ON THE ARITHMETIC TRANSFER CONJECTURE FOR EXOTIC SMOOTH FORMAL MODULI SPACES

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ABSTRACT. In the relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture, we formulate a local conjecture (arithmetic transfer) in the case of an exotic smooth formal moduli space of p-divisible groups, associated to a unitary group relative to a ramified quadratic extension of a p-adic field. We prove our conjecture in the case of a unitary group in three variables.

CONTENTS

1. INTRODUCTION

The theorem of Gross and Zagier [6] relates the Neron–Tate heights of Heegner points on modular curves to special values of derivatives of certain L-functions. This has been generalized in various ways to higher-dimensional Shimura varieties. One such generalization, which is still conjectural, has been proposed by Gan–Gross–Prasad [4] and the third-named author [32, 33].

This arithmetic Gan–Gross–Prasad conjecture is inspired by the (usual) Gan–Gross–Prasad conjecture relating period integrals on classical groups to special values of certain L-functions. In [8] Jacquet and Rallis proposed a relative trace formula approach to this last conjecture in the case of unitary groups, which led them to formulate two local conjectures in this context: a fundamental lemma (FL) conjecture, and a smooth transfer (ST) conjecture. Both of their local conjectures are now proved to a large extent, the first for $p \gg 0$ thanks to the work of Yun [30] (and Gordon [5]), and the second for arbitrary p-adic non-archimedean fields by the third-named author [35].

In [33] the third-named author proposed a relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture. In this context, he formulated the arithmetic fundamental lemma (AFL) conjecture, cf. [33, 21]. The AFL conjecturally relates the special value of the derivative an orbital integral to an arithmetic intersection number on a Rapoport–Zink formal moduli space of p-divisible groups attached to a unitary group. The AFL is proved for low ranks of the unitary group ($n = 2$ and 3) in [33], and for arbitrary rank n and *minuscule* group elements in [21]. A simplified proof for $n = 3$ appears in [12]. At present, the general case of the AFL seems out of reach, even though Yun has obtained interesting results concerning the function field analog [31].

In the present paper, we address an arithmetic transfer (AT) analog of the ST conjecture in the arithmetic context, in a very specific case; we refer to [19] for a more general context in which we expect such arithmetic analogs of ST. The special feature of the case at hand is that, despite the fact that we take the unitary group to be ramified, the corresponding RZ space is smooth, cf. [15]. For this reason we speak of *exotic smoothness.*

Now that we have explained the title of the paper, let us describe its contents in more detail.

Let p be an odd prime number, and let F_0 be a finite extension of $\mathbb{Q}_p$. Let F/F_0 be a quadratic field extension. We denote by $a \mapsto \overline{a}$ the non-trivial automorphism of F/F_0, and by $\eta = \eta_{F/F_0}$ the corresponding quadratic character on $F_0^\times$. Let $e := (0, \dots, 0, 1) \in F_0^n$, and let $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ be the natural embedding that identifies GL_{n-1} with the subgroup fixing e under left multiplication, and fixing the transposed vector ${}^t e$ under right multiplication. Let

$$S_n := \{\, s \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid s\overline{s} = 1 \,\},$$

with its action by conjugation of GL_{n-1}. On the other hand, let W_0 and W_1 be the respective split and non-split F/F_0-hermitian spaces of dimension n. For $i = 0$ and 1, fix a vector $u_i \in W_i$ of length 1, and denote by $W_i^\flat$ the orthogonal complement of the line spanned by u_i. The unitary group $\mathrm{U}(W_i^\flat)$ acts by conjugation on $\mathrm{U}(W_i)$.

We now explain the matching relation between regular semi-simple elements of $S_n(F_0)$ and of $\mathrm{U}(W_0)(F_0)$ and $\mathrm{U}(W_1)(F_0)$. Here an element of $S_n(F_0)$, resp. of $\mathrm{U}(W_i)(F_0)$, is called *regular semi-simple* (rs) if its orbit under GL_{n-1}, resp. $\mathrm{U}(W_i^\flat)$, is Zariski-closed of maximal dimension. For each i, choose a basis of W_i by first choosing a basis of $W_i^\flat$ and then appending u_i to it. This identifies $\mathrm{U}(W_i^\flat)(F_0)$ with a subgroup of $\mathrm{GL}_{n-1}(F)$ and $\mathrm{U}(W_i)(F_0)$ with a subgroup of $\mathrm{GL}_n(F)$. An element $\gamma \in S_n(F_0)_{\mathrm{rs}}$ is said to *match* an element $g \in \mathrm{U}(W_i)(F_0)_{\mathrm{rs}}$ if both elements are conjugate under $\mathrm{GL}_{n-1}(F)$ when considered as elements in $\mathrm{GL}_n(F)$. This matching relation induces a bijection

$$\big[\mathrm{U}(W_0)(F_0)_{\mathrm{rs}}\big] \amalg \big[\mathrm{U}(W_1)(F_0)_{\mathrm{rs}}\big] \simeq \big[S_n(F_0)_{\mathrm{rs}}\big],$$

cf. [33, §2], where the brackets indicate the sets of orbits under $\mathrm{U}(W_i^\flat)(F_0)$, resp. $\mathrm{GL}_{n-1}(F_0)$.

Dual to the matching of elements is the transfer of functions, which is defined through weighted, resp. ordinary, orbital integrals. For a function $f' \in C_c^\infty(S_n(F_0))$, an element $\gamma \in S_n(F_0)_{\mathrm{rs}}$, and a complex parameter $s \in \mathbb{C}$, we define the weighted orbital integral

$$\mathrm{Orb}(\gamma, f', s) := \int_{\mathrm{GL}_{n-1}(F_0)} f'(h^{-1}\gamma h) |\det h|^s \eta(\det h)\, dh,$$

as well as its special value

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0).$$

Here the Haar measure on $\mathrm{GL}_{n-1}(F_0)$ is normalized so that $\mathrm{vol}(\mathrm{GL}_{n-1}(O_{F_0})) = 1$. For a function $f_i \in C_c^\infty(\mathrm{U}(W_i)(F_0))$ and an element $g \in \mathrm{U}(W_i)(F_0)_{\mathrm{rs}}$, we define the orbital integral

$$\mathrm{Orb}(g, f_i) := \int_{\mathrm{U}(W_i^\flat)(F_0)} f_i(h^{-1}gh)\, dh.$$

Then the function $f' \in C_c^\infty(S_n(F_0))$ is said to *transfer* to the pair of functions (f_0, f_1) in $C_c^\infty(\mathrm{U}(W_0)(F_0)) \times C_c^\infty(\mathrm{U}(W_1)(F_0))$ if

$$\omega(\gamma)\,\mathrm{Orb}(\gamma, f') = \mathrm{Orb}(g, f_i)$$

whenever $\gamma \in S_n(F_0)_{\mathrm{rs}}$ matches the element $g \in \mathrm{U}(W_i)(F_0)_{\mathrm{rs}}$. Here

$$\omega\colon S_n(F_0)_{\mathrm{rs}} \longrightarrow \mathbb{C}^\times$$

is a fixed transfer factor [35, p. 988], and the Haar measures on $\mathrm{U}(W_i^\flat)(F_0)$ are fixed. The ST conjecture asserts that for any f', a transfer (f_0, f_1) exists (non-uniquely), and that any pair (f_0, f_1) arises as a transfer from some (non-unique) f'. The FL conjecture asserts a specific transfer relation in a completely unramified situation.

When F/F_0 is unramified, if one takes for $\omega\colon S_n(F_0)_{\mathrm{rs}} \to \{\pm 1\}$ the *natural* transfer factor (see [21, (1.5)]), and normalizes the Haar measure on $\mathrm{U}(W_0^\flat)(F_0)$ by giving a hyperspecial maximal compact subgroup volume one, then the FL conjecture asserts that $\mathbf{1}_{S_n(O_{F_0})}$ transfers to $(\mathbf{1}_{K_0}, 0)$, where $K_0 \subset \mathrm{U}(W_0)(F_0)$ denotes a hyperspecial maximal open subgroup.

By contrast, when F/F_0 is ramified, there is no natural choice of a conjugacy class of open compact subgroups K_0, no natural choice of a transfer factor, and no natural candidate for f' transferring to $(\mathbf{1}_{K_0}, 0)$.

We next pass to the AFL conjecture, which requires, just as in the FL conjecture, that F/F_0 is unramified. We take the same transfer factor as in the FL conjecture, and the same fixed Haar measure on $\mathrm{U}(W_0^\flat)(F_0)$. For $f' \in C_c^\infty(S_n(F_0))$ and $\gamma \in S_n(F_0)_{\mathrm{rs}}$, set

$$\partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s).$$

Then the AFL conjecture asserts that

$$\omega(\gamma)\,\partial\mathrm{Orb}\big(\gamma, \mathbf{1}_{S_n(O_{F_0})}\big) = -\mathrm{Int}(g)\cdot \log q, \tag{1.1}$$

whenever $\gamma \in S_n(F_0)_{\mathrm{rs}}$ matches $g \in \mathrm{U}(W_1)(F_0)_{\mathrm{rs}}$ (note that the FL conjecture asserts that $\mathrm{Orb}(\gamma, \mathbf{1}_{S_n(O_{F_0})}) = 0$ for such γ). Here q denotes the number of elements in the residue field of F_0.

The term $\mathrm{Int}(g)$ requires explanation. Let $\mathcal{N}_n = \mathcal{N}_{F/F_0,n}$ denote the formal scheme over $\mathrm{Spf}\, O_{\breve F}$ which represents the following functor on the category of $O_{\breve F}$-schemes S such that $p\cdot\mathcal{O}_S$ is a locally nilpotent ideal sheaf. The functor associates to S the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$ where X is a formal p-divisible O_{F_0}-module of relative height $2n$ and dimension n, where $\iota\colon O_F \to \mathrm{End}(X)$ is an action of O_F satisfying the Kottwitz condition of signature $(1, n-1)$ on $\mathrm{Lie}(X)$ (cf. [9, §2]), where λ is a principal polarization whose Rosati involution induces the automorphism $a \mapsto \overline{a}$ on $\iota(O_F)$, and where $\rho\colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\mathrm{Spec}\, \overline{k}} \overline{S}$ is a framing of the restriction of X to the special fiber $\overline{S}$ of S, compatible with ι and λ in a certain sense, cf. [21, §2]. Then $\mathcal{N}_n$ is formally smooth of relative formal dimension $n-1$ over $\mathrm{Spf}\, O_{\breve F}$. The automorphism group (in a certain sense) of the framing object $\mathbb{X}_n$ can be identified with $\mathrm{U}(W_1)(F_0)$; it acts on $\mathcal{N}_n$ by changing the framing. Let $\mathcal{E}$ be the canonical lifting of the formal O_F-module of relative height 1 and dimension 1 over $\mathrm{Spf}\, O_{\breve F}$, with its canonical O_F-action $\iota_{\mathcal{E}}$ and its natural polarization $\lambda_{\mathcal{E}}$. There is a natural closed embedding of $\mathcal{N}_{n-1}$ into $\mathcal{N}_n$,

$$\begin{aligned} \delta_{\mathcal{N}}\colon\ \mathcal{N}_{n-1} &\longrightarrow \mathcal{N}_n \\ Y &\longmapsto Y \times \overline{\mathcal{E}}. \end{aligned}$$

Here all auxiliary structure (O_F-action, polarization, framing) has been suppressed from the notation, and $\overline{\mathcal{E}}$ denotes $\mathcal{E}$, with $\iota_{\mathcal{E}}$ replaced by its conjugate. Let

$$\Delta \subset \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_n$$

denote the graph of $\delta_{\mathcal{N}}$. Then $\mathrm{Int}(g)$ is defined as the intersection number of Δ with its translate under the automorphism $1 \times g$ of $\mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_n$,

$$\mathrm{Int}(g) = \chi\big(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{(1\times g)\Delta}\big).$$

This concludes the statement of the AFL conjecture.

It should be true in the situation of the AFL that for *any* $f' \in C_c^\infty(S_n(F_0))$ with transfer $(\mathbf{1}_{K_0}, 0)$, there exists a function $f'_{\mathrm{corr}} \in C_c^\infty(S_n(F_0))$ such that

$$\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q + \omega(\gamma)\, \mathrm{Orb}(\gamma, f'_{\mathrm{corr}})$$

whenever $\gamma \in S_n(F_0)_{\mathrm{rs}}$ matches $g \in \mathrm{U}(W_1)(F_0)_{\mathrm{rs}}$. This would follow along the lines of Lemma 5.17 below from a conjectural *density principle* on weighted orbital integrals. See Conjecture 5.15 for the statement of the density principle in the setting of this paper.

Now we come to the formulation of our AT conjecture. We assume for this that F/F_0 is *ramified*. We then modify the definition of the formal moduli space $\mathcal{N}_{F/F_0,n} = \mathcal{N}_n$ by slightly changing the conditions on the tuples $(X, \iota, \lambda, \rho)$. Namely, in addition to the Kottwitz condition of signature $(1, n-1)$, we impose on ι the Pappas wedge condition of signature $(1, n-1)$, and the spin condition. The latter condition states that for a uniformizer π of F, the endomorphism $\iota(\pi) \mid \mathrm{Lie}(X)$ is nowhere zero on S. Furthermore, we change the condition that λ is principal to the condition

$$\mathrm{Ker}(\lambda) \subset X[\iota(\pi)] \quad \text{with} \quad |\mathrm{Ker}(\lambda)| = q^{2\lfloor n/2 \rfloor}.$$

It turns out that $\mathcal{N}_n$ is again formally smooth of relative formal dimension $n-1$ over $\mathrm{Spf}\, O_{\breve F}$, and is essentially proper when n is even. We stress that this result is quite surprising in the presence of ramification.

The morphism $\delta_{\mathcal{N}} \colon \mathcal{N}_{n-1} \to \mathcal{N}_n$ can be defined exactly as before when discussing the AFL setup, provided that n is odd, since then $2\lfloor \frac{n-1}{2} \rfloor = 2\lfloor \frac{n}{2} \rfloor$. We then define $\mathrm{Int}(g)$ as before. Our AT conjecture is as follows.

Conjecture 1.1. *Let F/F_0 be ramified, and let $n \geq 3$ be odd.*

(a) *There exists a function $f' \in C_c^\infty(S(F_0))$ with transfer $(\mathbf{1}_{K_0}, 0)$ such that*

$$2\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q \tag{1.2}$$

for any $\gamma \in S(F_0)_{\mathrm{rs}}$ matching an element $g \in \mathrm{U}(W_1)(F_0)_{\mathrm{rs}}$.

(b) *For any $f' \in C_c^\infty(S(F_0))$ with transfer $(\mathbf{1}_{K_0}, 0)$, there exists a function $f'_{\mathrm{corr}} \in C_c^\infty(S(F_0))$ such that*

$$2\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q + \omega(\gamma)\, \mathrm{Orb}(\gamma, f'_{\mathrm{corr}})$$

for any $\gamma \in S(F_0)_{\mathrm{rs}}$ matching an element $g \in \mathrm{U}(W_1)(F_0)_{\mathrm{rs}}$.

Here K_0 denotes the maximal compact subgroup stabilizing a *nearly π-modular lattice* Λ_0 in W_0 (see (5.3) below), and ω is the transfer factor defined in (5.5) below. The Haar measure on $\mathrm{U}(W_0^\flat)(F_0)$ is defined by $\mathrm{vol}\, K_0^\flat = 1$ for a special maximal compact subgroup $K_0^\flat$ of $\mathrm{U}(W_0^\flat)(F_0)$. We also formulate a "homogeneous" variant of the AT conjecture (Conjecture 5.3), which we show is equivalent to the above conjecture in §5.2. Note that between the statements of the AFL conjecture (1.1) and the AT conjecture (1.2), there is a discrepancy of a factor of 2. This is a genuine difference between the unramified and ramified cases, and we refer to [19] for an explanation by way of a global comparison between the height pairing and the derivative of a relative trace formula.

Our main result concerns the first non-trivial case $n = 3$ of the AT conjecture. More precisely, we prove the following.

Theorem 1.2. *Let $F_0 = \mathbb{Q}_p$, and let $n = 3$. Then Conjecture 1.1 holds true. In addition, for any $g \in \mathrm{U}(W_1)(F_0)_{\mathrm{rs}}$, the intersection of Δ and $(1 \times g)\Delta$, if non-empty, is an artinian scheme with two points, and* $\mathrm{Int}(g) = \mathrm{length}(\Delta \cap (1 \times g)\Delta)$.

We also prove a Lie algebra version of the above theorem; see Theorem 5.22.

Let us comment on the proof of Theorem 1.2. In those cases in which the AFL conjecture has been established, the proof proceeds by calculating explicitly both sides of the conjectured identity and comparing the results. This approach fails for the AT conjecture because the left-hand side of the identity is not well-determined by the pair of transfer functions $(\mathbf{1}_{K_0}, 0)$. Unlike the AFL situation, there is no canonical choice for the function f'; in fact, f' cannot come from the Iwahori Hecke algebra, cf. Remark 5.4(iii). In particular, we note that the characteristic function $\mathbf{1}_{S(O_{F_0})}$ has vanishing orbital integrals at *all* regular semi-simple elements in the ramified setting, i.e. it transfers to $(0,0)$. Instead we prove part (b) of Conjecture 1.1 (in the case $F_0 = \mathbb{Q}_p$ and $n = 3$) by showing that, for any f' as in the statement, the sum

$$2\omega(\gamma)\,\partial\mathrm{Orb}(\gamma, f') + \mathrm{Int}(g)\cdot \log q \tag{1.3}$$

is an *orbital integral function*, i.e. of the form $\omega(\gamma)\,\mathrm{Orb}(\gamma, f'_{\mathrm{corr}})$ for a suitable function f'_{corr}. Then part (a) of Conjecture 1.1 follows easily.

To prove that (1.3) is an orbital integral function, we first remark that (1.3) may be viewed as a function on an open subset of the categorical quotient of S_n by GL_{n-1}, and then use the fact [35] that the desired property may be checked locally on the base. To achieve this goal, we proceed in two steps. First, we determine explicitly $\mathrm{Int}(g)$. Second, we develop a *germ expansion* around each point of the categorical quotient, which is sufficiently explicit that it determines $\omega(\gamma)\,\partial\mathrm{Orb}(\gamma, f')$ up to a *local orbital integral function*. Putting these two steps together, we check that (1.3) is an orbital integral function. The description just given is inaccurate, insofar as we first perform a reduction to a Lie algebra analog. Here the *Cayley transform* from [35] plays a key role. For the first step we use, similarly to [9], the results of Gross and Keating [1] on *quasi-canonical liftings* (the reduction to [1] in [9, §8] is transposed here to the ramified case; in fact, we found a drastic simplification of the proof (due to Zink) in loc. cit., which applies to both the unramified and the ramified case). For the second step, we base ourselves on the results on local harmonic analysis in [34], which we complete and make more explicit in various ways.

Additionally, let us point out two group-theoretic features in the case $n = 3$ which seem to be important. The first is the exceptional isomorphism $\mathrm{SL}_2 \simeq \mathrm{SU}_2$. One geometric manifestation of this is that there is a natural isomorphism between $\mathcal{M}$ and a connected component of $\mathcal{N}_2$, where $\mathcal{M}$ is the Lubin–Tate deformation space over $\mathrm{Spf}\, O_{\breve{F}}$ of the formal O_{F_0}-module of dimension 1 and height 2; see Proposition 6.3. To state the second feature, we note that the aforementioned maximal compact subgroups K_0 and $K_0^\flat$ have symplectic reductions; see Remark 5.2. When $n = 3$, associated to the reduction of $K_0^\flat$ is a second exceptional isomorphism $\mathrm{Sp}_2 \simeq \mathrm{SL}_2$, which plays a role in reducing the conjecture to a Lie algebra version; see the proof of Lemma 11.1.

Let us also remark on the restriction to $F_0 = \mathbb{Q}_p$ in Theorem 1.2. While we certainly expect that the overall framework of this paper should be valid for any p-adic field F_0, there are a few instances, all of which occur when working with $\mathcal{N}_n$ or related formal schemes, where we need to appeal to results in the literature which are only established at the level of generality of $q = p$ or $F_0 = \mathbb{Q}_p$. Indeed, strictly speaking, this is already the case for the representability result in [23] which is needed to know that $\mathcal{N}_n$ is a formal scheme in the first place! In accordance with our expectations, we will use the general notation q and F_0 throughout the paper, but when working in a context where formal schemes are present, we will always tacitly take $q = p$ and $F_0 = \mathbb{Q}_p$. By contrast, the parts of this paper lying in the realm of harmonic analysis are valid without any restriction on F_0.

Now let us comment on the possibility of extending our main result to odd integers $n > 3$. The difficulties seem formidable. First, one would have to deal with degenerate intersections. Related to this is the fact that the reduction procedure to a Lie algebra analog breaks down. In fact, we are unable to even formulate a conjectural Lie algebra version of Conjecture 1.1, since we are lacking a reasonable definition of an intersection multiplicity in this context; see Conjecture 5.10 below, in which we have to assume that the intersection is artinian. The second difficulty is that our knowledge of local harmonic analysis when $n > 3$ is not advanced enough; even a germ expansion principle is missing beyond the case of $n = 3$ [34]. One possibility for making further progress would be to consider Conjecture 1.1 only for elements γ and g that satisfy certain simplifying restrictions, in the spirit of [21] (which considers only minuscule elements).

On the positive side, there are other instances of AT conjectures. Indeed, in [19], we formulate AT conjectures for F/F_0 ramified (as in the present paper) and n even, and also for F/F_0 unramified and any n. The methods developed in the present paper can be applied to some low-dimensional cases of them, cf. [19]. We also refer to loc. cit. for the global motivation behind all of these conjectures.

We now give an overview of the contents of this paper. The paper consists of four parts.

In Part 1, we give the group-theoretic setup (in its homogeneous, its inhomogeneous, and its Lie algebra versions); we define the formal moduli spaces of p-divisible groups, and establish some structural properties for them; and we define the arithmetic intersection numbers that enter into the formulation of our conjectures and results. In §5, we formulate our main results.

In Part 2, we explicitly calculate the arithmetic intersection numbers in the case $n = 3$, by reduction to the Lie algebra and by relating this case to the Gross–Keating formulas.

In Part 3, we tackle the left-hand side of the identities to be proved in Theorem 1.2. This is done by reducing the problem to one on the *reduced subset* of the Lie algebra. The rest of part 3 is devoted to explicitly evaluating the germ expansion of the orbital integral of a function f' with transfer $(\mathbf{1}_{K_0}, 0)$, and then making the comparison with the result of part 2. At the end of part 3, the proof of Theorem 1.2 is complete.

In Part 4, we prove the germ expansion of the orbital integral of a general function f'. This part of the paper can be read independently of the rest and lies squarely in the domain of local harmonic analysis for the Jacquet–Rallis relative trace formula approach to the Gan–Gross–Prasad conjecture.

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Notation. We list here some notation that we use throughout the whole paper. We denote by p an odd prime number.

F/F_0 is a ramified quadratic extension of finite extensions of $\mathbb{Q}_p$. We denote by $a \mapsto \overline{a}$ the nontrivial automorphism of F/F_0, and by $\eta = \eta_{F/F_0}$ the corresponding quadratic character on $F_0^\times$. When h is a square matrix with entries in F_0, we sometimes abbreviate $\eta(\det h)$ to $\eta(h)$. Since $p \neq 2$, we may and do choose uniformizers π of F and ϖ of F_0 such that $\pi^2 = \varpi$. We denote by $\overline{k}$ an algebraic closure of the common residue field k of F and F_0, and we set $q := \#k$. As stated above, when working in an algebro-geometric context where formal schemes are present, we always understand that $q = p$ and $F_0 = \mathbb{Q}_p$. We denote the group of norm 1 elements in $F^\times$ by

$$F^1 := \{\, a \in F \mid a\overline{a} = 1 \,\}.$$

We denote by $\breve{F}_0$ the completion of a maximal unramified extension of F_0, and by $\breve{F} := \breve{F}_0 \otimes_{F_0} F$ the analogous object for F.

A polarization on a p-divisible group X is an anti-symmetric isogeny $X \to X^\vee$, where $X^\vee$ denotes the dual. We use a superscript $\circ$ to denote the operation $- \otimes_{\mathbb{Z}} \mathbb{Q}$ on groups of homomorphisms, so that for example

$$\mathrm{Hom}^\circ(X, Y) := \mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where Y is another p-divisible group. For any quasi-isogeny $\rho\colon X \to Y$ and polarization λ on Y, we define the pullback polarization

$$\rho^*(\lambda) := \rho^\vee \circ \lambda \circ \rho.$$

We denote by $\mathbb{E}$ the unique (up to isomorphism) formal O_{F_0}-module of relative height 2 and dimension 1 over $\operatorname{Spec} \overline{k}$. We set

$$O_D := \mathrm{End}_{O_{F_0}}(\mathbb{E}) \quad \text{and} \quad D := O_D \otimes_{O_{F_0}} F_0.$$

Thus D is "the" quaternion division algebra over F_0, and O_D is its maximal order. Since F/F_0 is ramified, any F_0-embedding of F into D makes $\mathbb{E}$ into a formal O_F-module of relative height 1. We fix such an embedding

$$\iota_{\mathbb{E}}\colon F \longrightarrow D$$

once and for all, and we always understand $\mathbb{E}$ to be a formal O_F-module via $\iota_{\mathbb{E}}$. We denote by $\mathcal{E}$ the corresponding canonical lift of $\mathbb{E}$ over $\operatorname{Spf} O_{\breve{F}}$, equipped with its O_F-action $\iota_{\mathcal{E}}$ and O_F-linear framing isomorphism $\rho_{\mathcal{E}} \colon \mathcal{E}_{\overline{k}} \xrightarrow{\sim} \mathbb{E}$. We denote by $\overline{\mathbb{E}}$ the same object as $\mathbb{E}$, except where the O_F-action $\iota_{\overline{\mathbb{E}}}$ is equal to $\iota_{\mathbb{E}}$ precomposed by the nontrivial automorphism of F/F_0; and ditto for $\overline{\mathcal{E}}$ in relation to $\mathcal{E}$, which is furthermore equipped with the same framing $\rho_{\overline{\mathcal{E}}} := \rho_{\mathcal{E}}$ on the level of O_{F_0}-modules over $\operatorname{Spec} \overline{k}$. Of course $\overline{\mathbb{E}}$ is a formal O_F-module of relative height 1 in its own right, but note that $\overline{\mathcal{E}}$ is *not* its canonical lift. In §3.3 we will specify a principal polarization $\lambda_{\mathbb{E}}$ on $\mathbb{E}$, and we will denote by $\lambda_{\mathcal{E}}$ the principal polarization on $\mathcal{E}$ lifting $\lambda_{\mathbb{E}}$. We write $\lambda_{\overline{\mathbb{E}}} := \lambda_{\mathbb{E}}$ and $\lambda_{\overline{\mathcal{E}}} := \lambda_{\mathcal{E}}$ for the same polarizations when we regard them as defined on $\overline{\mathbb{E}}$ and $\overline{\mathcal{E}}$, respectively.

We denote the main involution on D by $c \mapsto \overline{c}$, and the reduced norm by N. We also write N for the norm map $F^\times \to F_0^\times$; of course, all of this is compatible with any embedding of F into D. We write v_D for the normalized valuation on D, and we use π as a uniformizer for D, via $\iota_{\mathbb{E}}$. We write v for the normalized (i.e. ϖ-adic) valuation on F_0. For $c \in D^\times$, we define the conjugate embedding

$$\begin{aligned} {}^c\iota_{\mathbb{E}} \colon \ F &\longrightarrow D \\ a &\longmapsto c\iota_{\mathbb{E}}(a)c^{-1}. \end{aligned} \tag{1.4}$$

We denote by ${}^cF$ the image of ${}^c\iota_{\mathbb{E}}$.

Given a variety V over $\operatorname{Spec} F_0$, we of course denote by $C_c^\infty(V(F_0))$ the set of locally constant, compactly supported functions on the space $V(F_0)$, endowed with its ϖ-adic topology; and we typically abbreviate this set to $C_c^\infty(V)$. We choose the Haar measures on F_0 and F ($F_0^\times$ and $F^\times$, resp.) such that the volume $\mathrm{vol}(O_{F_0}) = \mathrm{vol}(O_F) = 1$ ($\mathrm{vol}(O_{F_0}^\times) = \mathrm{vol}(O_F^\times) = 1$, resp.).

We write 1_n for the $n \times n$ identity matrix, and $\mathbb{A}$ for the affine line. We use a subscript S to denote base change to S, and when $S = \operatorname{Spec} A$, we often use a subscript A instead.

Part 1. The conjectures

In this first part of the paper we introduce the objects involved in the statements of our AT conjectures. In §5 we state the conjectures and our main results.

2. Group-theoretic setup and orbit matching

We begin by explaining the general group-theoretic setup in this paper and the attendant matching relation for regular semi-simple elements. We consider three cases: the homogeneous group setting, the inhomogeneous group setting, and the Lie algebra setting. In this section $n \geq 2$ is an integer.

2.1. **Homogenous setting.**

We begin with the algebraic group

$$G' := \mathrm{Res}_{F/F_0}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n)$$

over F_0. We consider the following two subgroups of G'. The first subgroup is

$$H_1' := \mathrm{Res}_{F/F_0} \mathrm{GL}_{n-1},$$

which is embedded diagonally, via the inclusion of GL_{n-1} in GL_n sending $A \mapsto \mathrm{diag}(A, 1)$. The second subgroup is

$$H_2' := \mathrm{GL}_{n-1} \times \mathrm{GL}_n,$$

with its obvious embedding into G'. Let

$$H_{1,2}' := H_1' \times H_2'.$$

Then $H_{1,2}'$ acts on G' by $(h_1, h_2) \colon \gamma \mapsto h_1^{-1}\gamma h_2$. We call an element $\gamma \in G'(F_0)$ *regular semi-simple* if it is regular semi-simple for the action of $H_{1,2}'$, i.e. its orbit under $H_{1,2}'$ is closed, and its stabilizer is of minimal dimension. In the case at hand, it is equivalent that γ have closed orbit and trivial stabilizer, which follows from [17, Th. 6.1]. We denote by $G'(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $G'(F_0)$.

We next consider F/F_0-hermitian spaces of dimension n. Up to isomorphism there are two of them, a split one W_0 and a non-split one W_1. They are distinguished by the rule

$$\eta\big((-1)^{n(n-1)/2} \det W_i\big) = (-1)^i, \tag{2.1}$$

where $\det W_i := \det J_i$ for any hermitian matrix J_i (relative to the choice of a basis) representing the hermitian form. Let $i \in \{0, 1\}$. Write

$$U_i := \mathrm{U}(W_i), \tag{2.2}$$

and let

$$u_i \in W_i$$

be a non-isotropic vector, which we call the *special vector*. We will choose u_0 and u_1 such that their norms are congruent mod $\mathrm{N}F^\times$, which can always be done since $n \geq 2$. Let $W_i^\flat$ denote the orthogonal complement in W_i of the line spanned by u_i, and let

$$H_i := \mathrm{U}(W_i^\flat).$$

Then H_i naturally embeds into U_i as the stabilizer of u_i. Since we assume that u_0 and u_1 have congruent norms mod $\mathrm{N}F^\times$, $W_0^\flat$ and $W_1^\flat$ are non-isomorphic as hermitian spaces.

To lighten notation, now set $W := W_i$, and define $W^\flat$, U, H, and u analogously. Let

$$G_W := H \times U,$$

and consider H as a subgroup of G_W, embedded diagonally. Then $H \times H$ acts on G_W via the rule

$$(h_1, h_2) \colon g \longmapsto h_1^{-1} g h_2.$$

An element $g \in G_W(F_0)$ is called *regular semi-simple* if it is regular semi-simple for the action of $H \times H$. We denote by $G_W(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $G_W(F_0)$.

We now recall the matching relation between regular semi-simple elements, as in [33]. Choose an F-basis for $W^\flat$ and complete it to a basis for W by adjoining u. This identifies $W^\flat$ with F^{n-1} and W with F^n in such a way that u corresponds to the column vector

$$e := (0, \dots, 0, 1)$$

in F^n, and hence determines embeddings of groups $U \hookrightarrow \mathrm{Res}_{F/F_0} \mathrm{GL}_n$ and $G_W \hookrightarrow G'$. We call the embeddings obtained in this way *special embeddings*. An element $\gamma \in G'(F_0)_{\mathrm{rs}}$ and an element $g \in G_W(F_0)_{\mathrm{rs}}$ are said to *match* if these two elements, when considered as elements in $G'(F_0)$, are conjugate under $H'_{1,2}(F_0)$. The matching relation is independent of the choice of special embedding and induces a bijection [33, §2][1]

$$\big[G_{W_0}(F_0)_{\mathrm{rs}}\big] \amalg \big[G_{W_1}(F_0)_{\mathrm{rs}}\big] \simeq \big[G'(F_0)_{\mathrm{rs}}\big].$$

Here on the left-hand side, the square brackets denote the sets of orbits under the respective actions of $H_0(F_0) \times H_0(F_0)$ and $H_1(F_0) \times H_1(F_0)$, whereas the brackets on the right-hand side denote the set of orbits under $H'_{1,2}(F_0)$.

2.2. **Inhomogeneous setting.** Now we pass to the inhomogeneous version of the previous subsection. Consider the following identifications of algebraic varieties over F_0. First,

$$H'_1 \backslash G' \xrightarrow{\sim} \mathrm{Res}_{F/F_0} \mathrm{GL}_n, \quad \gamma = (\gamma_1, \gamma_2) \longmapsto \gamma_1^{-1}\gamma_2.$$

Second, let

$$S := S_n := \{\, g \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid g\overline{g} = 1_n \,\}$$

and

$$H' := \mathrm{GL}_{n-1}.$$

Then

$$\mathrm{Res}_{F/F_0}(\mathrm{GL}_n)/\mathrm{GL}_n \xrightarrow{\sim} S, \quad \gamma \longmapsto \gamma\overline{\gamma}^{-1},$$

and the above two identifications induce an identification on F_0-rational points

$$G'(F_0)/H'_{1,2}(F_0) \simeq S(F_0)/H'(F_0). \tag{2.3}$$

Here the action of H' on S is through conjugation. In other words, the map

$$G'(F_0) \longrightarrow S(F_0), \quad \gamma = (\gamma_1, \gamma_2) \longmapsto s(\gamma) := (\gamma_1^{-1}\gamma_2)\big(\overline{\gamma_1^{-1}\gamma_2}\big)^{-1},$$

induces the bijection (2.3).

[1] In loc. cit. only the inhomogeneous version, which will be taken up in the next subsection, is considered. However, the inhomogeneous version easily implies the homogeneous version.

On the unitary group side, let $i \in \{0,1\}$ and use the notation W, U, H, etc. as in the previous subsection. Then we similarly have an identification of F_0-points of algebraic varieties over F_0,

$$G_W(F_0)/\big(H(F_0) \times H(F_0)\big) \simeq U(F_0)/H(F_0), \quad (g_1, g_2) \longmapsto g_1^{-1} g_2.$$

Here the action of $H(F_0)$ on $U(F_0)$ is by conjugation.

The definitions from the homogeneous setting readily transfer to the inhomogeneous setting. Thus an element $\gamma \in S(F_0)$ is called *regular semi-simple* if it is regular semi-simple for the action of GL_{n-1} on GL_n. As in the homogeneous setting, it is again equivalent that γ have closed orbit and trivial stabilizer, cf. [17, Th. 6.1]. We denote by $S(F_0)_{\mathrm{rs}}$ or $S_n(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $S(F_0)$. An element $g \in U(F_0)$ is *regular semi-simple* if it is regular semi-simple for the action of H on U; equivalently, when g is considered as an element in $\mathrm{GL}_n(F)$ upon choosing a special embedding for U as in the previous subsection, it is regular semi-simple for the action of GL_{n-1} on GL_n. We denote by $U(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $U(F_0)$.

An element $\gamma \in S(F_0)_{\mathrm{rs}}$ *matches* an element $g \in U(F_0)_{\mathrm{rs}}$ if these two elements are conjugate under $\mathrm{GL}_{n-1}(F)$ when considered as elements of $\mathrm{GL}_n(F)$, upon choosing a special embedding for U. The matching relation is again independent of the choice of special embedding, and induces a bijection [33, §2]

$$\big[U_0(F_0)_{\mathrm{rs}}\big] \amalg \big[U_1(F_0)_{\mathrm{rs}}\big] \simeq \big[S(F_0)_{\mathrm{rs}}\big],$$

where on the left-hand side are the respective sets of orbits under $H_0(F_0)$ and $H_1(F_0)$, and on the right-hand side the set of orbits under $\mathrm{GL}_{n-1}(F_0)$. In particular, there is a disjoint union decomposition

$$S(F_0)_{\mathrm{rs}} = S_{\mathrm{rs},0} \amalg S_{\mathrm{rs},1}, \tag{2.4}$$

where $S_{\mathrm{rs},i}$ denotes the set of elements in $S(F_0)_{\mathrm{rs}}$ that match with elements in $U_i(F_0)_{\mathrm{rs}}$.

2.3. **Lie algebra setting.** We also consider a Lie algebra version of the inhomogeneous setup. We introduce the Lie algebra version of S_n,

$$\mathfrak{s} := \mathfrak{s}_n := \big\{\, y \in \operatorname{Res}_{F/F_0} \mathrm{M}_n \;\big|\; y + \overline{y} = 0 \,\big\}. \tag{2.5}$$

Then $H' = \mathrm{GL}_{n-1}$ acts on $\mathfrak{s}$, and we call an element of $\mathfrak{s}(F_0)$ *regular semi-simple* if its H'-orbit is closed and of maximal dimension. It is again equivalent that the element have closed orbit and trivial stabilizer. We denote by $\mathfrak{s}(F_0)_{\mathrm{rs}} = \mathfrak{s}_n(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $\mathfrak{s}(F_0)$.

For $i \in \{0,1\}$, let

$$\mathfrak{u}_i := \operatorname{Lie} U_i.$$

An element of $\mathfrak{u}_i(F_0)$ is called *regular semi-simple* if its orbit under H_i is closed and of maximal dimension. We denote by $\mathfrak{u}_i(F_0)_{\mathrm{rs}}$ the set of regular semi-simple elements in $\mathfrak{u}_i(F_0)$.

As in §2.1, the choice of a basis for $W_i^\flat$, extended by u_i to a basis for W_i, determines an embedding $\mathfrak{u}_i \hookrightarrow \operatorname{Res}_{F/F_0} \mathrm{M}_n$, which we again call a *special embedding.* An element $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ *matches* an element $x \in \mathfrak{u}_i(F_0)_{\mathrm{rs}}$ if these two elements are conjugate under $\mathrm{GL}_{n-1}(F)$ when considered as elements of $\mathrm{M}_n(F)$. The matching relation is independent of the special embedding and induces a bijection [8, §5]

$$[\mathfrak{u}_0(F_0)_{\mathrm{rs}}] \amalg [\mathfrak{u}_1(F_0)_{\mathrm{rs}}] \simeq [\mathfrak{s}(F_0)_{\mathrm{rs}}],$$

where on the left-hand side are the respective sets of orbits under $H_0(F_0)$ and $H_1(F_0)$, and on the right-hand side the set of orbits under $\mathrm{GL}_{n-1}(F_0)$. As before, we get a disjoint union decomposition

$$\mathfrak{s}(F_0)_{\mathrm{rs}} = \mathfrak{s}_{\mathrm{rs},0} \amalg \mathfrak{s}_{\mathrm{rs},1},$$

where $\mathfrak{s}_{\mathrm{rs},i}$ denotes the set of elements in $\mathfrak{s}(F_0)_{\mathrm{rs}}$ that match with elements in $\mathfrak{u}_i(F_0)_{\mathrm{rs}}$.

2.4. **Linear algebra characterizations.** We now recall the linear algebra characterizations of regular semi-simple elements and of their matching, in the inhomogeneous group setting and the Lie algebra setting. First we introduce the *discriminant* Δ, which is a morphism of varieties

$$\begin{aligned}\Delta\colon \operatorname{Res}_{F/F_0}\mathrm{M}_n &\longrightarrow \operatorname{Res}_{F/F_0}\mathbb{A}\\ x &\longmapsto \det({}^t e x^{i+j} e)_{0\le i,j\le n-1}\end{aligned}\tag{2.6}$$

over F_0. We remark that beginning in §8 we will actually work with a rescaled version of Δ, cf. (8.12).

An element $\gamma \in S(F_0)$ is regular semi-simple if and only if, considering γ as an element of $\mathrm{GL}_n(F)$, the sets of vectors $\{\gamma^i e\}_{i=0}^{n-1}$ and $\{{}^t e\gamma^i\}_{i=0}^{n-1}$ are both linearly independent [17, Th. 6.1].[2] Equivalently, $\Delta(\gamma) \neq 0$. Hence the regular semi-simple locus inside S is the complement of the locus $\Delta = 0$, i.e. the Zariski-open subscheme

$$S_{\mathrm{rs}} := S_{n,\mathrm{rs}} := \bigl\{\, \gamma \in S \bigm| \Delta(\gamma) \neq 0 \,\bigr\}.$$

Thus we may write $S(F_0)_{\mathrm{rs}}$ and $S_{\mathrm{rs}}(F_0)$ interchangeably.

To $x \in \mathrm{M}_n(F)$ we associate the following numerical invariants: the n coefficients of the characteristic polynomial $\mathrm{char}_x(T) \in F[T]$, and the $n-1$ elements ${}^t e x^i e \in F$ for $i = 1, \dots, n-1$. Then two elements of $S(F_0)_{\mathrm{rs}}$ are conjugate under $\mathrm{GL}_{n-1}(F_0)$ if and only if they have the same numerical invariants when considered as elements of $\mathrm{M}_n(F)$, cf. [33].

For $i \in \{0,1\}$ and $U = U_i$, an element $g \in U(F_0)$ is regular semi-simple if and only if, when considered as an element of $\mathrm{GL}_n(F)$ via any special embedding, g satisfies the conditions above, i.e. the sets of vectors $\{g^i e\}_{i=0}^{n-1}$ and $\{{}^t e g^i\}_{i=0}^{n-1}$ are both linearly independent. Equivalently, $\Delta(g) \neq 0$, so that the regular semi-simple set is the Zariski open complement to the locus $\Delta(g) = 0$, and we write $U(F_0)_{\mathrm{rs}}$ and $U_{\mathrm{rs}}(F_0)$ interchangeably. Two elements $g \in U(F_0)_{\mathrm{rs}}$ and $\gamma \in S(F_0)_{\mathrm{rs}}$ are matched if and only if their numerical invariants, when both are considered as elements of $\mathrm{M}_n(F)$, coincide.

The theory in the Lie algebra setting is entirely analogous. An element $y \in \mathfrak{s}(F_0)$ is regular semi-simple if and only if, considering y as an element of $\mathrm{M}_n(F)$, the sets of vectors $\{y^i e\}_{i=0}^{n-1}$ and $\{{}^t e y^i\}_{i=0}^{n-1}$ are both linearly independent [17, Th. 6.1]. Equivalently, $\Delta(y) \neq 0$, i.e. the regular semi-simple set is the Zariski open complement to the locus $\Delta(y) = 0$, and we write $\mathfrak{s}(F_0)_{\mathrm{rs}}$ and $\mathfrak{s}_{\mathrm{rs}}(F_0)$ interchangeably. Furthermore, two elements of $\mathfrak{s}(F_0)_{\mathrm{rs}}$ are conjugate under $\mathrm{GL}_{n-1}(F_0)$ if and only if they have the same numerical invariants when considered as elements of $\mathrm{M}_n(F)$, cf. [17].

For $i \in \{0,1\}$ and $\mathfrak{u} = \mathfrak{u}_i$, an element $x \in \mathfrak{u}(F_0)$ is regular semi-simple if and only if, when considered as an element of $\mathrm{M}_n(F)$ via any special embedding, x satisfies the conditions above, i.e. the sets of vectors $\{x^i e\}_{i=0}^{n-1}$ and $\{{}^t e x^i\}_{i=0}^{n-1}$ are both linearly independent. Equivalently, $\Delta(x) \neq 0$, and we again write $\mathfrak{u}(F_0)_{\mathrm{rs}}$ and $\mathfrak{u}_{\mathrm{rs}}(F_0)$ interchangeably. Two elements $x \in \mathfrak{u}(F_0)_{\mathrm{rs}}$ and $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ are matched if and only if their numerical invariants, when both are considered as elements of $\mathrm{M}_n(F)$, coincide [8, 33].

From now on in the paper, we make the blanket assumption that

$$\text{\textit{the special vectors } } u_0 \in W_0 \text{ \textit{ and } } u_1 \in W_1 \text{ \textit{ have norm} } 1. \tag{2.7}$$

Under this assumption, we have the following simple formula to distinguish between W_0 and W_1 in terms of the discriminant.

Lemma 2.1. *For $i \in \{0,1\}$ and any $x \in \mathfrak{u}_i(F_0)_{\mathrm{rs}}$,*

$$\eta\bigl(\Delta(x)\bigr) = (-1)^i.$$

Proof. Let $u = u_i$ and $W = W_i$. Let h denote the hermitian form on W. By (2.7),

$${}^t e x^{i+j} e = h(u, x^{i+j} u).$$

Since x is in the Lie algebra $\mathfrak{u}_i$ we have $h(xv, w) = -h(v, xw)$ for all $v, w \in W$. Hence

$$h(u, x^{i+j} u) = (-1)^i h(x^i u, x^j u).$$

[2] In [17], the Lie algebra version is considered, but it is easy to deduce the group version from this.

Hence

$$\Delta(x) = \det\big((-1)^i h(x^i u, x^j u)_{0\le i,j\le n-1}\big) = (-1)^{n(n-1)/2} \det h(x^i u, x^j u)_{0\le i,j\le n-1}.$$

Since x is regular semi-simple, the vectors $u, xu, \dots, x^{n-1}u$ form a basis of W. Hence the lemma follows from (2.1). □

2.5. **Invariants in the Lie algebra setting.** Consider the $2n-1$ maps of varieties over F_0

$$\mathrm{Res}_{F/F_0} \mathrm{M}_n \longrightarrow \mathrm{Res}_{F/F_0} \mathbb{A}$$

defined on points by sending x to the quantities

$$\operatorname{tr} \wedge^i x, \quad 1 \le i \le n, \quad \text{and} \quad {}^t e x^j e, \quad 1 \le j \le n-1. \tag{2.8}$$

When restricted to the subscheme $\mathfrak{s} = \mathfrak{s}_n \subset \mathrm{Res}_{F/F_0} \mathrm{M}_n$, each of these maps factors through either $\mathbb{A} \subset \mathrm{Res}_{F/F_0} \mathbb{A}$ or $\mathfrak{s}_1 \subset \mathrm{Res}_{F/F_0} \mathbb{A}$ according as i, resp. j, is even or odd. Here we have allowed the case $n = 1$ in the definition (2.5) of $\mathfrak{s}_n$, i.e. $\mathfrak{s}_1$ denotes the scheme of points y in $\mathrm{Res}_{F/F_0} \mathbb{A}$ such that $\overline{y} = -y$. Upon choosing special embeddings $\mathfrak{u}_0, \mathfrak{u}_1 \hookrightarrow \mathrm{Res}_{F/F_0} \mathrm{M}_n$, the same statement is true of each of the maps in (2.8) when restricted to $\mathfrak{u}_0$ and $\mathfrak{u}_1$. Thus we obtain a map from each of $\mathfrak{s}$, $\mathfrak{u}_0$, and $\mathfrak{u}_1$ into the common $(2n-1)$-fold product of the corresponding $\mathbb{A}$'s and $\mathfrak{s}_1$'s; and in the case of $\mathfrak{u}_0$ and $\mathfrak{u}_1$, this map is independent of the choice of special embedding. These maps into the product of $\mathbb{A}$'s and $\mathfrak{s}_1$'s are invariant for the respective actions of H', H_0, and H_1 on $\mathfrak{s}$, $\mathfrak{u}_0$, and $\mathfrak{u}_1$, and it is shown in [35, Lem. 3.1] that they identify the target with the categorical quotients $\mathfrak{s}/H'$, $\mathfrak{u}_0/H_0$, and $\mathfrak{u}_1/H_1$. In other words, the ring of global invariants on each of $\mathfrak{s}$, $\mathfrak{u}_0$, and $\mathfrak{u}_1$ is a polynomial ring over F_0 generated by the $2n-1$ functions (2.8).

There is another set of polynomial generators, also given in loc. cit., which will be a little more convenient for us to work with. Write a point x in $\mathrm{Res}_{F/F_0} \mathrm{M}_n$ in the form

$$x = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}, \tag{2.9}$$

where the block decomposition is with respect to the special vector e. Then the functions

$$\operatorname{tr} \wedge^i A, \quad 1 \le i \le n-1, \quad \mathbf{c}A^j\mathbf{b}, \quad 0 \le j \le n-2, \quad \text{and} \quad d \tag{2.10}$$

realize the categorical quotients $\mathfrak{s}/H'$, $\mathfrak{u}_0/H_0$, and $\mathfrak{u}_1/H_1$ as above (again after choosing any special embeddings $\mathfrak{u}_0, \mathfrak{u}_1 \hookrightarrow \mathrm{Res}_{F/F_0} \mathrm{M}_n$ for the latter two). Explicitly, for $\mathfrak{b}$ any of these three quotients, the invariants (2.10) induce an isomorphism of schemes over F_0

$$\begin{aligned} \mathfrak{b} &\xrightarrow{\ \sim\ } \overbrace{\mathfrak{s}_1 \times \mathbb{A} \times \cdots}^{\substack{n-1\\ \text{alternating}\\ \text{factors}}} \times \overbrace{\mathbb{A} \times \mathfrak{s}_1 \times \cdots}^{\substack{n-1\\ \text{alternating}\\ \text{factors}}} \times \mathfrak{s}_1 \\ x &\longmapsto (\operatorname{tr} A, \operatorname{tr} \wedge^2 A, \dots, \mathbf{cb}, \mathbf{c}A\mathbf{b}, \dots, d). \end{aligned} \tag{2.11}$$

3. The moduli space

In this section we introduce the moduli space of p-divisible groups $\mathcal{N}_n$ over $\operatorname{Spf} O_{\breve{F}}$. It is an analog of spaces appearing in [9, 21, 27, 28, 33] in the unramified setting, but in the ramified setting a subtler definition is required to obtain a formally smooth space. In accordance with our convention in the Introduction, throughout this section we take $F_0 = \mathbb{Q}_p$. Let $n \ge 1$ be an integer.

3.1. **Unitary p-divisible groups.** Let S be a scheme over $\operatorname{Spf} O_{\breve{F}}$. A *unitary p-divisible group of signature* $(1, n-1)$ in the setting of this paper is a triple

$$(X, \iota_X, \lambda_X)$$

consisting of a p-divisible group X over S, a homomorphism

$$\iota_X : O_F \longrightarrow \mathrm{End}_S(X),$$

and a polarization

$$\lambda_X : X \longrightarrow X^\vee,$$

subject to the following constraints:

- (*Kottwitz condition*) for the action of O_F on $\operatorname{Lie} X$ induced by ι_X, there is an equality of polynomials

$$\operatorname{char}\big(\iota_X(\pi) \mid \operatorname{Lie} X\big) = (T-\pi)(T+\pi)^{n-1} \in \mathcal{O}_S[T];$$

- (*wedge condition*) $\bigwedge^2_{\mathcal{O}_S}\big(\iota_X(\pi)+\pi \mid \operatorname{Lie} X\big) = 0$;
- (*spin condition*) if $n > 1$, then for every point $s \in S$, the operator $\iota_X(\pi) \mid \operatorname{Lie} X_s$ is nonzero;
- the Rosati involution on $\operatorname{End}^\circ_S(X)$ attached to λ_X induces the nontrivial Galois automorphism on O_F; and
- if n is even, then $\operatorname{Ker}\lambda_X = X[\iota_X(\pi)]$; and if n is odd, then $\operatorname{Ker}\lambda_X \subset X[\iota_X(\pi)]$ of height $n-1$.

Let us make a few remarks on the definition. First note that our formulation of the Kottwitz condition implies that

$$\operatorname{char}\big(\iota_X(a) \mid \operatorname{Lie} X\big)= (T-a)(T+\overline{a})^{n-1} \in \mathcal{O}_S[T] \quad \text{for all} \quad a \in O_F,$$

and that $\operatorname{rank}_{\mathcal{O}_S}(\operatorname{Lie} X) = n$. Since X is isogenous to its dual, it follows that X has height $2n$. We also note that the Rosati condition on λ_X is equivalent to requiring that λ_X is O_F-linear, where O_F acts on the dual $X^\vee$ via the rule

$$\iota_{X^\vee}(a) = \iota_X(\overline{a})^\vee. \tag{3.1}$$

The wedge condition is due to Pappas [13]. There is another part to the wedge condition in loc. cit., which for signature $(1, n-1)$ is

$$\bigwedge\nolimits^n_{\mathcal{O}_S}\big(\iota_X(\pi)-\pi \mid \operatorname{Lie} X\big) = 0.$$

Since $\operatorname{Lie} X$ has rank n, this condition holds automatically by the Kottwitz condition. Similarly, the wedge condition as we have formulated it above is implied by the Kottwitz condition when $n \leq 2$.

The spin condition is based on the spin condition introduced in [15]; see Remark 3.11 below for further discussion. Note that the spin condition is a condition only on the underlying point set of S. Since S lies over $\operatorname{Spf} O_{\breve F}$, we have $\pi\cdot\kappa(s) = 0$ for every point $s \in S$. Thus in the presence of the wedge condition, the spin condition is equivalent to the condition that $\iota_X(\pi) \mid \operatorname{Lie} X_s$ has rank 1 for every $s \in S$.

3.2. **Serre tensor construction.** Before continuing, we pause to briefly review the Serre tensor construction, which will also play a central role in §6.

