi, \mu(\mathcal{T}_P)\rangle = c_1(\mathcal{T}_{P,\xi}) \in \mathbb{Z}\,, \qquad \forall \xi \in X^*(A'_P)\,,$$

where $\mathcal{T}_{P,\xi}$ is the line bundle on X deduced from $\mathcal{T}_P$ by push–out via $P \twoheadrightarrow A'_P \xrightarrow{\xi} GL_1$. Let us also recall that a P–bundle $\mathcal{T}_P$ on X is said to be *semi–stable* (see [Ra]) if, for each standard parabolic subgroup $Q \subset P$ such that $|\Delta_Q^P| = 1$ and for each Q–bundle $\mathcal{T}_Q$ on X such that

$$\mathcal{T}_P \cong \mathcal{T}_Q \times^Q P\,,$$

the slope $\mu(\mathcal{T}_Q) \in X_*(A'_Q) \subset \mathfrak{a}_Q$ satisfies

$$\langle \alpha_Q, \mu(\mathcal{T}_Q)\rangle \leq 0\,,$$

where α_Q is the unique element of Δ_Q^P.

Lemma 3.1. — *For each G–bundle $\mathcal{T}$ of slope ν'_G, there exist $(P, \nu'_P) \in \mathfrak{P}$ with $[\nu'_P]_G = \nu'_G$ and $[\nu'_P]^G \in \mathfrak{a}_P^{G+}$, a semi–stable P–bundle $\mathcal{T}_P$ of slope ν'_P and an isomorphism*

$$\iota : \mathcal{T}_P \times^P G \xrightarrow{\sim} \mathcal{T}\ .$$

Moreover, the pair (P, ν'_P) and the isomorphism class of the pair $(\mathcal{T}_P, \iota)$ are uniquely determined by $\mathcal{T}$.

The pair (P, ν'_P) is called the *Harder–Narasimhan type* of $\mathcal{T}$ and the pair $(\mathcal{T}_P, \iota)$ is called the *Harder–Narasimhan reduction* of $\mathcal{T}$.

Proof : See [HN] in the case of $G = GL_n$ and [AB] in general.

□

For each $\nu'_G \in X_*(A'_G)$, we wish to consider the Poincaré series

$$P_t^{\mathrm{ss}}(G, \nu'_G) \in \mathbb{Z}[[t]]$$

of the stack $\mathcal{M}^{\mathrm{ss}}(G, \nu'_G)$ of semi–stable G–bundles on X of slope ν'_G. There are at least three ways to make sense of this series. Harder and Narasimhan ([HN]) count (in a weighted way) in the case $k = \overline{\mathbb{F}}_p$ the number of semi–stable bundles which are defined over a finite subfield of k and obtain this series as a consequence of Deligne's purity theorem. Atiyah and Bott ([AB]) consider in the case $k = \mathbb{C}$ the action of a gauge group on the space of complex structures on a $\mathcal{C}^\infty$–bundle on the Riemann surface $X(\mathbb{C})$ and define $P_t^{\mathrm{ss}}(G, \nu'_G)$ as the Poincaré series for the equivariant cohomology of the semi–stable open subset. Bifet, Ghione and Letizia ([BGL]) consider an ind-variety of semi–stable matrix divisors and obtain $P_t^{\mathrm{ss}}(G,\nu'_G)$ in terms of its ℓ–adic cohomology. Most probably, $P_t^{\mathrm{ss}}(G,\nu'_G)$ is also the Poincaré series of the smooth algebraic stack of semi–stable G–bundles on X of slope ν'_G for the ℓ–adic cohomology.

We point out that this Poincaré series is not the Poincaré polynomial of the coarse moduli scheme of semi–stable G–bundles on X of slope ν'_G (for a relation in a special case, see section 4). In fact, it is not even a polynomial in general.