Quite generally, let S be any base scheme, A a commutative ring, M a finite projective A-module, and X a contravariant functor on the category of S-schemes, valued in A-modules. For T a scheme over S, define

$$(M \otimes_A X)(T) := M \otimes_A X(T).$$

When X is a scheme, so is $M\otimes_A X$, and moreover many properties of X are inherited by $M\otimes_A X$. See [3, §7]. It follows easily from loc. cit. that $M \otimes_A X$ is a p-divisible group when X is, which will be the case of interest to us. In this case, one furthermore has canonical isomorphisms

$$(M \otimes_A X)^\vee \cong M^\vee \otimes_A X^\vee,$$

where $M^\vee := \operatorname{Hom}_A(M, A)$ is the dual A-module; and

$$\operatorname{Lie}(M \otimes_A X) \cong M \otimes_A \operatorname{Lie} X.$$

3.3. **Framing objects.** In this subsection we consider unitary p-divisible groups of signature $(1, n-1)$ over $\operatorname{Spec}\overline{k}$. The first point of business is that, up to isogeny, there is only one whose underlying p-divisible group is supersingular, in the following sense.

Proposition 3.1. *Let (X, ι_X, λ_X) and (Y, ι_Y, λ_Y) be supersingular unitary p-divisible groups of signature $(1, n-1)$ over $\operatorname{Spec}\overline{k}$. Then there exists an O_F-linear quasi-isogeny*

$$\rho\colon X \longrightarrow Y$$

such that $\rho^*(\lambda_Y)$ is an $F_0^\times$-multiple of λ_X in $\operatorname{Hom}^\circ_{O_F}(X, X^\vee)$.

Proof. Let M denote the covariant Dieudonné module of X, endowed with its Frobenius operator $\underline{F}$ and Verschiebung $\underline{V}$. The polarization on X translates to an alternating form $\langle\ ,\ \rangle$ on M satisfying

$$\langle \underline{F}x, y\rangle = \langle x, \underline{V}y\rangle^\sigma \quad \text{for all} \quad x, y \in M,$$

where σ denotes the Frobenius operator on $W(\overline{k}) = O_{\breve{F}_0}$. The O_F-action on X translates to an O_F-action on M commuting with $\underline{F}$ and $\underline{V}$ and satisfying

$$\langle ax, y\rangle = \langle x, \overline{a}y\rangle$$

for all x, y.

Let $N := M \otimes_{O_{\breve{F}_0}} \breve{F}_0$ denote the rational Dieudonné module of X. Then $\langle\ ,\ \rangle$ extends to a nondegenerate alternating form on N. We must show that the rational Dieudonné module of Y is isomorphic to N as a polarized isocrystal (with the polarization taken up to scalar in $F_0^\times$) with F-action.

Let $\zeta \in O_{\breve{F}_0}^\times$ satisfy $\zeta^2\pi_0 = -p$. Since N is supersingular, all slopes of the σ-linear operator

$$\tau := \zeta\pi\underline{V}^{-1} \colon N \longrightarrow N \tag{3.2}$$

are 0. Hence

$$C := N^{\tau=1}$$

is an F_0-subspace of N such that

$$C \otimes_{F_0} \breve{F}_0 \xrightarrow{\sim} N;$$

and in this way $\mathrm{id}_C \otimes \sigma$ identifies with τ. Furthermore C is F-stable, the restriction of $\langle\ ,\ \rangle$ to C takes values in F_0, and the form

$$h(x, y) := \langle \pi x, y\rangle + \langle x, y\rangle\pi, \quad x, y \in C,$$

makes C into a nondegenerate F/F_0-hermitian space of dimension n, cf. [20, pp. 1170–1].[3]

Clearly, to classify N up to isomorphism as a polarized isocrystal with F-action is to classify C up to similarity as a hermitian space. When n is odd, all nondegenerate n-dimensional F/F_0-hermitian spaces are similar, which proves the lemma. When n is even, the two isomorphism types of nondegenerate n-dimensional hermitian spaces remain non-similar. By Dieudonné theory, the Lie algebra of X identifies with $M/\underline{V}M = M/\pi\tau^{-1}M$, and the spin condition is that $\dim_{\overline{k}}(\pi M + \pi\tau^{-1}M)/\pi\tau^{-1}M = 1$. The condition $\operatorname{Ker}\lambda_X = X[\iota_X(\pi)]$ translates to $M^\vee = \pi^{-1}M$ inside N, where the dual lattice $M^\vee$ is the set of $x \in N$ such that $\langle x, M\rangle \subset O_{\breve{F}_0}$, or equivalently such that $h(x, M) \subset O_{\breve{F}}$. The lemma is now a consequence of the general result in Lemma 3.3 below. ☐

We will need to prepare a little before coming to Lemma 3.3. Suppose that n is even, and let N be "the" n-dimensional $\breve{F}/\breve{F}_0$-hermitian space. Let M be a π-modular $O_{\breve{F}}$-lattice in N, which is to say that the dual lattice of M with respect to the hermitian form is $\pi^{-1}M$. Let $U := \mathrm{U}(N)$ denote the (quasi-split) unitary group of N over $\operatorname{Spec}\breve{F}_0$. Let K denote the stabilizer in $U(\breve{F}_0)$ of M. Then K is a special maximal parahoric subgroup of $U(\breve{F}_0)$, and all special maximal parahoric subgroups of $U(\breve{F}_0)$ are conjugate to K; see e.g. case (b) in [14, §4.a]. We also need the *Kottwitz homomorphism* on $U(\breve{F}_0)$, which is a homomorphism

$$\kappa \colon U(\breve{F}_0) \longrightarrow \{\pm 1\} \tag{3.3}$$

admitting the following simple description. For any $g \in U(\breve{F}_0)$, the determinant $\det_F(g)$ is a norm 1 element in $\breve{F}$, and $\kappa(g)$ takes the value ± 1 according as $\det_F(g) \equiv \pm 1 \bmod \pi$; this follows from e.g. [14, §§3.b.1, 4.a], or see [15, §1.2.3(b)] for the closely related case of quasi-split GU_n.

Lemma 3.2. *For n even and any $g \in U(\breve{F}_0)$, $\kappa(g)$ equals 1 or -1 according as the $O_{\breve{F}}$-length of $(M + gM)/M$ is even or odd.*

[3]Note that the quantity η in loc. cit. should be a square root of $-\epsilon^{-1}$, rather than a square root of ϵ^{-1}.

Proof. We use the Cartan decomposition. We can choose a split $\breve F$-basis $e_1, \dots, e_n$ for N (meaning that e_i and e_j pair to $\delta_{i,n+1-j}$ under the hermitian form) such that

$$M = O_{\breve F} e_1 + \cdots + O_{\breve F} e_{n/2} + O_{\breve F} \pi e_{n/2+1} + \cdots + O_{\breve F} \pi e_n.$$

Let T be the maximal torus in U whose $\breve F_0$-points are

$$T(\breve F_0) = \bigl\{\, \mathrm{diag}\bigl(a_1, \dots, a_{n/2}, \overline{a}_{n/2}^{-1}, \dots, \overline{a}_1^{-1}\bigr) \bigm| a_1, \dots, a_{n/2} \in \breve F^\times \,\bigr\}.$$

Since K is a special maximal parahoric subgroup, by the Cartan decomposition $g = k_1 t k_2$ for some $k_1, k_2 \in K$ and $t \in T(\breve F_0)$. Then $(M + gM)/M = (M + k_1 t M)/M$, which, multiplying by k_1^{-1}, is isomorphic to $(M + tM)/M$. Since k_1 and k_2 of course have trivial Kottwitz invariant, we have therefore reduced the lemma to the case $g = t$, where it is obvious. (It may be helpful to note that an element of the form $a/\overline{a}$, $a \in \breve F^\times$, is congruent to $(-1)^{\mathrm{ord}_\pi a} \bmod \pi$.) $\square$

Lemma 3.3. *Let n be even, let C be an F/F_0-hermitian space of dimension n, let $N := C \otimes_{F_0} \breve F_0$, and let $\tau := \mathrm{id}_C \otimes \sigma$. Let M be an $O_{\breve F}$-lattice in N which is π-modular with respect to the induced $\breve F/\breve F_0$-hermitian form. Then the $O_{\breve F}$-length of $(M + \tau^{-1}M)/\tau^{-1}M$ is even or odd according as C is a split or non-split hermitian space.*

Proof. The length in question is the same as the length of the module $(M + \tau M)/M$, which we will work with instead. Let U denote the unitary group of C over $\mathrm{Spec}\, F_0$. (In terms of the notation in the previous lemma, in this way we view the unitary group of N as defined over F_0.)

If C is split, then it contains a π-modular O_F-lattice Λ. Let $L := O_{\breve F} \cdot \Lambda \subset N$. Then $M = gL$ for some $g \in U(\breve F_0)$. Hence $\tau M = \sigma(g) L$, and

$$(M + \tau M)/M = \bigl(M + \sigma(g) g^{-1} M\bigr)/M.$$

Since $\sigma(g)$ and g have the same Kottwitz invariant, we conclude from Lemma 3.2 that the $O_{\breve F}$-length of the displayed module is even.

If C is non-split, then it can be expressed as an orthogonal direct sum of a non-split 2-dimensional space C_1 and a split $(n-2)$-dimensional space C_2. There exists a basis e_1, e_2 of C_1 such that the associated matrix of the hermitian form is of the form

$$\begin{bmatrix} 1 & \\ & -b \end{bmatrix}$$

for some $b \in O_{F_0}^\times \smallsetminus \mathrm{N} O_F^\times$. Let $\beta \in O_{\breve F_0}$ be a square root of b. Then the vectors

$$f_1 := e_1 + \beta^{-1} e_2, \quad f_2 := \frac{1}{2}(e_1 - \beta^{-1} e_2)$$

form a split basis of $N_1 := C_1 \otimes_{F_0} \breve F_0$. Hence $L_1 := O_{\breve F} f_1 + O_{\breve F} \pi f_2$ is a π-modular lattice in N_1. Since C_2 is split, $N_2 := C_2 \otimes_{F_0} \breve F_0$ contains a π-modular lattice L_2 which is stable under $\mathrm{id}_{C_2} \otimes \sigma$. Then the lattice $L := L_1 \oplus L_2$ in N is π-modular, and by inspection, L and τL differ by the transformation $g_0 \in U(\breve F_0)$ which interchanges f_1 and f_2 and is the identity on N_2. Then $\det(g_0) = -1$, and hence $\kappa(g_0) = -1$. Now we argue as in the case that C is split. Writing $M = gL$ for an appropriate $g \in U(\breve F_0)$, we have

$$(M + \tau M)/M = \bigl(gL + \sigma(g)\tau L\bigr)/gL \cong \bigl(L + g^{-1}\sigma(g) g_0 L\bigr)/L.$$

By Lemma 3.2, the length of the module on the right is odd. $\square$

In the rest of this subsection we are going to fix particular framing objects $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$, $n \geq 1$, over $\mathrm{Spec}\, \overline{k}$ for use in the rest of the paper. When $n = 1$, we define

$$(\mathbb{X}_1, \iota_{\mathbb{X}_1}) := (\mathbb{E}, \iota_{\mathbb{E}}),$$

where as in the Introduction, $\mathbb{E}$ is the unique (up to isomorphism) connected p-divisible group of dimension 1 and height 2 over $\mathrm{Spec}\, \overline{k}$, and $\iota_{\mathbb{E}}$ is an embedding

$$\iota_{\mathbb{E}} \colon O_F \hookrightarrow O_D = \mathrm{End}_{O_{F_0}}(\mathbb{E}),$$

which makes $\mathbb{E}$ into a formal π-divisible module of relative height 1. (Recall that in this section $F_0 = \mathbb{Q}_p$.) The Dieudonné module $\mathbb{M}$ of $\mathbb{E}$ can be identified with $W(\overline{k})^2 = O_{\breve{F}_0}^2$ endowed with the Frobenius operator given in matrix form by

$$\begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix} \sigma,$$

where σ denotes the usual Frobenius homomorphism on the Witt vectors. To give a polarization on $\mathbb{E}$ is to give an alternating bilinear pairing on $\mathbb{M}$ with associated matrix of the form

$$\begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix}$$

for $\delta \in O_{\breve{F}_0}$ satisfying $\sigma(\delta) = -\delta$. We define the (principal) polarization $\lambda_{\mathbb{X}_1} = \lambda_{\mathbb{E}}$ by fixing any such $\delta \in O_{\breve{F}_0}^\times$ once and for all. Note that any other principal polarization of $\mathbb{E}$ differs from $\lambda_{\mathbb{E}}$ by an $O_{F_0}^\times$-multiple. The F_0-algebra $D = \operatorname{End}^\circ_{O_{F_0}}(\mathbb{E})$ is the quaternion division algebra over F_0, and $O_D = \operatorname{End}_{O_{F_0}}(\mathbb{E})$ is its maximal order. The Rosati involution attached to $\lambda_{\mathbb{E}}$ is the main involution on D, and therefore it induces the nontrivial Galois automorphism on O_F.

When $n = 2$, we define

$$\mathbb{X}_2 := O_F \otimes_{O_{F_0}} \mathbb{E}$$

via the Serre tensor construction, with $\iota_{\mathbb{X}_2}$ given by the tautological O_F-action on the left tensor factor. Then canonically

$$\operatorname{Lie}(O_F \otimes_{O_{F_0}} \mathbb{E}) \cong O_F \otimes_{O_{F_0}} \operatorname{Lie} \mathbb{E}$$

as $(O_F \otimes_{O_{F_0}} \overline{k})$-modules. It is clear from this that $(\mathbb{X}_2, \iota_{\mathbb{X}_2})$ satisfies the Kottwitz and spin conditions. To define the polarization $\lambda_{\mathbb{X}_2}$, first note that canonically

$$\mathbb{X}_2^\vee \cong \overline{O_F^\vee} \otimes_{O_{F_0}} \mathbb{E}^\vee$$

as O_F-modules (where O_F acts on $\mathbb{X}_2^\vee$ as prescribed by (3.1)); here $\overline{O_F^\vee}$ is the O_{F_0}-linear dual of O_F, made into an O_F-module by the rule

$$(x \cdot f)(y) = f(\overline{x}y) \quad \text{for} \quad x, y \in O_F, \quad f \in O_F^\vee.$$

Define the (injective but not surjective) O_F-linear map

$$\varphi\colon O_F \longrightarrow \overline{O_F^\vee}, \quad x \longmapsto \big[y \mapsto \tfrac{1}{2} \operatorname{tr}_{F/F_0}(\overline{x}y)\big]. \tag{3.4}$$

Then we define $\lambda_{\mathbb{X}_2}$ to be the map

$$O_F \otimes_{O_{F_0}} \mathbb{E} \xrightarrow{\varphi \otimes \lambda_{\mathbb{E}}} \overline{O_F^\vee} \otimes_{O_{F_0}} \mathbb{E}^\vee \cong (O_F \otimes_{O_{F_0}} X)^\vee.$$

Note that $\lambda_{\mathbb{X}_2}$ is anti-symmetric because φ is symmetric and $\lambda_{\mathbb{E}}$ is anti-symmetric,[4] and one readily verifies that $\operatorname{Ker} \lambda_{\mathbb{X}_2} = \mathbb{X}_2[\iota_{\mathbb{X}_2}(\pi)]$.

The triple $(\mathbb{X}_2, \iota_{\mathbb{X}_2}, \lambda_{\mathbb{X}_2})$ can be expressed in more concrete terms after choosing an O_{F_0}-basis for O_F. Indeed the choice of basis 1, π induces an isomorphism of O_{F_0}-modules

$$O_F \otimes_{O_{F_0}} \mathbb{E} \simeq \mathbb{E} \times \mathbb{E}. \tag{3.5}$$

The action of π on the left-hand side of this isomorphism translates to the action of the matrix

$$\begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}$$

on the right-hand side. Using the dual basis to identify $\overline{O_F^\vee} \otimes_{O_{F_0}} \mathbb{E}^\vee \simeq (\mathbb{E}^\vee)^2$, the polarization $\lambda_{\mathbb{X}_2}$ is given by the matrix

$$\begin{bmatrix} \lambda_{\mathbb{E}} & 0 \\ 0 & -\varpi\lambda_{\mathbb{E}} \end{bmatrix}. \tag{3.6}$$

[4] There is a mistake in [9, (6.2)]: the δ^{-1} in loc. cit. should be eliminated to obtain an anti-symmetric homomorphism into the dual, rather than a symmetric one.

Remark 3.4. A framing object in our setting in the case $n = 2$ is also described in [11, §5 d)], but contrary to the claim made there, it does not give rise to a formally smooth moduli space. Indeed this object manifestly does not satisfy the spin condition; or one can check directly that the hermitian space corresponding to it (defined as in §3.4 below) is split. The object in loc. cit. should be replaced with $\mathbb{X}_2$ as we have just defined it.

Now we define framing objects for $n > 2$. As in the Introduction, we denote by $\overline{\mathbb{E}}$ the same O_{F_0}-module as $\mathbb{E}$, but with O_F-action $\iota_{\overline{\mathbb{E}}}$ equal to $\iota_{\mathbb{E}}$ precomposed by the nontrivial Galois automorphism. Then $\lambda_{\mathbb{E}}$ is still O_F-linear with respect to $\iota_{\overline{\mathbb{E}}}$, and we denote it by $\lambda_{\overline{\mathbb{E}}}$. If n is even, then we define

$$\begin{aligned}\mathbb{X}_n &:= \mathbb{X}_2 \times \overline{\mathbb{E}}^{n-2},\\ \iota_{\mathbb{X}_n} &:= \iota_{\mathbb{X}_2} \times \iota_{\overline{\mathbb{E}}}^{n-2},\end{aligned}$$

and, in matrix form,

$$\lambda_{\mathbb{X}_n} := \lambda_{\mathbb{X}_2} \times \operatorname{diag}\Bigg(\underbrace{\begin{bmatrix} 0 & \lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) \\ -\lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) \\ -\lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) & 0 \end{bmatrix}}_{(n-2)/2 \text{ times}}\Bigg).$$

Then indeed $\operatorname{Ker} \lambda_{\mathbb{X}_n} = \mathbb{X}_n[\iota_{\mathbb{X}_n}(\pi)]$. If n is odd, then we define

$$(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) := (\mathbb{X}_{n-1} \times \overline{\mathbb{E}}, \iota_{\mathbb{X}_{n-1}} \times \iota_{\overline{\mathbb{E}}}, \lambda_{\mathbb{X}_{n-1}} \times \lambda_{\overline{\mathbb{E}}}). \tag{3.7}$$

Then indeed $\operatorname{Ker} \lambda_{\mathbb{X}_n} \subset \mathbb{X}_n[\iota_{\mathbb{X}_n}(\pi)]$ of height $n-1$. For either parity of n, if $n > 1$, then it is clear that $\iota_{\mathbb{X}_n}(\pi)$ acts on $\operatorname{Lie} \mathbb{X}_n$ with rank 1. Hence $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ is a unitary p-divisible group of signature $(1, n-1)$ for all n.

3.4. **Automorphisms of framing objects.** For $n \geq 1$, let $g \mapsto g^\dagger$ denote the Rosati involution on $\operatorname{End}^\circ_{O_F}(\mathbb{X}_n)$ induced by $\lambda_{\mathbb{X}_n}$. Define

$$U(\mathbb{X}_n) := U(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) := \big\{\, g \in \operatorname{End}^\circ_{O_F}(\mathbb{X}_n) \bigm| gg^\dagger = \mathrm{id}_{\mathbb{X}_n} \,\big\}. \tag{3.8}$$

Thus $U(\mathbb{X}_n)$ is the group of O_F-linear self-quasi-isogenies of $\mathbb{X}_n$ which preserve $\lambda_{\mathbb{X}_n}$ on the nose.

Next define the space of *special quasi-homomorphisms*

$$\mathbb{V}_n := \operatorname{Hom}^\circ_{O_F}\big(\overline{\mathbb{E}}, \mathbb{X}_n\big); \tag{3.9}$$

cf. e.g. [9, Def. 3.1]. Then $\mathbb{V}_n$ is an n-dimensional F-vector space. It carries a natural F/F_0-hermitian form h: for $x, y \in \mathbb{V}_n$, the composite

$$\overline{\mathbb{E}} \xrightarrow{y} \mathbb{X}_n \xrightarrow{\lambda_{\mathbb{X}_n}} \mathbb{X}_n^\vee \xrightarrow{x^\vee} \overline{\mathbb{E}}^\vee \xrightarrow{\lambda_{\overline{\mathbb{E}}}^{-1}} \overline{\mathbb{E}}$$

lies in $\operatorname{End}^\circ_{O_F}(\overline{\mathbb{E}})$, and hence identifies with an element $h(x, y) \in F$ via the isomorphism

$$\iota_{\overline{\mathbb{E}}} \colon F \xrightarrow{\sim} \operatorname{End}^\circ_{O_F}(\overline{\mathbb{E}}).$$

Lemma 3.5. *The hermitian space $(\mathbb{V}_n, h)$ is non-split for all n.*

Proof. When $n = 1$, we have $\mathbb{V}_1 = \operatorname{Hom}^\circ_{O_F}(\overline{\mathbb{E}}, \mathbb{E})$ and the lemma is clear. When $n = 2$, using the O_{F_0}-linear isomorphism $\mathbb{X}_2 \simeq \mathbb{E} \times \mathbb{E}$ in (3.5), $\mathbb{V}_2$ identifies with

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathrm{M}_{2\times 1}(D) \;\middle|\; \begin{bmatrix} & \varpi \\ 1 & \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \iota_{\overline{\mathbb{E}}}(\pi) \right\} = \left\{ \begin{bmatrix} b\iota_{\overline{\mathbb{E}}}(\pi) \\ b \end{bmatrix} \;\middle|\; b \in D \right\}.$$

For $b \in D$, one computes from the explicit form (3.6) of the polarization that $\left[\begin{smallmatrix} b\iota_{\overline{\mathbb{E}}}(\pi) \\ b \end{smallmatrix}\right]$ pairs with itself under h to $-2\varpi \mathrm{N} b$. Thus $\mathbb{V}_2$ has no nonzero isotropic vectors, which characterizes it as the non-split hermitian space of dimension 2.

To complete the proof for higher n, note that the definition of $\mathbb{V}_n$ as a hermitian space makes sense for any polarized formal O_F-module in place of $\mathbb{X}_n$. Doing this for the pair $(\overline{\mathbb{E}}, \lambda_{\overline{\mathbb{E}}})$, we obtain a 1-dimensional space V_1 which is obviously split; and doing this for

$$\left(\overline{\mathbb{E}}^2, \begin{bmatrix} 0 & \lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) \\ -\lambda_{\overline{\mathbb{E}}}\iota_{\overline{\mathbb{E}}}(\pi) & 0 \end{bmatrix}\right),$$

we obtain a 2-dimensional space V_2 which is obviously split. When n is even, we conclude that

$$\mathbb{V}_n \cong \mathbb{V}_2 \oplus V_2^{(n-2)/2}$$

is non-split, because it is the orthogonal direct sum of a non-split space and an even-dimensional split space. When n is odd, we conclude that

$$\mathbb{V}_n \cong \mathbb{V}_{n-1} \oplus V_1$$

is non-split, because it is the orthogonal direct sum of an even-dimensional non-split space with a split space. □

The group $U(\mathbb{X}_n)$ acts naturally from the left on $\mathbb{V}_n$, and in this way identifies with $\mathrm{U}(h)$. Thus in terms of the notation (2.2), we may, and will, choose an isomorphism

$$U(\mathbb{X}_n) \simeq U_1(F_0) \tag{3.10}$$

for all n.

3.5. **The moduli space.** We now define the moduli space $\mathcal{N}_n$. For S a scheme over $\operatorname{Spf} O_{\breve F}$, let

$$\overline{S} := \operatorname{Spec} \mathcal{O}_S/\pi\mathcal{O}_S.$$

The S-points on $\mathcal{N}_n$ are isomorphism classes of quadruples

$$(X, \iota_X, \lambda_X, \rho_X),$$

where (X, ι_X, λ_X) is a unitary p-divisible group of signature $(1, n-1)$, and where

$$\rho_X : X \times_S \overline{S} \longrightarrow \mathbb{X}_n \times_{\operatorname{Spec} \overline{k}} \overline{S}$$

is an O_F-linear quasi-isogeny of height 0 such that $\rho_X^*(\lambda_{\mathbb{X}_n} \times_{\operatorname{Spec} \overline{k}} \overline{S})$ is locally an $O_{F_0}^\times$-multiple of $\lambda_X \times_S \overline{S}$ in $\operatorname{Hom}^\circ_{O_F}(X_{\overline{S}}, X_{\overline{S}}^\vee)$, i.e., locally on S,

$$\rho_X^*(\lambda_{\mathbb{X}_n} \times_{\operatorname{Spec} \overline{k}} \overline{S}) = c(\lambda_X) \cdot (\lambda_X \times_S \overline{S}), \quad c(\lambda_X) \in O_{F_0}^\times. \tag{3.11}$$

Here an isomorphism between quadruples $(X, \iota_X, \lambda_X, \rho_X) \xrightarrow{\sim} (Y, \iota_Y, \lambda_Y, \rho_Y)$ is an O_F-linear isomorphism of p-divisible groups $\alpha : X \xrightarrow{\sim} Y$ over S such that $\rho_Y \circ (\alpha \times_S \overline{S}) = \rho_X$ and such that $\alpha^*(\lambda_Y)$ is locally an $O_{F_0}^\times$-multiple of λ_X.

Each g in the group $U(\mathbb{X}_n)$ (3.8) is a quasi-isogeny of height 0, and therefore $U(\mathbb{X}_n)$ acts naturally on $\mathcal{N}_n$ on the left via the rule $g \cdot (X, \iota_X, \lambda_X, \rho_X) = (X, \iota_X, \lambda_X, g\rho_X)$.

Remark 3.6. We have formulated the moduli problem defining $\mathcal{N}_n$ in a way that conforms with other instances of RZ spaces in the literature, by taking the polarization λ_X up to scalar in $O_{F_0}^\times$. But the definition can be reformulated without reference to scalar factors: consider the moduli problem of quadruples $(X, \iota_X, \lambda_X, \rho_X)$ as above, except where $\rho_X^*(\lambda_{\mathbb{X}_n} \times_{\operatorname{Spec} \overline{k}} \overline{S})$ is required to equal $\lambda_X \times_S \overline{S}$ on the nose, and where isomorphisms α as above satisfy $\alpha^*(\lambda_Y) = \lambda_X$ on the nose. It is easy to see that the functor this defines is isomorphic to $\mathcal{N}_n$; and in some situations this version of the moduli problem is a little more convenient to work with.

Example 3.7 ($n = 1$)**.** Let us make the definition of $\mathcal{N}_n$ explicit in the case $n = 1$. The framing object $\mathbb{E} = \mathbb{X}_1$ is a formal π-divisible O_F-module of height 1 via $\iota_{\mathbb{E}}$. It is easy to see that any framing map ρ of height 0 into $\mathbb{E}$ as above must be an isomorphism, and it follows that $\mathcal{N}_1$ is the universal deformation space of $\mathbb{E}$ over $\operatorname{Spf} O_{\breve F}$, which is just $\operatorname{Spf} O_{\breve F}$ itself. We write $\mathcal{E}$ for the universal p-divisible group over $\mathcal{N}_1$, endowed with its O_F-action $\iota_{\mathcal{E}}$, principal polarization $\lambda_{\mathcal{E}}$, and framing $\rho_{\mathcal{E}} : \mathcal{E}_{\overline{k}} \xrightarrow{\sim} \mathbb{E}$. Of course, the triple $(\mathcal{E}, \iota_{\mathcal{E}}, \rho_{\mathcal{E}})$ is nothing but the canonical lift of $(\mathbb{E}, \iota_{\mathbb{E}})$ in the sense of Lubin–Tate theory, as in the Introduction.

3.6. **Formal smoothness and essential properness.** In this subsection we prove the following basic result on the geometry of $\mathcal{N}_n$.

Proposition 3.8. *The formal scheme $\mathcal{N}_n$ is formally locally of finite type and formally smooth over* $\operatorname{Spf} O_{\breve F}$ *of relative formal dimension $n-1$ at every point. If n is even, then $\mathcal{N}_n$ is essentially proper over* $\operatorname{Spf} O_{\breve F}$.

Here *essentially proper* means that every irreducible component of $(\mathcal{N}_n)_{\mathrm{red}}$ is proper over $\operatorname{Spec} \bar k$. The proof of the proposition is via the *local model*, whose definition we now recall in the situation at hand. Let

$$m := \lfloor n/2 \rfloor.$$

Let $e_1, \dots, e_n$ denote the standard basis in F^n. Let ϕ be the (split) F/F_0-hermitian form on F^n specified by

$$\phi(ae_i, be_j) = \overline{a} b \delta_{i,n+1-j} \quad \text{(Kronecker delta)}.$$

Let $\langle\ ,\ \rangle$ be the alternating F_0-bilinear form $F^n \times F^n \to F_0$ defined by

$$\langle x, y\rangle := \frac{1}{2} \operatorname{tr}_{F/F_0}\bigl(\pi^{-1}\phi(x,y)\bigr).$$

Then

$$\langle \pi x, y\rangle = -\langle x, \pi y\rangle \quad \text{for all} \quad x, y \in F^n. \tag{3.12}$$

For $i = bn + c$ with $0 \le c < n$, define the O_F-lattice

$$\Lambda_i := \sum_{j=1}^{c} \pi^{-b-1} O_F e_j + \sum_{j=c+1}^{n} \pi^{-b} O_F e_j \subset F^n.$$

For each i, the form $\langle\ ,\ \rangle$ induces a perfect O_{F_0}-bilinear pairing

$$\Lambda_i \times \Lambda_{-i} \longrightarrow O_{F_0}.$$

In this way, for fixed nonempty $I \subset \{0, \dots, m\}$, the set

$$\Lambda_I := \{\, \Lambda_i \mid i \in \pm I + n\mathbb{Z} \,\}$$

forms a polarized chain of O_F-lattices over O_{F_0} in the sense of [23, Def. 3.14]. The following definition of the naive local model is an alternative formulation of [23, Def. 3.27] in our situation, in the case $I = \{m\}$.

Definition 3.9. The *naive local model* M^{naive} is the scheme over $\operatorname{Spec} O_F$ representing the functor that sends each O_F-scheme S to the set of all families

$$(\mathcal{F}_i \subset \Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S)_{i \in \pm m + n\mathbb{Z}}$$

such that

(1) for all i, $\mathcal{F}_i$ is an $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S$ which is Zariski-locally on S an $\mathcal{O}_S$-direct summand of rank n;

(2) for all $i < j$, the natural arrow $\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S \to \Lambda_j \otimes_{O_{F_0}} \mathcal{O}_S$ carries $\mathcal{F}_i$ into $\mathcal{F}_j$;

(3) for all i, the isomorphism $\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S \xrightarrow[\sim]{\pi\otimes 1} \Lambda_{i-n} \otimes_{O_{F_0}} \mathcal{O}_S$ identifies $\mathcal{F}_i \xrightarrow{\sim} \mathcal{F}_{i-n}$;

(4) for all i, the perfect $\mathcal{O}_S$-bilinear pairing

$$(\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S) \times (\Lambda_{-i} \otimes_{O_{F_0}} \mathcal{O}_S) \xrightarrow{\langle\ ,\ \rangle \otimes \mathcal{O}_S} \mathcal{O}_S$$

identifies $\mathcal{F}_i^{\perp}$ with $\mathcal{F}_{-i}$ inside $\Lambda_{-i} \otimes_{O_{F_0}} \mathcal{O}_S$; and

(5) (*Kottwitz condition*) for all i, the section $\pi \otimes 1 \in O_F \otimes_{O_{F_0}} \mathcal{O}_S$ acts on $\mathcal{F}_i$ as an $\mathcal{O}_S$-linear endomorphism with characteristic polynomial

$$\operatorname{char}(\pi \otimes 1 \mid \mathcal{F}_i) = (T - \pi)^{n-1}(T + \pi) \in \mathcal{O}_S[T].$$

The *local model* M^{loc} is the subscheme of M^{naive} defined by the additional conditions

(6) (*wedge condition*) for all i,

$$\bigwedge\nolimits_{\mathcal{O}_S}^{2} (\pi \otimes 1 - 1 \otimes \pi \mid \mathcal{F}_i) = 0;$$

and

(7) if $n > 1$, then $\pi \otimes 1 \mid \mathcal{F}_i \otimes_{\mathcal{O}_S} \kappa(s)$ is nonvanishing for all $s \in S$ and all i.

Plainly M^{naive} is representable by a closed subscheme of a product of Grassmannians. Furthermore the inclusion $M^{\mathrm{loc}} \subset M^{\mathrm{naive}}$ is an isomorphism on generic fibers, which both identify naturally with $\mathbb{P}_F^{n-1}$ [15, §1.5.3].

Proposition 3.10. *M^{loc} is smooth over $\operatorname{Spec} O_F$ of relative dimension $n-1$ at every point, and a closed subscheme of M^{naive} when n is even.*

Proof. This is essentially a matter of collecting results in the literature. Let $M^{\wedge}$ denote the closed subscheme of M^{naive} defined by the wedge condition. Then M^{loc} is an open subscheme of $M^{\wedge}$. As is explained in [15, §3.3], the geometric special fiber $M_{\bar{k}}^{\mathrm{naive}}$ of the naive local model embeds into an affine flag variety for GU_n, where it and $M_{\bar{k}}^{\wedge}$ decompose (topologically) into unions of Schubert cells. In the present setting, the Schubert cells C_r that occur in $M_{\bar{k}}^{\mathrm{naive}}$ are described in §2.4 of loc. cit. They are indexed by the rank r of $\pi \otimes 1$ acting on $\mathcal{F}_m$ at each point. On the level of topological spaces, it follows from the definitions that $M_{\bar{k}}^{\wedge} = C_0 \cup C_1$ and $M_{\bar{k}}^{\mathrm{loc}} = C_1$. Now, Arzdorf [2, Prop. 3.2] and [15, §5.3] show respectively for odd and even n that $M^{\wedge}$ contains an open subscheme isomorphic to $\mathbb{A}_{O_F}^{n-1}$, and it is immediate from these references that this open subscheme is contained in M^{loc}.[5] Since M^{loc} has generic fiber $\mathbb{P}_F^{n-1}$, and since C_1 is an orbit for a group action, it follows that M^{loc} is everywhere smooth of relative dimension $n-1$.

Furthermore, when n is even, one readily verifies that the perfect pairing

$$\Lambda_m \times \Lambda_m \xrightarrow[\sim]{\mathrm{id}\times\pi} \Lambda_m \times \Lambda_{-m} \xrightarrow{\langle\, ,\, \rangle} O_{F_0}$$

is split symmetric. Hence by conditions (3) and (4) above, M^{naive} embeds into the orthogonal Grassmannian OGr of totally isotropic n-planes in $2n$-space; cf. [16, Rem. 2.32]. The scheme OGr has 2 connected components, and it is easy to see that these separate C_0, which consists of a single point in the special fiber, from the rest of $M^{\wedge}$. Hence $M^{\mathrm{loc}} = M^{\wedge} \smallsetminus C_0$ is closed in $M^{\wedge}$, and hence in M^{naive}. $\square$

Remark 3.11. When n is even, and in the presence of the other conditions in the definition of M^{loc}, condition (7) is equivalent to the *spin condition* formulated in [15, §7.2]. For a general signature and parahoric level structure, the spin condition in loc. cit. is not a purely topological condition, but in this special case it is; see again Rem. 2.32 and the paragraph following it in [16].

When n is odd, and again in the presence of the other conditions, condition (7) *implies* the spin condition in [15]. Indeed, the spin condition is a closed condition on M^{naive} which is satisfied on the generic fiber, and hence on the flat closure of the generic fiber, and we have just seen that M^{loc} is smooth, and hence flat, with the same generic fiber. However, (7) and the spin condition are not equivalent, since imposing the spin condition on $M^{\wedge}$ does not eliminate the point C_0.

The above relationships are the origin of the term "spin condition" in the definition of unitary p-divisible group in §3.1. But, for the reason just explained, note that this is a slight abuse of language when n is odd.

Remark 3.12. More is true when n is odd. Indeed, in this case let N denote the scheme-theoretic closure of the generic fiber of M^{naive} in M^{naive}. Then N contains M^{loc} as an open subscheme, and is itself smooth over $\operatorname{Spec} O_F$ by Richarz [2, Prop. 4.16]. The main result of [24] establishes a moduli description for N, by introducing a condition which is weaker than (7) above but stronger than the spin condition in [15]. However, this condition is much more complicated to state than (7), and for the purposes of this paper it suffices to work just with (7) instead.

[5] Strictly speaking, these references work with signature $(n-1,1)$, whereas we are working with the opposite signature $(1,n-1)$. But these situations are isomorphic: explicitly, the isomorphism $F^n \xrightarrow{\sim} F^n$ given by applying the nontrivial Galois automorphism on each factor induces an isomorphism of lattice chains $\Lambda_I \xrightarrow{\sim} \Lambda_I$, which in turn induces an isomorphism between the corresponding local models. Also, when n is odd, we note that the scheme denoted M^{loc} in [2] does not coincide with M^{loc} as we have defined it.

The link between Propositions 3.8 and 3.10 is via the *local model diagram* [23]. Let $\mathcal{N}_n^{\mathrm{naive}}$ denote the moduli functor over $\operatorname{Spf} O_{\breve{F}}$ defined in the same way as $\mathcal{N}_n$, except without the wedge and spin conditions. By [23, Th. 3.25], $\mathcal{N}_n^{\mathrm{naive}}$ is representable by a formal scheme which is formally locally of finite type over $\operatorname{Spf} O_{\breve{F}}$. We will see in the course of proving Proposition 3.8 below that the inclusion

$$\mathcal{N}_n \subset \mathcal{N}_n^{\mathrm{naive}}$$

is a closed immersion when n is even, and an immersion in general.

Let $\widetilde{\mathcal{N}}_n^{\mathrm{naive}}$ denote the functor that associates to each scheme S over $\operatorname{Spf} O_{\breve{F}}$ the set of isomorphism classes of quintuples

$$(X, \iota_X, \lambda_X, \rho_X, \gamma),$$

where $(X, \iota_X, \lambda_X, \rho_X) \in \mathcal{N}_n^{\mathrm{naive}}(S)$, and γ is an isomorphism of polarized chains of $O_F \otimes_{O_{F_0}} \mathcal{O}_S$-modules

$$[\cdots \xrightarrow{\iota_X(\pi)_*} M(X) \xrightarrow{\iota_X(\pi)_*} M(X) \xrightarrow{\iota_X(\pi)_*} \cdots] \xrightarrow[\sim]{\gamma} \Lambda_{\{m\}} \otimes_{O_{F_0}} \mathcal{O}_S$$

when n is even, and

$$[\cdots \longrightarrow M(X) \xrightarrow{(\lambda_X)_*} M(X^\vee) \xrightarrow{\mu_*} \cdots] \xrightarrow[\sim]{\gamma} \Lambda_{\{m\}} \otimes_{O_{F_0}} \mathcal{O}_S$$

when n is odd, in the terminology of [23]. Let us explain the notation. We have denoted by M the functor that assigns to a p-divisible group the Lie algebra of its universal vector extension. Our requirements for ι_X and λ_X imply that there is a unique (necessarily O_F-linear) isogeny $\mu\colon X^\vee \to X$ such that the composite

$$X \xrightarrow{\lambda_X} X^\vee \xrightarrow{\mu} X$$

is $\iota_X(\pi)$; thus μ is an isomorphism or of height 1 according as n is even or odd. Upon applying M, this diagram extends periodically to the chains appearing above (and it explains the source of the chain's polarization when n is even).

The functor $\widetilde{\mathcal{N}}_n^{\mathrm{naive}}$ is representable by a formal scheme over $\operatorname{Spf} O_{\breve{F}}$, and it sits in a diagram

$$\begin{tikzcd} & \widetilde{\mathcal{N}}_n^{\mathrm{naive}} \arrow[dl, "\varphi"'] \arrow[dr] & \\ \mathcal{N}_n^{\mathrm{naive}} & & M^{\mathrm{naive}}_{\operatorname{Spf} O_{\breve{F}}}. \end{tikzcd}$$

Here φ is the natural map that forgets γ; it is a torsor under the automorphism scheme of $\Lambda_{\{m\}}$ as a polarized O_F-lattice chain over O_{F_0}, which is smooth. The arrow on the right sends an S-point $(X, \iota_X, \lambda_X, \rho_X, \gamma)$ to the family $(\mathcal{F}_i \subset \Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S)_{i \in \pm m + n\mathbb{Z}}$, where for each i, $\mathcal{F}_i$ is the image under γ in $\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_S$ of the Fil^1 term in the covariant Hodge filtration for the p-divisible group. With this in hand, we now arrive at the proof of Proposition 3.8.

Proof of Proposition 3.8. The key point is that by [23, Prop. 3.33], after passing to an étale cover $\mathcal{U} \to \mathcal{N}_n^{\mathrm{naive}}$, the map φ admits a section s such that the composite $\mathcal{U} \to M^{\mathrm{naive}}_{\operatorname{Spf} O_{\breve{F}}}$ in the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "s"] \arrow[dr] & \widetilde{\mathcal{N}}_n^{\mathrm{naive}} \arrow[d, "\varphi"'] \arrow[dr] & \\ & \mathcal{N}_n^{\mathrm{naive}} & M^{\mathrm{naive}}_{\operatorname{Spf} O_{\breve{F}}} \end{tikzcd}$$

is formally étale. We claim that the respective inverse images of $\mathcal{N}_n$ and $M^{\mathrm{loc}}_{\operatorname{Spf} O_{\breve{F}}}$ in $\mathcal{U}$ coincide. In fact, we will show that the respective wedge conditions on the one hand, and the spin condition and condition (7) on the other hand, separately pull back to equivalent conditions on $\mathcal{U}$.

First consider the wedge conditions. Let $(\mathcal{F}_i \subset \Lambda_i \otimes_{O_{F_0}} \mathcal{O}_{M^{\mathrm{naive}}})_i$ denote the universal object over M^{naive}. For fixed i, consider the tautological exact sequence

$$0 \longrightarrow \mathcal{F}_i \longrightarrow \Lambda_i \otimes_{O_{F_0}} \mathcal{O}_{M^{\mathrm{naive}}} \longrightarrow (\Lambda_i \otimes_{O_{F_0}} \mathcal{O}_{M^{\mathrm{naive}}})/\mathcal{F}_i \longrightarrow 0. \tag{3.13}$$

It is easy to see that on the explicit affine charts on M^{naive} calculated in [15, §5.1–5.3],

$$\bigwedge\nolimits^2_{\mathcal{O}_{M^{\mathrm{naive}}}}(\pi\otimes 1-1\otimes\pi \mid \mathcal{F}_i)=0 \iff \bigwedge\nolimits^2_{\mathcal{O}_{M^{\mathrm{naive}}}}\big(\pi\otimes 1+1\otimes\pi \mid (\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})/\mathcal{F}_i\big)=0.$$

Since these charts meet every Schubert cell in the special fiber (after embedding the geometric special fiber into an affine flag variety, as discussed in the proof of Proposition 3.10), the equivalence in the display holds on all of M^{naive}.[6] Now consider the $\mathcal{O}_{M^{\mathrm{naive}}}$-linear dual of (3.13),

$$0\longrightarrow\big((\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})/\mathcal{F}_i\big)^\vee\longrightarrow(\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})^\vee\longrightarrow\mathcal{F}_i^\vee\longrightarrow 0.$$

The isomorphism $\Lambda_{-i}\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}}\xrightarrow{\sim}(\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})^\vee$ induced by $\langle\ ,\ \rangle$ identifies $\mathcal{F}_{-i}$ with $((\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})/\mathcal{F}_i)^\vee$ by condition (4), and it identifies the operator $\pi\otimes 1$ on $\Lambda_{-i}\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}}$ with $-\pi\otimes 1$ on $(\Lambda_i\otimes_{O_{F_0}}\mathcal{O}_{M^{\mathrm{naive}}})^\vee$ by (3.12). It follows from these observations and from condition (3) that the wedge condition on M^{naive} holds for all i as soon as it holds for a single i, and that it and the wedge condition on $\mathcal{N}_n^{\mathrm{naive}}$ pull back to equivalent conditions on $\mathcal{U}$.

Now consider condition (7) on M^{naive}. This condition is automatically satisfied in the generic fiber, so let S be the spectrum of a field κ which is an extension of k. Let $(\mathcal{F}_i\subset\Lambda_i\otimes_{O_{F_0}}\kappa)_i$ be an S-point on M^{naive}. Then

$$(\pi\otimes 1\mid\mathcal{F}_i)=0 \iff \mathcal{F}_i=(\pi\otimes 1)\cdot(\Lambda_i\otimes_{O_{\breve F}}\kappa) \iff \big(\pi\otimes 1\mid(\Lambda_i\otimes_{O_{F_0}}\kappa)/\mathcal{F}_i\big)=0.$$

Using conditions (3) and (4), it follows as above that (7) holds for all i as soon as it holds for a single i, and that it and the spin condition on $\mathcal{N}_n^{\mathrm{naive}}$ pull back to equivalent conditions on $\mathcal{U}$.

Thus we have shown that $\mathcal{N}_n$ and $M^{\mathrm{loc}}_{\operatorname{Spf}O_{\breve F}}$ have common inverse image in $\mathcal{U}$. We conclude that $\mathcal{N}_n\subset\mathcal{N}_n^{\mathrm{naive}}$ is an immersion in general and closed immersion when n is even, since the same is true of $M^{\mathrm{loc}}\subset M^{\mathrm{naive}}$. Thus $\mathcal{N}_n$ inherits the property of being formally locally of finite type from $\mathcal{N}_n^{\mathrm{naive}}$. By Proposition 3.10, it also follows that $\mathcal{N}_n$ is formally smooth over $\operatorname{Spf}O_{\breve F}$ of relative formal dimension $n-1$. Finally, by Prop. 2.32 and the proof of Th. 3.25 in [23], $\mathcal{N}_n^{\mathrm{naive}}$ is essentially proper over $\operatorname{Spf}O_{\breve F}$. Thus the same is true of $\mathcal{N}_n$ when n is even. □

Remark 3.13. In analogy with Remark 3.12, when n is odd, one can replace the spin condition in the definition of $\mathcal{N}_n$ with a weaker condition, based on the one introduced in [24], to obtain an essentially proper, formally smooth formal scheme containing $\mathcal{N}_n$ as an open formal subscheme. Although this formal scheme is in some sense a "better" object, its definition is much more complicated to state, and for the purposes of this paper it suffices to work with $\mathcal{N}_n$ as we have defined it.

4. Intersection numbers

In this section we define the intersection numbers that appear in our AT conjectures. Let $n\geq 3$ be an odd integer.

4.1. **The morphisms $\delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}}$.** We begin by introducing some closed embeddings of moduli spaces. As in the Introduction, over $O_{\breve F}$ we have the canonical lift $\mathcal{E}$ of $\mathbb{E}$ equipped with its action $\iota_{\mathcal{E}}\colon O_F\to\operatorname{End}(\mathcal{E})$, its principal polarization $\lambda_{\mathcal{E}}$, and its framing isomorphism $\rho_{\mathcal{E}}\colon\mathcal{E}_{\overline{k}}\xrightarrow{\sim}\mathbb{E}$; see also Example 3.7. We further have the quadruple $(\overline{\mathcal{E}},\iota_{\overline{\mathcal{E}}},\lambda_{\overline{\mathcal{E}}},\rho_{\overline{\mathcal{E}}})$, where $\overline{\mathcal{E}}$ is the same underlying p-divisible group, where the O_F-action $\iota_{\overline{\mathcal{E}}}$ is obtained by precomposing $\iota_{\mathcal{E}}$ with the nontrivial Galois automorphism on O_F, and where $\lambda_{\overline{\mathcal{E}}}=\lambda_{\mathcal{E}}$ and $\rho_{\overline{\mathcal{E}}}=\rho_{\mathcal{E}}$.

Using $(\overline{\mathcal{E}},\iota_{\overline{\mathcal{E}}},\lambda_{\overline{\mathcal{E}}},\rho_{\overline{\mathcal{E}}})$, we define a morphism of formal moduli schemes

$$\delta_{\mathcal{N}}\colon\mathcal{N}_{n-1}^{\mathrm{naive}}\longrightarrow\mathcal{N}_n^{\mathrm{naive}}\tag{4.1}$$

as follows. Let S be a scheme over $\operatorname{Spf}O_{\breve F}$, and let $(Y,\iota_Y,\lambda_Y,\rho_Y)\in\mathcal{N}_{n-1}^{\mathrm{naive}}(S)$, cf. §§3.5, 3.6. Then locally on S there exists $c_Y\in O_{F_0}^\times$ such that $\rho_Y^*(\lambda_{\mathbb{X}_{n-1}}\times_{\operatorname{Spec}\overline{k}}\overline{S})=c_Y\cdot(\lambda_Y\times_S\overline{S})$,

[6] Strictly speaking, when n is odd, [15] computes an affine chart for a different maximal parahoric level structure than the one we are using. But the calculations in [15] are easily adapted to our case. See also Arzdorf [2].

cf. (3.11). Recall that for odd n we have the framing object $\mathbb{X}_n = \mathbb{X}_{n-1} \times_{\operatorname{Spec} \bar{k}} \overline{\mathbb{E}}$ and its polarization $\lambda_{\mathbb{X}_n} = \lambda_{\mathbb{X}_{n-1}} \times \lambda_{\overline{\mathbb{E}}}$. We define

$$\delta_{\mathcal{N}} \colon (Y, \iota_Y, \lambda_Y, \rho_Y) \longmapsto \bigl(Y \times \overline{\mathcal{E}}, \iota_Y \times \iota_{\overline{\mathcal{E}}}, \lambda_Y \times c_Y^{-1} \lambda_{\overline{\mathcal{E}}}, \rho_Y \times \rho_{\overline{\mathcal{E}}}\bigr).$$

This is a well-defined functor. Indeed, $(\rho_Y \times \rho_{\overline{\mathcal{E}}})^*(\lambda_{\mathbb{X}_{n-1}} \times \lambda_{\overline{\mathbb{E}}}) = c_Y(\lambda_Y \times c_Y^{-1}\lambda_{\overline{\mathbb{E}}})$. Furthermore, if $(Y, \iota_Y, \lambda_Y, \rho_Y)$ is isomorphic to $(Y', \iota_{Y'}, \lambda_{Y'}, \rho_{Y'})$, then there exists an O_F-linear isomorphism $\alpha \colon Y' \xrightarrow{\sim} Y$ compatible with $\rho_{Y'}$ and ρ_Y, and such that, locally on the base, $\gamma\lambda_{Y'} = \alpha^*(\lambda_Y)$ for some $\gamma \in O_{F_0}^\times$. But then $c_{Y'} = \gamma c_Y$, from which one verifies that $\alpha \times \mathrm{id}_{\overline{\mathcal{E}}} \colon Y' \times \overline{\mathcal{E}} \xrightarrow{\sim} Y \times \overline{\mathcal{E}}$ defines an isomorphism from $\delta_{\mathcal{N}}(Y', \iota_{Y'}, \lambda_{Y'}, \rho_{Y'})$ to $\delta_{\mathcal{N}}(Y, \iota_Y, \lambda_Y, \rho_Y)$.

Remark 4.1. If one uses the alternative formulation of the moduli problem defining $\mathcal{N}_n$ given in Remark 3.6, where c_Y is required to equal 1, then the well-definedness of $\delta_{\mathcal{N}}$ is essentially obvious.

Lemma 4.2. *The map $\delta_{\mathcal{N}}$ induces a closed immersion of formal schemes*

$$\delta_{\mathcal{N}} \colon \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_n.$$

Proof. If $(Y, \iota, \lambda, \rho)$ is a point on $\mathcal{N}_{n-1}$, then $(\iota \times \iota_{\overline{\mathcal{E}}})(\pi) \mid \operatorname{Lie}(Y \times \overline{\mathcal{E}})$ is pointwise nonvanishing because $\iota(\pi) \mid \operatorname{Lie} Y$ is, and $\delta_{\mathcal{N}}(Y, \iota, \lambda, \rho)$ satisfies the wedge condition because $(Y, \iota, \lambda, \rho)$ does and because $(\iota_{\overline{\mathcal{E}}}(\pi) + \pi \mid \operatorname{Lie} \overline{\mathcal{E}}) = 0$. ☐

Using $\delta_{\mathcal{N}}$, we obtain a closed immersion of formal schemes,

$$\Delta_{\mathcal{N}} \colon \mathcal{N}_{n-1} \xrightarrow{(\mathrm{id}_{\mathcal{N}_{n-1}}, \delta_{\mathcal{N}})} \mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n.$$

Let us conclude this subsection by explaining the equivariance properties of the maps $\delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}}$. Since n is odd, the non-split hermitian space $\mathbb{V}_n = \operatorname{Hom}^\circ_{O_F}(\overline{\mathbb{E}}, \mathbb{X}_n)$ in (3.9) has a canonical special vector u of norm 1, namely

$$u := (0, \mathrm{id}_{\overline{\mathbb{E}}}) \in \operatorname{Hom}^\circ_{O_F}\bigl(\overline{\mathbb{E}}, \mathbb{X}_n\bigr) = \operatorname{Hom}^\circ_{O_F}\bigl(\overline{\mathbb{E}}, \mathbb{X}_{n-1} \times \overline{\mathbb{E}}\bigr). \tag{4.2}$$

The stabilizer of u in $U(\mathbb{X}_n) \cong \mathrm{U}(\mathbb{V}_n)$ identifies with $U(\mathbb{X}_{n-1})$. In this way we obtain an identification of

$$H_1(F_0) \subset U_1(F_0) \quad \text{with} \quad U(\mathbb{X}_{n-1}) \subset U(\mathbb{X}_n), \tag{4.3}$$

as in (3.10). Via these identifications, $H_1(F_0)$ acts on $\mathcal{N}_{n-1}$, $U_1(F_0)$ acts on $\mathcal{N}_n$, and the product $G_{W_1}(F_0) = H_1(F_0) \times U_1(F_0)$ acts on $\mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n$. The maps $\delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}}$ are then equivariant for the action of $H_1(F_0)$, i.e.

$$\begin{aligned} \delta_{\mathcal{N}}\bigl(h \cdot (Y, \iota_Y, \lambda_Y, \rho_Y)\bigr) &= h \cdot \delta_{\mathcal{N}}\bigl((Y, \iota_Y, \lambda_Y, \rho_Y)\bigr), \\ \Delta_{\mathcal{N}}\bigl(h \cdot (Y, \iota_Y, \lambda_Y, \rho_Y)\bigr) &= h \cdot \Delta_{\mathcal{N}}\bigl((Y, \iota_Y, \lambda_Y, \rho_Y)\bigr), \end{aligned} \tag{4.4}$$

where $H_1(F_0)$ acts via its diagonal embedding into $G_{W_1}(F_0)$ on the right-hand side of the second equation.