There is a recursion formula for $P_t^{\mathrm{ss}}(G,\nu'_G)$, as follows. For each $\nu'_G \in X_*(A'_G)$, we have the stack $\mathcal{M}(G,\nu'_G)$ of G–bundles on X of slope ν'_G. For each $(P,\nu'_P)\in\mathfrak{P}$ such that $[\nu'_P]_G=\nu'_G$ and $[\nu'_P]^G\in\mathfrak{a}_P^{G+}$, we also have the substack $\mathcal{M}(G,P,\nu'_P)\subset\mathcal{M}(G,\nu'_G)$ of G–bundles on X of slope ν'_G which admit (P,ν'_P) as Harder–Narasimhan type. The family of $\mathcal{M}(G,P,\nu'_P)$ is a stratification of $\mathcal{M}(G,\nu'_G)$, with $\mathcal{M}^{\mathrm{ss}}(G,\nu'_G)=\mathcal{M}(G,G,\nu'_G)$ as the open stratum. The codimension of the stratum $\mathcal{M}(G,P,\nu'_P)$ is equal to

$$\dim(N_P)(g-1)+2\langle\rho_P^G,\nu'_P\rangle\,,$$

where

$$\rho_P^G=\frac{1}{2}\sum_{\alpha\in\Phi_P^{G+}}\alpha\in\mathfrak{a}_P^{G*}\subset\mathfrak{a}_P^*\ .$$

We set

$$m(P,\nu'_P)=2\dim(N_P)(g-1)+4\langle\rho_P^G,\nu'_P\rangle\ .$$

We have the Poincaré series $P_t(G,\nu'_G)$ of $\mathcal{M}(G,\nu'_G)$ and also the Poincaré series $P_t(G,P,\nu'_P)$ of $\mathcal{M}(G,P,\nu'_P)$ for any Harder–Narasimhan type (P,ν'_P).

In all the above definitions we may replace G by the Levi component M_P of any standard parabolic subgroup P of G. For each Harder–Narasimhan type (P,ν'_P) we have a fibration

$$\mathcal{M}(G,P,\nu'_P)\to\mathcal{M}^{\mathrm{ss}}(M_P,\nu'_P)$$

given by $\mathcal{T}\mapsto\mathcal{T}_P/N_P$, where $(\mathcal{T}_P,\iota)$ is the Harder–Narasimhan reduction of $\mathcal{T}$.

Theorem 3.2 (Harder–Narasimhan ; Atiyah–Bott). — *The stratification of $\mathcal{M}(G,\nu'_G)$ by the $\mathcal{M}(G,P,\nu'_P)$ is perfect modulo torsion, so that for the Poincaré series we have*

$$P_t(G,\nu'_G)=\sum_{P\in\mathcal{P}}\sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_G=\nu'_G}}\tau_P^G([\nu'_P]^G)t^{m(P,\nu'_P)}P_t(G,P,\nu'_P)\ .$$

Moreover, for each Harder–Narasimhan type (P,ν'_P), the above fibration is acyclic and we have

$$P_t(G,P,\nu'_P)=P_t^{\mathrm{ss}}(M_P,\nu'_P)\ .$$

Again, in this theorem, we may replace G by the Levi component of any standard parabolic subgroup of G.

Proof : See [HN] and [AB] Theorem 10.10.

∎

The Weyl group W_0^G of A_0 in G acts on the real vector space $\mathfrak{a}_0^G$ and, therefore, on the graded algebra $\mathrm{Sym}(\mathfrak{a}_0^{G*})$ of polynomials on $\mathfrak{a}_0^G$. It is well–known that the algebra of invariants

$$\mathrm{Sym}(\mathfrak{a}_0^{G*})^{W_0^G}$$

(with its grading) is isomorphic to an algebra of polynomials

$$\mathbb{R}[I_1,\ldots,I_{\dim(\mathfrak{a}_0^G)}]\,,$$

where I_1, ..., $I_{\dim(\mathfrak{a}_0^G)}$ are algebraically independent homogeneous polynomials on $\mathfrak{a}_0^G$ of degree ≥ 2. Let us denote by $d_1(G),\ldots,d_{\dim(\mathfrak{a}_0^G)}(G)$ the degrees of these homogeneous polynomials. Up to a permutation, the sequence $(d_1(G),\ldots,d_{\dim(\mathfrak{a}_0^G)}(G))$ is canonically defined. If we set

$$\mathcal{W}_G(t)=\sum_{w\in W_0^G}t^{\ell(w)}=\prod_{\alpha\in\Phi_0^{G+}}\frac{t^{\langle\rho_0,\alpha^\vee\rangle+1}-1}{t^{\langle\rho_0,\alpha^\vee\rangle}-1}$$

($\ell : W_0^G\to\mathbb{Z}_{\geq 0}$ is the length function), we have

$$\mathcal{W}_G(t)=\prod_{i=1}^{\dim(\mathfrak{a}_0^G)}\frac{t^{d_i(G)}-1}{t-1}\,.$$

THEOREM 3.3. — *For any $\nu'_G\in X_*(A'_G)$, we have*

$$P_t(G,\nu'_G)=\Big(\frac{(1+t)^{2g}}{1-t^2}\Big)^{\dim(\mathfrak{a}_G)}\prod_{i=1}^{\dim(\mathfrak{a}_0^G)}\frac{(1+t^{2d_i(G)-1})^{2g}}{(1-t^{2d_i(G)-2})(1-t^{2d_i(G)})}\,.$$

In particular, $P_t(G,\nu'_G)$ does not depend on ν'_G.

Again, in this theorem, we may replace G by the Levi component of any standard parabolic subgroup of G.

Proof : See [AB] Theorem 2.15 for the case $G=GL_n$.

∎

THEOREM 3.4. — *For any $\nu'_G\in X_*(A'_G)$, the Poincaré series $P_t^{\mathrm{ss}}(G,\nu'_G)\in\mathbb{Z}[[t]]$ is equal to the expansion of the rational function*

$$\sum_{P\in\mathcal{P}}(-1)^{\dim(\mathfrak{a}_P^G)}\Big(\frac{(1+t)^{2g}}{1-t^2}\Big)^{\dim(\mathfrak{a}_P)}\Big(\prod_{i=1}^{\dim(\mathfrak{a}_0^P)}\frac{(1+t^{2d_i(M_P)-1})^{2g}}{(1-t^{2d_i(M_P)-2})(1-t^{2d_i(M_P)})}\Big)\cdot$$
$$\cdot t^{2\dim(N_P)(g-1)}\Big(\prod_{\alpha\in\Delta_P}\frac{1}{1-t^{4\langle\rho_P,\alpha^\vee\rangle}}\Big)\sum_{\substack{\lambda\in\Lambda_P^G\\ [\lambda]_G=\nu'_G}}t^{4\sum_{\alpha\in\Delta_P}\langle\rho_P,\alpha^\vee\rangle\langle\varpi_\alpha^G(\lambda)\rangle}$$

in $\mathbb{Q}(t)$.

Proof : Let

$$a_0 : \mathcal{P} \to \mathbb{Q}[[t]]$$

be the function defined by

$$a_0(P) = \Big(\frac{(1+t)^{2g}}{1-t^2}\Big)^{\dim(\mathfrak{a}_P)} \prod_{i=1}^{\dim(\mathfrak{a}_0^P)} \frac{(1+t^{2d_i(M_P)-1})^{2g}}{(1-t^{2d_i(M_P)-2})(1-t^{2d_i(M_P)})}\,.$$

Our function $m(P,\nu'_P)$ is of the form $n_P + \langle \delta_P^G, \nu'_P\rangle$ with $n_P = 2\dim(N_P)(g-1)$ and $\delta_P^G = 4\rho_P^G$ satisfying the hypotheses imposed in section 2. We may therefore apply theorem 2.4. Let b_0 be the unique function from $\mathfrak{P}$ to $\mathbb{Z}[[t]]$ which satisfies the relation

$$a_0(Q) = \sum_{\substack{P\in\mathcal{P}\\ P\subset Q}} \sum_{\substack{\nu'_P\in X_*(A'_P)\\ [\nu'_P]_Q=\nu'_Q}} \tau_P^Q([\nu'_P]^Q) b_0(P,\nu'_P) t^{m(P,\nu'_P)-m(Q,\nu'_Q)}\,,$$

for each $(Q,\nu'_Q)\in\mathfrak{P}$. It follows from theorems 3.2 and 3.3 that

$$b_0(G,\nu'_G) = P_t^{\mathrm{ss}}(G,\nu'_G)$$

and the theorem is proved.