4.2. **Homogeneous setting.** We now use $\Delta_{\mathcal{N}}$ to define intersection numbers. We start with the homogeneous case. Define the closed formal subscheme of $\mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n$,

$$\Delta := \Delta_{\mathcal{N}}(\mathcal{N}_{n-1}). \tag{4.5}$$

Of course this is not to be confused with the discriminant (2.6); throughout the paper context should suffice to make the meaning of Δ clear. For $g \in G_{W_1}(F_0)$, we define the translate

$$\Delta_g := g\Delta. \tag{4.6}$$

We then define the intersection number of Δ and Δ_g by the Euler-Poincaré characteristic of the derived tensor product (cf. [9]),

$$\operatorname{Int}(g) := \langle \Delta, \Delta_g \rangle := \chi\bigl(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_g}\bigr). \tag{4.7}$$

Note that, since $\mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n$ is a regular formal scheme by Proposition 3.8, this derived tensor product is represented by a finite complex of locally free coherent modules. Furthermore, the intersection $\Delta \cap \Delta_g$ can be identified with

$$\Delta \cap \Delta_g = \Delta_{\mathcal{N}}^{-1}(\Delta_g),$$

where Δ and Δ_g are closed formal subschemes of $\mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_n$. It follows that $\Delta \cap \Delta_g$ may be identified with a closed formal subscheme of $\mathcal{N}_{n-1}$. Since $n-1$ is an even integer, the formal scheme $\mathcal{N}_{n-1}$ is essentially proper over $\mathrm{Spf}\, O_{\breve F}$ by Proposition 3.8. It follows that $\mathrm{Int}(g)$ is finite, provided that the underlying reduced scheme of the intersection $\Delta \cap \Delta_g$ is of finite type over $\mathrm{Spec}\, O_{\breve F}$, and that the ideal of definition of this formal scheme is nilpotent. Indeed, under these conditions, $\Delta \cap \Delta_g$ will be a scheme X proper over $\mathrm{Spec}\, O_{\breve F}$ with support in the special fiber, and the cohomology groups of any perfect complex of $\mathcal{O}_X$-modules are finite length $O_{\breve F}$-modules, and there are only finitely many of them. For situations where we can make sure this happens, see Remark 4.5 below.

Remark 4.3. We note that $\mathrm{Int}(g)$ only depends on the double coset of g modulo $H_1(F_0)$ (with respect to the diagonal embedding $H_1(F_0) \subset G_{W_1}(F_0)$). Indeed, the equivariance property (4.4) implies that

$$h\Delta = \Delta \quad \text{for all} \quad h \in H_1(F_0).$$

Hence

$$\Delta_{gh_2} = gh_2\Delta = g\Delta = \Delta_g,$$

and

$$\langle \Delta, \Delta_{h_1 g h_2} \rangle = \langle \Delta, \Delta_{h_1 g} \rangle = \langle \Delta, h_1 \Delta_g \rangle = \langle h_1^{-1} \Delta, \Delta_g \rangle = \langle \Delta, \Delta_g \rangle.$$

4.3. **Inhomogeneous setting.** For the inhomogeneous case, we recycle notation by introducing, for $g \in U_1(F_0)$, the closed formal subscheme of $\mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_n$,

$$\Delta_g := (1 \times g)\Delta, \tag{4.8}$$

and setting

$$\mathrm{Int}(g) := \langle \Delta, \Delta_g \rangle = \chi(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_g}). \tag{4.9}$$

The same remarks as in the homogeneous case above apply to this definition.

Remark 4.4. It follows from Remark 4.3 that in the inhomogeneous setting, $\mathrm{Int}(g)$ only depends on the orbit of g under the conjugation action of $H_1(F_0)$.

Remark 4.5. Identify $U_1(F_0)$ with $U(\mathbb{X}_n)$, and $G_{W_1}(F_0)$ with $U(\mathbb{X}_{n-1}) \times U(\mathbb{X}_n)$, according to (4.3). We claim that for any $g \in U_{1,\mathrm{rs}}(F_0)$ in the inhomogeneous case, or any $g \in G_{W_1,\mathrm{rs}}(F_0)$ in the homogeneous case, the quantity $\mathrm{Int}(g)$ is finite. More precisely, we claim that in either case, the intersection of Δ and Δ_g inside $\mathcal{N}_{n-1} \times_{O_{\breve F}} \mathcal{N}_n$ is a *scheme* over $\mathrm{Spf}\, O_{\breve F}$ (i.e., any ideal of definition of this formal scheme is nilpotent) which is proper over $\mathrm{Spec}\, O_{\breve F}$.

Indeed, for simplicity, let us consider the inhomogeneous case. Then the proof of [33, Lem. 2.8] goes through, although we are lacking at the moment a suitable reference for the global facts used in loc. cit. The argument is based on the relation with KR divisors [9]. Namely, let $\mathcal{Z}(u)$ be the special cycle defined by the special vector $u = u_1$. Then there is the identification $\mathcal{Z}(u) = \delta_{\mathcal{N}}(\mathcal{N}_{n-1})$. Via the second projection there is an identification $\Delta \cap \Delta_g \subset \mathcal{N}_n^g$. Therefore we obtain an inclusion

$$\Delta \cap \Delta_g \subset \mathcal{Z}(u) \cap \mathcal{Z}(gu) \cap \cdots \cap \mathcal{Z}(g^{n-1}u).$$

But since g is regular semi-simple, the vectors $u, gu, \ldots, g^{n-1}u$ in $\mathbb{V}_n$ are linearly independent, cf. §2.4. In other words, the fundamental matrix of the special divisors $\mathcal{Z}(u), \ldots, \mathcal{Z}(g^{n-1}u)$ is non-singular. Now we approximate the vectors $u, gu, \ldots, g^{n-1}u$ by "global vectors" $v_1, \ldots, v_n$, and we imitate in the present ramified case the global argument of [33, Lem. 2.8], which shows that there is a chain of inclusions of schemes

$$\mathcal{Z}(u) \cap \mathcal{Z}(xu) \cap \cdots \cap \mathcal{Z}(x^{n-1}u) \subset \mathcal{Z}(v_1) \cap \mathcal{Z}(v_2) \cap \cdots \cap \mathcal{Z}(v_n) \subset \mathrm{Sh}^{\mathrm{ss}}.$$

Here $\mathrm{Sh}^{\mathrm{ss}}$ denotes the supersingular locus of the integral model of the global Shimura variety, and we are implicitly using nonarchimedean uniformization to make these identifications. (In the unramified situation the $\mathcal{Z}(v_i)$ are relative divisors on Sh, whereas in our ramified situation we can only assert that the underlying point set has codimension one.)

On the other hand, in the case $n = 3$, we will show directly in Theorem 5.20 below that $\Delta \cap \Delta_g$ is an artinian scheme whenever $g \in U_{1,\mathrm{rs}}(F_0)$. The proof of this does not use a global argument.

4.4. **Lie algebra setting.** It would be interesting to transpose the above to the Lie algebra case. To this end, first note that the closed formal subscheme (4.8) can be identified with the closed sublocus of points $((Y, \iota_Y, \lambda_Y, \rho_Y), (X, \iota_X, \lambda_X, \rho_X)) \in \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$ where

$$\textit{the quasi-endomorphism } g\colon \mathbb{X}_n \to \mathbb{X}_n \textit{ lifts to a homomorphism } Y \times \overline{\mathcal{E}} \to X \ . \tag{4.10}$$

Here we recall from (3.7) that $\mathbb{X}_n = \mathbb{X}_{n-1} \times \overline{\mathbb{E}}$ by definition, and by "lifts" we mean with respect to the framings ρ_Y of Y, resp. $\rho_{\overline{\mathcal{E}}}$ of $\overline{\mathcal{E}}$, resp. ρ_X of X.

Note that condition (4.10) still makes sense when g is replaced by any quasi-endomorphism $\mathbb{X}_n$. Hence we may replace the formal subspace $\Delta_g = (1 \times g) \cdot \Delta_{\mathcal{N}}(\mathcal{N}_{n-1})$ in the inhomogeneous version above by the analogous subspace Δ_x, for any x in the Lie algebra $\mathfrak{u}_1$ of U_1. Unfortunately, we do not know how to give a reasonable definition of $\mathrm{Int}(x) = \langle \Delta, \Delta_x \rangle$ in general, cf. our remarks in the Introduction. Here by "reasonable" we mean that at least, as in Remark 4.3, $\mathrm{Int}(x)$ only depends on the conjugation orbit of x under $H_1(F_0)$, and that if $\Delta \cap \Delta_x$ is an artinian *scheme*, then $\mathrm{Int}(x)$ coincides with the length of $\Delta \cap \Delta_x$. One problem, pointed out to us by A. Mihatsch, is that it may happen that the formal dimension of Δ_x is smaller than n.

Remark 4.6. For any quasi-endomorphism x of $\mathbb{X}_n$ and any $h_1, h_2 \in H_1(F_0)$, it is elementary to verify that

$$(h_2 \times h_1)\Delta_x = \Delta_{h_1 x h_2^{-1}}.$$

Taking $h_1 = h_2 =: h$, and taking $H_1(F_0)$ to act diagonally on $\mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$ as before, we conclude that the formal subspaces $\Delta \cap \Delta_x$ and $\Delta \cap \Delta_{hxh^{-1}} = \Delta \cap h\Delta_x = h(\Delta \cap \Delta_x)$ are isomorphic.

Remark 4.7. Similar to Remark 4.5, the intersection $\Delta \cap \Delta_x$ is a scheme proper over $\mathrm{Spec}\, O_{\breve{F}}$, provided that $x \in \mathfrak{u}_{1,\mathrm{rs}}(F_0)$. Indeed, the same proof works, once the following two observations are taken into account. First,

$$\Delta \cap \Delta_x \subset \Delta \cap \Delta_{x^i}, \quad i \geq 1.$$

Indeed, abusing notation in the obvious way, the left-hand side is the locus of points (Y, X) in $\mathcal{N}_{n-1} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$ where $Y \times \overline{\mathcal{E}} \simeq X$ and where x lifts to an endomorphism of X. It is clear that then also x^i lifts as an endomorphism of X, for any $i \geq 1$. Second, for any $y \in \mathrm{End}_{O_F}(\mathbb{X}_n)$,

$$\Delta_y \subset \mathcal{Z}(yu).$$

Indeed, the restriction of $y : Y \times \overline{\mathcal{E}} \to X$ to the second factor coincides with the homomorphism $yu : \overline{\mathcal{E}} \to X$, hence the locus where y extends to a homomorphism is contained in the locus $\mathcal{Z}(yu)$ where the homomorphism yu extends. We conclude that

$$\Delta \cap \Delta_x \subset \mathcal{Z}(u) \cap \mathcal{Z}(xu) \cap \cdots \cap \mathcal{Z}(x^{n-1}u),$$

and the proof proceeds from here as in the group case.

We also conjecture the following partial converse to Remarks 4.5 and 4.7.

Conjecture 4.8. *Let $g \in G_{W_1}(F_0)$ be semi-simple (i.e., its orbit under $H_1 \times H_1$ is Zariski-closed). If the intersection of Δ and Δ_g inside $\mathcal{N}_{n-1} \times_{O_{\breve{E}}} \mathcal{N}_n$ is a nonempty scheme proper over* $\mathrm{Spec}\, O_{\breve{F}}$*, then g is regular semi-simple. The inhomogeneous version for $g \in U_1(F_0)$ and the Lie algebra version for $x \in \mathfrak{u}_1(F_0)$ instead of g also hold true.*

5. Conjectures and main results

In this section, we formulate the general conjecture that is addressed in this paper. Following the case distinctions we have made earlier, we will formulate three variants: a homogeneous version, an inhomogeneous version, and a Lie algebra version. We continue to denote by F/F_0 a ramified quadratic extension (and assume $p \neq 2$). Throughout this section, $n \geq 3$ is an *odd* integer.

5.1. **Homogeneous setting.** For $\gamma \in G'(F_0)_{\mathrm{rs}}$, for a function $f' \in C_c^\infty(G')$, and for a complex parameter $s \in \mathbb{C}$, we introduce the weighted orbital integral

$$\mathrm{Orb}(\gamma, f', s) := \int_{H'_{1,2}(F_0)} f'(h_1^{-1}\gamma h_2)|\det h_1|^s \eta(h_2)\, dh_1\, dh_2. \tag{5.1}$$

Here $\eta = \eta_{F/F_0}$ is the quadratic character corresponding to F/F_0, and we are using a product Haar measure on $H'_{1,2}(F_0) = H'_1(F_0) \times H'_2(F_0)$. Also, for

$$h_2 = (h_2', h_2'') \in H_2'(F_0) = \mathrm{GL}_{n-1}(F_0) \times \mathrm{GL}_n(F_0),$$

we write $\eta(h_2)$ for $\eta(\det h_2')$. Note that $\det h_1$ is an element of F, and we are taking the normalized absolute value on F in (5.1). We also introduce

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0) \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s).$$

The integral in (5.1) is absolutely convergent, and

$$\mathrm{Orb}(h_1^{-1}\gamma h_2, f') = \eta(h_2)\,\mathrm{Orb}(\gamma, f') \quad \text{for} \quad (h_1, h_2) \in H'_{1,2}(F_0) = H'_1(F_0) \times H'_2(F_0).$$

Now let $i \in \{0, 1\}$, and set $W := W_i$ and $H := H_i$. For $g \in G_W(F_0)_{\mathrm{rs}}$ and for a function $f \in C_c^\infty(G_W)$, we introduce the orbital integral

$$\mathrm{Orb}(g, f) := \int_{H(F_0)\times H(F_0)} f(h_1^{-1} g h_2)\, dh_1\, dh_2.$$

Here on $H(F_0) \times H(F_0)$ we take a product measure of identical Haar measures on $H(F_0)$.

Dual to the matching of regular semi-simple elements discussed in §2 is the transfer relation of functions on $G'(F_0)$, $G_{W_0}(F_0)$, and $G_{W_1}(F_0)$. To define this, recall that a *transfer factor* is a function $\Omega\colon G'(F_0)_{\mathrm{rs}} \to \mathbb{C}^\times$ such that

$$\Omega(h_1^{-1}\gamma h_2) = \eta(h_2)\Omega(\gamma) \quad \text{for all} \quad (h_1, h_2) \in H'_{1,2}(F_0) = H'_1(F_0) \times H'_2(F_0).$$

Transfer factors always exist, and we will specify our particular choice below.

Definition 5.1. A function $f' \in C_c^\infty(G')$ and a pair of functions $(f_0, f_1) \in C_c^\infty(G_{W_0}) \times C_c^\infty(G_{W_1})$ are *transfers* of each other, or are *associated* (for the fixed choices of Haar measures and a fixed choice of transfer factor and a fixed choice of special vectors u_i in W_i), if for each $i \in \{0, 1\}$ and each $g \in G_{W_i}(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}(g, f_i) = \Omega(\gamma)\,\mathrm{Orb}(\gamma, f')$$

whenever $\gamma \in G'(F_0)_{\mathrm{rs}}$ matches g.

In the specific case at hand, we will take the transfer factor $\Omega = \omega$ to be given by the following formula. Let $\widetilde{\eta}$ be an extension of η from $F_0^\times$ to $F^\times$ (not necessarily of order 2). Then we take[7]

$$\omega(\gamma) := \widetilde{\eta}\big(\det(\widetilde{\gamma})^{-(n-1)/2} \det(\widetilde{\gamma}^i e)_{i=0,\dots,n-1}\big), \tag{5.2}$$

where for $\gamma = (\gamma_1, \gamma_2) \in G'(F_0)_{\mathrm{rs}}$ we set $\widetilde{\gamma} = s(\gamma) = (\gamma_1^{-1}\gamma_2)\big(\overline{\gamma_1^{-1}\gamma_2}\big)^{-1} \in S_n(F_0)$, and where we recall the column vector $e = (0, \dots, 0, 1) \in F^n$. We point out that

$$\det(\widetilde{\gamma})^{-(n-1)/2} \det(\widetilde{\gamma}^i e)_{i=0,1,\dots,n-1}$$

is an eigenvector under the Galois involution, i.e., it is in either F_0 or πF_0.

Let us fix the rest of the choices that go into the statement of our AT conjecture. We normalize the Haar measure in (5.1) by assigning the subgroup $H'_1(O_{F_0}) = \mathrm{GL}_{n-1}(O_F)$ measure 1, and by taking the product Haar measure on $H'_2(F_0) = \mathrm{GL}_{n-1}(F_0) \times \mathrm{GL}_n(F_0)$ such that $\mathrm{GL}_{n-1}(O_{F_0})$ and $\mathrm{GL}_n(O_{F_0})$ have measure 1. On the unitary side, recall from (2.7) that we assume that the special vectors $u_0 \in W_0$ and $u_1 \in W_1$ have norm 1. Since n is odd, it follows that the perp spaces $W_0^\flat$ and $W_1^\flat$ are the respective split and non-split hermitian spaces of dimension $n-1$. Since $W_0^\flat$ is furthermore even dimensional, it contains a *π-modular lattice* [7, Prop. 8.1(b)], that is, an O_F-lattice $\Lambda_0^\flat$ such that $(\Lambda_0^\flat)^\vee = \pi^{-1}\Lambda_0^\flat$, where $(\Lambda_0^\flat)^\vee \subset W_0^\flat$ denotes the set of elements that pair with $\Lambda_0^\flat$ to values in O_F under the hermitian form. We fix such a $\Lambda_0^\flat$ and denote by $K_0^\flat$

[7]Note that in this paper our choice of various transfer factors is slightly different from that in [35].

its stabilizer in $H_0(F_0)$. We normalize the Haar measure on $H_0(F_0)$ such that $K_0^\flat$ gets measure 1. The normalization of the Haar measure on $H_1(F_0)$ will not be important for us. We set

$$\Lambda_0 := \Lambda_0^\flat \oplus O_F u_0, \tag{5.3}$$

which is a *nearly π-modular lattice* in W_0, i.e. $\Lambda_0 \subset \Lambda_0^\vee \subset \pi^{-1}\Lambda_0$ with $\pi^{-1}\Lambda_0/\Lambda_0^\vee$ of length 1. We denote by K_0 the stabilizer of Λ_0 in $U_0(F_0)$.

Remark 5.2. The maximal compact open subgroup $K_0^\flat \subset H_0(F_0)$ is a special maximal parahoric subgroup in the sense of Bruhat–Tits theory. The maximal compact open subgroup $K_0 \subset U_0(F_0)$ is the full stabilizer in $U_0(F_0)$ of a special vertex in the (extended) building, and contains the associated maximal parahoric subgroup with index 2. See [14, §4.a]. These subgroups have symplectic reduction in the sense that the quotients $(\Lambda_0^\flat)^\vee/\Lambda_0^\flat$ and $\Lambda_0^\vee/\Lambda_0$ have dimension $n-1$ over the residue field k, and the hermitian form on the ambient space induces a non-degenerate alternating bilinear pairing on both.

We now state the homogeneous version of the arithmetic transfer conjecture in the case at hand. For $g \in G_{W_1}(F_0)$, recall the intersection number $\mathrm{Int}(g)$ from (4.7).

Conjecture 5.3 (Homogeneous AT conjecture)**.**

(a) *There exists a function $f' \in C_c^\infty(G')$ which transfers to $(\mathbf{1}_{K_0^\flat \times K_0}, 0) \in C_c^\infty(G_{W_0}) \times C_c^\infty(G_{W_1})$, and which satisfies the following identity for any $\gamma \in G'(F_0)_{\mathrm{rs}}$ matched with an element $g \in G_{W_1}(F_0)_{\mathrm{rs}}$:*

$$\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q.$$

(b) *For any $f' \in C_c^\infty(G')$ which transfers to $(\mathbf{1}_{K_0^\flat \times K_0}, 0) \in C_c^\infty(G_{W_0}) \times C_c^\infty(G_{W_1})$, there exists a function $f'_{\mathrm{corr}} \in C_c^\infty(G')$ such that for any $\gamma \in G'(F_0)_{\mathrm{rs}}$ matched with an element $g \in G_{W_1}(F_0)_{\mathrm{rs}}$,*

$$\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q + \omega(\gamma)\, \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}).$$

Remarks 5.4. (i) Both sides of these identities depend only on the respective orbits of γ and g: for the right-hand side this follows from Remark 4.3; and for the left-hand side, this follows from $\mathrm{Orb}(\gamma, f') = 0$ for any $\gamma \in G'(F_0)_{\mathrm{rs}}$ matched with an element $g \in G_{W_1}(F_0)_{\mathrm{rs}}$, which holds because f' transfers to 0 on $G_{W_1}(F_0)$.

(ii) Part (a) of Conjecture 5.3 follows from part (b); see Proposition 5.14 below. The converse (a) $\Longrightarrow$ (b) would follow from a conjectural density principle (Conjecture 5.15); see Lemma 5.17 below.

(iii) The function f' in Conjecture 5.3(a) cannot be taken to lie in the Iwahori Hecke algebra of $G'(F_0) = \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F)$ due to the presence of the ramified quadratic character η. It is tempting to guess that such an f' could be taken in the pro-unipotent Hecke algebra of $G'(F_0)$, i.e., to be bi-invariant under the pro-unipotent radical of an Iwahori subgroup. But even in the case $n = 3$ we do not know how to prove this.

5.2. **Inhomogeneous setting.** Now we give the inhomogeneous version of the conjecture. Recall the scheme $S = S_n = \{g \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid g\bar{g} = 1_n\}$. For $\gamma \in S(F_0)_{\mathrm{rs}}$, a function $f' \in C_c^\infty(S)$, and a complex parameter $s \in \mathbb{C}$, we introduce the weighted orbital integral

$$\mathrm{Orb}(\gamma, f', s) := \int_{H'(F_0)} f'(h^{-1}\gamma h) |\det h|^s \eta(h)\, dh, \tag{5.4}$$

as well as the special values

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0) \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s).$$

Here we write $\eta(h)$ for $\eta(\det h)$, as in the Introduction. Note that $\det h \in F_0$, and in (5.4) we are taking its absolute value for the normalized absolute value on F_0. As in the homogeneous setting, the integral defining $\mathrm{Orb}(\gamma, f', s)$ is absolutely convergent, and

$$\mathrm{Orb}(h^{-1}\gamma h, f') = \eta(h)\, \mathrm{Orb}(\gamma, f') \quad \text{for all} \quad h \in H'(F_0) = \mathrm{GL}_{n-1}(F_0).$$

For $i \in \{0,1\}$, set $U := U_i$ and $H := H_i$. For $g \in U(F_0)_{\mathrm{rs}}$ and a function $f \in C_c^\infty(U)$, we similarly introduce the orbital integral

$$\mathrm{Orb}(g, f) := \int_{H(F_0)} f(h^{-1}gh)\, dh.$$

A *transfer factor* on $S(F_0)_{\mathrm{rs}}$ is a function $\Omega\colon S(F_0)_{\mathrm{rs}} \to \mathbb{C}^\times$ such that $\Omega(h^{-1}\gamma h) = \eta(h)\Omega(\gamma)$ for all $\gamma \in S(F_0)_{\mathrm{rs}}$ and $h \in H'(F_0)$. Quite analogously to the homogeneous setting, in the case at hand we take the transfer factor $\Omega = \omega$ given by

$$\omega(\gamma) := \widetilde{\eta}\big(\det(\gamma)^{-(n-1)/2} \det(\gamma^i e)_{i=0,\dots,n-1}\big). \tag{5.5}$$

By definition, this is compatible with the transfer factor (5.2) on $G'(F_0)$ in the sense that, for $\gamma \in G'(F_0)$,

$$\omega(\gamma) = \omega\big(s(\gamma)\big). \tag{5.6}$$

We take the other normalizations (the length of the special vectors, the Haar measures on $H'(F_0)$ and $H(F_0)$) and the notation K_0 as in the homogeneous setting. The transfer relation for functions is the following.

Definition 5.5. A function $f' \in C_c^\infty(S)$ and a pair of functions $(f_0, f_1) \in C_c^\infty(U_0) \times C_c^\infty(U_1)$ are *transfers* of each other, or are *associated*, if for each $i \in \{0,1\}$ and each $g \in U_i(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}(g, f_i) = \omega(\gamma)\, \mathrm{Orb}(\gamma, f')$$

whenever $\gamma \in S(F_0)_{\mathrm{rs}}$ matches g.

The inhomogeneous version of the conjecture takes the following form.

Conjecture 5.6 (Inhomogeneous AT conjecture)**.**
(a) *There exists a function $f' \in C_c^\infty(S)$ which transfers to $(\mathbf{1}_{K_0}, 0) \in C_c^\infty(U_0) \times C_c^\infty(U_1)$, and which satisfies the following identity for any $\gamma \in S(F_0)_{\mathrm{rs}}$ matched with an element $g \in U_1(F_0)_{\mathrm{rs}}$:*

$$2\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q.$$

(b) *For any $f' \in C_c^\infty(S)$ which transfers to $(\mathbf{1}_{K_0}, 0) \in C_c^\infty(U_0) \times C_c^\infty(U_1)$, there exists a function $f'_{\mathrm{corr}} \in C_c^\infty(S)$ such that for any $\gamma \in S(F_0)_{\mathrm{rs}}$ matched with an element $g \in U_1(F_0)_{\mathrm{rs}}$,*

$$2\omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f') = -\mathrm{Int}(g) \cdot \log q + \omega(\gamma)\, \mathrm{Orb}(\gamma, f'_{\mathrm{corr}}).$$

Note that there is a factor of 2 in Conjecture 5.6 which is not present in Conjecture 5.3; this is due to the fact that the restriction to F_0 of the normalized absolute value of F is the square of the normalized absolute value of F_0, cf. Lemma 5.7 and its proof below.

In the rest of this subsection, we show that Conjectures 5.3 and 5.6 are equivalent.

Lemma 5.7. *Let $f' \in C_c^\infty(G')$, and define the function $\widetilde{f}'$ on $S(F_0)$ by, for $g \in \mathrm{GL}_n(F)$,*

$$\widetilde{f}'(g\overline{g}^{-1}) := \int_{\mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F_0)} f'(h_1, h_1 g h_2)\, dh_1\, dh_2. \tag{5.7}$$

(i) *$\widetilde{f}' \in C_c^\infty(S)$, and every element in $C_c^\infty(S)$ arises in this way.*
(ii) *For all $\gamma \in G'(F_0)_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, f') = \mathrm{Orb}\big(s(\gamma), \widetilde{f}'\big). \tag{5.8}$$

If moreover f' transfers to $(f_0, 0)$ for some $f_0 \in C_c^\infty(G_{W_0})$, then for any γ matching an element in $G_{W_1}(F_0)_{\mathrm{rs}}$,

$$\partial\mathrm{Orb}(\gamma, f') = 2\, \partial\mathrm{Orb}\big(s(\gamma), \widetilde{f}'\big). \tag{5.9}$$

Proof. Part (i) is clear; we show part (ii). Note that for $(\gamma_1, \gamma_2) \in G'(F_0) = \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_n(F)$,

$$\mathrm{Orb}\big((\gamma_1, \gamma_2), f', s\big) = |\gamma_1|^{-s}\, \mathrm{Orb}\big((1, \gamma_1^{-1}\gamma_2), f', s\big).$$

It is then clear that the left-hand sides of both (5.8) and (5.9) are invariant if we replace (γ_1, γ_2) by $(1, \gamma_1^{-1}\gamma_2)$ under the assumption of the lemma. Hence it suffices to consider elements of the form $(1, \gamma) \in G'(F_0)$. By definition,

$$\mathrm{Orb}\big((1, \gamma), f', s\big) = \int_{H'_{1,2}(F_0)} f'(h_1^{-1}h_2', h_1^{-1}\gamma h_2'')|\det h_1|^s \eta(h_2')\, dh_1\, dh_2'\, dh_2'',$$

where $h_1 \in H_1'(F_0) = \mathrm{GL}_{n-1}(F)$ and $(h_2', h_2'') \in H_2'(F_0) = \mathrm{GL}_{n-1}(F_0) \times \mathrm{GL}_n(F_0)$. Replacing h_1 by $h_2' h_1$, we have

$$\mathrm{Orb}\big((1,\gamma), f', s\big) = \int_{H'_{1,2}(F_0)} f'\big(h_1^{-1}, h_1^{-1}(h_2')^{-1}\gamma h_2''\big)|\det(h_2'h_1)|^s \eta(h_2')\, dh_1\, dh_2'\, dh_2''.$$

This is equal to the sum of

$$\int_{H'_{1,2}(F_0)} f'\big(h_1^{-1}, h_1^{-1}(h_2')^{-1}\gamma h_2''\big)|\det h_2'|^s \eta(h_2')\, dh_1\, dh_2'\, dh_2'' \tag{5.10}$$

and

$$\int_{H'_{1,2}(F_0)} f'\big(h_1^{-1}, h_1^{-1}(h_2')^{-1}\gamma h_2''\big)(|\det h_1|^s - 1)|\det h_2'|^s \eta(h_2')\, dh_1\, dh_2'\, dh_2''. \tag{5.11}$$

Comparing with the definition (5.4), we see that the term (5.10) is equal to

$$\int_{\mathrm{GL}_{n-1}(F_0)} \widetilde{f'}\big((h_2')^{-1}\gamma\overline{\gamma}^{-1}h_2'\big)|\det h_2'|_{F_0}^{2s}\, dh_2' = \mathrm{Orb}\big(\gamma\overline{\gamma}^{-1}, \widetilde{f'}, 2s\big).$$

The term (5.11) has an obvious zero at $s = 0$ for all regular semi-simple elements $(1,\gamma)$. This establishes (5.8). To prove (5.9), we note that, when $(1,\gamma)$ matches an element in $G_{W_1}(F_0)_{\mathrm{rs}}$, the term (5.11) has vanishing order at least two at $s = 0$. Indeed since $|\det h_1|^s - 1$ has a zero at $s = 0$, the first derivative evaluated at $s = 0$ is equal to

$$\int_{H'_{1,2}(F_0)} f'\big(h_1^{-1}, h_1^{-1}(h_2')^{-1}\gamma h_2''\big)\eta(h_2')\, dh_1\, dh_2'\, dh_2'' = \mathrm{Orb}\big((1,\gamma), f', 0\big) = 0,$$

since f' is assumed to transfer to some $(f_0, 0)$. This completes the proof. □

Lemma 5.8. *Conjectures 5.3(a) and 5.6(a) are equivalent. Similarly for Conjectures 5.3(b) and 5.6(b).*

Proof. We have

$$\mathrm{Int}(h_1 g h_2) = \mathrm{Int}(g) \quad \text{for all} \quad g \in G_{W_1}(F_0), \quad h_1, h_2 \in H_1(F_0).$$

Therefore it suffices to consider elements of the form $(1, g)$ for $g \in U_1(F_0)$. By (5.6), we have $\omega(\gamma) = \omega(s(\gamma))$ for $\gamma \in G'(F_0)$. The equivalence between Conjectures 5.3(a) and 5.6(a) is now clear by Lemma 5.7. Indeed, if $f' \in C_c^\infty(G')$ satisfies the conclusion of Conjecture 5.3(a), then the function $\widetilde{f'} \in C_c^\infty(S)$ will satisfy the conclusion of Conjecture 5.6(a). Conversely, if $f'' \in C_c^\infty(S)$ satisfies the conclusion of Conjecture 5.6(a), then we may choose $f' \in C_c^\infty(G')$ such that $\widetilde{f'} = f''$, and f' will then satisfy the conclusion of Conjecture 5.3(a). One may similarly show the equivalence between Conjectures 5.3(b) and 5.6(b), by taking the f'_{corr} in Conjecture 5.6(b) to be $\widetilde{f'_{\mathrm{corr}}}$ (cf. (5.7)) for the f'_{corr} in Conjecture 5.3(b). □

5.3. **Lie algebra setting.** We would like to formulate a Lie algebra version of Conjecture 5.6. At least the analytic side of the conjecture makes sense. Recall the Lie algebra version of the symmetric space $S = S_n$,

$$\mathfrak{s} = \big\{\, y \in \mathrm{Res}_{F/F_0} \mathrm{M}_n \;\big|\; y + \overline{y} = 0 \,\big\}.$$

For a regular semi-simple element $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$, a function $\phi' \in C_c^\infty(\mathfrak{s})$, and a complex parameter s, we define the weighted orbital integral

$$\mathrm{Orb}(y, \phi', s) := \int_{H'(F_0)} \phi'(h^{-1}yh)|\det h|^s \eta(h)\, dh,$$

as well as the special values

$$\mathrm{Orb}(y, \phi') := \mathrm{Orb}(y, \phi', 0) \quad \text{and} \quad \partial\mathrm{Orb}(y, \phi') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(y, \phi', s).$$

As in (5.4), here we are using the normalized absolute value on F_0. As before, the integral defining $\mathrm{Orb}(y, \phi', s)$ is absolutely convergent, and

$$\mathrm{Orb}(h^{-1}yh, \phi') = \eta(h)\, \mathrm{Orb}(y, \phi') \quad \text{for all} \quad h \in H'(F_0) = \mathrm{GL}_{n-1}(F_0).$$

For $i \in \{0,1\}$, set $\mathfrak{u} := \mathfrak{u}_i$ and $H := H_i$. For $x \in \mathfrak{u}(F_0)_{\mathrm{rs}}$ and a function $\phi \in C_c^\infty(\mathfrak{u})$, we similarly introduce the orbital integral

$$\mathrm{Orb}(x, \phi) := \int_{H(F_0)} \phi(h^{-1}xh)\, dh.$$

A *transfer factor* on $\mathfrak{s}(F_0)_{\mathrm{rs}}$ is a function $\Omega\colon \mathfrak{s}(F_0)_{\mathrm{rs}} \to \mathbb{C}^\times$ such that $\Omega(h^{-1}yh) = \eta(h)\Omega(y)$ for all $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ and $h \in H'(F_0)$. We take the transfer factor $\Omega = \omega$ on $\mathfrak{s}(F_0)_{\mathrm{rs}}$ given by

$$\omega(y) := \widetilde{\eta}\big(\det(y^i e)_{i=0,1,\dots,n-1}\big); \tag{5.12}$$

cf. [35, (3.7)]. Again we normalize the Haar measure on the symmetric space side such that $H'(O_{F_0}) = \mathrm{GL}_{n-1}(O_{F_0})$ has measure 1. On the unitary side, we denote by $\mathfrak{k}_0$ the stabilizer in $\mathfrak{u}_0(F_0)$ of the nearly π-modular lattice $\Lambda_0 \subset W_0$ defined in (5.3). Again we normalize the Haar measure on $H_0(F_0)$ such that the stabilizer $K_0^\flat$ of $\Lambda_0^\flat$ has measure 1. The transfer relation for functions extends readily to the Lie algebra setting as follows.

Definition 5.9. A function $\phi' \in C_c^\infty(\mathfrak{s})$ and a pair of functions $(\phi_0, \phi_1) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$ are *transfers* of each other, or are *associated*, if for each $i \in \{0,1\}$ and each $x \in \mathfrak{u}_i(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}(x, \phi_i) = \omega(y)\,\mathrm{Orb}(y, \phi')$$

whenever $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ matches x.

Now we state the Lie algebra version of the conjecture.

Conjecture 5.10 (Lie algebra AT conjecture)**.**
(a) *There exists a function $\phi' \in C_c^\infty(\mathfrak{s})$ which transfers to $(\mathbf{1}_{\mathfrak{k}_0}, 0) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$, and which satisfies the following identity for any $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ matched with an element $x \in \mathfrak{u}_1(F_0)_{\mathrm{rs}}$ for which the intersection $\Delta \cap \Delta_x$ is an artinian scheme:*

$$2\omega(y)\,\partial\mathrm{Orb}(y, \phi') = -\ell\text{-}\mathrm{Int}(x) \cdot \log q.$$

Here we set

$$\ell\text{-}\mathrm{Int}(x) := \mathrm{length}(\Delta \cap \Delta_x).$$

(b) *For any $\phi' \in C_c^\infty(\mathfrak{s})$ which transfers to $(\mathbf{1}_{\mathfrak{k}_0}, 0) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$, there exists a function $\phi'_{\mathrm{corr}} \in C_c^\infty(\mathfrak{s})$ such that for any $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ matched with an element $x \in \mathfrak{u}_1(F_0)_{\mathrm{rs}}$ for which the intersection $\Delta \cap \Delta_x$ is an artinian scheme,*

$$2\omega(y)\,\partial\mathrm{Orb}(y, \phi') = -\ell\text{-}\mathrm{Int}(x) \cdot \log q + \omega(y)\,\mathrm{Orb}(y, \phi'_{\mathrm{corr}}).$$

Remark 5.11. It is not currently clear to us how to formulate a conjecture without the hypothesis that $\Delta \cap \Delta_x$ is artinian; see our comments right before Remark 4.7.

5.4. **Relation between parts (a) and (b) of the conjectures.** In this subsection we explain how parts (a) and (b) in each of Conjectures 5.3, 5.6, and 5.10 are related.

The group $H'(F_0) = \mathrm{GL}_{n-1}(F_0)$ acts on $S(F_0)$ and hence on the space $C_c^\infty(S)$. For a function $f' \in C_c^\infty(S)$ and an element $h \in H'(F_0)$, we denote by ${}^h f' \in C_c^\infty(S)$ the function defined by

$${}^h f'\colon \gamma \longmapsto f'(h^{-1}\gamma h).$$

We also denote by ${}^{\eta(h)h-1} f' \in C_c^\infty(S)$ the function

$${}^{\eta(h)h-1} f'\colon \gamma \longmapsto \eta(h) f'(h^{-1}\gamma h) - f'(\gamma).$$

We use analogous notation in the Lie algebra setting, with $\mathfrak{s}$ in place of S.

Lemma 5.12. (i) *For any $f' \in C_c^\infty(S)$, $\gamma \in S(F_0)_{\mathrm{rs}}$, and $h \in H'(F_0)$,*

$$\mathrm{Orb}\big(\gamma, {}^{\eta(h)h-1} f'\big) = 0 \quad \textit{and} \quad \partial\mathrm{Orb}\big(\gamma, {}^{\eta(h)h-1} f'\big) = \log|\det h|\,\mathrm{Orb}(\gamma, f').$$

(ii) *For any $\phi' \in C_c^\infty(\mathfrak{s})$, $\gamma \in S(F_0)_{\mathrm{rs}}$, and $h \in H'(F_0)$,*

$$\mathrm{Orb}\big(\gamma, {}^{\eta(h)h-1} \phi'\big) = 0 \quad \textit{and} \quad \partial\mathrm{Orb}\big(\gamma, {}^{\eta(h)h-1} \phi'\big) = \log|\det h|\,\mathrm{Orb}(\gamma, \phi').$$

Proof. We prove (i); the proof of (ii) is analogous. The integral $\mathrm{Orb}(\gamma, f', s)$ transforms under the action of $H'(F_0)$ on f' by the character $\eta_s \circ \det$, where

$$\eta_s(a) := \eta(a)|a|^s, \quad a \in F_0^\times,$$

i.e.

$$\mathrm{Orb}\big(\gamma, {}^h f', s\big) = \eta_s(\det h)\,\mathrm{Orb}(\gamma, f', s).$$

It follows that

$$\mathrm{Orb}\big(\gamma, {}^{\eta(h)h-1}f', s\big) = (|\det h|^s - 1)\,\mathrm{Orb}(\gamma, f', s).$$

The proves the first desired identity, and the second follows by differentiating. □

Lemma 5.13. (i) *For any $f' \in C_c^\infty(S)$, there exists $f'^\sharp \in C_c^\infty(S)$ such that*

$$\mathrm{Orb}(\gamma, f') = \partial\mathrm{Orb}\big(\gamma, f'^\sharp\big) \quad \textit{for all} \quad \gamma \in S(F_0)_{\mathrm{rs}}.$$

Furthermore, we may choose $f'^\sharp$ such that it transfers to $(0,0) \in C_c^\infty(U_0) \times C_c^\infty(U_1)$.
(ii) *For any $\phi' \in C_c^\infty(\mathfrak{s})$, there exists $\phi'^\sharp \in C_c^\infty(\mathfrak{s})$ such that*

$$\mathrm{Orb}(y, \phi') = \partial\mathrm{Orb}\big(y, \phi'^\sharp\big) \quad \textit{for all} \quad y \in \mathfrak{s}(F_0)_{\mathrm{rs}}.$$

Furthermore, we may choose $\phi'^\sharp$ such that it transfers to $(0,0) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$.

Proof. Again we just prove (i). Choose any $h \in H'(F_0)$ with $\det h$ a non-unit, and set

$$f'^\sharp := \frac{{}^{\eta(h)h-1}f'}{\log|\det h|}.$$

Then $f'^\sharp$ has all desired properties by Lemma 5.12. □

Proposition 5.14. *Part (b) implies part (a) in each of Conjectures 5.3, 5.6, and 5.10.*

Proof. By the proof of the ST conjecture in [35, Th. 2.6] (resp. its Lie algebra analog in §4.5 of loc. cit.), there exists some function $f'' \in C_c^\infty(S)$ transferring to $(\mathbf{1}_{K_0}, 0)$ (resp. $\phi'' \in C_c^\infty(\mathfrak{s})$ transferring to $(\mathbf{1}_{\mathfrak{k}_0}, 0)$). Of course f'' and ϕ'' needn't satisfy the conclusion of part (a) in Conjectures 5.6 and 5.10, respectively, but assuming part (b) in these conjectures, we may use Lemma 5.13 to modify them into functions that do. The implication (b) $\Longrightarrow$ (a) for Conjecture 5.3 then follows from Lemma 5.8. □

For the converse direction from (a) to (b), we need the following.

Conjecture 5.15 (Density principle)**.** *The orbital integrals* $\mathrm{Orb}(\gamma, \cdot\,)$ *for all regular semi-simple γ span a weakly dense subspace in the space of $(H'(F_0), \eta)$-invariant distributions on $S(F_0)$. The same holds for $\mathfrak{s}(F_0)$.*

Remarks 5.16. (i) An equivalent statement to the above density principle for $S(F_0)$ is as follows: if a function $f' \in C_c^\infty(S)$ has vanishing orbital integrals $\mathrm{Orb}(\gamma, f') = 0$ for all regular semi-simple $\gamma \in S(F_0)$, then it lies in the subspace of $C_c^\infty(S)$ spanned by functions of the form ${}^{\eta(h)h-1}f''$ for $f'' \in C_c^\infty(S)$ and $h \in H'(F_0)$. (Of course the orbital integrals of ${}^{\eta(h)h-1}f''$ vanish by Lemma 5.12.)
(ii) It is easy to see that the density principles for $S(F_0)$ and $\mathfrak{s}(F_0)$ are equivalent.
(iii) The density principle holds for $\mathfrak{s}(F_0)$ when $n = 3$ by [34, Th. 1.1]; cf. Theorem 11.11 below. It is still open for $n \geq 4$.

Lemma 5.17. *Assume that Conjecture 5.15 holds. Then part (a) implies part (b) in each of Conjectures 5.3, 5.6, and 5.10.*

Proof. We show the implication (a) $\Longrightarrow$ (b) in Conjecture 5.6. The analogous implication for Conjecture 5.10 can be proved in a similar way, and that for Conjecture 5.3 follows by Lemma 5.8. Suppose that $f'_0 \in C_c^\infty(S)$ satisfies the conclusion of (a), and let $f' \in C_c^\infty(S)$ be any function transferring to $(\mathbf{1}_{K_0}, 0)$. Then the function $f' - f'_0$ has vanishing orbital integrals at all $\gamma \in S(F_0)_{\mathrm{rs}}$. By Conjecture 5.15, we may assume that $f' - f'_0$ is a sum of functions of the form ${}^{\eta(h)h-1}f''$ for $f'' \in C_c^\infty(S)$ and $h \in H'(F_0)$. By Lemma 5.12(i) (applied to f''), it follows that the function $\partial\mathrm{Orb}(\gamma, f' - f'_0)$ is of the form $\mathrm{Orb}(\gamma, f'_{\mathrm{corr}})$ for some $f'_{\mathrm{corr}} \in C_c^\infty(S)$, and hence (b) holds. This completes the proof. □

5.5. **Uniqueness of the function in part (a) of the conjectures.** In this subsection we explain the extent to which the function f' in Conjecture 5.6(a) is unique, assuming the density principle (Conjecture 5.15). An analogous statement holds for Conjecture 5.3(a). The statement does not carry over to Conjecture 5.10(a) as we have formulated it, owing to the additional hypothesis (artinian intersection) that we impose, cf. Remark 5.11.

Let $\epsilon := \eta(\det W_0^\flat) \in \{\pm 1\}$. Then $-\epsilon = \eta(\det W_1^\flat)$. Here we assume as usual that the norm of the special vector $u_i \in W_i$ is 1. Otherwise we may modify the definition of ϵ accordingly.

We consider the action of the transpose on $S(F_0)$ and on $C_c^\infty(S)$. For $f' \in C_c^\infty(S)$, we define the function f'^t by $f'^t(\gamma) := f'({}^t\gamma)$ for $\gamma \in S(F_0)$. Clearly, each f' can be written in a unique way as

$$f' = f'_+ + f'_-$$

where $f'^t_\pm = \pm\epsilon f'_\pm$. Now recall the decomposition $S(F_0)_{\mathrm{rs}} = S_{\mathrm{rs},0} \amalg S_{\mathrm{rs},1}$ from (2.4).

Lemma 5.18. *For any $f' \in C_c^\infty(S)$,*

$$\mathrm{Orb}(\gamma, f'_+) = 0 \quad \textit{for all} \quad \gamma \in S_{\mathrm{rs},1} \quad \textit{and} \quad \mathrm{Orb}(\gamma, f'_-) = 0 \quad \textit{for all} \quad \gamma \in S_{\mathrm{rs},0}.$$

Proof. This is a consequence of how Orb transforms with respect to the transpose operation. For $\gamma \in S(F_0)_{\mathrm{rs}}$, since ${}^t\gamma$ and γ have the same invariants, there exists a unique element $h_\gamma \in H'(F_0)$ such that

$$ {}^t\gamma = h_\gamma^{-1}\gamma h_\gamma. \tag{5.13}$$

Moreover, the element h_γ is symmetric, i.e. $h_\gamma = {}^th_\gamma$; and if $\gamma \in S_{\mathrm{rs},i}$, then h_γ defines a hermitian space isometric to $W_i^\flat$, i.e. $\eta(h_\gamma) = \eta(\det W_i^\flat)$ (cf. the proof of [33, Lem. 2.3]). Writing $\eta_s(h) := \eta_s(\det h)$ for $h \in H'(F_0)$, it follows from (5.13) and suitable substitutions that

$$\begin{aligned}
\mathrm{Orb}(\gamma, f'^t, s) &= \int_{H'(F_0)} f'({}^th\,{}^t\gamma\,{}^th^{-1})\eta_s(h)\,dh \\
&= \int_{H'(F_0)} f'(hh_\gamma^{-1}\gamma h_\gamma h^{-1})\eta_s(h)\,dh \\
&= \int_{H'(F_0)} f'(h^{-1}\gamma h)\eta_s(h^{-1}h_\gamma)\,dh \\
&= \eta_s(h_\gamma^{-1})\,\mathrm{Orb}(\gamma,\, f',s).
\end{aligned}$$

In particular,

$$\mathrm{Orb}(\gamma,\, f'^t) = \eta(h_\gamma)\,\mathrm{Orb}(\gamma, f').$$

The lemma follows from this. □

Lemma 5.19. *Assume that Conjecture 5.15 holds. Let $f' \in C_c^\infty(S)$ be a function satisfying the conclusion of Conjecture 5.6(a). Then f' is unique up to adding a linear combination of functions of the form*

(i) ${}^{\eta(h_1)h_1-1}({}^{\eta(h_2)h_2-1}\theta)$*;*

(ii) ${}^{\eta(h)h-1}(\Theta + \epsilon\Theta^t)$*; and*

(iii) $\sum_{i=1}^m {}^{\eta(h_i)h_i-1}f'_i$ *such that* $\sum_{i=1}^m \log|\det h_i| f'_i = 0$,

where $h, h_i \in H'(F_0)$ and $\theta, \Theta, f'_i \in C_c^\infty(S)$. Conversely, adding any function of the form (i)*,* (ii)*, or* (iii) *to f' gives a new function satisfying the conclusion of Conjecture 5.6(a).*

Proof. First consider the following two properties of a function $f'' \in C_c^\infty(S)$:

(1) $\mathrm{Orb}(\gamma, f'') = 0$ for all $\gamma \in S(F_0)_{\mathrm{rs}}$; and

(2) $\partial\mathrm{Orb}(\gamma, f'') = 0$ for all $\gamma \in S_{\mathrm{rs},1}$.

If f'' is of type (i), (ii), or (iii) above, then f'' satisfies (1) and (2) by Lemma 5.12(i), with an assist from Lemma 5.18 for property (2) when f'' is of type (ii). For such f'', the function $f' + f''$ therefore satisfies the conclusion of Conjecture 5.6(a) whenever f' does.

Now suppose that both f'_1 and f'_2 satisfy the conclusion of Conjecture 5.6(a), and set

$$f'' := f'_1 - f'_2.$$

Then f'' satisfies (1) and (2). By (1), it follows from Conjecture 5.15 that

$$f'' = \sum_{i=1}^{m} {}^{\eta(h_i)h_i - 1} f_i'' \quad \text{for some} \quad f_i'' \in C_c^\infty(S), \quad h_i \in H'(F_0).$$

If $\log|\det h_i| = 0$ for all i, then f'' is of type (iii) above, and we're done. If not, then say $\log|\det h_1| \neq 0$. By Lemma 5.12(i),

$$\partial \mathrm{Orb}(\gamma, f'') = \mathrm{Orb}(\gamma, \Theta) \quad \text{for} \quad \Theta := \sum_{i=1}^{m} \log|\det h_i| f_i''.$$

By (2), we have $\mathrm{Orb}(\gamma, \Theta) = 0$ for all $\gamma \in S_{\mathrm{rs},1}$. This and Lemma 5.18 imply that $\mathrm{Orb}(\gamma, \Theta_-) = 0$ for all $\gamma \in S(F_0)_{\mathrm{rs}}$. So by another application of Conjecture 5.15, we may write

$$\Theta_- = \sum_{j=1}^{k} {}^{\eta(h_j')h_j' - 1} \theta_j \quad \text{for some} \quad \theta_j \in C_c^\infty(S), \quad h_j' \in H'(F_0).$$

It follows that ${}^{\eta(h_1)h_1-1}\Theta = {}^{\eta(h_1)h_1-1}(\Theta_+ + \Theta_-)$ is a sum of functions type (i) and (ii). Since

$$f'' - \frac{{}^{\eta(h_1)h_1-1}\Theta}{\log|\det h_1|} = \sum_{i=2}^{m} \left({}^{\eta(h_i)h_i-1} f_i'' - \frac{\log|\det h_i|}{\log|\det h_1|} {}^{\eta(h_1)h_1-1} f_i'' \right)$$

is a function of type (iii), we conclude that f'' is of the asserted form. ☐

5.6. **Statement of the main results.** Now we state our main results. In the remainder of the paper, we will be concerned with the case $n = 3$, with (apart from §10) only occasional remarks about the case of general n. In this case, the following result gives a combined version of Remark 4.5, Remark 4.7, and Conjecture 4.8. Recall from (4.5) the notation $\Delta = \Delta_{\mathcal{N}}(\mathcal{N}_2)$, and from (4.6) (or from (4.10) in the Lie algebra case) the notation Δ_g. Note that, by the identification (3.10), we may view $U_1(F_0)$ and $\mathfrak{u}_1(F_0)$ as subsets of $\mathrm{End}^\circ_{O_F}(\mathbb{X}_n)$. Also, recall from the Introduction that, since the statements below involve the geometry of formal schemes, we are taking $F_0 = \mathbb{Q}_p$ throughout.

Theorem 5.20. *Let $n = 3$. Then for any $x \in \mathfrak{u}_1(F_0)$, the intersection $\Delta \cap \Delta_x$ is non-empty if and only if $x \in \mathrm{End}_{O_F}(\mathbb{X}_3)$, and the following three properties are equivalent.*

(i) $x \in \mathrm{End}_{O_F}(\mathbb{X}_3) \cap \mathfrak{u}_1(F_0)_{\mathrm{rs}}$.

(ii) *The intersection of Δ and Δ_x is a non-empty scheme proper over* $\mathrm{Spec}\, O_{\breve{F}}$.

(iii) *The intersection of Δ and Δ_x is non-empty artinian.*

The analog where $x \in \mathfrak{u}_1(F_0)$ is replaced by $g \in U_1(F_0)$ also holds true.

We complete the proof of Theorem 5.20 in §8.4, modulo some explicit calculations which we carry out in §9.

We also prove the AT conjectures formulated above in the case $n = 3$, namely the following theorems.

Theorem 5.21 (Group version). *Let $n = 3$. Then for any $g \in U_1(F_0)_{\mathrm{rs}}$, the intersection of Δ and Δ_g is an artinian scheme with two points unless it is empty, and there are no higher Tor terms in the expression* (4.9) *for* $\mathrm{Int}(g)$. *Furthermore, statements (a) and (b) of Conjecture 5.6 hold true.*

By Lemma 5.8, this theorem also implies the homogeneous group version, i.e. where $g \in G_{W_1}(F_0)_{\mathrm{rs}}$, cf. Conjecture 5.3.

Theorem 5.22 (Lie algebra version). *Let $n = 3$. Then for any $x \in \mathfrak{u}_1(F_0)_{\mathrm{rs}}$, the intersection of Δ and Δ_x is an artinian scheme with two points, unless it is empty. Furthermore, statements (a) and (b) of Conjecture 5.10 hold true.*

The proofs of Theorems 5.21 and 5.22 will occupy essentially the entire rest of the paper. We prove these theorems by a combination of methods from geometry and from local harmonic analysis, which in rough outline goes as follows. In §6 and §7, we relate the moduli space $\mathcal{N}_2$ to the formal deformation space of formal O_{F_0}-modules, and the special divisors on $\mathcal{N}_2$ (the analog of KR-divisors in the present setting of a ramified quadratic extension) to quasi-canonical divisors on this formal deformation space. In §8, using the Cayley transform, we reduce the computation of $\mathrm{Int}(g)$ to the calculation of ℓ-$\mathrm{Int}(x)$ in the Lie algebra, and even in the *reduced subset* of the Lie algebra. The calculation of ℓ-$\mathrm{Int}(x)$ for a reduced element x in the Lie algebra is then carried out in §9. This uses the calculation of intersection numbers of special cycles, and is based on the Gross-Keating formulas for the intersection numbers of quasi-canonical divisors. At this point the inputs to the proof from geometry are in place. The local harmonic analysis set-up used in our proof is explained in §10. Using this, we show in Proposition 11.14 that Theorem 5.21 follows from Theorem 5.22. We show in Proposition 12.4 that, in turn, Theorem 5.22 follows from Theorem 12.1, which is its analog for the reduced sets in the Lie algebra setting (cf. §8.2, §10.3, and §11.2). We then obtain Theorem 12.1 from a comparison of the result of the geometry side with the germ expansion of the orbital integral side. This part of the proof, carried out in §14 and §15, is explained in §13. The general germ expansion is established in Part 4, and has its own introduction in §16.

Part 2. Geometric side

In this part of the paper we address the geometric aspects of Theorems 5.21 and 5.22, including the geometric side of the identities to be proved in these theorems. Along the way we also prove Theorem 5.20. Throughout this part we take $n = 3$.

6. The Serre map

The main aim of this section is to describe the moduli space $\mathcal{N}_2$ in terms of the Serre tensor construction (see §3.2), and to use it to analyze special cycles.

6.1. The Serre map.

The first basic fact we need about $\mathcal{N}_2$ is the following.

Lemma 6.1. *$\mathcal{N}_2(\overline{k})$ consists of two points.*

Proof. Let N denote the rational covariant Dieudonné module of the framing object $\mathbb{X}_2$, and recycle the notation $\langle\ ,\ \rangle$, h, τ, and C from the proof of Proposition 3.1. Since $\mathbb{X}_2$ satisfies the spin condition, the hermitian space C is non-split by Lemma 3.3. By Dieudonné theory, $\mathcal{N}_2(\overline{k})$ identifies with the set of $O_{\breve{F}}$-lattices L in N such that

$$\varpi L \subset^2 \pi\tau L \subset^2 L, \tag{6.1}$$

where the superscripts indicate the $\overline{k}$-dimension of the corresponding quotients, and such that $L^\vee = \pi^{-1}L$, where $L^\vee$ denotes the dual lattice in N with respect to $\langle\ ,\ \rangle$, or equivalently with respect to h; cf. [22, Eg. 4.14], or [20, Prop. 2.2] for the variant where the polarization in the moduli problem is principal. Note that, since C is non-split, the p-divisible group corresponding to any such L automatically satisfies the spin condition by Lemma 3.3. Given such L, we have

$$L \subset^1 L + \tau L \subset^1 \pi^{-1}L. \tag{6.2}$$

By an obvious variant of [23, Prop. 2.17] (with $\breve{F}$ in place of $W_{\mathbb{Q}}$, π in place of p, etc.), the lattice $L+\tau L$ in the 2-dimensional $\breve{F}$-vector space N is τ-stable. This lattice is also self-dual, since the dual lattice

$$(L + \tau L)^\vee = L^\vee \cap (\tau L)^\vee = L^\vee \cap \tau(L^\vee) = \pi^{-1}(L \cap \tau L)$$

contains $L+\tau L$ by (6.1), and both $L+\tau L$ and $(L+\tau L)^\vee$ are contained in $\pi^{-1}L$ with codimension 1 by (6.2) (and the dual of the diagram (6.2)). Now, since C is non-split and 2-dimensional, C contains a *unique* self-dual O_F-lattice Λ. Hence $L + \tau L = O_{\breve{F}} \cdot \Lambda$ inside N. The hermitian form h induces a *symmetric* form on $V := \Lambda \otimes_{O_F} \overline{k}$, and the image of L in V is an isotropic line. Since V is 2-dimensional, there are exactly two isotropic lines in V, and these correspond to the two possibilities for L. $\square$

Remark 6.2. We record for later use that, in the notation of the proof, τ interchanges the two lattices in N corresponding to the two points in $\mathcal{N}_2(\overline{k})$.

Since $\mathcal{N}_2$ is formally locally of finite type over $\operatorname{Spf} O_{\breve{F}}$, we conclude that it consists of two connected components

$$\mathcal{N}_{2,+} \quad \text{and} \quad \mathcal{N}_{2,-},$$

each reduced to a point topologically. These components are distinguished as the respective loci in $\mathcal{N}_2$ where the framing map ρ is and is not an isomorphism. They are interchanged under the group action of $U(\mathbb{X}_2) \simeq U_1(F_0)$ by elements of nontrivial Kottwitz invariant, cf. (3.3). Thus to understand the structure of $\mathcal{N}_2$, it remains to understand either one of these components. We will do this in the rest of this subsection via the *Serre map*.

Forgetting the O_F-action, consider $\mathbb{E}$ as a connected p-divisible O_{F_0}-module of dimension 1 and height 2 over $\operatorname{Spec} \overline{k}$. Let $\mathcal{M}_{O_{\breve{F}_0}}$ denote its formal deformation space over $\operatorname{Spf} O_{\breve{F}_0}$. Thus for each scheme S over $\operatorname{Spf} O_{\breve{F}_0}$, $\mathcal{M}_{O_{\breve{F}_0}}(S)$ is the set of isomorphism classes of pairs (Y_0, β), where Y_0 is a p-divisible O_{F_0}-module over S and $\beta\colon Y_{0,\overline{S}} \xrightarrow{\sim} \mathbb{E}_{\overline{S}}$ is an O_{F_0}-linear isomorphism. An isomorphism between such pairs is an isomorphism between p-divisible O_{F_0}-modules over S which is compatible with the isomorphisms to $\mathbb{E}_{\overline{S}}$ in the obvious sense. By Lubin–Tate theory, $\mathcal{M}_{O_{\breve{F}_0}} \simeq \operatorname{Spf} O_{\breve{F}_0}[[t]]$. Set

$$\mathcal{M} := \mathcal{M}_{O_{\breve{F}_0}} \times_{\operatorname{Spf} O_{\breve{F}_0}} \operatorname{Spf} O_{\breve{F}}.$$

Proposition 6.3. *The Serre construction $Y_0 \mapsto O_F \otimes_{O_{F_0}} Y_0$ induces an isomorphism of formal schemes over* $\operatorname{Spf} O_{\breve{F}}$,

$$\mathcal{M} \xrightarrow{\sim} \mathcal{N}_{2,+}.$$

Proof. Given $(Y_0, \beta\colon Y_{0,\overline{S}} \xrightarrow{\sim} \mathbb{E}_{\overline{S}}) \in \mathcal{M}(S)$, we first have to explain how to define the rest of the quadruple $(O_F \otimes_{O_{F_0}} Y_0, \iota, \lambda, \rho) \in \mathcal{N}_{2,+}(S)$. For simplicity, we use the version of the moduli problem for $\mathcal{N}_2$ described in Remark 3.6. Of course, the notation signifies that for ι we take the tautological O_F-action on $O_F \otimes_{O_{F_0}} Y_0$. There is a canonical isomorphism

$$\operatorname{Lie}(O_F \otimes_{O_{F_0}} Y_0) \cong O_F \otimes_{O_{F_0}} \operatorname{Lie} Y_0$$

as $O_F \otimes_{O_{F_0}} O_S$-modules, from which it follows that $(O_F \otimes_{O_{F_0}} Y_0, \iota)$ satisfies the Kottwitz and spin conditions. We define the framing map ρ to be the isomorphism

$$\rho\colon O_F \otimes_{O_{F_0}} Y_{0,\overline{S}} \xrightarrow[\sim]{\mathrm{id}_{O_F} \otimes \beta} O_F \otimes_{O_{F_0}} \mathbb{E}_{\overline{S}} = \mathbb{X}_{2,\overline{S}}.$$

It remains to define the polarization λ. Since $\mathcal{M}$ is a formal scheme over $\operatorname{Spf} O_{\breve{F}}$ with a single $\overline{k}$-point, it suffices to assume that S is the spectrum of an Artin local ring with residue field $\overline{k}$. Let λ_0 be any principal polarization of Y_0; this exists, for example, because $\mathbb{E}$ is isomorphic to the p-divisible group of an elliptic curve, and hence so is Y_0 by the Serre–Tate theorem. Over $\overline{S} = \operatorname{Spec} \overline{k}$, the principal polarizations $\lambda_{0,\overline{k}}$ and $\beta^*(\lambda_{\mathbb{E}})$ of $Y_{0,\overline{k}}$ differ by an $O_{F_0}^\times$-multiple by our remarks in §3.3. Rescaling λ_0 as needed, we may assume that $\lambda_{0,\overline{k}} = \beta^*(\lambda_{\mathbb{E}})$. Then we define λ to be the polarization

$$\lambda\colon O_F \otimes_{O_{F_0}} Y_0 \xrightarrow{\varphi \otimes \lambda_0} \overline{O_F^\vee} \otimes_{O_{F_0}} Y_0^\vee \cong (O_F \otimes_{O_{F_0}} Y_0)^\vee,$$

where $\varphi\colon O_F \to \overline{O_F^\vee}$ is the symmetric O_F-linear map defined in (3.4). Just as when we defined $\mathbb{X}_2$, one readily verifies that $\operatorname{Ker} \lambda = (O_F \otimes_{O_{F_0}} Y_0)[\iota(\pi)]$. Furthermore

$$\lambda_{\overline{k}} = \varphi \otimes \lambda_{0,\overline{k}} = \varphi \otimes \beta^*(\lambda_{\mathbb{E}}) = \rho^*(\varphi \otimes \lambda_{\mathbb{E}}) = \rho^*(\lambda_{\mathbb{X}_2}).$$

Hence $(O_F \otimes_{O_{F_0}} Y_0, \iota, \lambda, \rho)$ gives a point in $\mathcal{N}_{2,+}(S)$, and its isomorphism class is clearly well-defined in terms of the isomorphism class of (Y_0, β). This defines the map $\mathcal{M} \to \mathcal{N}_{2,+}$.

Now we show that the map is an isomorphism. Since $\mathcal{M}$ and $\mathcal{N}_{2,+}$ both have a single $\overline{k}$-point and are formally smooth over $\operatorname{Spf} O_{\breve{F}}$ of relative formal dimension 1 (using Proposition 3.8 in the case of $\mathcal{N}_{2,+}$), it suffices to show that the induced map on tangent spaces is nonzero. For this we might as well prove the a priori stronger fact that $\mathcal{M}(S) \to \mathcal{N}_{2,+}(S)$ is an injection for any $\operatorname{Spf} O_{\breve{F}}$-scheme S. Let (Y_0, β) and (Y_0', β') be S-points on $\mathcal{M}$, and let $(O_F \otimes_{O_{F_0}} Y_0, \iota, \lambda, \rho)$ and $(O_F \otimes_{O_{F_0}} Y_0', \iota', \lambda', \rho')$ be the corresponding quadruples as defined above. With respect to the O_{F_0}-linear decompositions

$$O_F \otimes_{O_{F_0}} Y_0 = 1 \otimes Y_0 + \pi \otimes Y_0, \quad O_F \otimes_{O_{F_0}} Y_0' = 1 \otimes Y_0' + \pi \otimes Y_0', \quad \text{and} \quad \mathbb{X}_2 = 1 \otimes \mathbb{E} + \pi \otimes \mathbb{E},$$

the framings ρ and ρ' take the form

$$\rho = \mathrm{diag}(\beta, \beta) \quad \text{and} \quad \rho' = \mathrm{diag}(\beta', \beta').$$

Thus by ridigity $(\rho')^{-1} \circ \rho$ lifts to an isomorphism $O_F \otimes_{O_{F_0}} Y_0 \xrightarrow{\sim} O_F \otimes_{O_{F_0}} Y_0'$ if and only if $(\beta')^{-1} \circ \beta$ lifts to an isomorphism $Y_0 \xrightarrow{\sim} Y_0'$, which completes the proof. $\square$

Remark 6.4. Lemma 6.1 and Proposition 6.3 make precise and supply details for the Claim for the formal scheme denoted $\mathcal{M}_1$ in [22, Eg. 4.14]. Note however that loc. cit. uses the framing object described in [11, §5 d)]; this framing object should be replaced with our $\mathbb{X}_2$, as discussed in Remark 3.4.[8]

6.2. **Special divisors.** Recall the embedding $\iota_{\mathbb{E}} \colon O_F \hookrightarrow \mathrm{End}(\mathbb{E}) = O_D$ and the corresponding canonical lift $\mathcal{E}$ of $\mathbb{E}$ over $O_{\breve{F}}$, with its action $\iota_{\mathcal{E}} \colon O_F \to \mathrm{End}(\mathcal{E})$ and principal polarization $\lambda_{\mathcal{E}}$. Let $\mathcal{Y}_0$ denote the universal p-divisible group over $\mathcal{M}$, and let $c \in \mathrm{End}_{O_{F_0}}(\mathbb{E}) = \mathrm{End}_{O_{F_0}}(\overline{\mathbb{E}})$. Associated to c is the closed sublocus $\mathcal{T}_F(c)$ of $\mathcal{M}$ where c lifts to a homomorphism $\mathcal{Y}_0 \to \overline{\mathcal{E}}$. Note that the divisors $\mathcal{T}_F(c)$ are different from the divisors considered by Gross–Keating, i.e. the locus where c deforms to an endomorphism of $\mathcal{Y}_0$, or in other words a divisor of the form $\mathrm{Spf}\, W[[t]]/J$ in [18, top of p. 147].

Now let $\mathcal{Y}$ denote the universal p-divisible group over $\mathcal{N}_2$, and let $b \in \mathrm{Hom}_{O_F}(\mathbb{X}_2, \overline{\mathbb{E}})$. Associated to b is the closed sublocus $\mathcal{Z}(b)$ of $\mathcal{N}_{2,+}$ where the O_F-linear homomorphism $b \colon \mathbb{X}_2 \to \overline{\mathbb{E}}$ lifts to a homomorphism $\mathcal{Y} \to \overline{\mathcal{E}}$ (the analog in our present ramified setting of a *KR divisor* in the unramified setting [9][9]). By identifying, via Proposition 6.3, the restriction of $\mathcal{Y}$ to $\mathcal{N}_{2,+}$ with $O_F \otimes_{O_{F_0}} \mathcal{Y}_0$, adjunction implies the following lemma.

Lemma 6.5. *If c corresponds to b under the adjunction isomorphism*

$$\mathrm{Hom}_{O_{F_0}}\big(\mathbb{E}, \overline{\mathbb{E}}\big) \cong \mathrm{Hom}_{O_F}\big(O_F \otimes_{O_{F_0}} \mathbb{E}, \overline{\mathbb{E}}\big) = \mathrm{Hom}_{O_F}\big(\mathbb{X}_2, \overline{\mathbb{E}}\big),$$

then the Serre isomorphism $\mathcal{M} \cong \mathcal{N}_{2,+}$ identifies

$$\mathcal{T}_F(c) \cong \mathcal{Z}(b). \qquad \square$$

From now on, we often drop the field F from the notation $\mathcal{T}_F(c)$.

Proposition 6.6. *If b and c are nonzero, then both $\mathcal{Z}(b)$ and $\mathcal{T}(c)$ are relative divisors.*

Proof. Of course it suffices to prove this for $\mathcal{T}(c)$. Consider $\mathcal{S} := \mathcal{M} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{M}$, with its universal object $\mathcal{Y}_0 \times \mathcal{Y}_0'$. Recall from [29, Prop. 5.1] that the locus inside $\mathcal{S}$ where c lifts to a homomorphism $\mathcal{Y}_0 \to \mathcal{Y}_0'$ is a relative divisor $\mathcal{Z}$ in $\mathcal{S}$. Another divisor $\mathcal{D}$ inside $\mathcal{S}$ is given by the locus where $\mathcal{Y}_0' = \overline{\mathcal{E}}$. Now $\mathcal{D} \simeq \mathcal{M}$ is an irreducible divisor that is not contained in $\mathcal{Z}$ (otherwise c would lift to a homomorphism $\mathcal{Y}_0 \to \overline{\mathcal{E}}$ over all of $\mathcal{M}$, which is absurd). Hence $\mathcal{T}(c) = \mathcal{Z} \cap \mathcal{D}$ is a divisor on $\mathcal{M}$. It is a relative divisor because c does not lift to a homomorphism $\mathcal{Y}_0 \to \overline{\mathcal{E}}$ over the whole special fiber $\overline{\mathcal{M}} = \mathcal{M} \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathrm{Spec}\, \overline{k}$. $\square$

Remark 6.7. There are also quasi-canonical variants of this construction. Let $j \geq 1$. Let

$$O_j := O_{F,j} := O_{F_0} + \pi^j O_F$$

be the order of conductor j in O_F. Let W_j be the ring of integers of the ring class field extension of $\breve{F}$ corresponding to O_j, and let $\mathcal{W}_j := \mathrm{Spec}\, W_j$. Let $\mathcal{E}_j$ be the quasi-canonical lifting of level j over $\mathcal{W}_j$ [29, Def. 3.1]. In particular, $W_0 = O_{\breve{F}}$ and $\mathcal{E}_0 = \mathcal{E}$. Put

$$\mathcal{M}_j := \mathcal{M} \times_{\mathrm{Spec}\, O_{\breve{F}}} \mathcal{W}_j \quad \text{and} \quad \mathcal{N}_{2,+,j} := \mathcal{N}_{2,+} \times_{\mathrm{Spec}\, O_{\breve{F}}} \mathcal{W}_j.$$

Let $\overline{\mathcal{E}}_j$ denote the same object as $\mathcal{E}_j$, but where the O_j-action is precomposed by the nontrivial Galois automorphism. Inside $\mathcal{M}_j$, we have the locus $\mathcal{T}_{F,j}(c)$ where the endomorphism $c \in O_D$ lifts to a homomorphism $\mathcal{Y}_0 \to \overline{\mathcal{E}}_j$; and inside $\mathcal{N}_{2,+,j}$, we have the locus $\mathcal{Z}_j(b)$ where the O_F-linear homomorphism $b \colon \mathbb{X}_2 \to \overline{\mathbb{E}}$ lifts to an O_j-linear homomorphism $\mathcal{Y} \to \overline{\mathcal{E}}_j$. If c corresponds to b

[8]Note also that the claim concerning $O_F^\vee$ at the end of [22, Eg. 4.14] is obviously incorrect.

[9]Note however that in loc. cit. deformations of homomorphisms $\overline{\mathbb{E}} \to \mathbb{X}_2$ are considered.

under the adjunction isomorphism $\mathrm{Hom}_{O_{F_0}}(\mathbb{E}, \overline{\mathbb{E}}) \cong \mathrm{Hom}_{O_F}(O_F \otimes_{O_{F_0}} \mathbb{E}, \overline{\mathbb{E}}) = \mathrm{Hom}_{O_F}(\mathbb{X}_2, \overline{\mathbb{E}})$, then

$$\mathcal{T}_{F,j}(c) \cong \mathcal{Z}_j(b)$$

under the Serre isomorphism.

7. Special divisors as sums of quasi-canonical divisors

In this section we express the special divisor $\mathcal{T}(c)$ defined in §6.2 as a sum of quasi-canonical divisors, where $c \in O_D$ is nonzero. We identify F with its image in D via $\iota_{\mathbb{E}}$, but we also consider the conjugate embedding $\iota := {}^{\overline{c}}\iota_{\mathbb{E}} \colon F \hookrightarrow D$ and its image ${}^{\overline{c}}F$. Corresponding to ι, we have the quasi-canonical divisor $\mathcal{W}_{{}^{\overline{c}}F,j}$ on $\mathcal{M}$ and the quasi-canonical lift $\mathcal{E}_{\iota,j}$ of level j over $\mathcal{W}_{{}^{\overline{c}}F,j}$, as well as the canonical lift $\mathcal{E}_\iota$ over $\operatorname{Spf} O_{\breve{F}}$.

Proposition 7.1. *There is an equality of divisors on $\mathcal{M}$,*

$$\mathcal{T}(c) = \sum_{0 \le j \le v_D(c)} \mathcal{W}_{{}^{\overline{c}}F,j}.$$

Proof. Note that in the definition of $\mathcal{T}(c)$, it is harmless to replace $\overline{\mathbb{E}}$ with $\mathbb{E}$ and $\overline{\mathcal{E}}$ with $\mathcal{E}$, since in both cases the underlying O_{F_0}-modules are the same. Set $\gamma := v_D(c)$, and write $c = \pi^\gamma c_0$ with $c_0 \in O_D^\times$. Let $j \le \gamma$. The element $\pi^{\gamma-j}c_0 \in O_D$ conjugates ι into $\iota_{\mathbb{E}}$, and therefore lifts to a homomorphism $\mathcal{E}_\iota \to \mathcal{E}$. Over the locus $\mathcal{W}_{{}^{\overline{c}}F,j}$, the endomorphism $\iota(\pi^j)$ of $\mathbb{E}$ lifts to a homomorphism $\psi_j \colon \mathcal{E}_{\iota,j} \to \mathcal{E}_\iota$. Since $\iota = {}^{c_0^{-1}}\iota_{\mathbb{E}}$, we thus obtain a diagram

$$\begin{tikzcd}
\mathcal{E}_{\iota,j} \arrow[r, "\psi_j"] \arrow[d, dash] & \mathcal{E}_\iota \arrow[r] \arrow[d, dash] & \mathcal{E} \arrow[d, dash] \\
\mathbb{E} \arrow[r, "c_0^{-1}\pi^j c_0"] & \mathbb{E} \arrow[r, "\pi^{\gamma-j}c_0"] & \mathbb{E},
\end{tikzcd}$$

where the vertical lines indicate reduction to $\overline{k}$, and where the bottom row evidently composes to c. This shows that the divisor $\mathcal{W}_{{}^{\overline{c}}F,j}$ is a component of $\mathcal{T}(c)$. We therefore obtain an inequality of divisors on $\mathcal{M}$,

$$\sum_{j=0}^{\gamma} \mathcal{W}_{{}^{\overline{c}}F,j} \le \mathcal{T}(c).$$

The equality will follow by comparing the intersections of both sides with the special fiber $\overline{\mathcal{M}}$. For the left-hand side, we note that $(\mathcal{W}_{{}^{\overline{c}}F,j} \cdot \overline{\mathcal{M}}) = [W_j : O_{\breve{F}}] = q^j$. For the right-hand side, we apply the following lemma. $\square$

Lemma 7.2. *The intersection multiplicity of the cycle $\mathcal{T}(c)$ with the special fiber $\overline{\mathcal{M}}$ is*

$$\big(\mathcal{T}(c) \cdot \overline{\mathcal{M}}\big) = \textstyle\sum_{0 \le j \le v_D(c)} q^j.$$

Proof. We use the Kummer congruence, cf. [18, Th. 4.1]. Identify $\overline{\mathcal{M}}$ with $\operatorname{Spf} \overline{k}[[t]]$, and the product of $\overline{\mathcal{M}}$ with itself with $\operatorname{Spf} \overline{k}[[t,t']]$. Let $(\mathcal{Y}_0, \mathcal{Y}_0')$ be the universal p-divisible group over $\operatorname{Spf} \overline{k}[[t,t']]$. Let I be the ideal in $\overline{k}[[t,t']]$ describing the closed sublocus where $c \colon \mathbb{E} \to \mathbb{E}$ lifts to a homomorphism $\widetilde{c} \colon \mathcal{Y}_0 \to \mathcal{Y}_0'$. By loc. cit., the uniformizers t and t' may be chosen such that I is generated by the element

$$g := \big(t - (t')^{q^\gamma}\big)\big(t^q - (t')^{q^{\gamma-1}}\big) \cdots \big(t^{q^\gamma} - t'\big),$$

where as in the previous proof $\gamma = v_D(c)$. On the other hand, the locus where $\mathcal{Y}' = \mathcal{E}$ is defined by $t' = 0$. Hence the intersection multiplicity in question is equal to the length of the Artin ring

$$\overline{k}[[t,t']]/(t', g) = \overline{k}[[t]]/(t^{1+q+\cdots+q^\gamma}).$$

The claim follows. $\square$

Remarks 7.3. (i) Note that our convention that $q = p$ when working with formal schemes is in force in Lemma 7.2. Strictly speaking, we need it to appeal to the Kummer congruence.

(ii) The analog of Lemma 7.2 in the case when F/F_0 is unramified is [9, Prop. 8.2].[10] The proof of this analog in loc. cit., due to Th. Zink, uses displays and is difficult. The proof of Lemma 7.2 given here transposes in the obvious way to the unramified case, which gives a drastic simplification of loc. cit.

Conversely, one can reduce Lemma 7.2 to the unramified case in [9] as follows. Let F' denote the unramified quadratic extension of F_0. We first point out that the Serre isomorphism in Proposition 6.3 also holds in the unramified setting (with $\mathcal{M}_{O_{\breve{F}_0}}$ isomorphic to the entire space $\mathcal{N}_{F'/F_0,2}$), as does the compatibility of special divisors in Proposition 6.5. Now let $\mathcal{E}'/\operatorname{Spf} O_{\breve{F}_0}$ denote the canonical lifting of $\mathbb{E}$ for F'/F_0 (relative to any embedding of F' in D). Of course

$$\mathcal{E} \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \overline{k} \simeq \mathcal{E}' \times_{\operatorname{Spf} O_{\breve{F}_0}} \operatorname{Spec} \overline{k}$$

as formal O_{F_0}-modules over $\operatorname{Spec} \overline{k}$. Hence, identifying the special fibers of $\mathcal{M}$ and $\mathcal{M}_{O_{\breve{F}_0}}$, we get an identification of the corresponding divisors

$$\mathcal{T}_F(c) \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \overline{k} = \mathcal{T}_{F'}(c) \times_{\operatorname{Spf} O_{\breve{F}_0}} \operatorname{Spec} \overline{k}.$$

Therefore Lemma 7.2 follows from Proposition 8.2 of [9].

8. Reduction to the Lie algebra

In this section we lay the framework to reduce the geometric calculations in Theorem 5.21 (the group setting) to those in Theorem 5.22 (the Lie algebra setting). Modulo these calculations, which will be carried out in the next section, we also prove Theorem 5.20.

8.1. **Coordinates on U_1 and $\mathfrak{u}_1$.** We begin by presenting the unitary group $U_1(F_0) \simeq U(\mathbb{X}_3)$ and its Lie algebra $\mathfrak{u}_1(F_0)$ in terms of explicit coordinates. First recall the embedding

$$\iota_{\overline{\mathbb{E}}} \colon O_F \hookrightarrow \operatorname{End}_{O_{F_0}}(\overline{\mathbb{E}}) = O_D.$$

Except where stated to the contrary, from now on we will tacitly regard O_F as a subring of O_D via $\iota_{\overline{\mathbb{E}}}$, and likewise for $F \subset D$, and drop $\iota_{\overline{\mathbb{E}}}$ from the notation. Write

$$D = D_+ \oplus D_- \tag{8.1}$$

for the decomposition of D into its respective $+1$ and -1 eigenspaces under the conjugation action of π. Then $F = D_+$. For any $\alpha \in D$, we denote by α_+ and α_- its respective components with respect to this decomposition. Note that $\mathrm{N}\alpha = \mathrm{N}\alpha_+ + \mathrm{N}\alpha_-$. We also write $D^{\mathrm{tr}=0}$ for the set of traceless elements in D, and we analogously define $O_D^{\mathrm{tr}=0}$, $F^{\mathrm{tr}=0} = F_0\pi$, and $O_F^{\mathrm{tr}=0} = O_{F_0}\pi$.

Recall from §3.3 that we have defined the framing object

$$\mathbb{X}_3 = \mathbb{X}_2 \times \overline{\mathbb{E}} = (O_F \otimes_{O_{F_0}} \mathbb{E}) \times \overline{\mathbb{E}}. \tag{8.2}$$

Identifying $O_F \otimes_{O_{F_0}} \mathbb{E} \simeq \mathbb{E} \times \mathbb{E}$ as in (3.5), we obtain

$$\operatorname{End}_{O_{F_0}}(\mathbb{X}_3) \simeq \mathrm{M}_3(O_D). \tag{8.3}$$

In terms of this identification, the O_F-action $\iota_{\mathbb{X}_3}$ sends

$$\pi \longmapsto \begin{bmatrix} 0 & \varpi & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \pi \end{bmatrix}.$$

Hence we identify

$$\operatorname{End}_{O_F}(\mathbb{X}_3) = \left\{ \begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ c & \pi c & d \end{bmatrix} \,\middle|\, \alpha, \beta, b, c \in O_D,\ d \in O_F \right\}.$$

We emphasize that here and below the symbol π always means $\iota_{\overline{\mathbb{E}}}(\pi)$, in accordance with our convention.

With regard to the decomposition $\mathbb{X}_3 \simeq \mathbb{E} \times \mathbb{E} \times \overline{\mathbb{E}}$, the polarization $\lambda_{\mathbb{X}_3}$ is given by

$$\lambda_{\mathbb{X}_3} = \operatorname{diag}(\lambda_{\mathbb{E}}, -\varpi\lambda_{\mathbb{E}}, \lambda_{\overline{\mathbb{E}}});$$

[10] Note that the quantity v in loc. cit. should be replaced by half of its value.

see (3.6). Since the Rosati involution on D induced by $\lambda_{\mathbb{E}} = \lambda_{\overline{\mathbb{E}}}$ is the main involution $a \mapsto \overline{a}$, the Rosati involution $x \mapsto x^\dagger = \lambda_{\mathbb{X}_3}^{-1} \circ x^\vee \circ \lambda_{\mathbb{X}_3}$ on $\mathrm{End}^\circ_{O_F}(\mathbb{X}_3)$ is given by the formula

$$\begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ c & \pi c & d \end{bmatrix}^\dagger = \begin{bmatrix} \overline{\alpha} & -\overline{\beta}\varpi & \overline{c} \\ -\overline{\beta} & \overline{\alpha} & \overline{c}\pi^{-1} \\ -\pi\overline{b} & -\overline{b}\varpi & \overline{d} \end{bmatrix}.$$

Attached to $\mathbb{X}_3$ is the identification of unitary groups from (3.10),

$$\begin{aligned} U_1(F_0) \simeq U(\mathbb{X}_3) &= \bigl\{\, g \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_3) \bigm| gg^\dagger = 1 \,\bigr\} \\ &\subset \left\{ \begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ c & \pi c & d \end{bmatrix} \middle|\ \alpha, \beta, b, c \in D,\ d \in F \right\}. \end{aligned} \tag{8.4}$$

Thus we get an identification of Lie algebras,

$$\begin{aligned} \mathfrak{u}_1(F_0) &\simeq \bigl\{\, x \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_3) \bigm| x + x^\dagger = 0 \,\bigr\} \\ &= \left\{ \begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ \pi\overline{b} & \overline{b}\varpi & d \end{bmatrix} \middle|\ \alpha \in D^{\mathrm{tr}=0},\ \beta \in F_0,\ b \in D,\ d \in F^{\mathrm{tr}=0} \right\}. \end{aligned} \tag{8.5}$$

Intersecting with $\mathrm{End}_{O_F}(\mathbb{X}_3)$ gives a natural compact open subgroup in each,

$$K_1 := U_1(F_0) \cap \mathrm{End}_{O_F}(\mathbb{X}_3) = \bigl\{\, g \in \mathrm{End}_{O_F}(\mathbb{X}_3) \bigm| gg^\dagger = 1 \,\bigr\}$$

and

$$\begin{aligned} \mathfrak{k}_1 &:= \mathfrak{u}_1(F_0) \cap \mathrm{End}_{O_F}(\mathbb{X}_3) \\ &= \bigl\{\, x \in \mathrm{End}_{O_F}(\mathbb{X}_3) \bigm| x + x^\dagger = 0 \,\bigr\} \\ &= \left\{ \begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ \pi\overline{b} & \overline{b}\varpi & d \end{bmatrix} \middle|\ \alpha \in O_D^{\mathrm{tr}=0},\ \beta \in O_{F_0},\ b \in O_D,\ d \in O_F^{\mathrm{tr}=0} \right\}. \end{aligned} \tag{8.6}$$

Note that K_1 is the stabilizer in $U_1(F_0)$ of the standard basepoint in $\mathcal{N}_3$, i.e. the point

$$(\mathbb{X}_3, \iota_{\mathbb{X}_3}, \lambda_{\mathbb{X}_3}, \mathrm{id}_{\mathbb{X}_3}) \in \mathcal{N}_3\bigl(\overline{k}\bigr).$$

Furthermore we set

$$K_{1,\mathrm{rs}} := K_1 \cap U_{1,\mathrm{rs}}(F_0) \quad \text{and} \quad \mathfrak{k}_{1,\mathrm{rs}} := \mathfrak{k}_1 \cap \mathfrak{u}_{1,\mathrm{rs}}(F_0).$$

8.2. **Invariants on $\mathfrak{u}_1$ and reduced elements.** We now make explicit the invariants on $\mathfrak{u}_1$ discussed in §2.5 in terms of the coordinates just introduced. Let

$$x = \begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ \pi\overline{b} & \overline{b}\varpi & d \end{bmatrix} \in \mathfrak{u}_1(F_0)$$

be expressed in the form (8.5). Write

$$A' := \begin{bmatrix} \alpha & \beta\varpi \\ \beta & \alpha \end{bmatrix}, \quad \mathbf{b}' := \begin{bmatrix} b\pi \\ b \end{bmatrix}, \quad \mathbf{c}' := \begin{bmatrix} \pi\overline{b} & \overline{b}\varpi \end{bmatrix},$$

so that

$$x = \begin{bmatrix} A' & \mathbf{b}' \\ \mathbf{c}' & d \end{bmatrix}.$$

Note that this block decomposition for x is *not the same* as the earlier one in (2.9), since here we allow matrix entries in D. With respect to the identifications (8.2) and (8.3), we have

$$A' \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_2), \quad \mathbf{b}' \in \mathrm{Hom}^\circ_{O_F}\bigl(\overline{\mathbb{E}}, \mathbb{X}_2\bigr) = \mathbb{V}_2, \quad \text{and} \quad \mathbf{c}' \in \mathrm{Hom}^\circ_{O_F}\bigl(\mathbb{X}_2, \overline{\mathbb{E}}\bigr). \tag{8.7}$$

Using these identifications, one sees that the quantities

$$\lambda(x) := \det{}_F(A' \mid \mathbb{V}_2), \quad u(x) := \varpi^{-1}\mathbf{c}'\mathbf{b}', \quad w(x) := \varpi^{-1}\mathbf{c}'A'\mathbf{b}', \quad \mathrm{tr}_F(A' \mid \mathbb{V}_2), \quad d \tag{8.8}$$

are the five polynomial generators of the invariant ring listed in (2.10), except for the factor ϖ^{-1} in front of the second and third invariants, which we have inserted to give a more convenient normalization; see e.g. Lemma 11.3 below.

To make the first and fourth invariants in (8.8) explicit, note that the map

$$\mathbb{V}_2 = \left\{ \begin{bmatrix} b\pi \\ b \end{bmatrix} \;\middle|\; b \in D \right\} \longrightarrow D, \quad \begin{bmatrix} b\pi \\ b \end{bmatrix} \longmapsto b,$$

is an F-linear isomorphism, where F acts naturally on the right on source and target. In this way A' acting on $\mathbb{V}_2$ identifies with the F-linear map

$$b \longmapsto \alpha b + b\beta\pi$$

on D. Hence

$$\mathrm{tr}_F(A' \mid \mathbb{V}_2) = 2\beta\pi \quad \text{and} \quad \lambda(x) = \det_F(A' \mid \mathbb{V}_2) = \mathrm{N}\alpha + \beta^2\varpi.$$

Definition 8.1. An element $x \in \mathfrak{u}_1(F_0)$ written as above is called *reduced* if its invariants $\mathrm{tr}_F(A' \mid \mathbb{V}_2)$ and d are 0, that is, if $\beta = d = 0$. We denote by $\mathfrak{u}_{1,\mathrm{red}}(F_0)$ the subspace of reduced elements in $\mathfrak{u}_1(F_0)$.

Of course, the invariants $\mathrm{tr}_F(A' \mid \mathbb{V}_2)$ and d as written here arise from regular functions on $\mathfrak{u}_1$, and their vanishing therefore defines $\mathfrak{u}_{1,\mathrm{red}}$ as a closed subscheme of $\mathfrak{u}_1$; hence the notation. We also set

$$\mathfrak{u}_{1,\mathrm{red},\mathrm{rs}} := \mathfrak{u}_{1,\mathrm{red}} \cap \mathfrak{u}_{1,\mathrm{rs}}, \quad \mathfrak{k}_{1,\mathrm{red}} := \mathfrak{k}_1 \cap \mathfrak{u}_{1,\mathrm{red}}(F_0), \quad \text{and} \quad \mathfrak{k}_{1,\mathrm{red},\mathrm{rs}} := \mathfrak{k}_{1,\mathrm{red}} \cap \mathfrak{k}_{1,\mathrm{rs}}.$$

There is a natural map $\mathfrak{u}_1(F_0) \to \mathfrak{u}_{1,\mathrm{red}}(F_0)$, which we denote by $x \mapsto x_{\mathrm{red}}$, defined by

$$\begin{bmatrix} \alpha & \beta\varpi & b\pi \\ \beta & \alpha & b \\ \pi\bar{b} & \bar{b}\varpi & d \end{bmatrix} \longmapsto \begin{bmatrix} \alpha & 0 & b\pi \\ 0 & \alpha & b \\ \pi\bar{b} & \bar{b}\varpi & 0 \end{bmatrix}.$$

Taking this map together with the last two invariants in (8.8) gives a product decomposition

$$\begin{aligned} \mathfrak{u}_1(F_0) &\xrightarrow{\sim} \mathfrak{u}_{1,\mathrm{red}}(F_0) \times \mathfrak{s}_1(F_0) \times \mathfrak{s}_1(F_0) \\ x &\longmapsto (x_{\mathrm{red}}, 2\beta\pi, d). \end{aligned} \tag{8.9}$$

In §8.4 we are going to explain how to reduce the calculation of intersection numbers not just to the Lie algebra setting, but to *reduced* elements in $\mathfrak{u}_1(F_0)$. The first basic fact in this direction is the following.

Lemma 8.2. *An element $x \in \mathfrak{u}_1(F_0)$ is regular semi-simple if and only if x_{red} is.*

Proof. By the linear algebra characterization of regular semi-simple elements in §2.4, relative to the canonical special vector $u \in \mathbb{V}_3$ in (4.2), it suffices to show that the three vectors u, xu, x^2u are linearly independent over F if and only if the vectors $u, x_{\mathrm{red}}u, x_{\mathrm{red}}^2u$ are, and analogously for ${}^tu, {}^tux, {}^tux^2$ and ${}^tu, {}^tux_{\mathrm{red}}, {}^tux_{\mathrm{red}}^2$. For clarity we denote the F-action on $\mathbb{V}_3$ as a right action. Expressing x in terms of the coordinates (8.5), the first of these equivalences follows from the easily verified relations

$$xu = x_{\mathrm{red}}u + ud$$

and

$$x^2u = xx_{\mathrm{red}}u + xud = x_{\mathrm{red}}^2u + x_{\mathrm{red}}u\beta\pi + x_{\mathrm{red}}ud + ud^2 = x_{\mathrm{red}}^2u + x_{\mathrm{red}}u(\beta\pi + d) + ud^2.$$

The second, "transposed" equivalence is proved in a similar way. ☐

We conclude this subsection by giving a simple characterization of regular semi-simplicity for reduced elements. Let

$$x = \begin{bmatrix} \alpha & 0 & b\pi \\ 0 & \alpha & b \\ \pi\bar{b} & \bar{b}\varpi & 0 \end{bmatrix} \in \mathfrak{u}_{1,\mathrm{red}}(F_0), \quad \alpha \in D^{\mathrm{tr}=0}, \quad b \in D. \tag{8.10}$$

The first three invariants in (8.8) take the values on x,

$$\begin{aligned}\lambda(x) &= \mathrm{N}\alpha = \mathrm{N}\alpha_+ + \mathrm{N}\alpha_- = \mathrm{N}\alpha'_+ + \mathrm{N}\alpha'_-,\\ u(x) &= \varpi^{-1}(\pi\bar{b}b\pi + \bar{b}\varpi b) = 2\mathrm{N}b,\\ w(x) &= \varpi^{-1}(\pi\bar{b}\alpha b\pi + \bar{b}\varpi\alpha b) = \mathrm{N}b\cdot(\pi^{-1}\alpha'\pi + \alpha') = 2\mathrm{N}b\cdot\alpha'_+.\end{aligned} \tag{8.11}$$

Here in the expressions involving α', we have assumed that $b \neq 0$ and set

$$\alpha' := b^{-1}\alpha b;$$

recall that the subscripts $+$ and $-$ denote the components of an element with respect to the decomposition (8.1). Of course, if $b = 0$, then $u(x) = w(x) = 0$.

Now recall from §2.4 that x is regular semi-simple if and only if $\Delta(x) \neq 0$, where

$$\Delta(x) := -\varpi^{-2}\det({}^te x^{i+j}e)_{0\leq i,j\leq 2}. \tag{8.12}$$

Note that here we have rescaled the discriminant defined in (2.6), which will give us a more convenient normalization later on. From now on we will always understand Δ in the sense of (8.12).

Lemma 8.3. *A reduced element $x \in \mathfrak{u}_{1,\mathrm{red}}(F_0)$ in the form* (8.10) *is regular semi-simple if and only if $b \neq 0$ and $\alpha'_- \neq 0$.*

Proof. We calculate the discriminant as

$$\begin{aligned}\Delta(x) &= -\varpi^{-2}\det\begin{bmatrix}1 & 0 & \varpi u\\ 0 & \varpi u & \varpi w\\ \varpi u & \varpi w & -\lambda\varpi u + \varpi^2u^2\end{bmatrix}\\ &= \lambda u^2 + w^2\\ &= 4(\mathrm{N}b)^2\mathrm{N}\alpha'_-.\end{aligned} \tag{8.13}$$

□

8.3. **The Cayley transform.** Our main tool in passing from the group setting to the Lie algebra setting will be the Cayley transform $x \mapsto (1+x)(1-x)^{-1}$ from $\mathfrak{u}_1(F_0)$ to $U_1(F_0)$. More precisely, let

$$\mathfrak{u}_1^\circ(F_0) := \bigl\{\, x \in \mathfrak{u}_1(F_0) \bigm| 1 - x \text{ is invertible} \,\bigr\}. \tag{8.14}$$

Then the Cayley transform is defined on $\mathfrak{u}_1^\circ(F_0)$. In fact, we will need the variant

$$\mathfrak{c}_\xi \colon \mathfrak{u}_1^\circ(F_0) \longrightarrow U_1(F_0)$$

defined by

$$x \longmapsto \xi\frac{1+x}{1-x}, \tag{8.15}$$

where $\xi \in U_1(F_0)$ is a fixed element of the form $\mathrm{diag}(\pm 1_2, \pm 1)$, expressed in the presentation (8.4) of $U_1(F_0)$. We remark that in §10.2 we will give a slightly more general definition of the Cayley transform for $\mathfrak{u}_1$, and also define it for $\mathfrak{s}$ and $\mathfrak{u}_0$.

Lemma 8.4 (Cayley transform for $\mathfrak{k}_1$)**.** *There is an inclusion $\mathfrak{k}_1 \subset \mathfrak{u}_1^\circ(F_0)$, and hence the restriction of the Cayley map $\mathfrak{c}_\xi$ to $\mathfrak{k}_1$ is well-defined, where $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$. Furthermore, this restriction factors through K_1,*

$$\mathfrak{c}_\xi \colon \mathfrak{k}_1 \longrightarrow K_1,$$

and the images $\mathfrak{c}_\xi(\mathfrak{k}_1)$, as ξ varies over these four elements, cover K_1.

Proof. Let

$$x = \begin{bmatrix}\alpha & \beta\varpi & b\pi\\ \beta & \alpha & b\\ \pi\bar{b} & \bar{b}\varpi & d\end{bmatrix} \in \mathfrak{k}_1,$$

expressed in the form (8.6). Then $x \bmod \pi$ is an upper triangular block matrix of the form

$$\begin{bmatrix}\alpha & 0 & 0\\ \beta & \alpha & b\\ 0 & 0 & d\end{bmatrix} \mod \pi. \tag{8.16}$$

Since $\alpha \in O_D^{\mathrm{tr}=0}$ and $d \in O_F^{\mathrm{tr}=0}$, we have $1-\alpha, 1-d \in O_D^\times$. It follows that $1-x$ is invertible and that its inverse has entries in O_D. Hence $x \in \mathfrak{u}_1^\circ(F_0)$ and $\mathfrak{c}_\xi(x) \in \mathrm{End}_{O_F}(\mathbb{X}_3) \cap U_1(F_0) = K_1$.

To show that the images cover K_1, it suffices to show that for every $g \in K_1$, there exists $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$ such that $\mathfrak{c}_\xi^{-1}(g)$ is well-defined and has integral entries. Here

$$\mathfrak{c}_\xi^{-1}(g) = -\frac{1-\xi^{-1}g}{1+\xi^{-1}g}. \tag{8.17}$$

From the equation $gg^\dagger = 1$ it follows that $g \bmod \pi$ is also of the form (8.16). Since $1+\alpha$ and $1-\alpha$ sum to $2 \in O_D^\times$, at least one of them is in $O_D^\times$ too, and likewise for $1 \pm d$. Thus the desired ξ exists. ☐

Remark 8.5. Even though $\mathfrak{c}_\xi(\mathfrak{k}_1) \subset K_1$, there are elements $x \in \mathfrak{u}_1^\circ(F_0) \smallsetminus \mathfrak{k}_1$ with $\mathfrak{c}_\xi(x) \in K_1$.

In the case of $\mathfrak{c} := \mathfrak{c}_{\mathrm{id}_{\mathbb{X}_3}}$, we also have the following.

Lemma 8.6. *Let $x \in \mathfrak{k}_1$. Then there is an equality of subalgebras of $\mathrm{End}_{O_F}(\mathbb{X}_3)$,*

$$O_F[x] = O_F[\mathfrak{c}(x)].$$

Proof. Let $y := 1-x$. Then y is an automorphism of $\mathbb{X}_3$ by Lemma 8.4. It follows from the Cayley–Hamilton theorem that y^{-1} is expressible as a polynomial with coefficients in O_F in y, and hence in x. Hence $\mathfrak{c}(x) = (1+x)(1-x)^{-1}$ is a polynomial in x. Conversely, the same argument, using the inverse formula (8.17), shows that x is a polynomial in $\mathfrak{c}(x)$. ☐

Lemma 8.7. *Let $x \in \mathfrak{k}_1$ and $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$. Then x is regular semi-simple if and only if $\mathfrak{c}_\xi(x)$ is.*

Proof. Use the linear algebra characterization of regular semi-simple elements, as in the proof of Lemma 8.2, twice: first to deduce the lemma in the case $\xi = \mathrm{id}_{\mathbb{X}_3}$ from Lemma 8.6, and then to see that $\mathfrak{c}(x)$ is regular semi-simple if and only if $\mathfrak{c}_\xi(x) = \xi\mathfrak{c}(x)$ is (which is a simple exercise). ☐

8.4. **Relation to intersection numbers.** We now apply the material in the previous subsections to intersection numbers. In this subsection $\Delta = \Delta_{\mathcal{N}}(\mathcal{N}_2) \subset \mathcal{N}_2 \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_3$ (not to be confused with the discriminant!). We begin with a basic lemma on the geometry of intersections. For any quasi-endomorphism $x \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_3)$, recall the subspace Δ_x of $\mathcal{N}_2 \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_3$ defined by the condition (4.10).

Lemma 8.8. *For $x \in \mathrm{End}^\circ_{O_F}(\mathbb{X}_3)$, the following are equivalent.*

(i) $\Delta \cap \Delta_x$ *is nonempty.*

(ii) $(\Delta \cap \Delta_x)_{\mathrm{red}}$ *consists of two points.*

(iii) $x \in \mathrm{End}_{O_F}(\mathbb{X}_3)$.

Proof. By Lemma 6.1, $\mathcal{N}_2(\bar{k})$ consists of two points, the standard basepoint

$$z_+ := (\mathbb{X}_2, \iota_{\mathbb{X}_2}, \lambda_{\mathbb{X}_2}, \mathrm{id}_{\mathbb{X}_2})$$

and another point z_-. Thus Δ_{red} consists of the two points $(z_+, \delta_{\mathcal{N}}(z_+))$ and $(z_-, \delta_{\mathcal{N}}(z_-))$ in $\mathcal{N}_2 \times \mathcal{N}_3$. According to the definitions, $\delta_{\mathcal{N}}(z_+) = (\mathbb{X}_3, \iota_{\mathbb{X}_3}, \lambda_{\mathbb{X}_3}, \mathrm{id}_{\mathbb{X}_3})$. Thus what we have to show is that x either does or does not give an (honest) endomorphism of the framed p-divisible groups corresponding to $\delta_{\mathcal{N}}(z_+)$ and $\delta_{\mathcal{N}}(z_-)$ simultaneously. For this we translate the problem to Dieudonné modules. Let N_1, N_2, and N_3 denote the respective rational covariant Dieudonné modules of $\overline{\mathbb{E}}$, $\mathbb{X}_2$, and $\mathbb{X}_3$. We must show that the lattices in N_3 corresponding to $\delta_{\mathcal{N}}(z_+)$ and $\delta_{\mathcal{N}}(z_-)$ either are or are not simultaneously carried into themselves under the endomorphism of N_3 corresponding to x. For $i = 1, 2, 3$, let τ_i be the τ-operator on N_i defined in (3.2). Then

$$N_3 = N_2 \oplus N_1 \quad \text{and} \quad \tau_3 = \tau_2 \oplus \tau_1.$$

Since N_1 is 1-dimensional over $\breve{F}$, the Dieudonné lattice in N_1 corresponding to $\overline{\mathbb{E}}$ is τ_1-stable. By Remark 6.2, τ_2 interchanges the lattices in N_2 corresponding to z_+ and z_-. Therefore τ_3 interchanges the lattices in N_3 corresponding to $\delta_{\mathcal{N}}(z_+)$ and $\delta_{\mathcal{N}}(z_-)$. Since the endomorphism of N_3 induced by x commutes with τ_3, the lemma follows. ☐

Thus nonzero intersection numbers can only occur for $g \in K_1$ in the group setting, and for $x \in \mathfrak{k}_1$ in the Lie algebra setting.

Lemma 8.9. *For $x \in \mathfrak{k}_1$ and $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$, there are equalities of closed formal subschemes of $\mathcal{N}_2 \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_3$,*

$$\Delta \cap \Delta_x = \Delta \cap \Delta_{x_{\mathrm{red}}} = \Delta \cap \Delta_{\mathfrak{c}_\xi(x)}.$$

Proof. Let $(Y, \iota, \lambda, \rho)$ be a point on $\mathcal{N}_2$. It has to be shown that the endomorphism x of $\mathbb{X}_3$ lifts to an endomorphism of $Y \times \overline{\mathcal{E}}$ (via the framing $\rho \times \rho_{\overline{\mathcal{E}}}$) if and only if the endomorphism x_{red} lifts if and only if the endomorphism $\mathfrak{c}_\xi(x)$ lifts. Writing x in terms of the usual coordinates (8.6), the first two of these conditions are equivalent because the endomorphism

$$\iota_{\mathbb{X}_2}(\beta\pi) = \begin{bmatrix} 0 & \beta\varpi \\ \beta & 0 \end{bmatrix} \quad (\beta \in O_{F_0})$$

of $\mathbb{X}_2$ and the endomorphism $d \in O_F$ of $\overline{\mathbb{E}}$ automatically lift. Furthermore, since ξ obviously lifts, the equivalence of the first and third conditions is an immediate consequence of Lemma 8.6. $\square$

Combined with Lemma 8.8, the following proves Theorem 5.20.

Proposition 8.10. *The following three properties of $x \in \mathfrak{u}_1(F_0)$ are equivalent.*

(i) $x \in \mathfrak{k}_{1,\mathrm{rs}}$.

(ii) $\Delta \cap \Delta_x$ *is a nonempty scheme, proper over* $\mathrm{Spec}\, O_{\breve F}$.

(iii) $\Delta \cap \Delta_x$ *is artinian with two points.*

The analog where $x \in \mathfrak{u}_1(F_0)$ is replaced by $g \in U_1(F_0)$ and $\mathfrak{k}_{1,\mathrm{rs}}$ is replaced by $K_{1,\mathrm{rs}}$ is also true. Furthermore, in the group case, under these conditions, there are no higher Tor terms in the expression (4.9) *defining* $\mathrm{Int}(g)$.