$\square$

Remark 3.5. — It follows from this theorem and remark 2.5 that $P_t^{\mathrm{ss}}(G,\nu'_G)$ only depends on the class $\overline{\nu}'_G$ of ν'_G in $X_*(A'_G)/X_*(A_G)$. This can be viewed directly as follows. Let us arbitrarily choose a line bundle $\mathcal{L}$ of degree 1 on X. For any $\nu_G\in X_*(A_G)$, $\nu_{G*}\mathcal{L}$ is an A_G–bundle on X and the map

$$\mathcal{M}^{\mathrm{ss}}(G,\nu'_G) \to \mathcal{M}^{\mathrm{ss}}(G,\nu'_G+\nu_G)\,,\ \mathcal{T}\mapsto \mathcal{T}\times^{A_G}\nu_{G*}\mathcal{L}$$

is an isomorphism of algebraic stacks.

4. The case of vector bundles.

Let us consider the particular case $G=GL_n$. Let us take for P_0 the Borel subgroup of upper triangular matrices, so that $A_{P_0}=A'_{P_0}=(GL_1)^n$, $\mathfrak{a}_0=\mathbb{R}^n$ with standard coordinates $(H_1,\ldots,H_n)$, $\Phi_0=\{H_i-H_j\mid i\neq j\}$, $\Phi_0^+=\{H_i-H_j\mid i<j\}$ and $\Delta_0=\{H_i-H_{i+1}\mid i=1,\ldots,n-1\}$. Then, the standard parabolic subgroups of G are in one to one correspondence with the partitions of n.

Let P be a standard parabolic of G which corresponds to the partition $(n_1,\ldots,n_s)$. Then we have

$$\mathfrak{a}_P = \{H\in\mathbb{R}^n\mid H_1=\cdots=H_{n_1},\ldots,H_{n_1+\cdots+n_{s-1}+1}=\cdots=H_n\}\,,$$

$$\Delta_P = \{(H_{n_1+\cdots+n_j} - H_{n_1+\cdots+n_j+1})|\mathfrak{a}_P \mid j = 1, \ldots, s-1\}\,,$$

and, for any $\alpha = (H_{n_1+\cdots+n_j} - H_{n_1+\cdots+n_j+1})|\mathfrak{a}_P \in \Delta_P$,

$$\alpha^\vee = (0, \ldots, 0, \frac{1}{n_j}, \ldots, \frac{1}{n_j}, -\frac{1}{n_{j+1}}, \ldots, -\frac{1}{n_{j+1}}, 0, \ldots, 0)\,,$$

$$\varpi_\alpha^G = \big(H_1 + \cdots + H_{n_1+\cdots+n_j} - \frac{n_1 + \cdots + n_j}{n}(H_1 + \cdots + H_n)\big)|\mathfrak{a}_P$$

and

$$\langle \rho_P, \alpha^\vee \rangle = \frac{n_j + n_{j+1}}{2}\ .$$

The isomorphism

$$\mathfrak{a}_P \xrightarrow{\sim} \mathbb{R}^s\,,\ (H_1, \ldots, H_n) \mapsto (h_1, \ldots, h_s)$$

with $h_j = H_{n_1+\cdots+n_{j-1}+1} = \cdots = H_{n_1+\cdots+n_j}$ identifies

$$X_*(A_P) \subset X_*(A'_P) \subset \mathfrak{a}_P$$

with

$$\mathbb{Z}^s \subset \bigoplus_{j=1}^s \frac{1}{n_j}\mathbb{Z} \subset \mathbb{R}^s\,,$$

and $\alpha = (H_{n_1+\cdots+n_j} - H_{n_1+\cdots+n_j+1})|\mathfrak{a}_P \in \Delta_P$ with

$$h_j - h_{j+1}\,,$$

$\alpha^\vee$ with

$$(0, \ldots, 0, \frac{1}{n_j}, -\frac{1}{n_{j+1}}, 0, \ldots, 0)$$

and ϖ_α^G with

$$n_1h_1 + \cdots n_jh_j - \frac{n_1 + \cdots + n_j}{n}(n_1h_1 + \cdots + n_sh_s)\ .$$