Proof. Lemma 8.8 immediately gives the equivalence of (ii) and (iii) in both the Lie algebra and group cases; and in the proof of the rest of the proposition, it also allows us to assume that $x \in \mathfrak{k}_1$ in the Lie algebra case (resp. $g \in K_1$ in the group case) and that the intersection in question is nonempty. By Lemmas 8.2, 8.4, 8.7, and 8.9, the equivalence of (i) and (ii) in both the Lie algebra and group cases follows from their equivalence in the case that x is a reduced element in the Lie algebra. When $x \in \mathfrak{k}_{1,\mathrm{red}}$ is regular semi-simple, we will show in §9.2 that $\Delta \cap \Delta_x$ is an artinian scheme by explicitly computing its (finite) length. Thus (i) implies (ii).

To complete the proof of the equivalence of the three properties, we will show that if $x \in \mathfrak{k}_{1,\mathrm{red}}$ is not regular semi-simple, then the intersection $\Delta \cap \Delta_x$ is not a scheme. Write x in the form (8.10), with $\alpha \in O_D^{\mathrm{tr}=0}$ and $b \in O_D$. Then x is not regular semi-simple (if and) only if the elements αb and b are linearly dependent over F (which as before we take to act on D on the right).

First suppose that $b \neq 0$. Then $b^{-1}\alpha b \in F$ inside D. Or in other words, α is in the image of the conjugate embedding $\iota := {}^b\iota_{\mathbb{E}}$, in the notation of (1.4). Since F/F_0 is ramified, ι makes $\mathbb{E}$ into a formal O_F-module of height 1, and we denote by $\mathcal{E}_\iota$ the corresponding canonical lift of $\mathbb{E}$ over $\mathrm{Spf}\, O_{\breve F}$. Via the Serre construction, $O_F \otimes_{O_{F_0}} \mathcal{E}_\iota$ gives a $\mathrm{Spf}\, O_{\breve F}$-point on $\mathcal{N}_2$. Since α lifts to an endomorphism of $\mathcal{E}_\iota$, $\mathrm{diag}(\alpha, \alpha)$ lifts to an endomorphism of $O_F \otimes_{O_{F_0}} \mathcal{E}_\iota$. Since $b \in O_D$ conjugates $\iota_{\mathbb{E}}$ into ι, it lifts to a homomorphism $\mathcal{E} \to \mathcal{E}_\iota$. Hence $\left[\begin{smallmatrix} b\pi \\ b \end{smallmatrix}\right]$ lifts to a homomorphism $\overline{\mathcal{E}} \to O_F \otimes_{O_{F_0}} \mathcal{E}_\iota$. Similarly, since $\overline{b} = b^{-1}\mathrm{N}b$ conjugates ι into $\iota_{\mathbb{E}}$, $\left[\pi\overline{b} \quad \overline{b}\varpi\right]$ lifts to a homomorphism $O_F \otimes_{O_{F_0}} \mathcal{E}_\iota \to \overline{\mathcal{E}}$. This shows that x lifts to an endomorphism of $(O_F \otimes_{O_{F_0}} \mathcal{E}_\iota) \times \overline{\mathcal{E}}$. Thus we've constructed a $\mathrm{Spf}\, O_{\breve F}$-point on $\Delta \cap \Delta_x$, which shows that $\Delta \cap \Delta_x$ cannot be a scheme.

The case that $b = 0$ is even simpler: let $\mathcal{E}_0$ be any formal O_{F_0}-module over $\mathrm{Spf}\, O_{\breve F}$ which lifts $\mathbb{E}$ and for which the endomorphism α lifts. Then as before x lifts to an endomorphism of $(O_F \otimes_{O_{F_0}} \mathcal{E}_0) \times \overline{\mathcal{E}}$, so that we again obtain a $\mathrm{Spf}\, O_{\breve F}$-point on $\Delta \cap \Delta_x$.

Now let us prove the final assertion, i.e. that for $g \in U_{1,\mathrm{rs}}(F_0)$ we have

$$\mathrm{Int}(g) = \mathrm{length}(\Delta \cap \Delta_g). \tag{8.18}$$

We follow [25, Lem. 4.1] and [10, Prop. 11.6]. Let R be the local ring at a point $x \in \Delta \cap \Delta_g$ of $\mathcal{N}_2 \times \mathcal{N}_3$. Then Δ is defined in x by the ideal generated by a regular sequence f_1, f_2 of R.

Hence the Koszul complex $K(f_1, f_2)$ is a free resolution of the R-module $\mathcal{O}_{\Delta,x}$, and the complex $K(f_1, f_2) \otimes_R \mathcal{O}_{\Delta_g,x}$ represents $(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_g})_x$. But

$$K(f_1, f_2) \otimes_R \mathcal{O}_{\Delta_g} = K\big(\overline{f}_1, \overline{f}_2\big),$$

where $\overline{f}_i$ denotes the image of f_i in $\mathcal{O}_{\Delta_g}$, and where on the right-hand side appears the Koszul complex as $\mathcal{O}_{\Delta_g,x}$-module. Since Δ and Δ_g intersect properly, $\overline{f}_1, \overline{f}_2$ forms a regular sequence in $\mathcal{O}_{\Delta_g,x}$ which generates the ideal of $\Delta \cap \Delta_g$, we see that $K(\overline{f}_1, \overline{f}_2)$ is a free resolution of $\mathcal{O}_{\Delta\cap\Delta_g,x}$. Hence $(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_g})_x$ is represented by $\mathcal{O}_{\Delta\cap\Delta_g,x}$. The asserted equality (8.18) follows. ☐

Corollary 8.11. *For $x \in \mathfrak{k}_{1,\mathrm{rs}}$ and $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$,*

$$\ell\text{-Int}(x) = \ell\text{-Int}(x_{\mathrm{red}}) = \mathrm{length}\big(\Delta \cap \Delta_{\mathfrak{c}_\xi(x)}\big) = \mathrm{Int}\big(\mathfrak{c}_\xi(x)\big).$$

Proof. Lemma 8.9 gives the first two equalities, and the vanishing of higher Tor terms asserted in Proposition 8.10 gives the last one. ☐

9. Explicit calculations for the Lie algebra

By the results of the previous section, the calculation of intersection numbers in the situations of interest to us reduces to the calculation of ℓ-Int(x) for a reduced, regular semi-simple element $x \in \mathfrak{k}_1$. In this section we effect this calculation.

9.1. **Keating invariants.** To begin, we briefly recall the theorem of Keating [26, Th. 2.1] in the case that F/F_0 is ramified. Fix any F_0-embedding of F into D, and let $\psi \in O_D$. Let $\mathrm{dist}_j(\psi)$ be the "distance" of ψ to the order O_j of conductor j in F, i.e.

$$\mathrm{dist}_j(\psi) := \max\big\{\, v_D(x+\psi) \bigm| x \in O_j \,\big\}.$$

Equivalently, $\mathrm{dist}_j(\psi)$ is the positive integer ℓ such that

$$\psi \in \big(O_j + \pi^\ell O_D\big) \smallsetminus \big(O_j + \pi^{1+\ell} O_D\big).$$

(Recall that we use the uniformizer π of F as the uniformizer of O_D.) We may also describe the distance as the minimum

$$\mathrm{dist}_j(\psi) = \min\big\{\ell(\psi_-), \ell_j(\psi_+)\big\}, \tag{9.1}$$

where ψ_+ and ψ_- are the components of ψ with respect to the decomposition (8.1), and

$$\ell(\psi_-) = v_D(\psi_-) \quad \text{and} \quad \ell_j(\psi_+) = \begin{cases} v_D\big(\mathrm{Im}(\psi_+)\big), & v_D\big(\mathrm{Im}(\psi_+)\big) < 2j; \\ +\infty, & v_D\big(\mathrm{Im}(\psi_+)\big) \geq 2j. \end{cases}$$

Here $\mathrm{Im}(\psi_+) = \big(\psi_+ - \overline{\psi}_+\big)/2 \in F^{\mathrm{tr}=0}$ is the imaginary part; note that $v_D(\mathrm{Im}(\psi_+))$ is always odd. We adopt the usual convention that $v_D(0) = +\infty$.

Proposition 9.1 (Keating)**.** *Assume that F/F_0 is ramified. For $j \geq 0$, let $\ell = \mathrm{dist}_j(\psi)$ be defined as above. Then the length $n_j(\psi)$ of the locus inside the quasi-canonical divisor $\mathcal{W}_{F,j}$ where ψ lifts to an endomorphism of the corresponding quasi-canonical lifting is given by*

$$n_j(\psi) = \begin{cases} 2\displaystyle\sum_{i=0}^{\ell/2} q^i - q^{\ell/2}, & \ell \leq 2j \textit{ is even;} \\ 2\displaystyle\sum_{i=0}^{(\ell-1)/2} q^i = 2\frac{q^{(\ell+1)/2}-1}{q-1}, & \ell \leq 2j \textit{ is odd;} \\ 2\displaystyle\sum_{i=0}^{j-1} q^i + (\ell - 2j + 1)q^j, & \ell > 2j. \end{cases}$$

We refer to the third alternative in the statement as the *stable range for j relative to ℓ*, and the first two alternatives as the *unstable range for j relative to ℓ*.

9.2. **Calculation of ℓ-Int(x) for $x \in \mathfrak{k}_{1,\mathrm{red}}$.** Now let us return to our convention that F is embedded in D via $\iota_{\overline{\mathbb{E}}}$, as in §8. Let $x \in \mathfrak{k}_{1,\mathrm{red}}$ be regular semi-simple. By definition,

$$\ell\text{-Int}(x) = \mathrm{length}\big(\textit{locus in } \mathcal{N}_2 \textit{ where } x \textit{ lifts to an endomorphism of } \mathcal{Y} \times \overline{\mathcal{E}}\big), \tag{9.2}$$

where $\mathcal{Y}$ denotes the universal p-divisible group over $\mathcal{N}_2$. We are going to obtain an explicit expression for this length by pulling the calculation back to $\mathcal{M}$ via the Serre map (as in Proposition 6.3) and using Keating's theorem. Of course, to do so we have to account for the fact that $\mathcal{N}_2$ has two connected components, only one of which is identified with $\mathcal{M}$ under the Serre map. As in the proof of Lemma 8.8, let $z_\pm$ denote the two points in $\mathcal{N}_2(\bar{k})$, with z_+ the standard basepoint. Write ℓ-Int$_\pm(x)$ for the length of the locus occurring in (9.2) supported at $z_\pm$, so that ℓ-Int$(x) = \ell$-Int$_+(x) + \ell$-Int$_-(x)$. If $g \in H_1(F_0) = U(\mathbb{X}_2)$ interchanges z_+ and z_-, then via the inclusion $H_1(F_0) \subset U_1(F_0)$,

$$\ell\text{-Int}(x) = \ell\text{-Int}_+(x) + \ell\text{-Int}_+(gxg^{-1}). \tag{9.3}$$

We first consider the term ℓ-Int$_+(x)$ in (9.3). Write

$$x = \begin{bmatrix} \alpha & 0 & b\pi \\ 0 & \alpha & b \\ \pi\bar{b} & \varpi\bar{b} & 0 \end{bmatrix}, \quad \alpha \in O_D^{\mathrm{tr}=0}, \quad b \in O_D,$$

in the coordinates (8.6). Recall from §8.1 that the matrix entries are with respect to the O_{F_0}-linear decomposition of the framing object

$$\mathbb{X}_3 = \mathbb{X}_2 \times \overline{\mathbb{E}} = (O_F \otimes_{O_{F_0}} \mathbb{E}) \times \overline{\mathbb{E}} = (1 \otimes \mathbb{E} + \pi \otimes \mathbb{E}) \times \overline{\mathbb{E}} \simeq \mathbb{E} \times \mathbb{E} \times \overline{\mathbb{E}}.$$

By Lemma 8.3, since x is regular semi-simple, we have $b \neq 0$ and $\alpha'_- \neq 0$, where $\alpha' = b^{-1}\alpha b$ and the minus denotes the component of α' with respect to the decomposition (8.1) of D. Now, inside $\mathcal{M}$ is the special divisor $\mathcal{T}(\bar{b})$ where $\bar{b}$ lifts to a homomorphism $\mathcal{Y}_0 \to \overline{\mathcal{E}}$, where we recall that $\mathcal{Y}_0$ denotes the universal formal O_{F_0}-module over $\mathcal{M}$, cf. §6.2. Since $\lambda_{\mathbb{E}}$ lifts to the principal polarizations $\lambda_{\mathcal{Y}_0}$ of $\mathcal{Y}_0$ and $\lambda_{\overline{\mathcal{E}}}$ of $\overline{\mathcal{E}}$, and since the Rosati involution on D is the main involution, this is the same as the locus where b lifts to a homomorphism $\overline{\mathcal{E}} \to \mathcal{Y}_0$. Over the connected component $\mathcal{N}_{2,+} \subset \mathcal{N}_2$, the Serre map identifies $\mathcal{Y}$ with $O_F \otimes_{O_{F_0}} \mathcal{Y}_0 = 1 \otimes \mathcal{Y}_0 + \pi \otimes \mathcal{Y}_0$. Since $\pi \in O_D$ of course lifts to an endomorphism of $\overline{\mathcal{E}}$, we conclude that $\mathcal{T}(\bar{b})$ identifies with the locus in $\mathcal{N}_{2,+}$ where $\left[\begin{smallmatrix} b\pi \\ b \end{smallmatrix}\right]$ lifts to a homomorphism $\overline{\mathcal{E}} \to \mathcal{Y}$ and $\begin{bmatrix}\pi\bar{b} & \varpi\bar{b}\end{bmatrix}$ lifts to a homomorphism $\mathcal{Y} \to \overline{\mathcal{E}}$; and we further conclude that the locus in $\mathcal{T}(\bar{b})$ where α lifts to an endomorphism of $\mathcal{Y}_0$ identifies with the locus in $\mathcal{N}_{2,+}$ where x lifts to an endomorphism of $\mathcal{Y} \times \overline{\mathcal{E}}$. By Proposition 7.1, we can write the divisor $\mathcal{T}(\bar{b})$ as a sum of quasi-canonical divisors

$$\mathcal{T}(\bar{b}) = \sum_{0 \leq j \leq v_D(b)} \mathcal{W}_{{}^bF,j},$$

where we recall that ${}^bF$ denotes the image of F in D under the conjugate embedding ${}^b\iota_{\mathbb{E}}$. We obtain from Proposition 9.1

$$\ell\text{-Int}_+(x) = \sum_{0 \leq j \leq v_D(b)} n_j(\alpha'),$$

where the length $n_j(\alpha')$ depends, via Keating's formula, on the distance of α' to the original order in F (not to the conjugate order!).

Now consider the term ℓ-Int$_+(gxg^{-1})$ in (9.3). In terms of the coordinates (8.4) for $U_1(F_0)$, let us explicitly take

$$g := \begin{bmatrix} 0 & \pi & 0 \\ \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which indeed has nontrivial Kottwitz invariant. Then

$$gxg^{-1} = \begin{bmatrix} \pi\alpha\pi^{-1} & 0 & \pi b \\ 0 & \pi^{-1}\alpha\pi & \pi^{-1}b\pi \\ \varpi\bar{b}\pi^{-1} & \pi\bar{b}\pi & 0 \end{bmatrix} = \begin{bmatrix} {}^\pi\alpha & 0 & {}^\pi b\pi \\ 0 & {}^\pi\alpha & {}^\pi b \\ \pi\overline{{}^\pi b} & \varpi\overline{{}^\pi b} & 0 \end{bmatrix},$$

where the superscript π denotes conjugation of the base by π, which we have used is also the same as conjugation by π^{-1}, since π is traceless. Thus to compute ℓ-Int$_+(gxg^{-1})$, we can run through exactly the same analysis as above, with ${}^{\pi}b$ in place of b and ${}^{\pi}\alpha$ in place of α. Since the elements ${}^{\pi}b^{-1}\,{}^{\pi}\alpha\,{}^{\pi}b = {}^{\pi}(\alpha')$ and α' have the same distance to O_j, we conclude that $\mathrm{Int}_+(gxg^{-1}) = \ell\text{-}\mathrm{Int}_+(x)$. Hence

$$\ell\text{-}\mathrm{Int}(x) = 2\ell\text{-}\mathrm{Int}_+(x).$$

We now explicitly calculate this length via Keating's formula. Since $\mathrm{tr}(\alpha) = 0$, we also have $\mathrm{tr}(\alpha') = \mathrm{tr}(\alpha'_\pm) = 0$. Therefore the formula (9.1) for the distance gives

$$\mathrm{dist}_j(\alpha') = \begin{cases} \min\{v_D(\alpha'_-), v_D(\alpha'_+)\}, & v_D(\alpha'_+) < 2j; \\ v_D(\alpha'_-), & v_D(\alpha'_+) \geq 2j. \end{cases}$$

To lighten notation, set

$$\ell_- := v_D(\alpha'_-), \quad \ell_+ := v_D(\alpha'_+), \quad \text{and} \quad m := v_D(b).$$

Note that $\ell_+ = v_D(\alpha'_+)$ is odd, since $\alpha'_+ \in F$ is purely imaginary. Also note that these quantities depend only on the invariants $u(x)$, $w(x)$, and $\Delta(x)$ (which are given explicitly in (8.11) and (8.13)), via

$$v_D(u) = 2m, \quad v_D(w) = 2m + \ell_+, \quad \text{and} \quad v_D(\Delta) = 4m + 2\ell_-. \tag{9.4}$$

We will make use of the following ancillary calculation at several points below: for any $r \geq 0$, since

$$\sum_{j=0}^{r} jq^j = \frac{rq^{r+1}}{q-1} - \frac{q^{r+1}-q}{(q-1)^2} = \frac{rq^{r+2} - (r+1)q^{r+1} + q}{(q-1)^2},$$

we have

$$\begin{aligned}
&\sum_{j=0}^{r} \left(2\frac{q^j - 1}{q-1} + (\ell_- - 2j + 1)q^j \right) \\
&\quad = 2\frac{\frac{q^{r+1}-1}{q-1} - (r+1)}{q-1} + (\ell_- + 1)\frac{q^{r+1}-1}{q-1} - 2\frac{rq^{r+2} - (r+1)q^{r+1} + q}{(q-1)^2} \\
&\quad = q^{r+1}\frac{2 + (\ell_- + 1)(q-1) - 2(rq - r - 1)}{(q-1)^2} - \frac{\ell_- + 1 + 2(r+1)}{q-1} - \frac{2q+2}{(q-1)^2} \\
&\quad = q^{r+1}\frac{2(q+1) + (\ell_- - 2r - 1)(q-1)}{(q-1)^2} - \frac{\ell_- + 2r + 1}{q-1} - \frac{4q}{(q-1)^2}.
\end{aligned} \tag{9.5}$$

Finally we set

$$t := q^{-1}.$$

For the main calculation we now distinguish cases.

Case I: $\ell_- \leq \ell_+$. Then for any j,

$$\mathrm{dist}_j(\alpha') = \ell_-.$$

We divide this case further into three subcases.

(1) $\ell_- > 2m$. Then all j with $0 \leq j \leq m$ are in the stable range for $\mathrm{dist}_j(\alpha')$. Hence

$$\ell\text{-}\mathrm{Int}_+(x) = \sum_{j=0}^{m} \left(2\frac{q^j - 1}{q-1} + (\ell_- - 2j + 1)q^j \right).$$

Taking $r = m$ in (9.5), replacing q with t^{-1}, and multiplying by 2, we obtain

$$\ell\text{-}\mathrm{Int}(x) = 2t^{-m}\frac{2(1+t) + (\ell_- - 2m - 1)(1-t)}{(1-t)^2} - \frac{2(\ell_- + 2m + 1)t}{1-t} - \frac{8t}{(1-t)^2}.$$

(2) $\ell_- \leq 2m$ and ℓ_- odd. In this case, there are some j in the stable range and some in the unstable range. We get

$$\ell\text{-}\mathrm{Int}_+(x) = \sum_{j=0}^{(\ell_- - 1)/2} \left(2\frac{q^j - 1}{q-1} + (\ell_- - 2j + 1)q^j \right) + \sum_{j=(\ell_- + 1)/2}^{m} 2\frac{q^{(\ell_- + 1)/2} - 1}{q-1}. \tag{9.6}$$

Multiplying by 2 and using (9.5) with $r = (\ell_- - 1)/2$, we get

$$\begin{aligned}\ell\text{-Int}(x) &= 2q^{(\ell_-+1)/2}\frac{2(q+1)}{(q-1)^2} - 2\frac{2\ell_-}{q-1} - 2\frac{4q}{(q-1)^2} + 2\Big(m - \frac{\ell_-+1}{2} + 1\Big)\frac{2(q^{(\ell_-+1)/2}-1)}{q-1}\\ &= 2t^{-(\ell_--1)/2}\frac{(2m-\ell_-+3)-(2m-\ell_--1)t}{(1-t)^2} - \frac{2(\ell_-+2m+1)t}{1-t} - \frac{8t}{(1-t)^2}.\end{aligned}$$

(3) $\ell_- \le 2m$ and ℓ_- even. Again, there are stable and unstable j. Similarly to the previous subcase, we get

$$\begin{aligned}\ell\text{-Int}(x) &= 2\sum_{j=0}^{\ell_-/2-1}\Big(2\frac{q^j-1}{q-1} + (\ell_-+1-2j)q^j\Big) + 2\sum_{j=\ell_-/2}^{m}\Big(2\frac{q^{\ell_-/2+1}-1}{q-1} - q^{\ell_-/2}\Big)\\ &= 2q^{\ell_-/2}\frac{2(q+1)+q-1}{(q-1)^2} - 2\frac{2\ell_--1}{q-1} - 2\frac{4q}{(q-1)^2} + 2\Big(m - \frac{\ell_-}{2} + 1\Big)\frac{q^{\ell_-/2}(q+1)-2}{q-1}\\ &= 2t^{-\ell_-/2}\frac{(m-\ell_-/2+1)(1-t^2)+t(t+3)}{(1-t)^2} - \frac{2(\ell_-+2m+1)t}{1-t} - \frac{8t}{(1-t)^2}.\end{aligned}$$

Case II: $\ell_- > \ell_+$. In this case we have for the distance

$$\mathrm{dist}_j(\alpha') = \begin{cases}\ell_+, & \ell_+ < 2j;\\ \ell_-, & \ell_+ \ge 2j.\end{cases}$$

We consider the following two subcases.

(1) $\ell_+ \ge 2m$. Then for all j with $0 \le j \le m$, we have $\mathrm{dist}_j(\alpha') = \ell_- > 2m$. Hence all j lie in the stable range, and we get the same answer as in Case I(1),

$$\ell\text{-Int}(x) = 2t^{-m}\frac{2(1+t)+(\ell_--2m-1)(1-t)}{(1-t)^2} - \frac{2(\ell_-+2m+1)t}{1-t} - \frac{8t}{(1-t)^2}.$$

(2) $\ell_+ < 2m$. In this subcase, the relevant j can be in the stable range as well as in the unstable range. Note that ℓ_+ is odd. Hence we get, similarly to Case I(2),

$$\begin{aligned}\ell\text{-Int}(x) &= 2\sum_{j=0}^{(\ell_+-1)/2}\Big(2\frac{q^j-1}{q-1} + (\ell_-+1-2j)q^j\Big) + 2\sum_{j=(\ell_++1)/2}^{m} 2\frac{q^{(\ell_++1)/2}-1}{q-1}\\ &= 2q^{(\ell_++1)/2}\frac{2(q+1)+(\ell_--\ell_+)(q-1)}{(q-1)^2}\\ &\qquad - 2\frac{\ell_-+\ell_+}{q-1} - 2\frac{4q}{(q-1)^2} + 2\Big(m - \frac{\ell_++1}{2} + 1\Big)\frac{2(q^{(\ell_++1)/2}-1)}{q-1}\\ &= 2t^{-(\ell_+-1)/2}\frac{(\ell_--2\ell_++2m+3)(1-t)+4t}{(1-t)^2} - \frac{2(\ell_-+2m+1)t}{1-t} - \frac{8t}{(1-t)^2}.\end{aligned}$$

Part 3. Analytic side

In this part of the paper we turn to the analytic side of the identities to be proved in Theorems 5.21 and 5.22, and, modulo the material in Part 4 on germ expansions of orbital integrals, we complete the proofs of these theorems.

10. Inputs from local harmonic analysis

In this section we formulate some basic facts about harmonic analysis on the spaces in play in the Lie algebra and group settings. Except where noted to the contrary, we allow n to be arbitrary.

10.1. **Lie algebra setting.** Let

$$\pi_{\mathfrak{s}} \colon \mathfrak{s} \longrightarrow \mathfrak{b} \tag{10.1}$$

be the categorical quotient of $\mathfrak{s}$ by H' as discussed in §2.5, say by taking either set of invariants (2.8) or (2.10). Thus $\mathfrak{b}$ is an affine space over F_0 of dimension $2n-1$ (given explicitly in (2.11) in the case of the invariants (2.10)). Let $\mathfrak{b}_{\mathrm{rs}}$ be the image of $\mathfrak{s}_{\mathrm{rs}}$ in $\mathfrak{b}$. Then

$$\mathfrak{b}_{\mathrm{rs}} = \bigl\{\, x \in \mathfrak{b} \bigm| \Delta(x) \neq 0 \,\bigr\},$$

where Δ denotes the discriminant (8.12). Since this is a global function on $\mathfrak{s}$ which is H'-invariant, it descends to a global function on $\mathfrak{b}$.

For $\phi' \in C_c^\infty(\mathfrak{s})$, we note that the function $y \mapsto \omega(y)\operatorname{Orb}(y, \phi')$ descends to a function φ on $\mathfrak{b}_{\mathrm{rs}}(F_0)$. Let $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$ denote the space of locally constant functions on $\mathfrak{b}_{\mathrm{rs}}(F_0)$ whose support has compact closure in $\mathfrak{b}(F_0)$ (functions with *relatively compact support*). By [35, Lem. 3.12] the function φ lies in $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$. By a slight variant of [35, Prop. 3.8], we have the following.

Theorem 10.1. *Let φ be a function in $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$. The following properties are equivalent.*

(i) *There exists a function $\phi' \in C_c^\infty(\mathfrak{s})$ such that*

$$\varphi\bigl(\pi_{\mathfrak{s}}(y)\bigr) = \omega(y)\operatorname{Orb}(y, \phi') \quad \textit{for all} \quad y \in \mathfrak{s}_{\mathrm{rs}}(F_0).$$

(ii) *For every $x_0 \in \mathfrak{b}(F_0)$, there exists an open neighborhood V_{x_0} of x_0 and a function $\phi'_{x_0} \in C_c^\infty(\mathfrak{s})$ such that*

$$\varphi\bigl(\pi_{\mathfrak{s}}(y)\bigr) = \omega(y)\operatorname{Orb}(y, \phi'_{x_0}) \quad \textit{for all} \quad y \in \pi_{\mathfrak{s}}^{-1}\bigl(V_{x_0} \cap \mathfrak{b}_{\mathrm{rs}}(F_0)\bigr) = \pi_{\mathfrak{s}}^{-1}(V_{x_0}) \cap \mathfrak{s}_{\mathrm{rs}}(F_0).$$

□

A function $\varphi \in C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$ satisfying property (i) is called an *orbital integral function*; a function φ on $\mathfrak{b}_{\mathrm{rs}}(F_0)$ satisfying property (ii) is called a *local orbital integral function* [35, Def. 3.7]. More precisely, if φ satisfies property (ii) locally around x_0, then φ will be called an orbital integral function *locally around* x_0.

We note that the map $\pi_{\mathfrak{s}}$ in (10.1) induces a surjection on F_0-rational points, and we have a decomposition

$$\mathfrak{b}_{\mathrm{rs}}(F_0) = \mathfrak{b}_{\mathrm{rs},0} \amalg \mathfrak{b}_{\mathrm{rs},1} \tag{10.2}$$

into a disjoint union of two open (for the p-adic topology) subsets. Here $\mathfrak{b}_{\mathrm{rs},i}$ is the image under $\pi_{\mathfrak{s}}$ of the set $\mathfrak{s}_{\mathrm{rs},i}$ of elements in $\mathfrak{s}_{\mathrm{rs}}(F_0)$ which match with elements in $\mathfrak{u}_{i,\mathrm{rs}}(F_0)$.

This decomposition can also be explained by a similar picture on the unitary side. Taking the same invariants as used in realizing the quotient map (10.1), by §2.5 we obtain for $i \in \{0, 1\}$ a categorical quotient map

$$\pi_{\mathfrak{u}_i} \colon \mathfrak{u}_i \longrightarrow \mathfrak{b}.$$

Then $\mathfrak{b}_{\mathrm{rs},i}$ is the image under $\pi_{\mathfrak{u}_i}$ of $\mathfrak{u}_{i,\mathrm{rs}}(F_0)$. To be quite clear, the maps $\pi_{\mathfrak{u}_0}$ and $\pi_{\mathfrak{u}_1}$ do *not* induce surjections on F_0-rational points, cf. the remark after Lem. 3.1 in [35].

Proposition 10.2. *Let $n = 3$. For $i \in \{0, 1\}$, an element $x \in \mathfrak{b}_{\mathrm{rs}}(F_0)$ lies in $\mathfrak{b}_{\mathrm{rs},i}$ if and only if $\eta(-\Delta(x)) = (-1)^i$.*

Proof. This follows from Lemma 2.1, noting that we rescaled $\Delta(x)$ by the factor $-\varpi^{-2}$ in (8.12). (One can also use (8.13) for $i = 1$, taking note that $\eta(-\mathrm{N}\alpha'_-) = -1$, and the explicit coordinates given in (11.5) below for $i = 0$.) □

Of course, Lemma 2.1 works perfectly well to distinguish between the two summands in (10.2) for any n; we have stated the proposition for $n = 3$ only because we are now working with the rescaled version of Δ.

We conclude the subsection by noting the following.

Lemma 10.3. *Let $\phi' \in C_c^\infty(\mathfrak{s})$ be a function with transfer $(\phi, 0) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$. Then the function*

$$y \longmapsto \begin{cases} \omega(y)\, \partial\mathrm{Orb}(y, \phi'), & y \in \mathfrak{s}_{\mathrm{rs},1}; \\ 0, & y \in \mathfrak{s}_{\mathrm{rs},0} \end{cases}$$

descends to a function on $\mathfrak{b}_{\mathrm{rs}}(F_0)$ which lies in $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$.

Proof. By Remark 5.4(i) (or rather, its Lie algebra analog), the function descends to a function φ on $\mathfrak{b}_{\mathrm{rs}}(F_0)$. The map $\pi_{\mathfrak{s}}\colon \mathfrak{s}(F_0) \to \mathfrak{b}(F_0)$ is continuous and hence sends the support of ϕ' (a compact set) to a compact set in $\mathfrak{b}(F_0)$. The support of φ lies in the image of the support of ϕ' under $\pi_{\mathfrak{s}}$ and is therefore relatively compact. The local constancy of φ follows from the same argument as in [35, Lem. 3.12] (which is about the case $y \mapsto \omega(y) \operatorname{Orb}(y, \phi')$). $\square$

10.2. **Group setting.** We now translate the contents of §10.1 to the group setting. Let B denote the categorical quotient of S by H', and B_i the categorical quotient of U_i by H_i for $i = 0$ and 1. These are affine varieties with rings of global functions given by the ring of invariants. We denote by

$$\pi_S\colon S \longrightarrow B \quad \text{and} \quad \pi_{U_i}\colon U_i \longrightarrow B_{U_i}$$

the corresponding quotient morphisms, and by B_{rs}, resp. $B_{U_i,\mathrm{rs}}$, the images of S_{rs}, resp. $U_{i,\mathrm{rs}}$, under these morphisms. All of these are open subschemes defined by the non-vanishing of the discriminant function (which by equivariance drops to the categorical quotients). On the level of F_0-rational points, we write $B_{\mathrm{rs}}(F_0)$ and $B(F_0)_{\mathrm{rs}}$, resp. $B_{U_i,\mathrm{rs}}(F_0)$ and $B_{U_i}(F_0)_{\mathrm{rs}}$, interchangeably.

We are going to see that, in analogy with the Lie algebra case, the quotients B, B_{U_0}, and B_{U_1} can all be naturally identified. To facilitate the precise statement and proof of this result, we introduce the Cayley transform on each of our Lie algebra spaces (cf. §8.3 for the case of $\mathfrak{u}_1$ when $n = 3$). Let[11]

$$\mathfrak{s}^\circ := \bigl\{\, y \in \mathfrak{s} \bigm| \det(1 - y) \neq 0 \,\bigr\} \quad \text{and} \quad \mathfrak{u}_i^\circ := \bigl\{\, x \in \mathfrak{u}_i \bigm| \det(1 - x) \neq 0 \,\bigr\}, \quad i = 0 \text{ or } 1. \tag{10.3}$$

Then under the respective quotient maps $\pi_{\mathfrak{s}}$, $\pi_{\mathfrak{u}_0}$, and $\pi_{\mathfrak{u}_1}$, these open subschemes descend to a common open subscheme $\mathfrak{b}^\circ$ of $\mathfrak{b}$. Next recall from the Introduction the norm 1 subgroup $F^1 = \{\xi \in F \mid \mathrm{N}\xi = 1\}$, and define

$$S^1 := \bigl\{\, \mathrm{diag}(\xi_1 \cdot 1_{n-1}, \xi_2) \bigm| \xi_1, \xi_2 \in F^1 \,\bigr\} \subset S(F_0).$$

Then S^1 canonically identifies with a subgroup in both $U_0(F_0)$ and $U_1(F_0)$ (upon choosing any special embeddings of U_0 and U_1 as in §2.1).

For $\xi \in S^1$, we define the Cayley transform for $\mathfrak{u}_0$ and $\mathfrak{u}_1$ via the same formula as in (8.15),

$$\begin{aligned} \mathfrak{c}_\xi\colon\ \mathfrak{u}_i^\circ &\longrightarrow U_i, \qquad i = 0 \text{ or } 1. \\ x &\longmapsto \xi \frac{1 + x}{1 - x} \end{aligned} \tag{10.4}$$

Then $\mathfrak{c}_\xi$ is H_i-equivariant, and, abusing notation, we continue to denote by $\mathfrak{c}_\xi$ the induced map on the quotients

$$\mathfrak{c}_\xi\colon \mathfrak{b}^\circ \longrightarrow B_{U_i}, \quad i = 0 \text{ or } 1.$$

Remark 10.4. This definition of $\mathfrak{c}_\xi$ on $\mathfrak{u}_1^\circ$ generalizes the definition in §8.3 when $n = 3$ and $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$. But note that for more general ξ there is a small subtlety between the matrix notation we are currently using and the coordinates in §8: the element $\mathrm{diag}(\xi_1, \xi_1, \xi_2) \in S^1$ is expressed as

$$\begin{bmatrix} \alpha & \beta\varpi & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \xi_2 \end{bmatrix} \in U_1(F_0)$$

in the coordinates (8.4), where $\xi_1 = \alpha + \beta\pi$ with $\alpha, \beta \in F_0$.

To define the Cayley transform on $\mathfrak{s}^\circ$, note that the formula in (10.4) only gives a map into S when ξ is of the form $\xi_1 \cdot 1_n$. To define $\mathfrak{c}_\xi$ for an arbitrary $\xi = \mathrm{diag}(\xi_1 \cdot 1_{n-1}, \xi_2) \in S^1$, first choose $\nu_1, \nu_2 \in F^\times$ such that $\overline{\nu}_1/\nu_1 = \xi_1$ and $\overline{\nu}_2/\nu_2 = \xi_2$, and set

$$\nu := \mathrm{diag}(\nu_1 \cdot 1_{n-1}, \nu_2).$$

[11] Here by det we of course mean the usual $n \times n$ determinant $\mathrm{Res}_{F/F_0} \mathrm{M}_n \to \mathrm{Res}_{F/F_0} \mathbb{A}$, relative to any choice of basis for the hermitian space W_i in the case of $\mathfrak{u}_i$. This definition of $\mathfrak{u}_1^\circ$ is consistent with the notation (8.14) for $\mathfrak{u}_1^\circ(F_0)$, but note that the matrix representations used in §8 involve entries in D, and therefore care must be taken to correctly interpret their determinant. An analogous remark applies in the definition of $U_{1,\xi}$ below.

Then we define

$$\begin{aligned}\mathfrak{c}_\xi \colon \mathfrak{s}^\circ &\longrightarrow S,\\ y &\longmapsto \overline{\nu} \cdot \frac{1+y}{1-y} \cdot \nu^{-1}.\end{aligned} \tag{10.5}$$

As before, $\mathfrak{c}_\xi$ is H'-equivariant. Note that this definition depends on the choice of ν_1 and ν_2, but the induced map on the quotients (which we again abusively denote by $\mathfrak{c}_\xi$)

$$\mathfrak{c}_\xi \colon \mathfrak{b}^\circ \longrightarrow B$$

depends only on ξ. When ξ is a scalar matrix, by convention we always take $\nu_1 = \nu_2$; then the Cayley transform into S is given by the usual formula $y \mapsto \xi(1+y)(1-y)^{-1}$.

Lemma 10.5 ([35, Lem. 3.4])**.** *For $\xi \in S^1$, define the open subschemes $S_\xi^\circ \subset S$ and $U_{i,\xi}^\circ \subset U_i$ by*

$$S_\xi^\circ := \left\{\, \gamma \in S \;\middle|\; \det(\xi + \gamma) \neq 0 \,\right\} \quad \text{and} \quad U_{i,\xi}^\circ := \left\{\, g \in U_i \;\middle|\; \det(\xi + g) \neq 0 \,\right\}, \quad i = 0 \text{ or } 1.$$

(i) *The Cayley transform $\mathfrak{c}_\xi$ induces an H'-equivariant isomorphism*

$$\mathfrak{s}^\circ \xrightarrow{\sim} S_\xi^\circ.$$

Furthermore, let $\xi_1, \xi_2, \dots, \xi_{n+1}$ be $n+1$ distinct elements of F^1. Then, as j varies, the open subschemes $S^\circ_{\xi_j \cdot 1_n}$ cover S.

(ii) *Analogous statement for $\mathfrak{u}_0$ and U_0 (and H_0-equivariance) in place of $\mathfrak{s}$ and S.*

(iii) *Analogous statement for $\mathfrak{u}_1$ and U_1 (and H_1-equivariance) in place of $\mathfrak{s}$ and S.* □

The following generalizes Lemma 8.7.

Lemma 10.6. *For any $\xi \in S^1$, an element y in any of $\mathfrak{s}^\circ(F_0)$, $\mathfrak{u}_0^\circ(F_0)$, or $\mathfrak{u}_1^\circ(F_0)$ is regular semi-simple if and only if $\mathfrak{c}_\xi(y)$ is.*

Proof. We show that the sets of vectors $\{y^i e\}_{i=0}^{n-1}$ and $\{{}^t e y^i\}_{i=0}^{n-1}$ are linearly independent if and only if $\{\mathfrak{c}_\xi(y)^i e\}_{i=0}^{n-1}$ and $\{{}^t e \mathfrak{c}_\xi(y)^i\}_{i=0}^{n-1}$ are; see §2.4. Set $\mathfrak{c} := \mathfrak{c}_{1_n}$. First note that the same argument as in the proof of Lemma 8.6 shows that there is an equality of F-algebras $F[y] = F[\mathfrak{c}(y)]$, which proves the desired equivalence when $\xi = 1_n$.

For an arbitrary $\xi = \operatorname{diag}(\xi_1 \cdot 1_{n-1}, \xi_2)$, we next claim that $\{\mathfrak{c}_\xi(y)^i e\}_{i=0}^{n-1}$ is linearly independent if and only if $\{\mathfrak{c}(y)^i e\}_{i=0}^{n-1}$ is. To show this we make a little calculation which will also be useful later. Let $\zeta := \operatorname{diag}(1_{n-1}, \xi_2/\xi_1)$. By an easy induction argument, for any $A \in \mathrm{M}_n(F)$, there is an equality of matrices

$$\begin{bmatrix} e & \zeta A e & \dots & (\zeta A)^{n-1} e \end{bmatrix} = \begin{bmatrix} e & Ae & \dots & A^{n-1} e \end{bmatrix} \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$$

for some upper triangular unipotent matrix on the right. Hence for the Cayley transform on $\mathfrak{u}_0^\circ$ and $\mathfrak{u}_1^\circ$,

$$\begin{aligned} \det \Big[e \quad \mathfrak{c}_\xi(y) e \quad \dots \quad \big(\mathfrak{c}_\xi(y)\big)^{n-1} e \Big] &= \det \Big[e \quad \xi \mathfrak{c}(y) e \quad \dots \quad \big(\xi \mathfrak{c}(y)\big)^{n-1} e \Big] \\ &= \xi_1^{n(n-1)/2} \det \Big[e \quad \zeta \mathfrak{c}(y) e \quad \dots \quad \big(\zeta \mathfrak{c}(y)\big)^{n-1} e \Big] \\ &= \xi_1^{n(n-1)/2} \det \big[e \quad \mathfrak{c}(y) e \quad \dots \quad \mathfrak{c}(y)^{n-1} e \big]. \end{aligned} \tag{10.6}$$

For the Cayley transform on $\mathfrak{s}^\circ$, taking $\nu = \operatorname{diag}(\nu_1 \cdot 1_{n-1}, \nu_2)$ as in the definition of $\mathfrak{c}_\xi$, we have

$$\nu^{-1} \cdot \Big[e \quad \mathfrak{c}_\xi(y) e \quad \dots \quad \big(\mathfrak{c}_\xi(y)\big)^{n-1} e \Big] = \Big[\nu_2^{-1} e \quad \xi \mathfrak{c}(y) \nu_2^{-1} e \quad \dots \quad \big(\xi \mathfrak{c}(y)\big)^{n-1} \nu_2^{-1} e \Big].$$

Hence, calculating as in (10.6),

$$\begin{aligned} &\det \Big[e \quad \mathfrak{c}_\xi(y) e \quad \dots \quad \big(\mathfrak{c}_\xi(y)\big)^{n-1} e \Big] \\ &\quad = \det(\nu) \cdot \nu_2^{-n} \cdot \xi_1^{n(n-1)/2} \det \big[e \quad \mathfrak{c}(y) e \quad \dots \quad \mathfrak{c}(y)^{n-1} e \big] \\ &\quad = (\nu_1/\nu_2)^{n-1} \xi_1^{n(n-1)/2} \det \big[e \quad \mathfrak{c}(y) e \quad \dots \quad \mathfrak{c}(y)^{n-1} e \big]. \end{aligned} \tag{10.7}$$

The equalities (10.6) and (10.7) prove the claim in all cases. A similar calculation shows that $\{{}^t e\mathfrak{c}_\xi(y)^i\}_{i=0}^{n-1}$ is linearly independent if and only if $\{{}^t e\mathfrak{c}(y)^i\}_{i=0}^{n-1}$ is, which completes the proof. □

For $\xi \in S^1$, let B_ξ° denote the image of S_ξ° in B, and let $B_{U_i,\xi}^\circ$ denote the image of $U_{i,\xi}^\circ$ in B_{U_i}. Then the Cayley transforms drop to isomorphisms

$$\mathfrak{b}^\circ \xrightarrow{\sim} B_\xi^\circ \quad \text{and} \quad \mathfrak{b}^\circ \xrightarrow{\sim} B_{U_i,\xi}^\circ, \quad i = 0 \text{ or } 1.$$

Thus we obtain isomorphisms

$$\varphi_\xi \colon B_\xi^\circ \xrightarrow{\sim} B_{U_i,\xi}^\circ, \quad i = 0 \text{ or } 1. \tag{10.8}$$

Lemma 10.7. *Let $i = 0$ or $i = 1$. There is a unique isomorphism $B \xrightarrow{\sim} B_{U_i}$ which induces for each $\xi \in S^1$ the isomorphism* (10.8). *This isomorphism induces an identification of open subschemes $B_{\mathrm{rs}} \xrightarrow{\sim} B_{U_i,\mathrm{rs}}$.*

Proof. What has to be seen is that for any $\xi, \eta \in S^1$, the isomorphisms φ_ξ and φ_η coincide on the intersection $B_\xi^\circ \cap B_\eta^\circ$. Now, after base extension from F_0 to F, there are standard isomorphisms of algebraic varieties

$$S \otimes_{F_0} F \cong U_i \otimes_{F_0} F \cong \mathrm{GL}_{n,F} \quad \text{and} \quad \mathfrak{s} \otimes_{F_0} F \cong \mathfrak{u}_i \otimes_{F_0} F \cong \mathrm{M}_{n,F}.$$

The latter identification is compatible with the quotient maps $\pi_{\mathfrak{s}}$ and $\pi_{\mathfrak{u}_i}$ to $\mathfrak{b} \otimes_{F_0} F$, and it also identifies $\mathfrak{s}^\circ \otimes_{F_0} F \cong \mathfrak{u}_i^\circ \otimes_{F_0} F$. Similarly, $H' \otimes_{F_0} F \cong H_i \otimes_{F_0} F \cong \mathrm{GL}_{n-1,F}$. Since formation of the categorical quotient commutes with flat base change, the first isomorphism in the display induces an isomorphism of algebraic varieties over F,

$$B \otimes_{F_0} F \cong B_{U_i} \otimes_{F_0} F.$$

Under the above identifications, the Cayley transforms $\mathfrak{c}_\xi$ on $\mathfrak{s}^\circ \otimes_{F_0} F$ and on $\mathfrak{u}_i^\circ \otimes_{F_0} F$ need not coincide (indeed the Cayley transform for $\mathfrak{s}$ is not even well-defined in terms of ξ), but one readily checks that they are $\mathrm{GL}_{n-1}(F)$-conjugate. In other words, under these identifications, the base change to F of the isomorphism φ_ξ becomes simply the identity morphism. Now the assertion is obvious. □

Via the lemma, we regard B as the common categorical quotient of S by H', of U_0 by H_0, and of U_1 by H_1. Analogously to (10.2), we obtain a disjoint union decomposition

$$B_{\mathrm{rs}}(F_0) = B_{\mathrm{rs},0} \amalg B_{\mathrm{rs},1}, \tag{10.9}$$

where $B_{\mathrm{rs},i}$ is the image under π_{U_i} of $U_{i,\mathrm{rs}}(F_0)$. Equivalently, $B_{\mathrm{rs},i}$ is the image under π_S of the set $S_{\mathrm{rs},i}$ of elements in $S_{\mathrm{rs}}(F_0)$ which match with elements in $U_{i,\mathrm{rs}}(F_0)$. We have

$$S_{\mathrm{rs}} = \pi_S^{-1}(B_{\mathrm{rs}}), \quad S_{\mathrm{rs},0} = \pi_S^{-1}(B_{\mathrm{rs},0}), \quad \text{and} \quad S_{\mathrm{rs},1} = \pi_S^{-1}(B_{\mathrm{rs},1}).$$

Of course, setting

$$\mathfrak{b}_{\mathrm{rs}}^\circ := \mathfrak{b}_{\mathrm{rs}} \cap \mathfrak{b}^\circ, \quad \mathfrak{b}_{\mathrm{rs},0}^\circ := \mathfrak{b}^\circ(F_0) \cap \mathfrak{b}_{\mathrm{rs},0}, \quad \text{and} \quad \mathfrak{b}_{\mathrm{rs},1}^\circ := \mathfrak{b}^\circ(F_0) \cap \mathfrak{b}_{\mathrm{rs},1},$$

the decomposition

$$\mathfrak{b}_{\mathrm{rs}}^\circ(F_0) = \mathfrak{b}_{\mathrm{rs},0}^\circ \amalg \mathfrak{b}_{\mathrm{rs},1}^\circ$$

is compatible with the decomposition (10.9) under the Cayley transform $\mathfrak{c}_\xi$ for any $\xi \in S^1$.

For $f' \in C_c^\infty(S)$, we note that the function $\gamma \mapsto \omega(\gamma)\operatorname{Orb}(\gamma, f')$ descends to a function φ on $B_{\mathrm{rs}}(F_0)$. Just as in the Lie algebra case, φ is locally constant with relatively compact support on $B_{\mathrm{rs}}(F_0)$. We denote the space of such functions by $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$. By Prop. 3.8 and the remarks before Lem. 3.6 in [35], we have the following.

Theorem 10.8. *Let φ be a function in $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$. The following properties are equivalent.*

(i) *There exists a function $f' \in C_c^\infty(S)$ such that*

$$\varphi\big(\pi_S(\gamma)\big) = \omega(\gamma)\operatorname{Orb}(\gamma, f') \quad \textit{for all} \quad \gamma \in S_{\mathrm{rs}}(F_0).$$

(ii) *For every $x_0 \in B(F_0)$, there exists an open neighborhood V_{x_0} of x_0 and a function $f'_{x_0} \in C_c^\infty(S)$ such that*

$$\varphi\big(\pi_S(\gamma)\big) = \omega(\gamma)\operatorname{Orb}(\gamma, f'_{x_0}) \quad \textit{for all} \quad \gamma \in \pi_S^{-1}\big(V_{x_0} \cap B_{\mathrm{rs}}(F_0)\big) = \pi_S^{-1}(V_{x_0}) \cap S_{\mathrm{rs}}(F_0).$$

□

As in the Lie algebra setting, we call a function $\varphi \in C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$ satisfying (ii) a *local orbital integral function*, and if φ satisfies (ii) locally around x_0, then we call φ an orbital integral function *locally around* x_0.

Similarly to Lemma 10.3, we have the following.

Lemma 10.9. *Let $f' \in C_c^\infty(S)$ be a function with transfer $(f, 0) \in C_c^\infty(U_0) \times C_c^\infty(U_1)$. Then the function*

$$\gamma \longmapsto \begin{cases} \omega(\gamma)\, \partial\mathrm{Orb}(\gamma, f'), & \gamma \in S_{\mathrm{rs},1}; \\ 0, & \gamma \in S_{\mathrm{rs},0} \end{cases}$$

descends to a function on $B_{\mathrm{rs}}(F_0)$ which lies in $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$. □

Proof. The same argument as in the proof Lemma 10.3 shows that the function descends to a function with relatively compact support. The local constancy of the descended function reduces to Lemma 10.3 by a partition of unity argument (cf. the proof of [35, Lem. 3.6]). □

10.3. **Reduced Lie algebra setting.** In this subsection, we formulate a variant of the Lie algebra version which eliminates "trivial" factors from $\mathfrak{s}$.

For y a point on $\mathfrak{s}$, write y in the block form

$$y = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \in \mathrm{Res}_{F/F_0} \mathrm{M}_n,$$

as in (2.9). In analogy with Definition 8.1, let $\mathfrak{s}_{\mathrm{red}}$ denote the closed subscheme of *reduced* points in $\mathfrak{s}$, defined by

$$\mathfrak{s}_{\mathrm{red}} := \{\, y \in \mathfrak{s} \mid \mathrm{tr}\, A = 0 \text{ and } d = 0 \,\}.$$

Then $\mathfrak{s}_{\mathrm{red}}$ is an H'-invariant subscheme of $\mathfrak{s}$. Define $\mathfrak{b}_{\mathrm{red}}$ to be the product of the middle $2n-3$ factors in the target in (2.11),

$$\mathfrak{b}_{\mathrm{red}} := \overbrace{\mathbb{A} \times \mathfrak{s}_1 \times \cdots}^{\substack{n-2 \\ \text{alternating} \\ \text{factors}}} \times \overbrace{\mathbb{A} \times \mathfrak{s}_1 \times \cdots}^{\substack{n-1 \\ \text{alternating} \\ \text{factors}}}.$$

Then the composite map

$$\begin{aligned} \pi_{\mathrm{red}} \colon\ \mathfrak{s}_{\mathrm{red}} \hookrightarrow \mathfrak{s} &\xrightarrow{\quad \pi'_{\mathfrak{s}} \quad} \mathfrak{b}_{\mathrm{red}} \\ y &\longmapsto (\mathrm{tr} \wedge^2 A, \ldots, \mathrm{tr} \wedge^{n-1} A, \mathbf{c}\mathbf{b}, \ldots, \mathbf{c}A^{n-2}\mathbf{b}) \end{aligned}$$

is a categorical quotient for $\mathfrak{s}_{\mathrm{red}}$ by H'. In terms of this notation, we also realize the quotient map (10.1) for $\mathfrak{s}$ by taking $\mathfrak{b} = \mathfrak{b}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1$ and

$$\begin{aligned} \pi_{\mathfrak{s}} \colon\ \mathfrak{s} &\longrightarrow \mathfrak{b}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \\ y &\longmapsto \big(\pi'_{\mathfrak{s}}(y), \mathrm{tr}\, A, d\big) \end{aligned}$$

(of course this is nothing but a reordering of the factors in (2.11)).

There is a natural H'-equivariant map $\mathfrak{s} \to \mathfrak{s}_{\mathrm{red}}$, $y \mapsto y_{\mathrm{red}}$, sending

$$\begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \longmapsto \begin{bmatrix} A - \frac{\mathrm{tr}\, A}{n-1} \cdot 1_{n-1} & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix}.$$

This induces an evident H'-equivariant product decomposition $\mathfrak{s} \cong \mathfrak{s}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1$, where the map onto the last two factors is given by taking $\mathrm{tr}\, A$ and d. We obtain a commutative diagram

$$\begin{tikzcd} \mathfrak{s} \arrow[r, "\sim"] \arrow[d, "\pi_{\mathfrak{s}}"'] & \mathfrak{s}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \arrow[d, "\pi_{\mathrm{red}} \times \mathrm{id} \times \mathrm{id}"] \\ \mathfrak{b} \arrow[r, equal] & \mathfrak{b}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1. \end{tikzcd} \tag{10.10}$$

We also denote by $x \mapsto x_{\mathrm{red}}$ the natural projection $\mathfrak{b} \to \mathfrak{b}_{\mathrm{red}}$, and we regard $\mathfrak{b}_{\mathrm{red}}$ as a closed subscheme of $\mathfrak{b}$ via the 0 section.

The following extends both the statement and proof of Lemma 8.2 to the case of $\mathfrak{s}$.

Lemma 10.10. *An element $y \in \mathfrak{s}(F_0)$ is regular semi-simple if and only if y_{red} is.*

Proof. By an easy induction argument, there is an equality of matrices

$$\begin{bmatrix} e & y_{\mathrm{red}}e & \dots & y_{\mathrm{red}}^{n-1}e \end{bmatrix} = \begin{bmatrix} e & ye & \dots & y^{n-1}e \end{bmatrix} \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \tag{10.11}$$

for some upper triangular unipotent matrix on the right. Similarly, the matrices

$$\begin{bmatrix} {}^te \\ {}^tey_{\mathrm{red}} \\ \vdots \\ {}^tey_{\mathrm{red}}^{n-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} {}^te \\ {}^tey \\ \vdots \\ {}^tey^{n-1} \end{bmatrix}$$

differ by left multiplication by a lower triangular unipotent matrix. The conclusion now follows from the linear algebra characterization of regular semi-simple elements in §2.4. ☐

All concepts introduced in §10.1 in the Lie algebra context have obvious analogs in the "reduced" setting. The analog of Theorem 10.1 for the reduced set is the following. The proof is essentially the same. Set

$$\mathfrak{s}_{\mathrm{red,rs}} := \mathfrak{s}_{\mathrm{red}} \cap \mathfrak{s}_{\mathrm{rs}} \quad \text{and} \quad \mathfrak{b}_{\mathrm{red,rs}} := \mathfrak{b}_{\mathrm{red}} \cap \mathfrak{b}_{\mathrm{rs}}.$$

Theorem 10.11. *Let φ be a function in $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{red,rs}})$. The following properties are equivalent.*

(i) *There exists a function $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ such that*

$$\varphi\big(\pi_{\mathrm{red}}(y)\big) = \omega(y)\operatorname{Orb}(y,\phi') \quad \textit{for all} \quad y \in \mathfrak{s}_{\mathrm{red,rs}}(F_0).$$

(ii) *For every $x_0 \in \mathfrak{b}_{\mathrm{red}}(F_0)$, there exists an open neighborhood V_{x_0} of x_0 and a function $\phi'_{x_0} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ such that*

$$\varphi\big(\pi_{\mathrm{red}}(y)\big) = \omega(y)\operatorname{Orb}(y,\phi'_{x_0}) \quad \textit{for all} \quad y \in \pi_{\mathrm{red}}^{-1}\big(V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs}}(F_0)\big) = \pi_{\mathrm{red}}^{-1}(V_{x_0}) \cap \mathfrak{s}_{\mathrm{red,rs}}(F_0).$$

☐

11. Reduction to the Lie algebra

The main aim of this section is to show that Theorem 5.21 (the main group theorem) follows from Theorem 5.22 (the main Lie algebra theorem). Except where noted to the contrary, from now on we specialize to the case $n = 3$.

11.1. **Renormalized invariants on $\mathfrak{s}_3$.** We first fix a slight renormalization of the categorical quotient map $\pi_{\mathfrak{s}}$ in §10.3 when $n = 3$, in analogy with the renormalization of the invariants on $\mathfrak{u}_1$ given in (8.8). As in §10.3, we take

$$\mathfrak{b} = \mathbb{A} \times \mathbb{A} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \times \mathfrak{s}_1 \quad \text{and} \quad \mathfrak{b}_{\mathrm{red}} = \mathbb{A} \times \mathbb{A} \times \mathfrak{s}_1$$

over F_0. Then we realize the quotient maps $\pi_{\mathfrak{s}}$ and π_{red} as

$$\begin{aligned} \pi_{\mathfrak{s}} \colon \mathfrak{s} &\longrightarrow \mathfrak{b} \\ y &\longmapsto \big(\lambda(y), u(y), w(y), \operatorname{tr} A, d\big) \end{aligned} \quad \text{and} \quad \begin{aligned} \pi_{\mathrm{red}} \colon \mathfrak{s}_{\mathrm{red}} &\longrightarrow \mathfrak{b}_{\mathrm{red}} \\ y &\longmapsto \big(\lambda(y), u(y), w(y)\big), \end{aligned}$$

where we write the point y in the usual block form

$$y = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \in \operatorname{Res}_{F/F_0} \mathrm{M}_3,$$

and where

$$\lambda(y) := \det A, \quad u(y) := \varpi^{-1}\mathbf{cb}, \quad \text{and} \quad w(y) := \varpi^{-1}\mathbf{c}A\mathbf{b}. \tag{11.1}$$

Of course, the invariants used in this definition of $\pi_{\mathfrak{s}}$, regarded as defined on $\operatorname{Res}_{F/F_0} \mathrm{M}_3$ in the obvious way, give exactly the invariants (8.8) on $\mathfrak{u}_1$, relative to any special embedding of $\mathfrak{u}_1$ in the sense of §2.3. From now on we realize the quotient map $\pi_{\mathfrak{u}_1} \to \mathfrak{b}$ via these invariants.

11.2. **Coordinates on U_0 and $\mathfrak{u}_0$.** In parallel with §8.1, we now describe the unitary group U_0 and its Lie algebra $\mathfrak{u}_0$ in terms of explicit coordinates. Define the F/F_0-hermitian matrices

$$J_0^\flat := \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \in \mathrm{M}_2(F) \quad \text{and} \quad J_0 := \begin{bmatrix} J_0^\flat & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{M}_3(F).$$

Then $J_0^\flat$ and J_0 determine split F/F_0-hermitian spaces of dimensions 2 and 3, respectively. We take

$$H_0 = \mathrm{U}(J_0^\flat) \quad \text{and} \quad U_0 = \mathrm{U}(J_0).$$

Explicitly,

$$U_0(F_0) = \big\{\, g \in \mathrm{M}_3(F) \bigm| gg^\dagger = 1 \,\big\},$$

where the adjoint $g^\dagger = J_0^{-1}\,{}^t\overline{g}J_0$ is given in coordinates by

$$\begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ c_1 & c_2 & d \end{bmatrix}^\dagger = \begin{bmatrix} \overline{a}_4 & -\overline{a}_2 & -\pi^{-1}\overline{c}_2 \\ -\overline{a}_3 & \overline{a}_1 & \pi^{-1}\overline{c}_1 \\ -\pi\overline{b}_2 & \pi\overline{b}_1 & \overline{d} \end{bmatrix}. \tag{11.2}$$

Of course, H_0 is described similarly in terms of $J_0^\flat$, and it embeds in U_0, via the rule $h \mapsto \mathrm{diag}(h, 1)$, as the stabilizer of the special vector $e = (0, 0, 1)$. We also note that under these coordinates, the tautological embedding of SL_2 into $\mathrm{Res}_{F/F_0} \mathrm{M}_2$ identifies

$$\mathrm{SL}_2 \xrightarrow{\sim} \mathrm{SU}(J_0^\flat). \tag{11.3}$$

The Lie algebra $\mathfrak{u}_0$ is given in these coordinates by

$$\begin{aligned} \mathfrak{u}_0(F_0) &= \big\{\, x \in \mathrm{M}_3(F) \bigm| x + x^\dagger = 0 \,\big\} \\ &= \left\{ \begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & -\overline{a}_1 & b_2 \\ \pi\overline{b}_2 & -\pi\overline{b}_1 & d \end{bmatrix} \,\middle|\, a_1, b_1, b_2 \in F,\ a_2, a_3 \in F_0,\ d \in F^{\mathrm{tr}=0} \right\}. \end{aligned}$$

Recall that our formulation of the AT conjecture in §5 involved the choice of a π-modular lattice $\Lambda_0^\flat \subset W_0^\flat = F^2$. We now take for $\Lambda_0^\flat$ the standard lattice $O_F^2 \subset F^2$, which is indeed π-modular for $J_0^\flat$. As in (5.3), we then take $\Lambda_0 = O_F^3 \subset F^3$, and K_0 and $\mathfrak{k}_0$ are the respective subgroups of $U_0(F_0)$ and $\mathfrak{u}_0(F_0)$ stabilizing Λ_0:

$$K_0 = U_0(F_0) \cap \mathrm{M}_3(O_F) = \big\{\, g \in \mathrm{M}_3(O_F) \bigm| gg^\dagger = 1 \,\big\}$$

and

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{u}_0(F_0) \cap \mathrm{M}_3(O_F) \\ &= \left\{ \begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & -\overline{a}_1 & b_2 \\ \overline{b}_2\pi & -\overline{b}_1\pi & d \end{bmatrix} \,\middle|\, a_1, b_1, b_2 \in O_F,\ a_2, a_3 \in O_{F_0},\ d \in O_F^{\mathrm{tr}=0} \right\}. \end{aligned}$$

We also set

$$\mathfrak{k}_{0,\mathrm{rs}} := \mathfrak{k}_0 \cap \mathfrak{u}_{0,\mathrm{rs}}(F_0).$$

Given a point x in $\mathfrak{u}_0$, write x in the block form

$$x = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}.$$

We realize the categorical quotient of $\mathfrak{u}_0$ by H_0 by taking the same invariants as in the previous subsection,

$$\begin{aligned} \pi_{\mathfrak{u}_0} \colon \mathfrak{u}_0 &\longrightarrow \mathfrak{b} = \mathbb{A} \times \mathbb{A} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \times \mathfrak{s}_1 \\ x &\longmapsto \big(\lambda(x), u(x), w(x), \mathrm{tr}\, A, d\big), \end{aligned} \tag{11.4}$$

where λ, u, and w are as defined in (11.1).

As in the cases of $\mathfrak{u}_1$ and $\mathfrak{s}$, we say that x is *reduced* if $\mathrm{tr}\, A = d = 0$. We write $\mathfrak{u}_{0,\mathrm{red}}$ for the closed subscheme of reduced points in $\mathfrak{u}_0$. In terms of explicit coordinates,

$$\mathfrak{u}_{0,\mathrm{red}}(F_0) = \left\{ \begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & -a_1 & b_2 \\ \overline{b}_2\pi & -\overline{b}_1\pi & 0 \end{bmatrix} \,\middle|\, a_1, a_2, a_3 \in F_0,\ b_1, b_2 \in F \right\}. \tag{11.5}$$

As in previous cases, there is a natural map $\mathfrak{u}_0 \to \mathfrak{u}_{0,\mathrm{red}}$, $x \mapsto x_{\mathrm{red}}$, sending

$$\begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \longmapsto \begin{bmatrix} A - \frac{1}{2}(\operatorname{tr} A)\cdot 1_2 & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix},$$

and this gives rise to an H_0-equivariant product decomposition

$$\begin{aligned} \mathfrak{u}_0 &\xrightarrow{\ \sim\ } \mathfrak{u}_{0,\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \\ x &\longmapsto (x_{\mathrm{red}}, \operatorname{tr} A, d). \end{aligned} \tag{11.6}$$

Just as in Lemma 8.2 for $\mathfrak{u}_1$ and Lemma 10.10 for $\mathfrak{s}$, an element $x \in \mathfrak{u}_0(F_0)$ is regular semi-simple if and only if x_{red} is. We set

$$\mathfrak{u}_{0,\mathrm{red,rs}} := \mathfrak{u}_{0,\mathrm{red}} \cap \mathfrak{u}_{0,\mathrm{rs}}, \quad \mathfrak{k}_{0,\mathrm{red}} := \mathfrak{k}_0 \cap \mathfrak{u}_{0,\mathrm{red}}(F_0), \quad \text{and} \quad \mathfrak{k}_{0,\mathrm{red,rs}} := \mathfrak{k}_{0,\mathrm{red}} \cap \mathfrak{k}_{0,\mathrm{rs}}.$$

11.3. **Integral Cayley transform on $\mathfrak{u}_0$.** In this subsection we prove an analog for $\mathfrak{u}_0$ of Lemma 8.4, which pertained to the Cayley transform on $\mathfrak{u}_1$. Let $\xi \in S^1$. Recall from (10.3) the open subscheme $\mathfrak{u}_0^\circ \subset \mathfrak{u}_0$, which is the locus where the Cayley transform $\mathfrak{c}_\xi$ is defined, and recall from Lemma 10.5 its image $U_{0,\xi}^\circ \subset U_0$. Define the sets of F_0-rational points

$$\mathfrak{u}_0^{\circ\circ} := \{\, x \in \mathfrak{u}_0^\circ(F_0) \mid \det(1-x) \in O_F^\times \,\} \quad \text{and} \quad U_{0,\xi}^{\circ\circ} := \{\, g \in U_{0,\xi}^\circ(F_0) \mid \det(\xi + g) \in O_F^\times \,\}.$$

It is trivial to verify that $\mathfrak{c}_\xi$ carries $\mathfrak{u}_0^{\circ\circ}$ isomorphically onto $U_{0,\xi}^{\circ\circ}$, and we then have the following.

Lemma 11.1 (Cayley transform for $\mathfrak{k}_0$)**.** *For any $\xi \in S^1$, the restriction of the Cayley map to $\mathfrak{k}_0 \cap \mathfrak{u}_0^{\circ\circ}$ induces an isomorphism*

$$\begin{aligned} \mathfrak{c}_\xi \colon\ \mathfrak{k}_0 \cap \mathfrak{u}_0^{\circ\circ} &\xrightarrow{\ \sim\ } K_0 \cap U_{0,\xi}^{\circ\circ} \\ x &\longmapsto \xi\frac{1+x}{1-x}. \end{aligned}$$

Furthermore, the sets $K_0 \cap U_{0,\xi}^{\circ\circ}$, as ξ varies over the four elements $\operatorname{diag}(\pm 1_2, \pm 1)$*, cover K_0.*

Proof. Clearly $\mathfrak{c}_\xi(\mathfrak{k}_0 \cap \mathfrak{u}_0^{\circ\circ}) \subset K_0 \cap U_{0,\xi}^{\circ\circ}$, and it is also clear from the inverse formula

$$\mathfrak{c}_\xi^{-1}(g) = \frac{\xi^{-1}g - 1}{\xi^{-1}g + 1}$$

that $\mathfrak{c}_\xi^{-1}(K_0 \cap U_{0,\xi}^{\circ\circ}) \subset \mathfrak{k}_0 \cap \mathfrak{u}_0^{\circ\circ}$. This proves the first assertion. It remains to show that if $g \in K_0$, then $\det(\xi + g) \in O_F^\times$ for some $\xi = \operatorname{diag}(\pm 1_2, \pm 1)$. In terms of the coordinates in the previous subsection, write g in the block form

$$g = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}.$$

Since $g \in K_0 = U_0(F_0) \cap \mathrm{GL}_3(O_F)$, we may reduce the entries of g mod π. Since $g^\dagger = g^{-1}$ also has integral entries, (11.2) shows that that g_k is of the form

$$g_k = \begin{bmatrix} A_k & \mathbf{b}_k \\ 0 & \pm 1 \end{bmatrix},$$

where the subscript k everywhere denotes reduction mod π. Since K_0 has symplectic reduction in the sense of Remark 5.2, $A_k \in \mathrm{Sp}_2(k) = \mathrm{SL}_2(k)$. Hence $A_k + 1_2$ or $A_k - 1_2$ is invertible, since otherwise A_k has eigenvalues 1 and -1, contradicting $A_k \in \mathrm{SL}_2(k)$. The lemma follows. $\square$

Remark 11.2. Note that $\mathfrak{k}_0$ is not contained in $\mathfrak{u}_0^{\circ\circ}$, nor even in $\mathfrak{u}_0^\circ(F_0)$, i.e. $\mathfrak{c}_\xi$ is not defined on all of $\mathfrak{k}_0$. This differs from the situation for $\mathfrak{k}_1$. Indeed, defining $\mathfrak{u}_1^{\circ\circ} \subset \mathfrak{u}_1(F_0)$ in the obvious way (by[12] $\det(1-x) \in O_F^\times$), the proof of Lemma 8.4 shows that $\mathfrak{k}_1 \subset \mathfrak{u}_1^{\circ\circ}$.

[12]Meaning the determinant of the F-linear endomorphism $1-x$ acting on the hermitian space W_1; as before, care is required when working with the coordinates for $\mathfrak{u}_1$ in §8.

11.4. **Integral points on $\mathfrak{b}$.** We now collect some facts related to integral points on the quotient space $\mathfrak{b} = \mathbb{A} \times \mathbb{A} \times \mathfrak{s}_1 \times \mathfrak{s}_1 \times \mathfrak{s}_1$. Note that this space is naturally defined over O_{F_0}, and

$$\mathfrak{b}(O_{F_0}) = O_{F_0} \times O_{F_0} \times O_F^{\mathrm{tr}=0} \times O_F^{\mathrm{tr}=0} \times O_F^{\mathrm{tr}=0}.$$

We claim that

$$\mathfrak{b}(O_{F_0}) = \pi_{\mathfrak{u}_0}(\mathfrak{k}_0) \cup \pi_{\mathfrak{u}_1}(\mathfrak{k}_1). \tag{11.7}$$

Indeed, the reverse inclusion is obvious from the explicit form of the invariants (11.4) and (8.8) on $\mathfrak{u}_0$ and $\mathfrak{u}_1$, respectively. For the forward inclusion, we give the following more precise lemma. Recall the decomposition $\mathfrak{b}_{\mathrm{rs}}(F_0) = \mathfrak{b}_{\mathrm{rs},0} \amalg \mathfrak{b}_{\mathrm{rs},1}$ from (10.2), and for $i \in \{0,1\}$ set

$$\mathfrak{b}(O_{F_0})_{\mathrm{rs},i} := \mathfrak{b}(O_{F_0}) \cap \mathfrak{b}_{\mathrm{rs},i}, \quad \mathfrak{b}_{\mathrm{red,rs},i} := \mathfrak{b}_{\mathrm{red}}(F_0) \cap \mathfrak{b}_{\mathrm{rs},i}, \quad \mathfrak{b}(O_{F_0})_{\mathrm{red,rs},i} := \mathfrak{b}(O_{F_0}) \cap \mathfrak{b}_{\mathrm{red,rs},i}.$$

Lemma 11.3. (i) *For $i \in \{0,1\}$,*

$$\mathfrak{b}(O_{F_0})_{\mathrm{rs},i} = \pi_{\mathfrak{u}_i}(\mathfrak{k}_{i,\mathrm{rs}}).$$

(ii) *Let $x \in \mathfrak{b}(O_{F_0}) \smallsetminus \mathfrak{b}_{\mathrm{rs}}(F_0)$. Then*

(a) $x \in \pi_{\mathfrak{u}_0}(\mathfrak{k}_0)$.

(b) $x \in \pi_{\mathfrak{u}_1}(\mathfrak{k}_1)$ *unless* $x_{\mathrm{red}} = (\lambda, u, w)$ *with* $-\lambda \in F_0^{\times,2}$.

Furthermore, if $x \in \mathfrak{b}(F_0)$ lies in the closure of $\mathfrak{b}_{\mathrm{rs},1}$, then x is not exceptional in the sense of (ii)(b).

Proof. The statements are immediately reduced to the corresponding ones for the reduced sets.

To prove (i), first let $x = (\lambda, u, w) \in \mathfrak{b}(O_{F_0})_{\mathrm{red,rs},1}$. We will show the existence of an element in $\mathfrak{k}_{1,\mathrm{red,rs}}$ in terms of the explicit coordinates (8.10) whose image is x. We use the formulas (8.11) for the invariants. By Lemma 8.3, $u \neq 0$ since $x \in \mathfrak{b}_{\mathrm{rs},1}$. Choose any $b \in D$ such that $\mathrm{N}b = u/2$. Then $b \in O_D$, since $p \neq 2$. Let $\alpha'_+ := w/u$. We claim that α'_+ is integral. Indeed, if instead $|w/u| > 1$, then $|w^2| > |\lambda u^2|$, since λ is integral. Then by (8.13) and the fact that w is traceless,

$$\eta(-\Delta) = \eta\big(-(\lambda u^2 + w^2)\big) = \eta(-w^2) = 1,$$

a contradiction to Proposition 10.2. Finally choose $\alpha'_- \in D$ of norm Δ/u^2. Then the same argument shows that α'_- is integral, and (8.13) shows that this suffices to solve our problem. A similar analysis, using the coordinates in (11.5), shows that $\mathfrak{b}(O_{F_0})_{\mathrm{red,rs},0} = \pi_{\mathfrak{u}_0}(\mathfrak{k}_{0,\mathrm{red,rs}})$.

To prove (ii), part (a) follows from the explicit construction of elements in §18.1 below. Part (b) is straightforward, again using the explicit coordinates (8.10) and the formulas for the invariants (8.11). For example, the condition $-\lambda \notin F_0^{\times,2}$ holds on all of $\pi_{\mathfrak{u}_1}(\mathfrak{u}_{1,\mathrm{red}}(F_0))$ by virtue of the facts $\lambda = \mathrm{N}\alpha$ and $\alpha \in D^{\mathrm{tr}=0}$.

Finally, suppose $x = (\lambda, u, w) \in \mathfrak{b}_{\mathrm{red}}(F_0)$ lies in the closure of $\mathfrak{b}_{\mathrm{red,rs},1}$, with $\lambda \neq 0$. Then for $x' = (\lambda', u', w') \in \mathfrak{b}_{\mathrm{red,rs},1}$ sufficiently close to x, λ and λ' will lie in the same class in $F_0^\times/F_0^{\times,2}$. So $-\lambda \notin F_0^{\times,2}$ by the fact just cited. $\square$

Remark 11.4. Although $\mathfrak{s}$ is also naturally defined over O_{F_0}, the map $\mathfrak{s}(O_{F_0}) \to \mathfrak{b}(O_{F_0})$ is not a surjection, in contrast to the map $\mathfrak{s}(F_0) \twoheadrightarrow \mathfrak{b}(F_0)$ on F_0-rational points.

For the next statement, note that the function $x \mapsto \det(1-x)$ descends from each of $\mathfrak{s}$, $\mathfrak{u}_0$, and $\mathfrak{u}_1$ to a common function on $\mathfrak{b}$, and define

$$\mathfrak{b}^{\circ\circ} := \big\{\, x \in \mathfrak{b}^\circ(F_0) \;\big|\; \det(1-x) \in O_F^\times \,\big\}.$$

Lemma 11.5. *Let $x \in \mathfrak{b}(O_{F_0})$, and suppose that x lies in the closure of $\mathfrak{b}_{\mathrm{rs},1}$. Then $x \in \mathfrak{b}^{\circ\circ}$.*

Proof. By Lemma 11.3, $x \in \pi_{\mathfrak{u}_1}(\mathfrak{k}_1)$. The conclusion then follows from Remark 11.2. $\square$

Now recall our general discussion of the Cayley transform from §10.2, and that we regard B as the common categorical quotient of S, U_0, and U_1 via Lemma 10.7.

Lemma 11.6. *There is an inclusion of subsets of $B(F_0)$,*

$$\pi_{U_1}(K_1) \smallsetminus B_{\mathrm{rs}}(F_0) \subset \pi_{U_0}(K_0).$$

Proof. Suppose that $x^\natural \in \pi_{U_1}(K_1)$ is not regular semi-simple. By Lemma 8.4, we may choose $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$ such that $x := \mathfrak{c}_\xi^{-1}(x^\natural)$ is defined and contained in $\pi_{\mathfrak{u}_1}(\mathfrak{k}_1) \subset \mathfrak{b}(O_{F_0})$. Then $x \in \pi_{\mathfrak{u}_0}(\mathfrak{k}_0)$ by Lemma 11.3(ii), and $x \in \mathfrak{b}^{\circ\circ}$ by Lemma 11.5. Hence $x^\natural = \mathfrak{c}_\xi(x) \in K_0$ by Lemma 11.1. $\square$

11.5. **Intersection numbers as a function on the quotient.** In this subsection we consider intersection numbers as a function on the categorical quotient in both the group and Lie algebra settings. Recall the decomposition $B_{\mathrm{rs}}(F_0) = B_{\mathrm{rs},0} \amalg B_{\mathrm{rs},1}$ from (10.9). For any odd n, by Remark 4.4 the function $g \mapsto \mathrm{Int}(g)$ on $U_1(F_0)_{\mathrm{rs}}$ descends to a function on $B_{\mathrm{rs},1}$. We extend it by zero to $B_{\mathrm{rs},0}$, and still denote by Int the resulting function on $B_{\mathrm{rs}}(F_0)$. By Remark 4.6, the same applies to the function $x \mapsto \ell\text{-}\mathrm{Int}(x)$ on $\mathfrak{u}_{1,\mathrm{rs}}$, which we consider as a function on $\mathfrak{b}_{\mathrm{rs}}(F_0)$. Note that, by Remark 4.5 and Remark 4.7 respectively, or by Lemma 8.8 and Proposition 8.10 when $n = 3$, these are finite-valued functions.

Proposition 11.7. *Let $n = 3$. The function* Int *belongs to $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$; similarly, the function* ℓ-Int *belongs to $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$.*

Proof. To prove that the closure of the support of Int is compact, it suffices to prove that the closure inside $U_1(F_0)$ of the support of the function $g \mapsto \mathrm{Int}(g)$ on $U_1(F_0)_{\mathrm{rs}}$ is compact. But by Lemma 8.8, this support is contained in the compact subgroup K_1, and hence the assertion is clear. A similar argument applies to ℓ-Int.

Now we prove that the function ℓ-Int on $\mathfrak{b}_{\mathrm{rs},1}$ is locally constant. By Corollary 8.11, this assertion follows from the corresponding statement for the function ℓ-Int on $\mathfrak{b}_{\mathrm{red},\mathrm{rs},1}$. But this follows in turn from the expressions for this function in §9.2 in terms of the quantities ℓ_-, ℓ_+, m or, equivalently by (9.4), in terms of the functions u, w, Δ on $\mathfrak{b}_{\mathrm{red}}(F_0)$, all of which are locally constant on $\mathfrak{b}_{\mathrm{red},\mathrm{rs}}(F_0)$.

The fact that Int is locally constant on $B_{\mathrm{rs},1}$ now follows from Lemma 8.4, Corollary 8.11, and the fact that the Cayley transform $\mathfrak{c}_\xi$, for any $\xi \in S^1$, is a local homeomorphism (on its domain of definition). $\square$

Remark 11.8. We conjecture that the preceding assertion regarding the function Int on $B_{\mathrm{rs}}(F_0)$ continues to hold for arbitrary odd n.

11.6. **The reduction step.** In this subsection we complete the proof that Theorem 5.21 follows from Theorem 5.22. We begin by noting the following compatibility between transfer factors under the Cayley transform for $\mathfrak{s}$, which holds for all odd n. Set $\mathfrak{s}_{\mathrm{rs}}^\circ := \mathfrak{s}_{\mathrm{rs}} \cap \mathfrak{s}^\circ$.

Lemma 11.9 ([35, Lem. 3.5][13]). *Let n be odd. Then for any $y \in \mathfrak{s}_{\mathrm{rs}}^\circ(F_0)$ and $\xi \in S^1$,*

$$\omega\bigl(\mathfrak{c}_\xi(y)\bigr) = \widetilde{\eta}\bigl(2^{n(n-1)/2}\bigr)\omega(y).$$

Proof. Let $\gamma := \mathfrak{c}_\xi(y) \in S_{\mathrm{rs}}(F_0)$. Write $\xi = \mathrm{diag}(\xi_1 \cdot 1_{n-1}, \xi_2)$, and choose $\nu = \mathrm{diag}(\nu_1 \cdot 1_{n-1}, \nu_2)$ with $\overline{\nu}\nu^{-1} = \xi$ as in the definition (10.5) of $\mathfrak{c}_\xi$. By the definition (5.5) of ω on $S_{\mathrm{rs}}(F_0)$,

$$\begin{aligned}\omega(\gamma) &= \widetilde{\eta}\bigl(\det(\gamma)^{-(n-1)/2}\det(\gamma^i e)_{i=0,\dots,n-1}\bigr)\\ &= \widetilde{\eta}\Bigl(\det\bigl(\overline{\nu}\mathfrak{c}_{1_n}(y)\nu^{-1}\bigr)^{-(n-1)/2}(\nu_1/\nu_2)^{n-1}\xi_1^{n(n-1)/2}\det(\mathfrak{c}_{1_n}(y)^i e)_{i=0,\dots,n-1}\Bigr) \quad \text{(by (10.7))}\\ &= \widetilde{\eta}\bigl((\xi_1/\xi_2)^{(n-1)/2}(\nu_1/\nu_2)^{n-1}\bigr)\omega\bigl(\mathfrak{c}_{1_n}(y)\bigr).\end{aligned}$$

Since $\xi_1\nu_1^2 = \overline{\nu}_1\nu_1$ and $\xi_2\nu_2^2 = \overline{\nu}_2\nu_2$ are both norms, we conclude that $\omega(\gamma) = \omega(\mathfrak{c}_{1_n}(y))$. Therefore we reduce to the case $\xi = 1$. Then

$$\gamma = (1+y)(1-y)^{-1} = -1 + T,$$

[13] The proof of loc. cit. contains some miscalculations and should be corrected accordingly. This does not affect the results in loc. cit.

where we set $T := 2(1-y)^{-1}$. We compute

$$\begin{aligned}\det(\gamma^i e)_{i=0,1,\dots,n-1} &= \det\bigl((-1+T)^i e\bigr)_{i=0,1,\dots,n-1}\\ &= \det(T^i e)_{i=0,1,\dots,n-1}\\ &= 2^{n(n-1)/2}\det(1-y)^{1-n}\det\bigl((1-y)^i e\bigr)_{i=n-1,n-2,\dots,0}\\ &= 2^{n(n-1)/2}\det(1-y)^{1-n}\det(y^i e)_{i=0,1,\dots,n-1}.\end{aligned}$$

Note that $1-y = 1+\overline{y}$, since $y \in \mathfrak{s}(F_0)$. Hence $\det(1+y)\det(1-y) \in \mathrm{N}F^\times$, and

$$\widetilde{\eta}\bigl(\det(1+y)\det(1-y)\bigr) = 1.$$

Recalling the definition (5.12) of ω on $\mathfrak{s}_{\mathrm{rs}}(F_0)$, we conclude that

$$\begin{aligned}\omega(\gamma) &= \widetilde{\eta}\Bigl(\det\bigl((1+y)(1-y)^{-1}\bigr)^{-\frac{n-1}{2}}2^{n(n-1)/2}\det(1-y)^{1-n}\det(y^i e)_{i=0,1,\dots,n-1}\Bigr)\\ &= \widetilde{\eta}\bigl(2^{n(n-1)/2}\bigr)\widetilde{\eta}\bigl(\det(1+y)^{-\frac{n-1}{2}}\det(1-y)^{-\frac{n-1}{2}}\bigr)\omega(y)\\ &= \widetilde{\eta}\bigl(2^{n(n-1)/2}\bigr)\omega(y).\end{aligned}$$ □

Now we return to $n=3$. The main part in the reduction step is given by the following lemma. Recall from Lemma 10.3 that if a function $\phi' \in C_c^\infty(\mathfrak{s})$ has vanishing orbital integrals $\mathrm{Orb}(y,\phi') = 0$ for all $y \in \mathfrak{s}_{\mathrm{rs},1}$, then the function $y \mapsto \omega(y)\,\partial\mathrm{Orb}(y,\phi')$ on $\mathfrak{s}_{\mathrm{rs},1}$ descends to a function on $\mathfrak{b}_{\mathrm{rs},1}$ which, when extended by zero to $\mathfrak{b}_{\mathrm{rs},0}$, lies in $C_{\mathrm{rc}}^\infty(\mathfrak{b}_{\mathrm{rs}})$; and recall from Lemma 10.9 that if $f' \in C_c^\infty(S)$ is such that $\mathrm{Orb}(\gamma, f') = 0$ for all $\gamma \in S_{\mathrm{rs},1}$, then the function $\gamma \mapsto \omega(\gamma)\,\partial\mathrm{Orb}(\gamma,f')$ on $S_{\mathrm{rs},1}$ analogously descends to an element of $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$.

Lemma 11.10. *Let $f' \in C_c^\infty(S)$ transfer to $(\mathbf{1}_{K_0}, 0) \in C_c^\infty(U_0)\times C_c^\infty(U_1)$, and let $\phi' \in C_c^\infty(\mathfrak{s})$ transfer to $(\mathbf{1}_{\mathfrak{k}_0}, 0) \in C_c^\infty(\mathfrak{u}_0)\times C_c^\infty(\mathfrak{u}_1)$. Fix $\xi \in S^1$. Let $x_0 \in \mathfrak{b}^\circ(F_0)$ be an element with the property that if $x_0 \notin \mathfrak{b}(O_{F_0})$, then $\mathfrak{c}_\xi(x_0) \notin \pi_{U_0}(K_0)$. Then the difference function*

$$x \longmapsto \begin{cases}\omega\bigl(\mathfrak{c}_\xi(y)\bigr)\,\partial\mathrm{Orb}\bigl(\mathfrak{c}_\xi(y), f'\bigr) - \omega(y)\,\partial\mathrm{Orb}(y,\phi'), & x = \pi_{\mathfrak{s}}(y) \in \mathfrak{b}^\circ_{\mathrm{rs},1};\\ 0, & x \in \mathfrak{b}_{\mathrm{rs},0}\end{cases}$$

is locally around x_0 an orbital integral function.[14]

Proof. Let $f := \mathbf{1}_{K_0}$ and $\phi := \mathbf{1}_{\mathfrak{k}_0}$. We may assume that x_0 lies in the closure of $\mathfrak{b}_{\mathrm{rs},1}$. Choose an open and compact (hence closed) neighborhood V_{x_0} of x_0 contained in $\mathfrak{b}^\circ(F_0)$. Consider the functions

$$\phi'_{x_0} := \phi'\cdot\mathbf{1}_{\pi_{\mathfrak{s}}^{-1}(V_{x_0})} \in C_c^\infty(\mathfrak{s}) \quad\text{and}\quad \phi_{x_0} := \phi\cdot\mathbf{1}_{\pi_{\mathfrak{u}_0}^{-1}(V_{x_0})} \in C_c^\infty(\mathfrak{u}_0),$$

and similarly

$$f'_{x_0} := f'\cdot\mathbf{1}_{\pi_S^{-1}(\mathfrak{c}_\xi(V_{x_0}))} \in C_c^\infty(S) \quad\text{and}\quad f_{x_0} := f\cdot\mathbf{1}_{\pi_{U_0}^{-1}(\mathfrak{c}_\xi(V_{x_0}))} \in C_c^\infty(U_0).$$

Then $\phi'_{x_0}(y) = \phi'(y)$ for all $y \in \pi_{\mathfrak{s}}^{-1}(V_{x_0})$, so that $\mathrm{Orb}(y,\phi'_{x_0},s) = \mathrm{Orb}(y,\phi',s)$ for all such y; and similarly for f'_{x_0} and f'. The assertion of the theorem is therefore reduced to the same assertion for f'_{x_0} and ϕ'_{x_0}.

We claim that, after possibly shrinking V_{x_0}, we have with respect to the Cayley transform $\mathfrak{c}_\xi\colon \mathfrak{u}_0^\circ \to U_0$,

$$\phi_{x_0} = \mathfrak{c}_\xi^*(f_{x_0}) \tag{11.8}$$

(where the right-hand side is extended by 0 from $\mathfrak{u}_0^\circ(F_0)$ to $\mathfrak{u}_0(F_0)$). Indeed, first suppose that $x_0 \in \mathfrak{b}(O_{F_0})$ is integral. We have

$$\operatorname{supp}\phi_{x_0} = \mathfrak{k}_0 \cap \pi_{\mathfrak{u}_0}^{-1}(V_{x_0}) \quad\text{and}\quad \operatorname{supp} f_{x_0} = K_0 \cap \pi_{U_0}^{-1}\bigl(\mathfrak{c}_\xi(V_{x_0})\bigr).$$

Since x_0 lies in the closure of $\mathfrak{b}_{\mathrm{rs},1}$, we have $x_0 \in \mathfrak{b}^{\circ\circ}$ by Lemma 11.5. Shrinking V_{x_0} if necessary, we may therefore assume that $V_{x_0} \subset \mathfrak{b}(O_{F_0}) \cap \mathfrak{b}^{\circ\circ}$. Then by Lemma 11.1, $\mathfrak{c}_\xi$ carries the left-hand set in the display isomorphically onto the right-hand set, which proves (11.8).

If x_0 is not integral, then by hypothesis $\mathfrak{c}_\xi(x_0) \notin \pi_{U_0}(K_0)$. Hence, after possibly shrinking V_{x_0}, both functions ϕ_{x_0} and f_{x_0} vanish identically, so that again identity (11.8) is satisfied.

[14] Note that this function is not defined on all of $\mathfrak{b}_{\mathrm{rs}}(F_0)$, but this raises no issue since the conclusion concerns only the local behavior of the function near the point $x_0 \in \mathfrak{b}^\circ(F_0)$.

By (11.8), for all $y \in \mathfrak{s}_{\mathrm{rs}}^\circ(F_0)$ matched with an element $x \in \mathfrak{u}_0^\circ(F_0)$, we have

$$\omega(y)\operatorname{Orb}(y, \phi'_{x_0}) = \operatorname{Orb}(x, \phi_{x_0}) = \operatorname{Orb}\bigl(\mathfrak{c}_\xi(x), f_{x_0}\bigr) = \omega\bigl(\mathfrak{c}_\xi(y)\bigr)\operatorname{Orb}\bigl(\mathfrak{c}_\xi(y), f'_{x_0}\bigr).$$

By Lemma 11.9, we have $\omega(\mathfrak{c}_\xi(y)) = c \cdot \omega(y)$ for the constant $c := \widetilde{\eta}(2^{n(n-1)/2})$. Hence the difference function $c \cdot \mathfrak{c}_\xi^*(f'_{x_0}) - \phi'_{x_0}$ (viewed as an element in $C_c^\infty(\mathfrak{s})$) has identically vanishing orbital integrals on $\mathfrak{s}_{\mathrm{rs},0}$. The same trivially holds on $\mathfrak{s}_{\mathrm{rs},1}$, since ϕ'_{x_0} transfers to $(\phi_{x_0}, 0)$ and f'_{x_0} transfers to $(f_{x_0}, 0)$. Now the assertion follows from Corollary 11.12 below. $\square$

The proof of Corollary 11.12 is based on the following theorem, which is the $n = 3$ case of Conjecture 5.15 (see also Remarks 5.16). Recall from §5.4 that for $\phi' \in C_c^\infty(\mathfrak{s})$ and $h \in H'(F_0) = \mathrm{GL}_{n-1}(F_0)$, we define

$$ {}^h\phi'(y) = \phi'(h^{-1}yh) \quad \text{and} \quad {}^{\eta(h)h-1}\phi'(y) = \eta(h)\phi'(h^{-1}yh) - \phi'(y).$$

Theorem 11.11 (Density principle)**.** *Let $n = 3$. Let $\phi' \in C_c^\infty(\mathfrak{s})$ be such that $\operatorname{Orb}(y, \phi') = 0$ for all $y \in \mathfrak{s}_{\mathrm{rs}}(F_0)$. Then ϕ' is in the kernel of the natural projection $C_c^\infty(\mathfrak{s}) \to C_c^\infty(\mathfrak{s})_{H',\eta}$, i.e. ϕ' is a linear combination of functions of the form ${}^{\eta(h)h-1}\phi''$ for $\phi'' \in C_c^\infty(\mathfrak{s})$ and $h \in H'(F_0)$.*

Proof. By [34, Th. 1.1], the set of orbital integrals of regular semi-simple elements spans a weakly dense subspace of the space all $(H'(F_0), \eta)$-invariant distributions on $\mathfrak{s}(F_0)$.[15] Therefore if a test function $\phi' \in C_c^\infty(\mathfrak{s})$ is such that $\operatorname{Orb}(y, \phi') = 0$ for all $y \in \mathfrak{s}_{\mathrm{rs}}(F_0)$, then ϕ' is annihilated by all $(H'(F_0), \eta)$-invariant distributions on $\mathfrak{s}(F_0)$. This implies that ϕ' lies in the kernel of the natural projection $C_c^\infty(\mathfrak{s}) \to C_c^\infty(\mathfrak{s})_{H',\eta}$. $\square$

Corollary 11.12. *Let $\phi' \in C_c^\infty(\mathfrak{s})$ be such that $\operatorname{Orb}(y, \phi') = 0$ for all $y \in \mathfrak{s}_{\mathrm{rs}}(F_0)$. Then there exists a function $\phi'^\flat \in C_c^\infty(\mathfrak{s})$ such that*

$$\omega(y)\,\partial\mathrm{Orb}(y, \phi') = \omega(y)\operatorname{Orb}\bigl(y, \phi'^\flat\bigr) \quad \textit{for all} \quad y \in \mathfrak{s}_{\mathrm{rs}}(F_0).$$

In other words, $y \mapsto \omega(y)\,\partial\mathrm{Orb}(y, \phi')$ is an orbital integral function on $\mathfrak{s}_{\mathrm{rs}}(F_0)$.

Proof. By the density principle, we may assume that ϕ' is of the form ${}^{\eta(h)h-1}\phi''$ for some $\phi'' \in C_c^\infty(\mathfrak{s})$ and $h \in H'(F_0)$. By Lemma 5.12(ii),

$$\partial\mathrm{Orb}\bigl(\gamma, {}^{\eta(h)h-1}\phi''\bigr) = \log|\det h| \operatorname{Orb}(\gamma, \phi'').$$

Setting $\phi'^\flat := \log|\det h| \cdot \phi''$ completes the proof. $\square$

Remark 11.13. Note that this corollary is essentially a converse to Lemma 5.13(ii). Indeed, in Corollary 11.12 above, we are given ϕ', and are writing $\partial\mathrm{Orb}(y, \phi')$ as an orbital integral; in Lemma 5.13(ii), we are given ϕ', and are writing $\operatorname{Orb}(y, \phi')$ as a derivative of an orbital integral.

Proposition 11.14. *Theorem 5.22 implies Theorem 5.21.*

Proof. What we need to show is that Conjecture 5.10 implies Conjecture 5.6(b) when $n = 3$; Proposition 5.14, Lemma 8.8, and Proposition 8.10 then take care of the rest. Suppose that $f' \in C_c^\infty(S)$ transfers to $(\mathbf{1}_{K_0}, 0)$, and let $\phi' \in C_c^\infty(\mathfrak{s})$ be a function satisfying the conclusion of Conjecture 5.10(a).

As in §11.5, we consider Int as a function in $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$ which vanishes identically on $B_{\mathrm{rs},0}$, and ℓ-Int as a function in $C_{\mathrm{rs}}^\infty(\mathfrak{b}_{\mathrm{rs}})$ which vanishes identically on $\mathfrak{b}_{\mathrm{rs},0}$. As in Lemmas 10.9 and 10.3, respectively, we consider the function $\gamma \mapsto \omega(\gamma)\,\partial\mathrm{Orb}(\gamma, f')$ as an element in $C_{\mathrm{rc}}^\infty(B_{\mathrm{rs}})$ which vanishes identically on $B_{\mathrm{rs},0}$, and the function $y \mapsto \omega(y)\,\partial\mathrm{Orb}(y, \phi')$ as an element in $C_{\mathrm{rs}}^\infty(\mathfrak{b}_{\mathrm{rs}})$ which vanishes identically on $\mathfrak{b}_{\mathrm{rs},0}$. Our task is to prove that the sum

$$\operatorname{Int}(g) \cdot \log q + 2\omega(\gamma)\,\partial\mathrm{Orb}(\gamma, f'), \tag{11.9}$$

regarded in this way as a function on $B_{\mathrm{rs}}(F_0)$, is an orbital integral function locally around $x_0^\natural$ for all $x_0^\natural \in B(F_0)$. If $x_0^\natural$ is either in $B_{\mathrm{rs},1}$ or outside the closure of $B_{\mathrm{rs},1}$, then (11.9) is constant (identically 0 in the latter case) in a neighborhood of $x_0^\natural$, and the conclusion follows. So let us assume for the rest of the proof that $x_0^\natural$ lies in the closure of $B_{\mathrm{rs},1}$ but is not itself regular semi-simple.

[15] In [34] this is proved for $\mathfrak{s}_{\mathrm{red}}$, but it is trivial to extend the result to $\mathfrak{s}$.

By Lemma 10.5(i), we may choose $\xi \in S^1$ such that the inverse Cayley transform is defined at $x_0^\natural$. Set $x_0 := \mathfrak{c}_\xi^{-1}(x_0^\natural) \in \mathfrak{b}^\circ(F_0)$. If $x_0^\natural \in \pi_{U_0}(K_0)$, then by Lemma 11.1 and (11.7), we may furthermore choose $\xi = \mathrm{diag}(\pm 1_2, \pm 1)$ such that $x_0 \in \mathfrak{b}(O_{F_0}) \cap \mathfrak{b}^\circ(F_0)$. Note that this in fact gives us $x_0 \in \mathfrak{b}(O_{F_0}) \iff x_0^\natural \in \pi_{U_0}(K_0)$, since if $x_0 \in \mathfrak{b}(O_{F_0})$, then $x_0 \in \pi_{\mathfrak{u}_0}(\mathfrak{k}_0) \cap \mathfrak{b}^{\circ\circ}$ by Lemmas 11.3(ii) and 11.5, and hence $x_0^\natural \in \pi_{U_0}(K_0)$ by Lemma 11.1.

We claim that for all $x \in \mathfrak{b}_{\mathrm{rs}}^\circ(F_0)$ contained in a sufficiently small neighborhood of x_0,

$$\ell\text{-Int}(x) = \mathrm{Int}\big(\mathfrak{c}_\xi(x)\big). \tag{11.10}$$

Of course this holds trivially for $x \in \mathfrak{b}_{\mathrm{rs},0}$. If $x_0 \in \mathfrak{b}(O_{F_0})$, then any $x \in \mathfrak{b}_{\mathrm{rs},1}$ which is sufficiently near x_0 will be contained in $\mathfrak{b}(O_{F_0})_{\mathrm{rs},1}$, which equals $\pi_{\mathfrak{u}_1}(\mathfrak{k}_{1,\mathrm{rs}})$ by Lemma 11.3(i). Hence (11.10) holds for such x by Corollary 8.11. If $x_0 \notin \mathfrak{b}(O_{F_0})$, then $x_0^\natural \notin \pi_{U_0}(K_0)$ by our choice of ξ. Since $x_0^\natural$ is not regular semi-simple, $x_0^\natural \notin \pi_{U_1}(K_1)$ by Lemma 11.6. Hence by Lemma 8.8 both sides of (11.10) vanish for $x \in \mathfrak{b}_{\mathrm{rs},1}$ sufficiently near x_0. This proves the claim.

We conclude that for all $x \in \mathfrak{b}_{\mathrm{rs}}(F_0)$ near x_0,

$$\begin{aligned}
&\mathrm{Int}\big(\mathfrak{c}_\xi(x)\big) \cdot \log q + 2\omega\big(\mathfrak{c}_\xi(x)\big)\, \partial\mathrm{Orb}\big(\mathfrak{c}_\xi(x), f'\big) \\
&\quad = \ell\text{-Int}(x) \cdot \log q + 2\omega(x)\, \partial\mathrm{Orb}(x, \phi') + 2\Big(\omega\big(\mathfrak{c}_\xi(x)\big)\, \partial\mathrm{Orb}\big(\mathfrak{c}_\xi(x), f'\big) - \omega(x)\, \partial\mathrm{Orb}(x, \phi')\Big).
\end{aligned}$$

By Theorem 5.22 and the choice of ϕ', the first two terms on the right-hand side cancel. By Lemma 11.10, the remaining expression on the right is an orbital integral function on $\mathfrak{b}_{\mathrm{rs}}(F_0)$ locally around x_0. The proposition follows. □

12. Reduction to the reduced set

We continue to take $n = 3$. In this section we enact a further reduction step: we reduce the main Lie algebra result Theorem 5.22 to an analog for the reduced sets $\mathfrak{s}_{\mathrm{red}}(F_0)$, $\mathfrak{u}_{0,\mathrm{red}}(F_0)$, and $\mathfrak{u}_{1,\mathrm{red}}(F_0)$. Recall from (10.10), (11.6), and (8.9) that we have product decompositions[16]

$$\mathfrak{s} \cong \mathfrak{s}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1, \quad \mathfrak{u}_0 \cong \mathfrak{u}_{0,\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1, \quad \text{and} \quad \mathfrak{u}_1 \cong \mathfrak{u}_{1,\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1, \tag{12.1}$$

all of which are compatible with the categorical quotient maps to $\mathfrak{b} = \mathfrak{b}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1$. The matching relation for regular semi-simple elements in §2.3 obviously respects reducedness on both sides. Since each of the above reduced sets is stable under the group action on the ambient space, the formulas for orbital integrals in §5.3 make sense in the obvious way for reduced regular semi-simple elements and functions on the reduced set (and of course we keep the same normalizations). The transfer relation for functions then extends in the obvious way to the reduced setting: a function $\phi'_{\mathrm{red}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ and a pair $(\phi_{0,\mathrm{red}}, \phi_{1,\mathrm{red}}) \in C_c^\infty(\mathfrak{u}_{0,\mathrm{red}}) \times C_c^\infty(\mathfrak{u}_{1,\mathrm{red}})$ are *transfers* of each other if for each $i \in \{0, 1\}$ and each $x \in \mathfrak{u}_{i,\mathrm{red},\mathrm{rs}}(F_0)$,

$$\mathrm{Orb}(x, \phi_{i,\mathrm{red}}) = \omega(y)\, \mathrm{Orb}(y, \phi'_{\mathrm{red}})$$

whenever $y \in \mathfrak{s}_{\mathrm{red},\mathrm{rs}}(F_0)$ matches x. Here the transfer factor ω is the obvious one, namely the restriction of (5.12) to $\mathfrak{s}_{\mathrm{red},\mathrm{rs}}(F_0)$. We are going to reduce Theorem 5.22 to the following statement.

Theorem 12.1. *Let $n = 3$. Then for any function $\phi'_{\mathrm{red}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ which transfers to the pair $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0) \in C_c^\infty(\mathfrak{u}_{0,\mathrm{red}}) \times C_c^\infty(\mathfrak{u}_{1,\mathrm{red}})$, there exists a function $\phi'_{\mathrm{red},\mathrm{corr}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ such that for any $y \in \mathfrak{s}_{\mathrm{red},\mathrm{rs}}(F_0)$ matched with an element $x \in \mathfrak{u}_{1,\mathrm{red},\mathrm{rs}}(F_0)$,*

$$2\omega(y)\, \partial\mathrm{Orb}(y, \phi'_{\mathrm{red}}) = -\ell\text{-Int}(x) \cdot \log q + \omega(y)\, \mathrm{Orb}(y, \phi'_{\mathrm{red},\mathrm{corr}}).$$

We will obtain Theorem 12.1 as a consequence of Theorems 15.1 and 16.5 below. To see that Theorem 12.1 implies Theorem 5.22, let us first note the following straightforward lemma.

Lemma 12.2. *If $\phi'_{\mathrm{red}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ transfers to $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0)$, then the function*

$$\phi'_{\mathrm{red}} \otimes \mathbf{1}_{\mathfrak{s}_1(O_{F_0}) \times \mathfrak{s}_1(O_{F_0})} \in C_c^\infty(\mathfrak{s}) \cong C_c^\infty(\mathfrak{s}_{\mathrm{red}} \times \mathfrak{s}_1 \times \mathfrak{s}_1)$$

transfers to $(\mathbf{1}_{\mathfrak{k}_0}, 0) \in C_c^\infty(\mathfrak{u}_0) \times C_c^\infty(\mathfrak{u}_1)$. □

[16] Strictly speaking, (8.9) only gives the product decomposition for $\mathfrak{u}_1$ on the level of F_0-rational points, but this obviously extends to an isomorphism of schemes.

We also record the following fact about the transfer factor ω, which holds for any odd n. It is an immediate consequence of the equation (10.11).

Lemma 12.3. *Let n be odd. Then for any $y \in \mathfrak{s}_{\mathrm{rs}}(F_0)$,*

$$\omega(y) = \omega(y_{\mathrm{red}}). \qquad \square$$

Now we return to the case $n = 3$.

Proposition 12.4. *Theorem 12.1 implies Theorem 5.22.*

Proof. The setup is parallel to the proof of Proposition 11.14: on account of Proposition 5.14, Lemma 8.8, and Proposition 8.10, what we need to show is that Theorem 12.1 implies Conjecture 5.10(b) when $n = 3$. Suppose that $\phi' \in C_c^\infty(\mathfrak{s})$ transfers to $(\mathbf{1}_{\mathfrak{k}_0}, 0)$. Then $\phi'_{\mathrm{red}} := \phi'|_{\mathfrak{s}_{\mathrm{red}}(F_0)}$ transfers to $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0)$. Let $\phi'_{\mathrm{red,corr}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ be a function which satisfies the conclusion of Theorem 12.1 for ϕ'_{red}. Set

$$\phi'' := \phi'_{\mathrm{red}} \otimes \mathbf{1}_{\mathfrak{s}_1^2(O_{F_0})} \quad \text{and} \quad \phi''_{\mathrm{corr}} := \phi'_{\mathrm{red,corr}} \otimes \mathbf{1}_{\mathfrak{s}_1^2(O_{F_0})}.$$

We claim that for any $y \in \mathfrak{s}(F_0)_{\mathrm{rs}}$ matched with an element $x \in \mathfrak{u}_1(F_0)_{\mathrm{rs}}$,

$$2\omega(y)\, \partial\mathrm{Orb}(y, \phi'') = -\ell\text{-}\mathrm{Int}(x) \cdot \log q + \omega(y)\, \mathrm{Orb}(y, \phi''_{\mathrm{corr}}). \tag{12.2}$$

Before proving the claim, let us explain how it implies the proposition. By Lemma 12.2, ϕ'' transfers to $(\mathbf{1}_{\mathfrak{k}_0}, 0)$. Hence $\phi' - \phi''$ has vanishing orbital integrals at all regular semi-simple elements. Hence the conclusion of Conjecture 5.10(b) for ϕ' follows from the claim and from Corollary 11.12.