Moreover, the composite map

$$\frac{1}{n_s}\mathbb{Z} \hookrightarrow \bigoplus_{j=1}^s \frac{1}{n_j}\mathbb{Z} \cong X_*(A'_P) \twoheadrightarrow \Lambda_P^G$$

is an isomorphism and, for any $\lambda = \frac{m}{n_s} \in \frac{1}{n_s}\mathbb{Z} \cong \Lambda_P^G$, we have

$$[\lambda]_G = \frac{m}{n} \in \frac{1}{n}\mathbb{Z} \cong X_*(A'_G)$$

and

$$\varpi_\alpha^G(\lambda) = -\frac{n_1 + \cdots + n_j}{n}m \in \mathbb{R}/\mathbb{Z}\,,\ \forall \alpha = h_j - h_{j+1} \in \Delta_P\ .$$

Therefore, we have

$$\prod_{\alpha\in\Delta_P}\frac{1}{1-t^{4\langle\rho_P,\alpha^\vee\rangle}}=\prod_{j=1}^{s-1}\frac{1}{1-t^{2(n_j+n_{j+1})}}$$

and, if $\nu'_G=\frac{d}{n}\in\frac{1}{n}\mathbb{Z}\cong X_*(A'_G)$, we have

$$\sum_{\substack{\lambda\in\Lambda_P^G\\ [\lambda]_G=\nu'_G}} t^{4\sum_{\alpha\in\Delta_P}\langle\rho_P,\alpha^\vee\rangle\langle\varpi_\alpha^G(\lambda)\rangle}=t^{2\sum_{j=1}^{s-1}(n_j+n_{j+1})\langle-\frac{n_1+\cdots+n_j}{n}d\rangle}$$

(it is easy to check directly that $\sum_{j=1}^{s-1}(n_j+n_{j+1})\langle-\frac{n_1+\cdots+n_j}{n}d\rangle\in\mathbb{Z}$).

The degrees of the invariant polynomials for $W_0^G\cong\mathfrak{S}_n$ acting on $\mathfrak{a}_0^G\cong\mathbb{R}^{n-1}$ are

$$2,3,\ldots,n\ .$$

Therefore, for any $\nu'_G\in X_*(A'_G)$, we have

$$P_t(G,\nu'_G)=\frac{(1+t)^{2g}}{1-t^2}\prod_{i=1}^{n}\frac{(1+t^{2i+1})^{2g}}{(1-t^{2i})(1-t^{2i+2})}\ .$$

From theorem 3.4, we conclude that the Poincaré series $P_t^{\mathrm{ss}}(GL_n,d/n)$ of the algebraic stack of semi–stable vector bundles of rank n and degree d on the curve X of genus $g\geq 2$ is equal to

$$\sum_{s=1}^{n}(-1)^{s-1}\Big(\frac{(1+t)^{2g}}{1-t^2}\Big)^s\sum_{\substack{n_1,\ldots,n_s\geq 1\\ n_1+\cdots+n_s=n}}\Big(\prod_{j=1}^{s}\prod_{i=1}^{n_j-1}\frac{(1+t^{2i+1})^{2g}}{(1-t^{2i})(1-t^{2i+2})}\Big)\cdot$$
$$\cdot t^{2\sum_{1\leq i<j\leq s}n_in_j(g-1)}\Big(\prod_{j=1}^{s-1}\frac{1}{1-t^{2(n_j+n_{j+1})}}\Big)t^{2\sum_{j=1}^{s-1}(n_j+n_{j+1})\langle-\frac{n_1+\cdots+n_j}{n}d\rangle}\ .$$

We may also consider the stack $\mathcal{M}^{\mathrm{s}}(GL_n,d/n)$ of stable vector bundles of rank n and degree d on the curve X (of genus $g\geq 2$). It is an open substack of $\mathcal{M}^{\mathrm{ss}}(GL_n,d/n)$, which is almost a smooth quasi–projective variety over k. More precisely, there exists a smooth quasi–projective variety $M^{\mathrm{s}}(GL_n,d/n)$ of dimension $(n^2-1)(g-1)$ over k and a morphism of stacks $\mathcal{M}^{\mathrm{s}}(GL_n,d/n)\to M^{\mathrm{s}}(GL_n,d/n)$ which is a gerb with fibers all isomorphic to BGL_1.