Now we prove the claim. Since the matching elements y and x have the same invariants, their last two components with respect to the respective product decompositions in (12.1) are the same. If either of these two common components does not lie in $\mathfrak{s}_1(O_{F_0})$, then ϕ'' and ϕ''_{corr} vanish identically on the $H'(F_0)$-orbit of y, and $x \notin \mathfrak{k}_1$. Hence, using Lemma 8.8 for the ℓ-Int term, every term in (12.2) vanishes, and the claim holds trivially.

Now suppose that the last two components of y and x do lie in $\mathfrak{s}_1(O_{F_0})$. Then

$$\partial\mathrm{Orb}(y, \phi'') = \partial\mathrm{Orb}(y_{\mathrm{red}}, \phi'_{\mathrm{red}}) \quad \text{and} \quad \mathrm{Orb}(y, \phi''_{\mathrm{corr}}) = \mathrm{Orb}(y_{\mathrm{red}}, \phi'_{\mathrm{red,corr}}).$$

Furthermore, in this case $x \in \mathfrak{k}_1$ if and only if $x_{\mathrm{red}} \in \mathfrak{k}_1$. Hence by Lemma 8.8 and Corollary 8.11,

$$\ell\text{-}\mathrm{Int}(x) = \ell\text{-}\mathrm{Int}(x_{\mathrm{red}}).$$

Since y_{red} and x_{red} match, and since $\omega(y) = \omega(y_{\mathrm{red}})$ by Lemma 12.3, we conclude that (12.2) holds because $\phi'_{\mathrm{red,corr}}$ satisfies the conclusion of Theorem 12.1 for the function ϕ'_{red}. This completes the proof. $\square$

Remark 12.5. We have formulated the notion of transfer above with respect to the particular transfer factor ω, but this notion of course make sense relative to any transfer factor, as in Definition 5.1 in the homogeneous group setting. In particular, for $c \in \mathbb{C}^\times$, note that a function $\phi'_{\mathrm{red}} \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ transfers to $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0)$ with respect to ω if and only if the function $c^{-1}\phi'_{\mathrm{red}}$ transfers to $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0)$ with respect to the transfer factor $c\omega$. It is easy to see from this that the truth of Theorem 12.1 is unaffected when we replace ω by a nonzero constant multiple.

13. Application of the germ expansion principle

In the rest of Part 3 of the paper, we suppress the subscript in the notation ϕ'_{red}, and simply call this function ϕ'. Also, from now on we systematically abuse notation and suppress the expression (F_0) when referring to sets of F_0-rational points, so that $\mathfrak{b}$ means $\mathfrak{b}(F_0)$, $\mathfrak{u}_i$ means $\mathfrak{u}_i(F_0)$, etc.

In Part 4 we prove a germ expansion of orbital integrals around each element $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$. The rough form of this germ expansion is as in Theorem 16.1. Taking the derivative at $s = 0$ of both sides, we obtain a sum decomposition as in (16.3),

$$\omega\big(\sigma(x)\big)\, \partial\mathrm{Orb}\big(\sigma(x), \phi'\big) = \omega\big(\sigma(x)\big)\, \partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) + \omega\big(\sigma(x)\big)\, \partial\mathrm{Orb}_2\big(\sigma(x), \phi'\big).$$

Here $\sigma(x)$ is some explicit element in $\mathfrak{s}_{\mathrm{red}}$ above $x \in \mathfrak{b}_{\mathrm{red,rs}}$. We have

$$\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) = \sum_{n \in \pi_{\mathrm{red}}^{-1}(x_0)/H'} \partial\Gamma_n(x)\,\mathrm{Orb}(n, \phi', 0), \tag{13.1}$$

where

$$\partial\Gamma_n(x) := \frac{d}{ds}\Big|_{s=0} \Gamma_n(x, s).$$

The explicit forms of the germ functions $\Gamma_n(x, s)$, and the definitions of the orbital integrals $\mathrm{Orb}(n, \phi', s)$ associated to elements which are not regular semi-simple, are given in Part 4.

Now we assume that ϕ' transfers to (ϕ_0, ϕ_1) where $\phi_i \in C_c^\infty(\mathfrak{u}_{i,\mathrm{red}})$. Throughout the rest of Part 3, we will assume that $\phi_1 = 0$. We will be concerned with the function on $\mathfrak{b}_{\mathrm{red,rs}}$,

$$x \longmapsto \begin{cases} \partial\mathrm{Orb}(\sigma(x), \phi'), & x \in \mathfrak{b}_{\mathrm{red,rs},1}; \\ 0, & x \in \mathfrak{b}_{\mathrm{red,rs},0}. \end{cases}$$

By Theorem 16.5, the term $\omega(\sigma(x))\,\partial\mathrm{Orb}_2(\sigma(x), \phi')$ is an orbital integral function. We will calculate $\partial\mathrm{Orb}_1(\sigma(x), \phi')$. To use the formula (13.1), we need to calculate $\partial\Gamma_n(x)$ and $\mathrm{Orb}(n, \phi', 0)$ for all $n \in \pi_{\mathrm{red}}^{-1}(x_0)/H'$. Their values are summarized by the following table. Let us explain some of the notation we use.

(1) Here $F' = F_0[X]/(X^2 + \lambda_0)$ is a quadratic F_0-algebra for $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}}$.
(2) $\Delta = \Delta(x)$ for $x \in \mathfrak{b}_{\mathrm{red,rs}}$.
(3) The ♣ values are omitted since the corresponding values of $\mathrm{Orb}(n, \phi')$ vanish in our case.
(4) The ♠ values are not needed in our case since we will only need those $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ which are also in the closure of $\mathfrak{b}_{\mathrm{red,rs},1}$ (cf. Lemma 11.3).

Element $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$	Orbit representative $n \in \mathfrak{s}_{\mathrm{red}}$ over x_0	Reference for orbit representative	Value of $\partial\Gamma_n(x)$	Reference for $\mathrm{Orb}(n, \phi', 0)$
0	$n(\mu)$, $\mu \in F_0$ $n_{0,+}$ $n_{0,-}$	(17.1) (17.2)	Theorem 17.1 0 $\log\lvert\Delta/\varpi\rvert$	Lemma 14.2 Lemma 14.1
$(\lambda_0, 0, 0)$ for $\lambda_0 \neq 0$ and $F' \not\simeq F, F_0 \times F_0$	y_+ y_-	(18.2)	0 $\eta(-\lambda_0)\log\lvert\Delta/\lambda_0\rvert$	Lemma 14.7(a)
$(\lambda_0, 0, 0)$ for $\lambda_0 \neq 0$ and $F' \simeq F_0 \times F_0$	y_0 $y_\pm$	(18.1) (18.2)	Corollary 18.4	♠ ♠
$(\lambda_0, 0, 0)$ for $\lambda_0 \neq 0$ and $F' \simeq F$	y_{++} y_{+-} y_{--} y_{-+}	(18.3) (18.4)	0 ♣ ♣ $\eta(-1)\log\lvert\Delta/\lambda_0\rvert$	Lemma 14.7(b)
(λ_0, u_0, w_0) for $u_0 \neq 0$	y_+ y_-	(18.5)	0 $\log\big\lvert\Delta/(u_0^2\varpi)\big\rvert$	Lemma 14.8

14. The germ expansion of a function with special transfer

Throughout this section, we fix a function $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ with transfer $(\mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}}, 0)$. We set

$$\phi := \mathbf{1}_{\mathfrak{k}_{0,\mathrm{red}}} \in C_c^\infty(\mathfrak{u}_{0,\mathrm{red}}).$$

In this section, we will calculate $\partial\mathrm{Orb}_1(\sigma(x), \phi')$. Note that, by Remark 16.4, the result is independent of the choice of the non-unique matching function ϕ'.

14.1. **Nilpotent orbital integrals.** In this section, we determine the nilpotent integrals of ϕ'. The nilpotent orbits are listed in §17.1. We start with the two regular nilpotent orbits. We denote $\zeta(s) = \zeta_{F_0}(s) = (1 - q^{-s})^{-1}$.

Lemma 14.1. *We have*

$$\mathrm{Orb}(n_{0,-},\phi') = -q^{-1}\zeta(1) \quad \textit{and} \quad \mathrm{Orb}(n_{0,+},\phi') = -\eta(-1)q^{-1}\zeta(1).$$

Proof. Since ϕ' transfers to $(\phi, 0)$, this follows from Theorem 19.4 and (19.3). $\square$

Next we calculate the nilpotent orbital integrals $\mathrm{Orb}(n(\mu), \phi')$ where $n(\mu) \in \mathfrak{s}_{\mathrm{red}}$ is the family of nilpotent elements parametrized by $\mu \in F_0$, given by (17.1). When we consider $\mathrm{Orb}(n(\mu), \phi')$ as a function on F_0, we denote it by $\mathrm{Orb}_{\phi'}$,

$$\mathrm{Orb}_{\phi'}(\mu) := \mathrm{Orb}\big(n(\mu), \phi'\big).$$

Lemma 14.2.

$$\mathrm{Orb}\big(n(\mu),\phi'\big) = q\frac{\eta(-1)}{\log q}\cdot\begin{cases} 0, & |\mu| \leq 1; \\ \dfrac{\eta(\mu)\log|\mu|}{|\mu|}, & |\mu| > 1. \end{cases}$$

Proof. We first calculate the orbital integrals of ϕ over the family of nilpotent elements in $\mathfrak{u}_0$ given by (19.1). As in [34, §2.1], we use the Iwasawa decomposition $H_0(F_0) = KAN$ for K the special parahoric subgroup (hence K contains $\mathrm{SL}_2(O_{F_0})$ and all diagonal elements $\mathrm{diag}(a, \overline{a}^{-1})$ for $a \in O_F^\times$, and $\mathrm{vol}(K) = 1$). Write

$$h = k\begin{bmatrix} z & \\ & \overline{z}^{-1}\end{bmatrix}\begin{bmatrix} 1 & t \\ & 1\end{bmatrix}, \quad dh = \zeta(1)\, dk\, dz\, dt. \tag{14.1}$$

By (19.2) we then have

$$\begin{aligned} \mathrm{Orb}\big(n(\mu),\phi\big) &= \zeta(1)\int_F \phi_K\left(\pi\begin{bmatrix} 0 & \mu\pi z\overline{z} & z \\ 0 & 0 & 0 \\ 0 & \pi\overline{z} & 0\end{bmatrix}\right) dz \\ &= \zeta(1)|\pi|_F^{-1}\int_F \phi_K\left(\begin{bmatrix} 0 & -\mu z\overline{z} & z \\ 0 & 0 & 0 \\ 0 & -\pi\overline{z} & 0\end{bmatrix}\right) dz \\ &= q\zeta(1)\cdot\mathrm{vol}\big\{\, z\in F \bigm| |z|\leq 1 \text{ and } |\mu z\overline{z}| \leq 1 \,\big\}. \end{aligned}$$

Note now that F/F_0 is *ramified.* We obtain

$$\mathrm{Orb}\big(n(\mu),\phi\big) = q\zeta(1)\cdot\begin{cases} 1, & |\mu|\leq 1; \\ 1/|\mu|, & |\mu| > 1. \end{cases}$$

Recall from the proof of [34, Prop. 4.3] the functions

$$\phi_0 := \mathbf{1}_{O_{F_0}}, \qquad \phi_1(x) := \begin{cases} \dfrac{\eta(x)}{|x|}, & |x| > 1; \\ 0, & |x| \leq 1, \end{cases}$$

$$\phi_2(x) := \begin{cases} \dfrac{1}{|x|}, & |x| > 1; \\ 0, & |x| \leq 1, \end{cases} \qquad \phi_3(x) := \begin{cases} \dfrac{\eta(x)\log|x|}{|x|}, & |x| > 1; \\ 0, & |x| \leq 1. \end{cases}$$

Then we have the following table for their extended Fourier transforms, cf. loc. cit.,

$$\widetilde{\phi}_0(v) = \begin{cases} 0, & |v|\leq 1; \\ \dfrac{\eta(-v)}{|v|}, & |v| > 1. \end{cases} \qquad \widetilde{\phi}_1(v) = \begin{cases} q^{-1}, & |v| \leq 1; \\ 0, & |v| > 1. \end{cases}$$

$$\widetilde{\phi}_2(v) = \begin{cases} 0, & |v| \leq 1; \\ \dfrac{\eta(-1)}{\log q\,\zeta(1)}\dfrac{\eta(v)\log|v|}{|v|} - \dfrac{\eta(-v)}{|v|}, & |v| > 1. \end{cases} \qquad \widetilde{\phi}_3(v) = \begin{cases} -\dfrac{\zeta'(1)}{\zeta(1)}, & |v|\leq 1; \\ -\dfrac{\zeta'(1)}{\zeta(1)}\dfrac{1}{|v|}, & |v| > 1. \end{cases}$$

In terms of the four fundamental functions ϕ_i, we may rewrite the nilpotent orbital integral as

$$\mathrm{Orb}_\phi = q\zeta(1)\cdot(\phi_0 + \phi_2).$$

Note that $-\frac{\zeta'(1)}{\zeta(1)} = \zeta(1)q^{-1}\log q$. Hence, by Theorem 19.4, we obtain for the orbital integral of ϕ',

$$\mathrm{Orb}_{\phi'} = q\frac{\eta(-1)}{\log q}\phi_3.$$

This completes the proof. $\square$

14.2. **Germ expansion of $\partial\mathrm{Orb}_1(\sigma(x), \phi')$ around $x_0 = 0$.** We now calculate the germ expansion of $\partial\mathrm{Orb}_1(\sigma(x), \phi')$ around $x_0 = 0$. In the sequel, we denote by v the normalized valuation for F_0.

We are using the section σ of $\pi_{\mathfrak{s}_{\mathrm{red}}}|_{\mathfrak{s}_{\mathrm{red,rs}}}$ in a neighborhood of $x_0 = 0$ introduced in (17.3).

Theorem 14.3. *For $x = (\lambda, u, w) \in \mathfrak{b}_{\mathrm{red,rs},1}$ near zero,*

$$\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) = \Phi(x) - q^{-1}\zeta(1)\log|\Delta(x)|,$$

where the function $\Phi(x)$ is as follows. Set $t = q^{-1}$ below.

Case I: $|\Delta| \geq |w|^2$

(1) If $v(\Delta) > 4v(u)$, then

$$\Phi(x) = -t^{-v(u)}\frac{2(1+t) + \big(v(\Delta/u^4) - 1\big)(1-t)}{(1-t)^2}\cdot\log q.$$

(2) If $v(\Delta) \leq 4v(u)$ and $v(\Delta)$ is odd, then

$$\Phi(x) = -t^{\frac{2v(u)-v(\Delta)+1}{2}}\frac{\big(4v(u) - v(\Delta) + 3\big) - \big(4v(u) - v(\Delta) - 1\big)t}{(1-t)^2}\cdot\log q.$$

(3) If $v(\Delta) \leq 4v(u)$ and $v(\Delta)$ is even, then

$$\Phi(x) = -t^{\frac{2v(u)-v(\Delta)}{2}}\frac{\big(2v(u) - \frac{v(\Delta)}{2} + 1\big)(1-t^2) + t(3+t)}{(1-t)^2}\cdot\log q.$$

Case II: $|\Delta| < |w|^2$

(1) If $v(w/\pi) \geq 2v(u)$, then

$$\Phi(x) = -t^{-v(u)}\frac{2(1+t) + \big(v(\Delta/u^4) - 1\big)(1-t)}{(1-t)^2}\cdot\log q.$$

(2) If $v(w/\pi) < 2v(u)$, then

$$\Phi(x) = -t^{v(u)-v(w/\pi)}\frac{4t + \big(v(\Delta) + 4v(u) - 4v(w/\pi) + 1\big)(1-t)}{(1-t)^2}\cdot\log q.$$

The proof of this theorem will occupy the entire subsection. The term $\partial\mathrm{Orb}_1(\sigma(x), \phi')$ is a sum of two parts, one from the two regular nilpotent orbits $n_{0\pm}$ and the other from the one-dimensional nilpotent family $n(\mu)$.

We first determine the contribution of the two regular nilpotents. We use Lemma 14.1. Noting that $\eta(\Delta/\varpi) = -1$ when $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ (cf. Proposition 10.2), and taking into account the values of $\partial\Gamma(n_{0\pm})$ (see table above), the total contribution of the two regular nilpotents to $\partial\mathrm{Orb}_1(\sigma(x), \phi')$ is equal to

$$-\eta(\Delta/\varpi)\log|\Delta|\,\mathrm{Orb}(n_{0,-}, \phi') = \eta(\Delta/\varpi)q^{-1}\zeta(1)\log|\Delta| = -q^{-1}\zeta(1)\log|\Delta|.$$

We now calculate the contribution from the one-dimensional family $n(\mu)$, which we denote by

$$\Phi(x) = \int_{F_0}\partial\Gamma_{n(\mu)}(x)\,\mathrm{Orb}\big(n(\mu), \phi', 0\big)\,d\mu.$$

It is easier to use an equivalent formula, namely (17.5) :

$$\Phi(x) = -\eta(-1)|u|^{-1}\int_{F_0}\mathrm{Orb}\Big(n\big(u^{-2}(t + t^{-1}\Delta/\varpi + 2w/\pi)\big), \phi'\Big)\eta(t)\log|t|\,\frac{dt}{|t|}. \tag{14.2}$$

Set

$$w' = w/u^2, \quad \lambda' = \lambda/u^2, \quad \Delta' = \Delta/u^4 = \lambda' + w'^2.$$

Note that $\eta(-\Delta) = -1$ (cf. Proposition 10.2). Denote

$$\eta_1(t) = \eta(t)|t|^{-1}, \quad t \in F_0^\times.$$

By the formula for $\mathrm{Orb}_{\phi'}$ in Lemma 14.2, and substituting $t \mapsto tu^2$, the integral (14.2) is equal to

$$\Phi(x) = -q^{-1}(\log q)^{-1}|u|^{-1} \cdot \Xi(x), \tag{14.3}$$

where

$$\Xi(x) = \int_F \log|t + \Delta'/(\varpi t) + 2w'/\pi|\, \eta_1(t^2 + \Delta'/\varpi + 2w't/\pi) \log|t|\, dt, \tag{14.4}$$

where the integrand is subject to the condition

$$|t + \Delta'/(\varpi t) + 2w'/\pi| > 1.$$

A simple observation is that the contribution of t with $|t| = |\Delta'/(\varpi t)|$ is always zero, since we may pair t with $\Delta'/(\varpi t)$ to see that the sum is canceled (use $\eta(-\varpi) = 1$, $\eta(-\Delta') = -1$ by Proposition 10.2). Hence the integral in (14.4) is equal to

$$\Xi(x) = \int \log|t + \Delta'/(\varpi t) + 2w'/\pi|\, \eta_1(t^2 + \Delta'/\varpi + 2w't/\pi)(2\log|t| - \log|\Delta'/\varpi|)\, dt, \tag{14.5}$$

where the integrand is subject to the conditions

$$|t| > |\Delta'/\varpi t|, \quad |t + \Delta'/\varpi t + 2w'/\pi| > 1.$$

We distinguish two cases.

Case I: $|\Delta'| \geq |w'|^2$. Then the last integral (14.5) is equal to

$$\begin{aligned}\Xi(x) &= \int \log|t + \Delta'/(\varpi t)|\, \eta_1(t^2)(2\log|t| - \log|\Delta'/\varpi|)\, dt \\ &= \int \log|t|\, \eta_1(t^2)(2\log|t| - \log|\Delta'/\varpi|)\, dt,\end{aligned}$$

where the integrand is subject to the conditions

$$t| > |\Delta'/\varpi t| \quad \text{and} \quad |t + \Delta'/\varpi t + 2w'/\pi| = |t| > 1.$$

The integral is equal to

$$\Xi(x) = \int_{|t| > \max\{1, |\Delta'/\varpi|^{1/2}\}} \log|t|\, \eta_1(t^2)(2\log|t| - \log|\Delta'/\varpi|)\, dt.$$

Making the substitution $t \mapsto t^{-1}$, we obtain

$$\Xi(x) = \int_{|t| < \min\{1, |\Delta'/\varpi|^{-1/2}\}} \log|t|(2\log|t| + \log|\Delta'/\varpi|)\, dt.$$

Set $n := 1 + \max\{0, [-v(\Delta'/\varpi)/2]\} > 0$. Then the integral is equal to

$$\Xi(x) = \zeta(1)^{-1}\left(\sum_{i \geq n}\big(-2i - v(\Delta'/\varpi)\big)(-i)q^{-i}\right) \cdot (\log q)^2. \tag{14.6}$$

For later use we tabulate the following elementary formulas:

$$\begin{aligned}\sum_{i \geq n} i t^{i-1} &= \frac{nt^{n-1} - (n-1)t^n}{(1-t)^2}, \\ \sum_{i \geq n} i(i+1)t^i &= \frac{t\big(n(n-1)t^{n+1} - 2(n^2-1)t^n + n(n+1)t^{n-1}\big)}{(1-t)^3}, \\ \sum_{i \geq n} i^2 t^i &= \frac{(n-1)^2 t^{n+2} - (2n^2 - 2n - 1)t^{n+1} + n^2 t^n}{(1-t)^3}.\end{aligned}$$

We now see that the integral (14.6) is given as follows.

(1) If $v(\Delta')>0$, then $n=1$ and

$$\Xi(x)=\zeta(1)^{-1}\left(\sum_{i=1}^{\infty}\big(2i+v(\Delta'/\varpi)\big)it^i\right)\cdot(\log q)^2=\frac{t\big(2(1+t)+v(\Delta'/\varpi)(1-t)\big)}{(1-t)^2}\cdot(\log q)^2.$$

(2) If $v(\Delta')\leq 0$ is odd, then $-v(\Delta')=2n-3$ and

$$\Xi(x)=\zeta(1)^{-1}\left(\sum_{i=n}^{\infty}(2i-2n+2)it^i\right)\cdot(\log q)^2=\frac{2t^n(-n-2t+nt)}{(1-t)^2}\cdot(\log q)^2.$$

(3) If $v(\Delta')\leq 0$ is even, then $-v(\Delta')=2n-2$ and

$$\Xi(x)=\zeta(1)^{-1}\left(\sum_{i=n}^{\infty}(2i-2n+1)it^i\right)\cdot(\log q)^2=\frac{t^n(n+3t+t^2-nt^2)}{(1-t)^2}\cdot(\log q)^2.$$

Case II: $|\Delta'|<|w'|^2$. In this case $-\lambda'/w'^2\in 1+\pi O_F$ (note that $-\Delta'=-w'^2-\lambda'$ is not a norm). Since $w'\in\pi F_0$, it follows that $-\lambda'/\varpi$ is a square and hence the following equation has two roots

$$t^2+2tw'/\pi+\Delta'/\varpi=0,\quad t=-w'/\pi\pm\sqrt{-\lambda'/\varpi}\in F_0.$$

We label t_0 as the unique root such that

$$|t_0|=|w'/\pi|=|\lambda'/\varpi|^{1/2},\tag{14.7}$$

and label t_1 the other root. Then $|t_0|>|t_1|$ and the integral (14.5) is equal to

$$\Xi(x)=\int\log\big|(t-t_0)(t-t_1)/t\big|\,\eta_1\big((t-t_0)(t-t_1)\big)(2\log|t|-\log|\Delta'/\varpi|)\,dt,$$

where the integrand is subject to

$$|t|>|\Delta'/\varpi t|,\quad \big|(t-t_0)(t-t_1)/t\big|>1.$$

When $|t|>|\Delta'/\varpi t|$, we always have $|t|>|t_1|$. Hence the integral is reduced to

$$\Xi(x)=\int\log|t-t_0|\,\eta_1\big((t-t_0)t\big)(2\log|t|-\log|\Delta'/\varpi|)\,dt,\tag{14.8}$$

subject to

$$|t|>|\Delta'/\varpi t|,\quad |t-t_0|>1.\tag{14.9}$$

We break the integral up as a sum of three pieces according to whether $|t|$ is less than, greater than, or equal to $|w'/\pi|$.

Lemma 14.4. *When $|t|<|w'/\pi|$, the contribution to* (14.8) *is zero.*

Proof. Now we have $|t-t_0|=|t_0|$ and hence the contribution is zero unless $|w'/\pi|>1$, which we assume now. Then the contribution to (14.8) is

$$\log|w'/\pi|\int_{|w'/\pi|>|t|>|\Delta'/\varpi t|}\eta_1(t_0t)\,(2\log|t|-\log|\Delta'/\varpi|)\,dt=0,$$

since η is *ramified*! ☐

Lemma 14.5. *When $|t|>|w'/\pi|$, the contribution to* (14.8) *is*

$$\int_{|t|<\min\{1,|w'/\pi|^{-1}\}}(2\log|t|+\log|\Delta'/\varpi|)\,\log|t|\,dt.\tag{14.10}$$

Proof. When $|t|>|w'/\pi|$, we have

$$\int_{|t|>\max\{1,|w'/\pi|\}}\log|t|\,(2\log|t|-\log|\Delta'/\varpi|)\,|t|^{-2}\,dt.$$

Substituting $t\to 1/t$, this becomes

$$\int_{|t|^{-1}>\max\{1,|w'/\pi|\}}\log|t|\,(2\log|t|+\log|\Delta'/\varpi|)\,dt.$$

This completes the proof. ☐

Lemma 14.6. *When $|t| = |w'/\pi|$, the contribution to* (14.8) *is*

$$(\log |w'/\pi|)\,(\log |\Delta'/w'^2|)\,|w'/\pi|^{-1}\,q^{-1},$$

when $|w'/\pi| > 1$, and zero otherwise.

Proof. The constraint $|t| > |\Delta'/\varpi t|$ in (14.9) is now superfluous. Under the assumption $|t| = |w'/\pi| = |t_0|$ (cf. (14.7)), we divide the integral (14.8) into two cases: $|t-t_0| < |t_0|$ and $|t-t_0| = |t_0|$.

First we show that when $|t-t_0| < |t_0|$, the contribution to (14.8) is zero. Let $x = t - t_0$ and we see that the integral (14.8) becomes

$$\int \log |x|\, \eta_1(xt_0)\,(2\log|t_0| - \log|\Delta'/\varpi|)\,dx,$$

where x satisfies conditions coming from (14.9)

$$|x| < |t_0|, \quad |x| > 1.$$

This integral is equal to

$$\eta_1(t_0)(2\log|t_0| - \log|\Delta'/\varpi|)\int_{1<|x|<|t_0|} \log|x|\,\eta_1(x)\,dx.$$

Since η is ramified, we have

$$\int_{1<|x|<|t_0|} \log|x|\,\eta_1(x)\,dx = 0.$$

It remains to consider the contribution when $|t-t_0| = |t_0|$. We hence have $|t-t_0| = |t_0| = |t| = |w'/\pi|$ (cf. (14.7)). The integral (14.8) becomes

$$\log|w'/\pi|\,(2\log|w'/\pi| - \log|\Delta'/\varpi|)\int \eta_1\big((t-t_0)t\big)\,dt,$$

subject to

$$|t-t_0| = |t| = |t_0|, \quad |t-t_0| = |t_0| > 1.$$

The integral is zero unless $|t_0| = |w'/\pi| > 1$, in which case we have

$$\int_{|t-t_0|=|t|=|w'/\pi|} \eta_1\big((t-t_0)t\big)\,dt = |w'/\pi|^{-2}\int_{|t-t_0|=|t|=|w'/\pi|} \eta(1-t_0/t)\,dt = -q^{-1}|w'/\pi|^{-1}.$$

This completes the proof. $\square$

Set $n := 1 + \max\{0, -v(w'/\pi)\}$. From the last three lemmas, we find that the integral (14.8) is given as follows.

(1) If $v(w'/\pi) \geq 0$, then $n = 1$ and the integral (14.8) is equal to

$$\begin{aligned}\Xi(x) &= \zeta(1)^{-1}\left(\sum_{i=1}^{\infty}\big(-2i - v(\Delta'/\varpi)\big)(-i)t^i\right)\cdot(\log q)^2\\ &= |u|^{-1}\frac{t\big(2(1+t) + v(\Delta'/\varpi)(1-t)\big)}{(1-t)^2}\cdot(\log q)^2.\end{aligned}$$

(2) If $v(w'/\pi) < 0$, then $n = -v(w'/\pi) + 1$. The integral (14.10) is equal to $(\log q)^2$ times the factor

$$\begin{aligned}&\zeta(1)^{-1}\sum_{i=n}^{\infty}(2i + v(\Delta'/\varpi))it^i\\ &\quad= -\frac{t^n(-dn - 2n^2 - 2t - dt - 4nt + 2dnt + 4n^2t - 2t^2 + dt^2 + 4nt^2 - dnt^2 - 2n^2t^2)}{(1-t)^2},\end{aligned}$$

where $d = v(\Delta'/\varpi)$. The contribution from the last lemma is

$$(\log|w'/\pi|)(\log|\Delta'/w'^2|)|w'/\pi|^{-1}q^{-1} = -(n-1)(d+2n-2)t^n\cdot(\log q)^2.$$

It follows that the sum (14.8) is given by

$$\Xi(x) = t^n \frac{4t + (d+4n-2)(1-t)}{(1-t)^2} \cdot (\log q)^2.$$

Note that $d+4n-2 = v(\Delta) + 4v(u) - 4v(w/\pi) + 1$. Hence, by (14.3), (14.4) and **Case I** (1), (2), (3), **Case II** (1), (2), we complete the proof of Theorem 14.3.

14.3. **x_0-nilpotent integrals for $x_0 \neq 0$.** In this subsection, we consider $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ with $x_0 \neq 0$, and calculate the orbital integrals of the x_0-nilpotent elements in $\mathfrak{s}_{\mathrm{red}}$, i.e., of the elements in $\mathfrak{s}_{\mathrm{red}}$ mapping to x_0 under $\pi_{\mathfrak{s}}$, cf. §16. These elements are listed in §18.1.

Lemma 14.7. *Let $x_0 = (\lambda_0, 0, 0) \in \mathfrak{b}_{\mathrm{red}}$, $\lambda_0 \in F_0^\times$. Assume that $-\lambda_0 \notin F_0^{\times,2}$. Set $F' = F_0[X]/(X^2+\lambda_0)$.*
(a) *In case (0i) (i.e., $F' \not\simeq F$),*

$$\mathrm{Orb}(y_+, \phi') = \eta(-\lambda_0)\,\mathrm{Orb}(y_-, \phi') = \frac{1}{2}\zeta(1) \cdot \begin{cases} -2q^{-1} + q^{\frac{v(\lambda_0)}{2}}(1+q^{-1}), & 2 \mid v(\lambda_0), \\ 2q^{-1}(q^{\frac{v(\lambda_0)+1}{2}} - 1), & 2 \nmid v(\lambda_0), \end{cases}$$

when $\lambda_0 \in O_{F_0}$, and $\mathrm{Orb}(y_+, \phi') = \mathrm{Orb}(y_-, \phi') = 0$ when $\lambda_0 \notin O_{F_0}$.
(b) *In case (0ii) (i.e., $F' \simeq F$),*

$$\mathrm{Orb}(y_{-+}, \phi') = \eta(-1)\,\mathrm{Orb}(y_{+-}, \phi') = 0.$$

Furthermore,

$$\mathrm{Orb}(y_{++}, \phi') = \eta(-1)\,\mathrm{Orb}(y_{--}, \phi') = \frac{1}{2}\eta(-\alpha)\zeta(1) \cdot \begin{cases} -2q^{-1} + q^{\frac{v(\lambda_0)}{2}}(1+q^{-1}), & 2 \mid v(\lambda_0), \\ 2q^{-1}(q^{\frac{v(\lambda_0)+1}{2}} - 1), & 2 \nmid v(\lambda_0), \end{cases}$$

when $\lambda_0 \in O_{F_0}$, and $\mathrm{Orb}(y_{++}, \phi') = \mathrm{Orb}(y_{--}, \phi') = 0$ when $\lambda_0 \notin O_{F_0}$.

Proof. Note that ϕ' matches the pair $(\phi_0 = \phi, \phi_1 = 0)$. By Theorem 19.5, we have case by case the following.
(a) In case (0i), we have

$$\mathrm{Orb}(y_+, \phi') = \eta(-\lambda_0)\,\mathrm{Orb}(y_-, \phi') = \frac{1}{2}\,\mathrm{Orb}(y_0, \phi_0),$$

where y_0 is any representative of the unique semi-simple orbit in $\mathfrak{u}_{0,\mathrm{red}}$ above $x_0 \in \mathfrak{b}_{\mathrm{red}}$.
(b) In case (0ii), since $\mathrm{Orb}(y_\pm, \phi_1) = 0$, we have

$$\mathrm{Orb}(y_{++}, \phi') = \eta(-1)\,\mathrm{Orb}(y_{--}, \phi') = \frac{1}{4}\eta(-\alpha)\big(\mathrm{Orb}(y_+, \phi_0) + \mathrm{Orb}(y_-, \phi_0)\big)$$

and

$$\mathrm{Orb}(y_{-+}, \phi') = \eta(-1)\,\mathrm{Orb}(y_{+-}, \phi') = \frac{1}{4}\eta(-\alpha)\big(\mathrm{Orb}(y_+, \phi_0) - \mathrm{Orb}(y_-, \phi_0)\big),$$

where $y_\pm \in \mathfrak{u}_{0,\mathrm{red}}$ are representatives of the two semi-simple orbits in $\mathfrak{u}_{0,\mathrm{red}}$ above $x_0 \in \mathfrak{b}_{\mathrm{red}}$.

Therefore it suffices to show the following in both cases: *let $y_0 \in \mathfrak{u}_{\mathrm{red}}$ be any semi-simple element with invariants x_0. Then we have*

$$\mathrm{Orb}(y_0, \phi) = \zeta(1) \cdot \begin{cases} -2q^{-1} + q^{\frac{v(\lambda_0)}{2}}(1+q^{-1}), & 2 \mid v(\lambda_0), \\ 2q^{-1}\big(q^{\frac{v(\lambda_0)+1}{2}} - 1\big), & 2 \nmid v(\lambda_0), \end{cases}$$

when $\lambda_0 \in O_{F_0}$, and $\mathrm{Orb}(y_0, \phi) = 0$ when $\lambda_0 \notin O_{F_0}$. In particular, the independence on the choice of y_0 implies that $\mathrm{Orb}(y_+, \phi_0) = \mathrm{Orb}(y_-, \phi_0)$ in case (0ii).

We may assume that y_0 is of the form

$$y_0 = \begin{bmatrix} 0 & -\lambda_0/\epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad |\epsilon| = 1.$$

We use the Iwasawa decomposition (14.1) as in the proof of Lemma 14.2. Note that ϕ is K-invariant. Then we have by definition

$$\mathrm{Orb}(y_0,\phi)=\zeta(1)\int_{z\in F,t\in F_0}\phi\left(\begin{bmatrix}\epsilon t & z\bar z(-\lambda_0/\epsilon-\epsilon t^2) & 0\\ \epsilon/z\bar z & -\epsilon t & 0\\ 0 & 0 & 0\end{bmatrix}\right)dz\,dt,$$

where the integrand is constrained by

$$|t|\le 1,\quad |(z\bar z)^{-1}|\le 1,\quad |z\bar z(-\lambda_0-\epsilon^2t^2)|\le 1.$$

Therefore we see that $|\lambda_0|\le 1$ or the integral vanishes. Note that $-\lambda_0$ is not a square by assumption. The integral is then equal to

$$\int_{|t^2|\le|\lambda_0|,1\le|z\bar z|\le|\lambda_0^{-1}|}dz\,dt+\int_{|\lambda_0|<|t^2|\le 1,1\le|z\bar z|\le|t^{-2}|}dz\,dt. \tag{14.11}$$

The first integral is equal to

$$q^{-\lfloor\frac{v(\lambda_0)+1}{2}\rfloor}(|\lambda_0|^{-1}-q^{-1}),$$

and the second one is equal to

$$\begin{aligned}&\int_{|\lambda_0|<|t^2|\le 1}(|t|^{-2}-q^{-1})dt\\ =&\int_{|\lambda_0|<|t^{-2}|\le 1}dt-q^{-1}\int_{|\lambda_0|<|t^2|\le 1}dt.\end{aligned}$$

We distinguish two cases.

(i) $v(\lambda_0)$ *is even.* Then the integral (14.11) is equal to

$$q^{-\frac{v(\lambda_0)}{2}}(q^{v(\lambda_0)}-q^{-1})+((q^{\frac{v(\lambda_0)-2}{2}}-q^{-1})-q^{-1}(1-q^{-\frac{v(\lambda_0)}{2}}))=-2q^{-1}+q^{\frac{v(\lambda_0)}{2}}(1+q^{-1}).$$

(ii) $v(\lambda_0)$ *is odd.* Then the integral (14.11) is equal to

$$q^{-\frac{v(\lambda_0)+1}{2}}(q^{v(\lambda_0)}-q^{-1})+((q^{\frac{v(\lambda_0)-1}{2}}-q^{-1})-q^{-1}(1-q^{-\frac{v(\lambda_0)+1}{2}}))=2q^{-1}(q^{\frac{v(\lambda_0)+1}{2}}-1).$$

Noting the factor $\zeta(1)$, this completes the proof. $\square$

Lemma 14.8. *Let $x_0=(\lambda_0,u_0,w_0)\in\mathfrak{b}_{\mathrm{red}}\smallsetminus\mathfrak{b}_{\mathrm{red,rs}}$ with $u_0\neq 0$. Then*

$$\mathrm{Orb}(y_+,\phi')=\mathrm{Orb}(y_-,\phi')=\frac12\zeta(1)\cdot\begin{cases}2q^{-1}(q^{v(u_0)+1}-1), & |\lambda_0|<|u_0|^2,\\ 2q^{-1}(q^{\frac{v(\lambda_0)+1}{2}}-1), & |\lambda_0|>|u_0|^2,\end{cases}$$

when $\lambda_0,u_0\in O_{F_0},w_0\in O_F$, and $\mathrm{Orb}(y_+,\phi')=\mathrm{Orb}(y_-,\phi')=0$ otherwise.

Proof. By Theorem 19.5 (item (c), i.e., case (1)), we have

$$\mathrm{Orb}(y_+,\phi')=\mathrm{Orb}(y_-,\phi')=\frac12\,\mathrm{Orb}(y_0,\phi_0),$$

where y_0 is any representative of the unique semi-simple orbit in $\mathfrak{u}_{0,\mathrm{red}}$ above x_0. It remains to show that

$$\mathrm{Orb}(y_0,\phi)=\zeta(1)\cdot\begin{cases}2q^{-1}(q^{v(u_0)+1}-1), & |\lambda_0|<|u_0|^2,\\ 2q^{-1}(q^{\frac{v(\lambda_0)+1}{2}}-1), & |\lambda_0|>|u_0|^2,\end{cases}$$

when $\lambda_0,u_0\in O_{F_0},w_0\in O_F$, and $\mathrm{Orb}(y_0,\phi)=0$ otherwise.

First we assume $\lambda_0\neq 0$. We use the representative with invariants (λ_0,u_0,w_0) (cf. (19.6) and (19.7))

$$y_0=\begin{bmatrix}0 & -\lambda_0 & \pi\alpha b\\ 1 & 0 & b\\ \cdots & \cdots & 0\end{bmatrix},$$

where $\alpha^2 = -\lambda_0/\varpi, \alpha \in F_0^\times$, and $-2b\bar{b}\alpha = u_0$. Again by the Iwasawa decomposition we arrive at

$$\mathrm{Orb}(y_0, \phi) = \zeta(1) \int_{z\in F, t\in F_0} \phi\left(\begin{bmatrix} t & z\overline{z}(-\lambda_0 - t^2) & bz(\pi\alpha + t) \\ 1/z\overline{z} & -t & b/\overline{z} \\ \cdots & \cdots & 0 \end{bmatrix}\right) dz\, dt.$$

It is easy to see that $\mathrm{Orb}(y, \phi) = 0$ unless $\lambda_0, u_0, w_0 \in O_F$ which we assume from now on. The integrand is constrained by

$$|t| \leq 1, \quad |(z\overline{z})^{-1}| \leq 1, \quad |z\overline{z}(-\lambda_0 - t^2)| \leq 1,$$

and

$$|bz(\pi\alpha + t)| \leq 1, \quad |b/\overline{z}| \leq 1.$$

Note that $-\lambda_0 = \alpha^2\varpi$ is not a square in F_0. We distinguish two subcases:

- *Case* $|\alpha| \geq |u_0|$, *(i.e.,* $|\lambda_0| > |u_0|^2$*)*. Then we have $|b| \leq 1$. The conditions $|bz(\pi\alpha + t)| \leq 1$ and $|b/\overline{z}| \leq 1$ are redundant by $|b| \leq 1$, $|z\overline{z}(\lambda_0 - t^2)| \leq 1$, $|(z\overline{z})^{-1}| \leq 1$. In this case we may simply apply the previous lemma, noting that $v(\lambda_0)$ is odd.
- *Case* $|\alpha| < |u_0|$, *(i.e.,* $|\lambda_0| < |u_0|^2$*)*. Then we have $|b| > 1$. In this case the constraints are reduced to

$$|bzt| \leq 1, \quad |z| \geq |b|, \quad |bz\pi\alpha| \leq 1.$$

Let v_F denote the normalized valuation on F. We calculate the orbital integral according to $v_F(z)$, which varies from $v_F(b)$ to $-v_F(b\pi\alpha)$:

$$\mathrm{Orb}(y_0, \phi) = \zeta(1) \int_{|b|\leq|z|\leq 1/|b\pi\alpha|} \left(\int_{|t|_F \leq 1/|bz|} dt\right) dz. \tag{14.12}$$

The double integral in (14.12) is equal to

$$(1 - q^{-1})\big(q^{v_F(b)}q^{-v_F(b)} + q^{v_F(b)-1}(q^{-v_F(b)+1} + q^{-v_F(b)+2}) + \cdots + q^{-v_F(\alpha)/2-1}q^{v_F(b)+1+v_F(\alpha)}\big).$$

We hence arrive at

$$\begin{aligned}\mathrm{Orb}(y_0, \phi) &= \zeta(1)(1 - q^{-1})\Big(1 + 1 + q + q + \cdots + q^{v_F(b)+\frac{v_F(\alpha)}{2}} + q^{v_F(b)+\frac{v_F(\alpha)}{2}}\Big) \\ &= 2\zeta(1)(1 - q^{-1})\frac{q^{v_F(b)+\frac{v_F(\alpha)}{2}+1} - 1}{q - 1} \\ &= 2\zeta(1)q^{-1}(q^{v(u_0)+1} - 1).\end{aligned}$$

Here we used that $v_F(u_0) = v_F(\alpha) + 2v_F(b)$ and $v_F(u_0) = 2v(u_0)$.

The case $\lambda_0 = 0$ is similar to the case $|\lambda_0| < |u_0|^2$ above, and we omit the details. This completes the proof. $\square$

14.4. **Germ expansion of $\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big)$ around $x_0 \neq 0$.** The following theorem together with the values in Lemma 14.7 and Lemma 14.8 give the germ expansion around nonzero elements $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$. We use the classification of such elements in §18.1.

Theorem 14.9. *Let $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ be a nonzero element. Assume that, if $u_0 = w_0 = 0$ then $-\lambda_0 \notin F_0^{\times,2}$, and set $F' = F_0[X]/(X^2 + \lambda_0)$. Let $x = (\lambda, u, w) \in \mathfrak{b}_{\mathrm{red,rs},1}$ be in a small neighborhood of x_0.*

(a) *In case* (0*i*) *(i.e., $u_0 = w_0 = 0$ and $F' \not\simeq F$),*

$$\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) = \eta(-\lambda)\,\mathrm{Orb}(y_-, \phi')\log|\Delta(x)| + C_1;$$

(b) *In case* (0*ii*) *(i.e., $u_0 = w_0 = 0$ and $F' \simeq F$),*

$$\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) = \eta(-1)\,\mathrm{Orb}(y_{--}, \phi')\log|\Delta(x)| + C_2;$$

(c) *In case* (1) *(i.e., $u_0 \neq 0$),*

$$\partial\mathrm{Orb}_1\big(\sigma(x), \phi'\big) = \mathrm{Orb}(y_-, \phi')\log|\Delta(x)/u(x)^2| + C_3;$$

where C_1, C_2, C_3 are constants (depending on x_0 only, independent of the choice of ϕ'). Here $\sigma(x)$ is the section defined by (17.3) *in cases* (0*i*) *and* (1)*, and by* (18.6) *in case* (0*ii*)*.*

Proof. We simply denote Δ for $\Delta(x)$. Note that $\eta(-\Delta) = -1$ for $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ (cf. Proposition 10.2). We apply Theorem 18.2 and use the notation in its statement. Case $(0i)$ follows from the fact $|\lambda|$ is a constant when x is near x_0. In case $(0ii)$, by Lemma 14.7, we have $\mathrm{Orb}(y_{-+}, \phi') = \mathrm{Orb}(y_{+-}, \phi') = 0$. This case then follows easily by $\eta(z_1 z_2) = \eta(\Delta)$ and since $|z_1 z_2/\Delta|$ is a constant when x is near x_0. Case (1) follows from the fact that $|u(x)|$ is a constant when x is near x_0. □

15. Comparison

15.1. **Statement of the theorem.** We denote by φ the function on $\mathfrak{b}_{\mathrm{red,rs}}$,

$$\varphi(x) := \begin{cases} 2\omega(y)\,\partial\mathrm{Orb}(y, \phi') + \ell\text{-}\mathrm{Int}(x) \cdot \log q, & y \in \mathfrak{s}_{\mathrm{red}},\ \pi_{\mathfrak{s}}(y) = x \in \mathfrak{b}_{\mathrm{red,rs},1}, \\ 0, & x \in \mathfrak{b}_{\mathrm{red,rs},0}. \end{cases}$$

We define φ_1 to be the analogous function where we replace $\partial\mathrm{Orb}$ by $\partial\mathrm{Orb}_1$, and define φ_2 to be

$$\varphi_2(x) := \begin{cases} 2\omega(y)\,\partial\mathrm{Orb}_2(y, \phi'), & y \in \mathfrak{s}_{\mathrm{red}},\ \pi_{\mathfrak{s}}(y) = x \in \mathfrak{b}_{\mathrm{red,rs},1}, \\ 0, & x \in \mathfrak{b}_{\mathrm{red,rs},0}. \end{cases}$$

Hence we have

$$\varphi = \varphi_1 + \varphi_2.$$

By Theorem 16.5, φ_2 is an orbital integral function. Therefore, to show Theorem 12.1, it suffices to show that φ_1 is an orbital integral function. Indeed, we will show the following stronger result.

Theorem 15.1. *For every $x_0 \in \mathfrak{b}_{\mathrm{red}}$, there exists an open neighborhood V_{x_0} of x_0 such that $\varphi_1|_{V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs},1}}$ is a constant function.*

The only non-trivial case is when x_0 lies in the closure of $\mathfrak{b}_{\mathrm{red,rs},1}$, but not in $\mathfrak{b}_{\mathrm{red,rs},1}$ itself. So, let us assume this. The proof of Theorem 15.1 will occupy the rest of this section. We first treat the case $x_0 = 0$ and then move on to the case where $x_0 \neq 0$, in the order appearing in the table in §13.

Before proceeding, we recall from (9.4) that

$$v_D(u) = 2m, \quad v_D(w) = 2m + \ell_+, \quad v_D(\Delta) = 4m + 2\ell_-,$$

and

$$v(\lambda) = \min\{\ell_+, \ell_-\}$$

The last identity follows because in the expression $\lambda = \mathrm{N}\alpha'_+ + \mathrm{N}\alpha'_-$, when the valuations of both summands are identical, there can be no cancellation.

15.2. **The case $x_0 = 0$.** Theorem 14.3 calculates $\omega(y)\,\partial\mathrm{Orb}(y, \phi')$ for $y = \sigma(x)$ the explicit section introduced in (17.3). Recall that in Theorem 14.3, the valuation is taken to be the normalized valuation for F_0 and therefore $v_D(\,) = 2v(\,)$.

We compare the formulas in Theorem 14.3 and the formulas for $\mathrm{Int}(x)$ in §9. We see that the case distinctions in both formulas are identical; we furthermore see case-by-case that, when $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ is close to 0,

$$\varphi_1(x) = \frac{-8t}{(1-t)^2} \cdot \log q$$

is constant. Here $t = q^{-1}$ for the rest of this section. Hence Theorem 15.1 follows in this case.

15.3. **The case $x_0 \neq 0$.** The theorem holds trivially if x_0 is not integral, so we assume from now on that x_0 is integral. We first consider the case $x_0 = (\lambda_0, 0, 0)$, $\lambda_0 \neq 0$, and derive from §9 a convenient expression for the quantity $\ell\text{-}\mathrm{Int}(x)$ in a small neighborhood of x_0.

Lemma 15.2. *Let $x_0 = (\lambda_0, 0, 0)$, $\lambda_0 \neq 0$. For $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ in a small neighborhood of x_0,*

$$\ell\text{-}\mathrm{Int}(x) = \begin{cases} v(\Delta/\lambda)\dfrac{t^{-v(\lambda_0)/2}(1+t) - 2t}{1-t} + C_1\big(v(\lambda_0)\big), & 2 \mid v(\lambda_0), \\[2ex] v(\Delta/\lambda)\dfrac{2t(t^{-(v(\lambda_0)+1)/2} - 1)}{1-t} + C_2\big(v(\lambda_0)\big), & 2 \nmid v(\lambda_0), \end{cases}$$

where $C_1(n)$ and $C_2(n)$ are explicit polynomials in n.

Proof. For $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ near x_0, we may assume that $|\lambda| = |\lambda_0|$ is fixed, while $v_D(u) = 2m$ and $v_D(w) = 2m + \ell_+$ are very large. Hence we may assume that m is larger than at least one of ℓ_+ or ℓ_-. To make the comparison, we will rewrite ℓ-Int(x) from **Case I** (2), (3) and **Case II** (2) in §9.

• If $v(\lambda_0)$ is even (and so is $v(\lambda)$), then the minimum among ℓ_+, ℓ_- must be ℓ_- since ℓ_+ is odd. Hence we may assume $\ell_- = \ell_{-0} = v(\lambda_0)$, and we are in case **Case I** (3). Hence the intersection number is given by

$$\begin{aligned}
\ell\text{-Int}(x) &= 2t^{-\ell_-/2}\frac{(m-\ell_-/2+1)(1-t^2)+t(t+3)}{(1-t)^2} + \frac{-2(\ell_-+2m+1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= 2m\frac{t^{-\ell_-/2}(1+t)-2t}{1-t}\\
&\quad + 2t^{-\ell_-/2}\frac{(-\ell_-/2+1)(1-t^2)+t(t+3)}{(1-t)^2} - \frac{2(\ell_-+1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= v(\Delta/\lambda)\frac{t^{-v(\lambda_0)/2}(1+t)-2t}{1-t} + C_1\big(v(\lambda_0)\big),
\end{aligned}$$

where $C_1(n)$ is an explicit function of n defined by the last equality.

• If $v(\lambda_0)$ is odd (and so is $v(\lambda)$), we are in **Case I** (2) or **Case II** (2), depending on whether $\ell_- \le \ell_+$ or not (equivalently, depending on whether $|\Delta| \ge |w|^2$ or not). In other words, this gives a partition of the intersection of a neighborhood of x_0 with the regular semi-simple set as a disjoint union of two sets.

In **Case I** (2) ($\ell_- \le \ell_+$, $\ell_- \le 2m$ and ℓ_- odd), we have

$$v_D(\lambda) = 2\ell_-, \quad v_D(\Delta/\lambda) = 4m,$$

and we may assume that $\ell_- = \ell_{-0} = v(\lambda_0)$. Then we have

$$\begin{aligned}
\ell\text{-Int}(x) &= 2t^{-(\ell_--1)/2}\frac{(2m-\ell_-+3)-(2m-\ell_--1)t}{(1-t)^2} + \frac{-2(2m+\ell_-+1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= 4m\frac{t(t^{-(\ell_-+1)/2}-1)}{1-t} + 2t^{-(\ell_--1)/2}\frac{(-\ell_-+3)+(\ell_-+1)t}{(1-t)^2} - \frac{2(\ell_-+1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= v(\Delta/\lambda)\frac{2t(t^{-(v(\lambda_0)+1)/2}-1)}{1-t} + C_2\big(v(\lambda_0)\big),
\end{aligned}$$

where $C_2(n)$ is an explicit function of n defined by the last equality.

In **Case II** (2) ($\ell_- > \ell_+$, $\ell_+ < 2m$), we have

$$v_D(\lambda) = 2\ell_+, \quad v_D(\Delta/\lambda) = 4m + 2\ell_- - 2\ell_+.$$

Hence

$$\begin{aligned}
\ell\text{-Int}(x) &= 2t^{-(\ell_+-1)/2}\frac{(\ell_--2\ell_++2m+3)(1-t)+4t}{(1-t)^2} + \frac{-2(\ell_-+2m+1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= 2(2m+\ell_--\ell_+)\frac{t(t^{-(\ell_++1)/2}-1)}{1-t}\\
&\quad + 2t^{-(\ell_+-1)/2}\frac{(-\ell_++3)(1-t)+4t}{(1-t)^2} - \frac{2(\ell_++1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= 2(2m+\ell_--\ell_+)\frac{t(t^{-(\ell_++1)/2}-1)}{1-t}\\
&\quad + 2t^{-(\ell_+-1)/2}\frac{(-\ell_++3)+(\ell_++1)t}{(1-t)^2} - \frac{2(\ell_++1)t}{1-t} + \frac{-8t}{(1-t)^2}\\
&= v(\Delta/\lambda)\frac{2t(t^{-(v(\lambda_0)+1)/2}-1)}{1-t} + C_2\big(v(\lambda_0)\big),
\end{aligned}$$

where $C_2(n)$ is the same function of n as in the last case (this is crucial). $\square$

We return to the proof of Theorem 15.1 for $x_0 = (\lambda_0, 0, 0)$, $\lambda_0 \neq 0$. Note that $-\lambda_0 \notin F_0^{\times,2}$ since x_0 is in the image of $\mathfrak{u}_{1,\mathrm{red}}$, cf. Lemma 11.3. Hence we can apply Theorem 14.9. We have the two subcases (a) and (b) of that theorem.