If d is prime to n, we have $\mathcal{M}^{\mathrm{s}}(GL_n,d/n)=\mathcal{M}^{\mathrm{ss}}(GL_n,d/n)$ and $M^{\mathrm{s}}(GL_n,d/n)$ is projective over k. Let us denote by $Q_t^{\mathrm{s}}(GL_n,d/n)$ the Poincaré polynomial of $M^{\mathrm{s}}(GL_n,d/n)$ in this case. We have

$$Q_t^{\mathrm{s}}(GL_n,d/n)=(1-t^2)P_t^{\mathrm{ss}}(GL_n,d/n)\ .$$

Therefore, we have proved :

Theorem 4.1. — *For each integer d prime to n, the Poincaré polynomial $Q_t^{\mathrm{s}}(GL_n, d/n)$ of the moduli space $M^{\mathrm{s}}(GL_n, d/n)$ of stable vector bundles of rank n and degree d on the curve X (of genus $g \geq 2$) is equal to*

$$\sum_{s=1}^{n}(-1)^{s-1}\frac{(1+t)^{2gs}}{(1-t^2)^{s-1}}\sum_{\substack{n_1,\ldots,n_s\geq 1\\ n_1+\cdots+n_s=n}}\Big(\prod_{j=1}^{s}\prod_{i=1}^{n_j-1}\frac{(1+t^{2i+1})^{2g}}{(1-t^{2i})(1-t^{2i+2})}\Big)\cdot$$
$$\cdot t^{2\sum_{1\leq i<j\leq s}n_in_j(g-1)}\Big(\prod_{j=1}^{s-1}\frac{1}{1-t^{2(n_j+n_{j+1})}}\Big)t^{2\sum_{j=1}^{s-1}(n_j+n_{j+1})\langle-\frac{n_1+\cdots+n_j}{n}d\rangle}\ .$$

∎

This last formula is equivalent to the following expression for the Poincaré polynomial of the moduli space of stable vector bundles of rank n having as determinant a fixed line bundle of degree d, prime to n, on the curve X (of genus $g \geq 2$)

$$\sum_{s=1}^{n}(-1)^{s-1}\Big(\frac{(1+t)^{2g}}{1-t^2}\Big)^{s-1}\sum_{\substack{n_1,\ldots,n_s\geq 1\\ n_1+\cdots+n_s=n}}\Big(\prod_{j=1}^{s}\prod_{i=1}^{n_j-1}\frac{(1+t^{2i+1})^{2g}}{(1-t^{2i})(1-t^{2i+2})}\Big)\cdot$$
$$\cdot t^{2\sum_{1\leq i<j\leq s}n_in_j(g-1)}\Big(\prod_{j=1}^{s-1}\frac{1}{1-t^{2(n_j+n_{j+1})}}\Big)t^{2\sum_{j=1}^{s-1}(n_j+n_{j+1})\langle-\frac{n_1+\cdots+n_j}{n}d\rangle}\ .$$

This was proved earlier by Zagier using different arguments (see [Za]). As he has remarked in loc. cit., it is not at all clear that the right hand sides of the last two formulas are polynomials.

Let us also point out that, if $n \geq 2$, the right hand side of the last formula vanishes at $t = -1$ (the order of vanishing at $t = -1$ of each summand of the double sum is

$$(2g-1)(s-1)+\sum_{j=1}^{s}\sum_{i=1}^{n_j-1}(2g-2)+\sum_{i=1}^{s-1}(-1) = (n-1)(2g-2)\,)\ .$$

This gives a new proof of the following result of Narasimhan and Ramanan (see [NR]) :

Corollary 4.2 (Narasimhan and Ramanan). — *The Euler–Poincaré characteristic of the moduli space of stable vector bundles of rank $n \geq 2$ having as determinant a fixed line bundle of degree d prime to n on the curve X (of genus $g \geq 2$) is equal to* 0.

∎

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