(a) *Case* (0*i*): $F' \not\simeq F$. By Theorem 14.9 and Lemma 14.7, we have the values of the orbital integrals $\partial\mathrm{Orb}_1(\sigma(x), \phi')$. By comparison with Lemma 15.2, we find that

$$\varphi_1(x) = \Big(C_1 + C_i\big(v(\lambda_0)\big)\Big) \cdot \log q, \quad x \in \mathfrak{b}_{\mathrm{red,rs},1},$$

is a constant, where C_1 is the constant in Theorem 14.9, and $C_i(n)$ is the polynomial of n for i with the same parity as $v(\lambda_0)$.

(b) *Case* (0*ii*): $F' \simeq F$ (so that $v(\lambda_0)$ is odd by $F' = F_0[\sqrt{-\lambda_0}] \simeq F_0[\sqrt{\varpi}]$). Similarly we find by Theorem 14.9 and Lemma 14.7, and by comparing with Lemma 15.2, that

$$\varphi_1(x) = \Big(C_2 + C_1\big(v(\lambda_0)\big)\Big) \cdot \log q, \quad x \in \mathfrak{b}_{\mathrm{red,rs},1},$$

is a constant, where C_2 is the constant in Theorem 14.9, and $C_1(n)$ is the polynomial of n in Lemma 15.2.

Hence Theorem 15.1 is proved for $x_0 = (\lambda_0, 0, 0)$.

We now consider the case $x_0 = (\lambda_0, u_0, w_0)$, $u_0 \neq 0$. Again, we first derive a convenient expression for the quantity $\ell\text{-Int}(x)$ in a small neighborhood of x_0.

Lemma 15.3. *Let $x_0 = (\lambda_0, u_0, w_0)$, where $u_0 \neq 0$. Then for $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ in a small neighborhood of x_0,*

$$\ell\text{-Int}(x) = \begin{cases} 2v(\Delta/u^2)\dfrac{t(t^{-v(u_0)-1}-1)}{(1-t)} + C_1\big(v(u_0)\big), & |\lambda| < |u|^2, \\ 2v(\Delta/u^2)\dfrac{t(t^{-(v(\lambda_0)+1)/2}-1)}{(1-t)} + C_2\big(v(\lambda_0), v(u_0)\big), & |\lambda| > |u|^2, \end{cases}$$

where C_1, C_2 are explicit polynomials.

Proof. First we assume that $\lambda_0 \neq 0$. Then for x near x_0, we have $|\lambda| = |\lambda_0|$, $|u| = |u_0|$, $|w| = |w_0|$, and ℓ_- is very large.

- In **Case II** (1)($\ell_- > \ell_+$, $\ell_+ > 2m$, i.e., $|\lambda| < |u|^2$) we have

$$\begin{aligned} \ell\text{-Int}(x) &= 2t^{-m}\frac{2(1+t)+(\ell_- - 2m-1)(1-t)}{(1-t)^2} + \frac{-2(\ell_- + 2m+1)t}{1-t} + \frac{-8t}{(1-t)^2} \\ &= 2\ell_- \frac{t(t^{-m-1}-1)}{(1-t)} + 2\left(t^{-m}\frac{2(1+t)+(-2m-1)(1-t)}{(1-t)^2} + \frac{-(2m+1)t}{1-t} + \frac{-4t}{(1-t)^2}\right) \\ &= 2\ell_- \frac{t(t^{-m_0-1}-1)}{(1-t)} + C_1(m_0). \end{aligned}$$

- In **Case II** (2) ($\ell_- > \ell_+$, $\ell_+ < 2m$, i.e., $|\lambda| > |u|^2$) we have

$$\begin{aligned} \ell\text{-Int}(x) &= 2t^{-(\ell_+-1)/2}\frac{(\ell_- - 2\ell_+ + 2m+3)(1-t)+4t}{(1-t)^2} + \frac{-2(\ell_- + 2m+1)t}{1-t} + \frac{-8t}{(1-t)^2} \\ &= 2\ell_-\left(t^{-(\ell_{+0}-1)/2}\frac{1}{(1-t)} - \frac{t}{1-t}\right) + C_2(\ell_{+0}, m_0) \\ &= 2\ell_- \frac{t(t^{-(\ell_{+0}+1)/2}-1)}{(1-t)} + C_2(\ell_{+0}, m_0). \end{aligned}$$

Now we assume that $\lambda_0 = 0$. Then, since $0 \neq x_0 \notin \mathfrak{b}_{\mathrm{red,rs}}$, it follows that x_0 has the form $x_0 = (0, u_0, 0)$, $u_0 \neq 0$. Then for x near x_0 we have $|u| = |u_0|$ and ℓ_+, ℓ_- are very large. Then the asymptotic behavior of $\ell\text{-Int}(x)$ follows from **Case I** (1) and **Case II** (1) in §9.2,

$$\ell\text{-Int}(x) = 2\ell_- \frac{t(t^{-m_0-1}-1)}{(1-t)} + C_1(m_0). \qquad \square$$

Now we can finish the proof of Theorem 15.1 for $x_0 = (\lambda_0, u_0, w_0)$ with $u_0 \neq 0$. We are in case (c) of Theorem 14.9. Noting that $v(\Delta/u^2) = \ell_-$, by comparing Lemma 15.3 with Theorem 14.9 and Lemma 14.8, we find that $\varphi_1(x)$ is a constant (explicitly depending on x_0) when $x \in \mathfrak{b}_{\mathrm{red,rs},1}$ is near x_0.

In view of the table in §13 (and the explanation), we have considered all $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ in the closure of $\mathfrak{b}_{\mathrm{red,rs},1}$. This completes the proof of Theorem 15.1.

Part 4. Germ expansion

In this part of the paper, F/F_0 is any quadratic extension of non-archimedean local fields of characteristic not equal to 2 (not necessarily ramified nor of odd residue characteristic). We write $F = F_0[\pi]$ for $\pi = \sqrt{\varpi}$, $\varpi \in F_0^\times$.

16. Statement of the germ expansion

Recall from §11.1 we have

$$\begin{aligned} \pi_{\mathrm{red}}\colon\ \mathfrak{s}_{\mathrm{red}} &\longrightarrow \mathfrak{b}_{\mathrm{red}} = \mathbb{A} \times \mathbb{A} \times \mathfrak{s}_1 \\ y &\longmapsto \big(\lambda(y), u(y), w(y)\big) \end{aligned}$$

where we write y in the block form

$$y = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix},$$

and

$$\lambda(y) = \det A, \quad u(y) = \varpi^{-1}\mathbf{c}\mathbf{b}, \quad \text{and} \quad w(y) = \varpi^{-1}\mathbf{c}A\mathbf{b}.$$

We also have

$$\Delta(y) = \lambda(y)u(y)^2 + w(y)^2.$$

Let y be *any* element in $\mathfrak{s}_{\mathrm{red}}$ (not necessarily semi-simple nor regular). We say that y is *relevant* if its stabilizer H'_y is contained in SL_2. For a relevant element y, the determinant det is well-defined on $H'_y \backslash H'$, and so is $|\det|^s$ for $s \in \mathbb{C}$. Let $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$, and let $y \in \mathfrak{s}_{\mathrm{red}}$ be relevant. We consider the integral

$$\mathrm{Orb}(y, \phi', s) = \tau(H'_y) \int_{H'_y \backslash H'} \phi'(h^{-1}yh)\eta(\det h)|\det h|^s \, dh, \tag{16.1}$$

where $\tau(H'_y) = \mathrm{vol}(H'_y)$ if H'_y is compact and $\tau(H'_y) = 1$ otherwise. In all cases except the one in Lemma 18.1, the integral (16.1) is absolutely convergent when $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of $s \in \mathbb{C}$. Even in the exceptional case, Lemma 18.1 defines $\mathrm{Orb}(y, \phi', s)$ as a meromorphic function. When the integral has no pole at $s = 0$, we use the notation

$$\mathrm{Orb}(y, \phi') := \mathrm{Orb}(y, \phi', 0).$$

For $x_0 \in \mathfrak{b}_{\mathrm{red}}$, the elements of $\mathfrak{s}_{\mathrm{red}}$ in $\pi_{\mathfrak{s}}^{-1}(x_0)$ will be called *x_0-nilpotent.* When $x_0 = 0$, we use the term *nilpotent* instead of 0-nilpotent.

Theorem 16.1. *Let $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ and $x_0 \in \mathfrak{b}_{\mathrm{red}}$. There exist an open neighborhood V_{x_0} of x_0, a section $\sigma\colon \mathfrak{b}_{\mathrm{red,rs}} \to \mathfrak{s}_{\mathrm{red}}$ defined on $V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs}}$, and (explicit) continuous functions $\Gamma_n(x, s)$ on $V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs}}$ such that, as meromorphic functions in the complex variable $s \in \mathbb{C}$,*

$$\mathrm{Orb}\big(\sigma(x), \phi', s\big) = \sum_{n \in \pi_{\mathfrak{s}}^{-1}(x_0)/H'} \Gamma_n(x, s)\, \mathrm{Orb}(n, \phi', s),$$

where the sum runs over an explicit set of representatives of relevant H'-orbits of x_0-nilpotent elements, and where the sum should be replaced by an integral with respect to a suitable measure when there is a continuous family of orbits $n(\mu)$, $\mu \in F_0$ (which only occurs when $x_0 = 0$).

Proof. When $x_0 \in \mathfrak{b}_{\mathrm{red,rs}}$, by (17.4) below it is easy to see that we have

$$\mathrm{Orb}\big(\sigma(x), \phi', s\big) = \mathrm{Orb}\big(\sigma(x_0), \phi', s\big),$$

when x is near x_0, which proves Theorem 16.1 in this case. When $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$, Theorem 16.1 will follow from the explicit germ expansion given by Theorem 17.1 for the case $x_0 = 0$, and by Theorem 18.2 for $x_0 \neq 0$. ∎

We also need a converse to the theorem above, specialized to $s = 0$, proved in §20. Let $\mathcal{C}_1(F_0)$ be the space defined in §19.3.

Theorem 16.2. *Let $\varphi \in C^\infty_{\mathrm{rc}}(\mathfrak{b}_{\mathrm{red,rs}})$. Then φ is an orbital integral function if and only if, for every $x_0 \in \mathfrak{b}_{\mathrm{red}}$, there exists an open neighborhood V_{x_0} of x_0, such that*

$$\varphi(x) = \omega\bigl(\sigma(x)\bigr) \sum_{n \in \pi_{\mathrm{red}}^{-1}(x_0)/H'} \Gamma_n(x,0)\varphi_{x_0}(n) \quad \text{for all} \quad x \in V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs}}, \tag{16.2}$$

where $\varphi_{x_0}(n) \in \mathbb{C}$; when $x_0 = 0$, the sum is to be interpreted as an integral for the one-dimensional family of nilpotent orbits $n(\mu), \mu \in F_0$, and the function $\mu \mapsto \varphi_{x_0}(n(\mu))$ is required to define an element in $\mathcal{C}_1(F_0)$; when $x_0 = (\lambda_0, 0, 0)$ with $F_0[\sqrt{-\lambda_0}] \simeq F$, this is understood as a function of the form $\phi_0 \log|\Delta(x)| + \phi_1$ for constants ϕ_0 and ϕ_1 (cf. (18.15)).

Corollary 16.3. *Let $\varphi \in C^\infty_{\mathrm{rc}}(\mathfrak{b}_{\mathrm{red,rs}})$ be such that the restriction of φ to $V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs,0}}$ is zero. Assume that, for every $x_0 \in \mathfrak{b}_{\mathrm{red}}$, there exists an open neighborhood V_{x_0} of x_0, such that the restriction of φ to $V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs,1}}$ is constant . Then φ is an orbital integral function.*

Proof. It suffices to verify that such φ is of the form (16.2) in Theorem 16.2 for every $x_0 \in \mathfrak{b}_{\mathrm{red}}$. We first consider $x_0 = 0$. In this case the nilpotent orbits in (16.2) are as in Theorem 17.1. By Theorem 17.1 the function $\Gamma_{n_{0,-}}(x,0)|_{V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs,0}}} = -\Gamma_{n_{0,-}}(x,0)|_{V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs,1}}}$ is a (nonzero) constant and $\Gamma_{n_{0,+}}(x,0)|_{V_{x_0} \cap \mathfrak{b}_{\mathrm{red,rs}}}$ is a (nonzero) constant. Therefore, by suitably choosing $\varphi_{x_0}(n_{0,-})$ and $\varphi_{x_0}(n_{0,+})$, we see that φ is of the form

$$\Gamma_{n_{0,+}}(x,0)\varphi_{x_0}(n_{0,+}) + \Gamma_{n_{0,-}}(x,0)\varphi_{x_0}(n_{0,-}).$$

Setting $\varphi_{x_0}(n)$ to be zero for n in the one-dimensional family of nilpotents, we see that φ is of the form (16.2) for $x_0 = 0 \in \mathfrak{b}_{\mathrm{red}}$.

For $x_0 \neq 0$, the proof is similar using Theorem 18.2. ∎

We will be interested in the first derivative of the orbital integral $\partial \operatorname{Orb}(\sigma(x), \phi')$ at $s = 0$ in a neighborhood of x_0. We have a decomposition according to the Leibniz rule

$$\partial \mathrm{Orb}\bigl(\sigma(x), \phi'\bigr) = \partial \mathrm{Orb}_1\bigl(\sigma(x), \phi'\bigr) + \partial \mathrm{Orb}_2\bigl(\sigma(x), \phi'\bigr), \tag{16.3}$$

where we define the two terms as

$$\partial \mathrm{Orb}_1\bigl(\sigma(x), \phi'\bigr) := \sum_{n \in \pi_{\mathrm{red}}^{-1}(x_0)/H'} \left(\frac{d}{ds}\Big|_{s=0} \Gamma_n(x,s)\right) \mathrm{Orb}(n, \phi', 0)$$

and

$$\partial \mathrm{Orb}_2\bigl(\sigma(x), \phi'\bigr) := \sum_{n \in \pi_{\mathrm{red}}^{-1}(x_0)/H'} \Gamma_n(x,0) \left(\frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(n, \phi', s)\right).$$

Remark 16.4. We point out that the first term $\partial\mathrm{Orb}_1(\sigma(x), \phi')$ depends only on the quantities $\mathrm{Orb}(n, \phi') = \mathrm{Orb}(n, \phi', 0)$ (and the intrinsically defined functions $\Gamma_n(x,s)$). In particular, the values of $\mathrm{Orb}(\sigma(x), \phi')$, for regular semi-simple x already determine $\partial\mathrm{Orb}_1(\sigma(x), \phi')$. This observation does not hold for $\partial\mathrm{Orb}_2(\sigma(x), \phi')$.

The explicit germ expansion also shows the following result, proved in §20 below.

Theorem 16.5. *Fix $i \in \{0,1\}$. Let $\phi' \in C^\infty_c(\mathfrak{s}_{\mathrm{red}})$ match $(\phi, 0)$ where $\phi \in C^\infty_c(\mathfrak{u}_{i,\mathrm{red}})$. Then the function*

$$\varphi(x) = \begin{cases} \omega\bigl(\sigma(x)\bigr)\, \partial\mathrm{Orb}_2\bigl(\sigma(x), \phi'\bigr), & x \in \mathfrak{b}_{\mathrm{red,rs},1-i}; \\ 0, & x \in \mathfrak{b}_{\mathrm{red,rs},i} \end{cases}$$

is an orbital integral function.

17. Germ expansion around $x_0 = 0$

In this section, we give the explicit germ expansion around $x_0 \in \mathfrak{b}_{\mathrm{red}}$ when $x_0 = 0$.

17.1. **Statement of the theorem.** The nilpotent orbits in $\mathfrak{s}_{\mathrm{red}}$ are classified in [34, §2.1]. We list here only the relevant orbits.

- a continuous family with representatives

$$n(\mu) := \pi \begin{bmatrix} 0 & \mu & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mu \in F_0 \tag{17.1}$$

with stabilizer N, the upper-triangular unipotent matrices, and

- two regular (i.e., with trivial stabilizer) nilpotents with representatives

$$n_{0,+} = \pi \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad n_{0,-} = {}^t n_{0,+}. \tag{17.2}$$

The nilpotent orbital integrals for the continuous family $n(\mu)$ and for the regular nilpotent orbits $n_{0,\pm}$ are defined by (16.1) and are both holomorphic at $s = 0$, cf. [34, Lem. 2.1]. Note that we choose a Haar measure on $N(F_0)$ by transporting the one on F_0.

We define a section

$$\sigma\colon \mathfrak{b}_{\mathrm{red}} \longrightarrow \mathfrak{s}_{\mathrm{red}}$$

by

$$\sigma(x) = \pi \begin{bmatrix} 0 & -\lambda/\pi^2 & 1 \\ 1 & 0 & 0 \\ u & w/\pi & 0 \end{bmatrix}, \quad x = (\lambda, u, w) \in \mathfrak{b}_{\mathrm{red}}. \tag{17.3}$$

Theorem 17.1. *For $x_0 = 0$ and $x = (\lambda, u, w) \in \mathfrak{b}_{\mathrm{red,rs}}$ near x_0, there is a germ expansion of $\mathrm{Orb}\big(\sigma(x), \phi', s\big)$ as*

$$\int_{F_0} \Gamma_{n(\mu)}(x,s)\,\mathrm{Orb}\big(n(\mu), \phi', s\big)\,d\mu + \eta(\Delta/\varpi)|\Delta/\varpi|^s\,\mathrm{Orb}(n_{0,-}, \phi', s) + \eta(-1)\,\mathrm{Orb}(n_{0,+}, \phi', s),$$

where

$$\Gamma_{n(\mu)}(x,s) = \begin{cases} 0, & \text{if } (u^2\mu - 2w/\pi)^2 - 4\Delta/\varpi \notin F_0^{\times,2}; \\ \dfrac{\eta(-\nu)\big(|\nu|^{-s} + \eta(\Delta/\varpi)|\Delta/\varpi|^{-s}\nu^s\big)}{\big|(u^2\mu - 2w/\pi)^2 - 4\Delta/\varpi\big|^{1/2}}, & \text{if } (u^2\mu - 2w/\pi)^2 - 4\Delta/\varpi \in F_0^{\times,2}. \end{cases}$$

Here ν denotes one of the two roots of

$$u^2\mu = \nu + \frac{\Delta/\varpi}{\nu} + 2w/\pi$$

($\Gamma_{n(\mu)}(x,s)$ is independent of the choice of ν).

Corollary 17.2. *If $x \in \mathfrak{b}_{\mathrm{red,rs,1}}$, then*

$$\Gamma_{n(\mu)}(x, 0) = 0.$$

Proof. When $x \in \mathfrak{b}_{\mathrm{red,rs,1}}$ we have $\eta(\Delta/\varpi) = -1$, cf. Proposition 10.2. The result follows from the formula above. □

The proof of this theorem will occupy the rest of this section. We will rely on [34, §3]. At this point we warn the reader that we will use slightly different notation from [34]. Let $\mathfrak{sl}_{\mathrm{red}}$ be the subspace $\pi^{-1}\mathfrak{s}_{\mathrm{red}}$ of $\mathfrak{gl}_3 = \mathrm{M}_{3,F_0}$. We fix an isomorphism as representations of $H' = \mathrm{GL}_2$,

$$\begin{aligned} \mathfrak{s}_{\mathrm{red}} &\xrightarrow{\sim} \mathfrak{sl}_{\mathrm{red}} \\ x &\longmapsto \pi^{-1}x. \end{aligned}$$

17.2. **Proof of Theorem 17.1.** The proof of Theorem 17.1 is essentially the same as that of Theorem 2.7 in [34] except for some notational changes. We often write $h\cdot x = h^{-1}xh$, for $h \in H'$ and $x \in \mathfrak{s}_{\mathrm{red}}$. To be consistent with [34], for $x \in \mathfrak{b}_{\mathrm{rs}}$ we rewrite (16.1) as

$$\mathrm{Orb}(x, \phi', s) = \int_{H'/H'_x} \phi'(hxh^{-1})\eta(\det h)|\det h|^{-s}\, dh.$$

We use the Iwasawa decomposition $H'(F_0) = KAN$ for $K = \mathrm{SL}_2(O_{F_0})$,

$$h = k \begin{bmatrix} b & \\ & a^{-1}b \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}, \quad dg = dk\, \frac{da\, db\, dt}{|b|}.$$

We may write $\mathrm{Orb}\big(\sigma(x), \phi', s\big)$ as (cf. [34, (3.1)][17])

$$\int_{a,b,t \in F_0} \phi'_K \left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & b \\ 1/a & -t & 0 \\ b^{-1}u & (w/\pi - ut)ab^{-1} & 0 \end{bmatrix} \right) \eta(a)|a^{-1}b^2|^{-s}\, \frac{da\, db\, dt}{|b|}. \tag{17.4}$$

Here we used the K-invariant function

$$\phi'_K(x) := \int_K \phi'(kxk^{-1})dk.$$

Without loss of generality, we may assume that ϕ', and hence ϕ'_K, is invariant under translation by $\mathfrak{s}_{\mathrm{red}}(O_{F_0}) := \pi\mathfrak{gl}_3(O_{F_0}) \cap \mathfrak{s}$.

Now all the equations from (3.1) to (3.20) in [34] hold for the orbital integral $\mathrm{Orb}(\sigma(x), \phi', s)$ (instead of simply its value at $s = 0$). By this we mean to interpret them as the integral against $\eta(a)|a^{-1}b^2|^s \frac{da\, db\, dt}{|b|}$. Indeed, all of these equations are derived from partitioning the domain of integration into various pieces, and the fact that the function ϕ_K is invariant under translation by $\mathfrak{s}_{\mathrm{red}}(O_{F_0})$ with compact support. It is never used in [34, §3] that $s = 0$.

Lemma 17.3. *The term (3.4) in* [34] *is equal to*

$$\int_{F_0} \Gamma_{n(\mu)}(x, s)\, \mathrm{Orb}\big(n(\mu), \phi', s\big)\, d\mu.$$

Proof. After a suitable substitution, we may write [34, (3.4)] as

$$\eta(-1)|u|^{-1} \int \phi'_K \left(\pi \begin{bmatrix} 0 & \frac{(t+w/\pi)^2 + u^2\lambda/\varpi}{u^2 t} ab & b \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \right) \eta(abt)|a^{-1}bt|^{-s}\, \frac{da\, db\, dt}{|t|}.$$

This is equal to

$$\eta(-1)|u|^{-1} \int_{F_0} \mathrm{Orb}\left(n \left(\frac{(t + w/\pi)^2 + u^2\lambda/\varpi}{u^2 t} \right), \phi', s \right) \eta(t)|t|^{-s}\, \frac{dt}{|t|}.$$

We rewrite this as

$$\eta(-1)|u|^{-1} \int_{F_0} \mathrm{Orb}\Big(n\big(u^{-2}(t + t^{-1}\Delta/\varpi + 2w/\pi)\big), \phi', s \Big) \eta(t)|t|^{-s}\, \frac{dt}{|t|}. \tag{17.5}$$

This is the form of the germ expansion we will use to do calculations later on. The rest is the same as the proof of [34, Lem. 3.1]. $\square$

Lemmas 3.2 and 3.3 of [34] remain unchanged. We recall them and indicate the necessary changes in the proof.

Lemma 17.4. *The sum of (3.3) and (3.5) in* [34] *is zero. The sum of (3.6) and (3.15) in* [34] *is zero.*

[17] We note the following notational changes: 1) a sign difference in λ; 2) a, b in [34] become $u, w/\pi$; and 3) the measure $dx\, dy\, du$ becomes $da\, db\, dt$.

Proof. The proof in [34] can actually be simplified. In both sums, it suffices to show that

$$\int_{|y|\leq 1} |y|^{2s-1} dy + \int_{|y|>1} |y|^{2s-1} dy = 0, \tag{17.6}$$

where we understand both integrals as meromorphic functions of $s \in \mathbb{C}$ obtained by analytic continuation as follows. The first integral converges when $\mathrm{Re}(s) > 0$,

$$\int_{|y|\leq 1} |y|^{2s-1} dy = \frac{1}{1-q^{-s}} \int_{|y|=1} \frac{dy}{|y|},$$

and the second one converges when $\mathrm{Re}(s) < 0$,

$$\int_{|y|>1} |y|^{2s-1} dy = \frac{q^s}{1-q^s} \int_{|y|=1} \frac{dy}{|y|}.$$

These equalities give the claimed analytic continuation. The sum of the two terms is then obviously zero. $\square$

Lemma 17.5. *The sum of the following terms in* [34] *is zero: (3.2), (3.8), (3.10), (3.16), (3.17), (3.20).*

Proof. The same proof as that of Lem. 3.3 in [34] still works, noting that the only ingredient is the identity (17.6) above (for instance, in loc. cit. one only uses (17.6), when splitting the term (II) into (3.12) plus the term following it). $\square$

Lemma 17.6. *The sum of (3.12) and (3.19) in* [34] *is zero.*

Proof. We work with the corresponding function ϕ' on $\mathfrak{sl}_{\mathrm{red}}$. The integral (3.12) is equal to

$$\begin{aligned}\int_{H'} \phi'\left(h \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ u & w & 0 \end{bmatrix} h^{-1}\right) \eta(\det h)|\det h|^{-s}\, dh \\ = \int_{H'} \phi'\left(h \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} h^{-1}\right) \eta(\det h)|\det h|^{-s}\, dh.\end{aligned}$$

After a substitution, the integral (3.19) is equal to

$$\begin{aligned}-\int_{H'} \phi'\left(h \begin{bmatrix} 0 & -(w/u)^2 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{bmatrix} h^{-1}\right) \eta(\det h)|\det h|^{-s}\, dh \\ = -\int_{H'} \phi'\left(h \begin{bmatrix} 0 & 0 & 0 \\ (w/u)^2 & 0 & 0 \\ 0 & u & 0 \end{bmatrix} h^{-1}\right) \eta(\det h)|\det h|^{-s}\, dh \\ = -\int_{H'} \phi'\left(h \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} h^{-1}\right) \eta(\det h)|\det h|^{-s}\, dh,\end{aligned}$$

where the first and the last equality follow, respectively, from the substitutions

$$h \longmapsto h \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad h \longmapsto h \begin{bmatrix} w/u & 0 \\ 0 & u/w \end{bmatrix}.$$

$\square$

To complete the proof, it suffices to treat the remaining terms: (3.7) and (3.9) in [34]. The term (3.7) in [34] yields, after a suitable substitution,

$$\begin{aligned}\int \phi'_K\left(\pi\begin{bmatrix} t & -at^2 & b\\ 1/a & -t & 0\\ 0 & 0 & 0\end{bmatrix}\right)\eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|}\\ &= \int_{H'}\phi'\left(h\cdot\pi\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix}h^{-1}\right)\eta(\det h)|\det h|^{-s}\,dh\\ &= \eta(-1)\operatorname{Orb}(n_{0,+},\phi',s).\end{aligned}$$

The term (3.9) in [34] yields, after suitable substitutions,

$$\begin{aligned}\int \phi'_K\left(\pi\begin{bmatrix} 0 & (-\lambda/\varpi-(w/u\pi)^2)a & 0\\ 0 & 0 & 0\\ b^{-1}u & (w/\pi-at)ab^{-1} & 0\end{bmatrix}\right)\eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|}\\ &= \int_{H'}\phi'\left(h\cdot\pi\begin{bmatrix}0&-\lambda/\varpi-(w/u\pi)^2&0\\0&0&0\\u&0&0\end{bmatrix}h^{-1}\right)\eta(\det h)|\det h|^{-s}\,dh\\ &= \eta(\Delta/\varpi)|\Delta/\varpi|^{-s}\operatorname{Orb}(n_{0,-},\phi',s).\end{aligned}$$

This completes the proof of Theorem 17.1.

18. Germ expansion around $x_0 \neq 0$

In this section we consider the germ expansion near a non-zero element $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$.

18.1. **Orbits in $\mathfrak{s}_{\mathrm{red}} \smallsetminus \mathfrak{s}_{\mathrm{red,rs}}$.** We need to classify the orbits in the fiber of $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$. Let

$$y_0 = \begin{bmatrix} A & \mathbf{b}\\ \mathbf{c} & 0\end{bmatrix} \in \mathfrak{s}_{\mathrm{red}}$$

be a semi-simple element in the fiber of x_0. It follows that the dimension r of the subspace spanned by $\mathbf{b}, A\mathbf{b}$ is then either zero or one.

Case $r = 0$. Then $\mathbf{b} = 0$ and $x_0 = (\lambda_0, 0, 0)$ where $\lambda_0 \neq 0$. We introduce the semi-simple quadratic F_0-algebra,

$$F' = F_0[X]/(X^2 + \lambda_0).$$

In the fiber of $x_0 = (\lambda_0, 0, 0)$, there is one semi-simple orbit, and there are two or four non-semi-simple orbits, depending on whether F' is isomorphic to F or not.

Subcase 0i: $F' \not\simeq F$. In the fiber of such x_0, there is one semi-simple orbit with representative

$$y_0 = \pi\begin{bmatrix}0 & -\lambda_0/\varpi & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{18.1}$$

There are two non-semi-simple orbits with representatives

$$y_+ = \pi\begin{bmatrix}0 & -\lambda_0/\varpi & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad y_- = \pi\begin{bmatrix}0 & -\lambda_0/\varpi & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \tag{18.2}$$

with trivial stabilizer.

Subcase 0ii: $F' \simeq F$. In this case there is one semi-simple orbit with representative

$$y_0 = \pi\begin{bmatrix}\alpha & 0 & 0\\ 0 & -\alpha & 0\\ 0 & 0 & 0\end{bmatrix},\quad \alpha^2 = -\lambda_0/\varpi,$$

and there are four non-semi-simple with representatives

$$y_{++} = \pi \begin{bmatrix} \alpha & 0 & 1 \\ 0 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad y_{+-} = \pi \begin{bmatrix} \alpha & 0 & 1 \\ 0 & -\alpha & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{18.3}$$

and

$$y_{--} = {}^t y_{++}, \quad y_{-+} = {}^t y_{+-}, \tag{18.4}$$

with trivial stabilizer.

Case $r = 1$. Then $\mathbf{b} \neq 0$ and $A\mathbf{b}$ is a multiple of $\mathbf{b}$, i.e., $\mathbf{b}$ is an eigenvector of A, and $x_0 = (\lambda_0, u_0, w_0)$ with $u_0 \neq 0$. We may choose a semi-simple representative as

$$y_0 = \pi \begin{bmatrix} \alpha & 0 & 1 \\ 0 & -\alpha & 0 \\ u_0 & 0 & 0 \end{bmatrix}, \quad \alpha \in F_0, \quad u_0 \neq 0, \quad \alpha^2 = -\lambda_0/\varpi.$$

The stabilizer of y_0 is GL_1 sitting inside GL_2 in the upper left corner. However, the character $\eta \circ \det$ is nontrivial on the stabilizer and hence it is not a relevant orbit. The non-semi-simple representatives are

$$y_{+} = \pi \begin{bmatrix} \alpha & 0 & 1 \\ 1 & -\alpha & 0 \\ u_0 & 0 & 0 \end{bmatrix}, \quad y_{-} = \pi \begin{bmatrix} \alpha & 1 & 1 \\ 0 & -\alpha & 0 \\ u_0 & 0 & 0 \end{bmatrix}, \tag{18.5}$$

and they have trivial stabilizer.

18.2. **Orbital integrals.** We first define the orbital integrals of the x_0-nilpotent elements. All of them are defined by (16.1) with one exceptional case:

Lemma 18.1. *Let $x_0 = (\lambda_0, 0, 0)$ with $\lambda_0 \neq 0$. Assume that $F' \simeq F$, i.e., Case 0ii. Let $y = y_{\pm\pm}$ be a non-semi-simple element on $\mathfrak{s}_{\mathrm{red}}$ mapping to x_0. Then for $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$, the integral*

$$\mathrm{Orb}(y, \phi, s_1, s_2) = \int_{a,b \in F_0^\times, t \in F_0} \phi'_K\left(\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} a & \\ & b \end{bmatrix} \cdot y\right) \eta(ab)|a|^{s_1}|b|^{s_2}\, d^\times a\, d^\times b\, dt$$

is absolutely convergent when $\mathrm{Re}(s_1) \gg 0, \mathrm{Re}(s_2) \gg 0$. *It has a meromorphic continuation to $(s_1, s_2) \in \mathbb{C}^2$; its restriction to the diagonal $s_1 = s_2$ is meromorphic and is holomorphic at $(s_1, s_2) = (0, 0)$.*

Proof. The proof is analogous to that of [34, Lem. 2.1] by the method of Tate's thesis. We omit the details. □

In this case we define the orbital integral $\mathrm{Orb}(y, \phi', s)$ to be the value at $s_1 = s_2 = s$. By Lemma 18.1, this is a meromorphic function of s.

Theorem 18.2. *Fix $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ with $x_0 \neq 0$. For $x \in \mathfrak{b}_{\mathrm{red,rs}}$ in a small neighborhood of x_0, the orbital integral $\mathrm{Orb}(\sigma(x), \phi', s)$ is equal*

(a) *in case* (0*i*) *to*

$$\mathrm{Orb}(y_+, \phi', s) + \eta(\lambda^{-1}\Delta)|\lambda^{-1}\Delta|^{-s}\,\mathrm{Orb}(y_-, \phi', s),$$

where $\sigma(x)$ is the section defined in (17.3)*;*

(b) *in case* (0*ii*) *to*

$$\mathrm{Orb}(y_{++}, \phi', s) + \eta(z_1)|z_1|^{-s}\,\mathrm{Orb}(y_{-+}, \phi', s) \\ + \eta(z_2)|z_2|^{-s}\,\mathrm{Orb}(y_{+-}, \phi', s) + \eta(z_1z_2)|z_1z_2|^{-s}\,\mathrm{Orb}(y_{--}, \phi', s),$$

where the section $\sigma \colon \mathfrak{b}_{\mathrm{red,rs}} \to \mathfrak{s}_{\mathrm{red,rs}}$ in a neighborhood of x_0 is defined by

$$\sigma(x) = \pi \begin{bmatrix} \alpha & 0 & 1 \\ 0 & -\alpha & 1 \\ z_1 & z_2 & 0 \end{bmatrix}, \tag{18.6}$$

with entries defined by the following identities,

$$\lambda = -\alpha^2\varpi, \quad u = z_1 + z_2, \quad w = \alpha(z_1 - z_2)\pi, \quad \Delta = \lambda u^2 + w^2 = -4\alpha^2\varpi z_1 z_2;$$

(c) *in case* (1) *to*

$$\mathrm{Orb}(y_+, \phi', s) + \eta(\Delta/\varpi)|u^{-2}\Delta/\varpi|^{-s}\,\mathrm{Orb}(y_-, \phi', s).$$

where $\sigma(x)$ *is the section defined in* (17.3).

Moreover, all the orbital integrals above are holomorphic at $s = 0$ *except in case* $(0i)$ *when* $F' = F_0 \times F_0$, *in which case* $\mathrm{Orb}(y_+, \phi', s)$ *and* $\mathrm{Orb}(y_-, \phi', s)$ *both have a simple pole at* $s = 0$.

18.3. **Proof of Theorem 18.2.** Let $x_0 = (\lambda_0, u_0, w_0) \neq 0 \in \mathfrak{b}_{\mathrm{red}}$. We distinguish three cases, labeled by $(0i)$, $(0ii)$ and (1), according to the case distinction in subsection 18.1. Fix a real number R such that the support of ϕ' is contained in the set

$$\bigl\{y = \pi \cdot (y_{ij}) \in \mathfrak{s}_{\mathrm{red}} \bigm| |y_{ij}| \leq R\bigr\}. \tag{18.7}$$

Case $(0i)$ (i.e., $r = 0$, $\lambda_0 \neq 0$ and $F' \neq F$): Then $x_0 = (\lambda_0, 0, 0)$ with $\lambda_0 \neq 0$. We use (17.4) to express the integral $\mathrm{Orb}(\sigma(x), \phi', s)$ for $x = (\lambda, u, v) \in \mathfrak{b}_{\mathrm{red}}$ in a small neighborhood of $(\lambda_0, 0, 0)$. We therefore assume that $|\lambda| = |\lambda_0| \neq 0$ and hence the integrands have the following constraints,

$$|t| \leq R, \quad |a| \leq R/|\lambda_0|.$$

We split the integral over b as a sum of two pieces, according as $|b| > 1$ or $|b| \leq 1$. The contribution for $|b| > 1$ is equal, when (u, w) is small enough, to

$$\int_{|b|>1, a, t\in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & b \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|}.$$

This can be written as

$$\int_{a,b\in F, t\in F} \cdots |\det h|^s dh - \int_{|b|\leq 1, a\in F, t\in F} \cdots |\det h|^s\,dh,$$

where both integrals converge absolutely when $\mathrm{Re}(s) > 0$. The first term extends to a meromorphic function

$$\mathrm{Orb}(y_+, \phi', s) = \int_{a,b,t\in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & b \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|},$$

while the second term is

$$\begin{aligned}
&\int_{|b|\leq 1, a, t\in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & 0 \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|} \\
&\quad = \left(\int_{|b|\leq 1} |b|^{2s}\frac{db}{|b|}\right) \int_{a,t\in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & 0 \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \eta(a)|a|^s\,da\,dt \\
&\quad = \zeta(-2s)Q(s),
\end{aligned} \tag{18.8}$$

where we denote by $Q(s)$ the second integral in the next-to-last equation. From $F' \not\simeq F$ it follows that $-\lambda_0/\varpi$ (hence $-\lambda/\varpi$) is not a square. The norms $|t|, |a|$ and $|a|^{-1}$ are all bounded independent of s and hence $Q(s)$ is an entire function. Again by $F_0[\sqrt{-\lambda_0}] \not\simeq F$, the quadratic character η is nontrivial on the stabilizer of the semi-simple representative y_0. It follows easily that $Q(0) = 0$, and hence the second term (18.8) is holomorphic at $s = 0$ and hence so is $\mathrm{Orb}(y_+, \phi', s)$.

Now we consider the contribution from the piece $|b| \leq 1$,

$$\int_{|b|\leq 1, a, t\in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & 0 \\ 1/a & -t & 0 \\ b^{-1}u & (w/\pi - ut)ab^{-1} & 0 \end{bmatrix}\right) \eta(a)|\det h|^{-s}\,\frac{da\,db\,dt}{|b|}.$$

This can be written as

$$\int_{a,b\in F, t\in F} \cdots |\det h|^{-s}\,dh - \int_{|b|>1, a\in F, t\in F} \cdots |\det h|^{-s}\,dh,$$

where both integrals converge absolutely when $\mathrm{Re}(s) < 0$. The second term is equal to

$$\int_{|b|>1, a, t \in F_0} \phi'_K \left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - t^2) & 0 \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \eta(a) |a^{-1} b^2|^{-s} \frac{da\, db\, dt}{|b|}$$
$$= \left(\int_{|b|>1} |b|^{-2s} \frac{db}{|b|} \right) Q(s)$$
$$= q^{-2s} \zeta(2s) Q(s).$$

Now note that $\zeta(-2s) + q^{-2s}\zeta(2s) = 0$. We see that this last term cancels (18.8). The first term can be rewritten as

$$\mathrm{Orb}\left(\begin{bmatrix} 0 & -\lambda/\varpi & 0 \\ 1 & 0 & 0 \\ u & w/\pi & 0 \end{bmatrix}, \phi', s \right) = \int_{H'} \phi' \left(h^{-1} \cdot \pi \begin{bmatrix} 0 & -\lambda/\varpi & 0 \\ 1 & 0 & 0 \\ u & w/\pi & 0 \end{bmatrix} h \right) \eta(\det h) |\det h|^s \, dh.$$

Since u and w cannot be simultaneously zero, we may make a change of variables $h \mapsto h_0 h$, where

$$h_0 := \frac{1}{\det h_1} h_1 \quad \text{with} \quad h_1 := \begin{bmatrix} u & -w/\pi \\ w\pi/\lambda & u \end{bmatrix}.$$

The integral becomes

$$\eta(h_0) |\det h_0|^s \int_H \phi' \left(h^{-1} \cdot \pi \begin{bmatrix} 0 & -\lambda/\varpi & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} h \right) \eta(h) |\det h|^s \, dh$$
$$= \eta(h_0) |\det h_0|^s \, \mathrm{Orb}(y_-, \phi', s).$$

Now note that $\det h_1 = \lambda^{-1} w^2 + u^2 = \lambda^{-1}\Delta$, and $\det h_0 = \det(h_1)^{-1}$. In summary we have

$$\mathrm{Orb}(y, \phi', s) = \mathrm{Orb}(y_+, \phi', s) + \eta(\lambda^{-1}\Delta) |\lambda^{-1}\Delta|^{-s} \, \mathrm{Orb}(y_-, \phi', s), \tag{18.9}$$

which proves Theorem 18.2 in this case.

Case $(0ii)$ (i.e., $r = 0$ and $F' = F$): In this case we may assume that

$$-\lambda_0/\varpi = \alpha_0^2 \neq 0.$$

We use the section $\sigma(x)$ defined by (18.6). When $x = (\lambda, u, w)$ is in a small neighborhood of $(\lambda_0, 0, 0)$, we may assume $|\alpha| = |\alpha_0|$. We use a variant of the Iwasawa decomposition

$$h = k \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & b \end{bmatrix}, \quad dg = dk \, \frac{da\, db\, dt}{|a||b|}.$$

Consider

$$\phi'_{KN}(y) = \int_{KN} \phi'(kn \cdot x) \, dk\, dn, \quad y = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix},$$

which may be viewed as a function of $(\mathbf{b}, \mathbf{c})$ on $M_{1,2}(F_0) \times M_{2,1}(F_0)$. For $\alpha \neq 0$, we see that the integral is absolutely convergent. Then we have

$$\mathrm{Orb}(y, \phi', s) = \int \phi'_{KN} \left(\pi \begin{bmatrix} \alpha & 0 & a \\ 0 & -\alpha & b \\ z_1 a^{-1} & z_2 b^{-1} & 0 \end{bmatrix} \right) \eta(ab) |ab|^{-s} \frac{da\, db}{|a||b|}.$$

Before we proceed, we consider a toy model: let $F_0^\times$ act on $F_0 \times F_0$ by $a \cdot (x, y) = (a^{-1}x, ay)$ and let η be a quadratic character (possibly trivial).

Lemma 18.3. *Let $\phi \in C_c^\infty(F_0 \times F_0)$, and define for $x \in F_0^\times$,*

$$\Phi(x, s) = \int_{F_0^\times} \phi(x/a, a) \eta(a) |a|^s \, d^\times a,$$

which is absolutely convergent when $\mathrm{Re}(s) \gg 0$. *For* x *in a small neighborhood of* 0 *(depending on* ϕ*),*

$$\Phi(x,s) = \int_{F_0^\times} \phi(0,a)\eta(a)|a|^s\, d^\times a + \eta(x)|x|^s \int_{F_0^\times} \phi(1/a,0)\eta(a)|a|^s\, d^\times a.$$

The identity is understood after an analytic continuation of each term on the right-hand side as a meromorphic function of $s \in \mathbb{C}$ *(without pole at* $s = 0$*, if* η *is non-trivial).*

Proof. The proof is again analogous to that of [34, Lem. 2.1]. We omit the details. □

Using this lemma, we obtain that $\mathrm{Orb}(\sigma(x), \phi', s)$ is, in a neighborhood of x_0, the value of the following sum when $s_1 = s_2 = s$,

$$\begin{aligned}\mathrm{Orb}(y_{++}, \phi', s_1, s_2) &+ \eta(z_1)|z_1|^s \,\mathrm{Orb}(y_{-+}, \phi', s_1, s_2)\\ &+ \eta(z_2)|z_2|^s \,\mathrm{Orb}(y_{+-}, \phi', s_1, s_2) + \eta(z_1 z_2)|z_1 z_2|^s \,\mathrm{Orb}(y_{--}, \phi', s_1, s_2).\end{aligned} \tag{18.10}$$

Theorem 18.2 in case $(0ii)$ now follows easily from this equality.

Case (1) (i.e., $r = 1$): In this case $u_0 \neq 0$. We now use the expression (17.4). We first observe that the integrand is constrained by

$$|t| \leq R, \quad |u|/R \leq |b| \leq R,$$

where R is as in (18.7). We only consider the case $\lambda_0 = 0$ (hence $w_0 = 0$). Otherwise, the proof of the previous cases still applies.

We break the integral over a up into two pieces: $|a|$ is large or small. Choose a constant C. Note that we will consider λ, w close to zero, and $|u| = |u_0| \neq 0$. When $|a| \leq C$, so that $|a\lambda| \leq 1$ and $|aw|R \leq 1$, we have

$$\int_{|a|\leq C, b, t \in F_0} \phi'_K\left(\pi \begin{bmatrix} t & -at^2 & b\\ 1/a & -t & 0\\ b^{-1}u & -utab^{-1} & 0\end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s} \frac{da\, db\, dt}{|b|}. \tag{18.11}$$

Now consider $|a| > C$. Substitute $t \to t + w/u$, and note that we may assume $|w/u| < 1$:

$$\int_{|a|> C, b, t \in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda/\varpi - (t - w/\pi u)^2) & b\\ 1/a & -t & 0\\ b^{-1}u & -utab^{-1} & 0\end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s} \frac{da\, db\, dt}{|b|}.$$

The condition $|-utab^{-1}| < R$ implies that

$$|at| < \frac{R^2}{|u|} \quad \text{and} \quad \left|at^2\right| = \left|\frac{(at)^2}{a}\right| < \frac{R^4}{|u|^2 C}.$$

Furthermore, for C sufficiently large, we have $|at^2| < \frac{R^2}{|u|} < 1$. Hence the last integral becomes

$$\int_{|a|> C, b, t \in F_0} \phi'_K\left(\pi \begin{bmatrix} 0 & a(-\lambda/\varpi - (w/\pi u)^2) & b\\ 0 & 0 & 0\\ b^{-1}u & -utab^{-1} & 0\end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s} \frac{da\, db\, dt}{|b|}. \tag{18.12}$$

A similar argument as in case $(0i)$ shows that the sum of (18.11) and (18.12) can be written as

$$\mathrm{Orb}\left(\pi \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ u & 0 & 0\end{bmatrix}, \phi', s\right) + \mathrm{Orb}\left(\pi \begin{bmatrix} 0 & -\lambda/\varpi - (w/\pi u)^2 & 1\\ 0 & 0 & 0\\ u & 0 & 0\end{bmatrix}, \phi', s\right). \tag{18.13}$$

Note that $\Delta = \lambda u^2 + w^2$. It follows that the second summand in (18.13) is equal to

$$\eta(\Delta/\varpi)|\Delta/\varpi u^2|^{-s}\, \mathrm{Orb}\left(\pi \begin{bmatrix} 0 & 1 & 1\\ 0 & 0 & 0\\ u & 0 & 0\end{bmatrix}, \phi', s\right).$$

In summary we have proved that in the case $r = 1$, when $x = (\lambda, u, w)$ is close to (λ_0, u_0, w_0), the integral $\mathrm{Orb}(\sigma(x), \phi', s)$ is equal to

$$\mathrm{Orb}(y_+, \phi', s) + \eta(\Delta/\varpi)|u^{-2}\Delta/\varpi|^{-s}\, \mathrm{Orb}(y_-, \phi', s). \tag{18.14}$$

The equations (18.9), (18.10), and (18.14) together complete the proof of Theorem 18.2 in this case.

18.4. **The exceptional case.** Theorem 18.2 gives the germ expansion for $\mathrm{Orb}(\sigma(x), \phi', s)$ at $s = 0$, except in case $(0i)$ when $F' \simeq F_0 \oplus F_0$, in which case both $\mathrm{Orb}(y_+, \phi', s)$ and $\mathrm{Orb}(y_-, \phi', s)$ have a pole at $s = 0$.

Corollary 18.4. *Fix $x_0 = (\lambda_0, 0, 0) \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$ with $\lambda_0 \neq 0$. Assume that $F' \simeq F_0 \times F_0$.*

(a) *The sum $\mathrm{Orb}(y_+, \phi', s) + \mathrm{Orb}(y_-, \phi', s)$ is holomorphic at $s = 0$. Denote by $\mathrm{Orb}(y_\pm, \phi')$ its value at $s = 0$.*

(b) *For $x \in \mathfrak{b}_{\mathrm{red,rs}}$ in a small neighborhood of x_0, the orbital integral $\mathrm{Orb}\big(\sigma(x), \phi', 0\big)$ (for $\sigma(x)$ defined by (17.3)) is equal to*

$$\mathrm{Orb}(y_\pm, \phi') - \frac{\log|\lambda^{-1}\Delta|}{2\log q}\,\mathrm{Orb}(y_0, \phi'), \tag{18.15}$$

and

$$\mathrm{Orb}(y_0, \phi') = \int_{\mathrm{PGL}_2(F_0)} \phi'(h^{-1} y_0 h)\eta(h)\,dh, \quad y_0 = \pi \begin{bmatrix} 0 & -\lambda_0/\varpi & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the measure on $\mathrm{PGL}_2(F_0)$ is the quotient measure on $\mathrm{GL}_2(F_0)$ divided by that of $F_0^\times$ with $\mathrm{vol}(O_{F_0}^\times) = 1$.

Proof. We first claim that *the meromorphic functions $\mathrm{Orb}(y_+, \phi', s)$, resp. $\mathrm{Orb}(y_-, \phi', s)$, have a simple pole at $s = 0$ with residue*

$$\mp\,\mathrm{Orb}(y_0, \phi')\frac{1}{2\log q}.$$

We now prove the corollary assuming the claim. The claim immediately implies part (a). To show (b), we note that the assumption $F' \simeq F_0 \times F_0$ is equivalent to $-\lambda_0 \in F_0^{\times,2}$. In this case, for all $x \in \mathfrak{b}_{\mathrm{red,rs}}$ near x_0, we always have $x \in \mathfrak{b}_{\mathrm{red,rs},0}$ (cf. Lemma 11.3), and hence $\eta(-\Delta(x)) = 1$ (cf. Proposition 10.2). Hence $\eta(\Delta(x)/\lambda(x)) = \eta(-\Delta(x))/\eta(-\lambda_0) = 1$. By Theorem 18.2, we have for regular semi-simple x near x_0,

$$\mathrm{Orb}\big(\sigma(x), \phi', s\big) = \big(\mathrm{Orb}(y_+, \phi', s) + \mathrm{Orb}(y_-, \phi', s)\big) + (|\lambda^{-1}\Delta|^{-s} - 1)\,\mathrm{Orb}(y_-, \phi', s),$$

where both terms in the right-hand side are holomorphic at $s = 0$ by the claim. Hence

$$\mathrm{Orb}\big(\sigma(x), \phi', 0\big) = \big(\mathrm{Orb}(y_+, \phi', s) + \mathrm{Orb}(y_-, \phi', s)\big)\big|_{s=0} + (|\lambda^{-1}\Delta|^{-s} - 1)\,\mathrm{Orb}(y_-, \phi', s)\big|_{s=0}.$$

The first term is now $\mathrm{Orb}(y_\pm, \phi')$ by (a). The second term is given by $-\log|\lambda^{-1}\Delta|$ times the residue of $\mathrm{Orb}(y_-, \phi', s)$ at $s = 0$. By the claim we complete the proof of (b).

We now prove the claim. We only treat $\mathrm{Orb}(y_+, \phi', s)$, since the other case is similar. By (17.4), $\mathrm{Orb}(y_+, \phi', s)$ is equal to

$$\int_{a,b,t \in F_0} \phi'_K\left(\pi \begin{bmatrix} t & a(-\lambda_0/\varpi - t^2) & b \\ 1/a & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \eta(a)|a^{-1}b^2|^{-s}\,\frac{da\,db\,dt}{|b|}.$$

We may view this as an integral of the form

$$\mathrm{Orb}(y_+, \phi', s) = \int_{b \in F_0} \Phi_s(b)|b|^{-2s}\,\frac{db}{|b|}, \tag{18.16}$$

where $\Phi_s(b)$ extends to an entire function in $s \in \mathbb{C}$. We may find the value

$$\begin{aligned}\Phi_0(0) &= \int_{a,t\in F_0} \phi'_K\left(\pi\begin{bmatrix} t & a(-\lambda_0/\varpi - t^2) & b\\ 1/a & -t & 0\\ 0 & 0 & 0\end{bmatrix}\right)\eta(a)\,da\,dt\\ &= (1-q^{-1})^{-1}\int_{h\in \mathrm{PGL}_2(F_0)} \phi'\left(h^{-1}\pi\begin{bmatrix} 0 & -\lambda_0/\varpi & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}h\right)\eta(\det h)\,dh\\ &= (1-q^{-1})^{-1}\operatorname{Orb}(y_0,\phi'),\end{aligned}$$

where the extra factor is due to the different choice of measures.

By Tate's thesis, the integral (18.16) has a simple pole at $s = 0$ with residue given by $\Phi_0(0)$ times the residue of

$$\int_{|b|\le 1} |b|^{-2s}\frac{db}{|b|} = \sum_{i\ge 0} q^{2is}(1-q^{-1}) = \frac{1-q^{-1}}{1-q^{2s}}.$$

This last term has residue $-(1-q^{-1})/2\log q$. This shows that the function $\operatorname{Orb}(y_+,\phi',s)$ has a simple pole at $s = 0$ with residue

$$-\operatorname{Orb}(y_0,\phi')\frac{1}{2\log q}.$$

This completes the proof of the claim. $\square$

19. Germ expansion for $\mathfrak{u}_{\mathrm{red}}$, and matching

In this section we present the germ expansion for the orbital integrals on $\mathfrak{u}_{\mathrm{red}}$ for $\mathfrak{u} = \mathfrak{u}(W)$, where W is either W_0 or W_1, i.e., $\mathfrak{u} = \mathfrak{u}_0$ or $\mathfrak{u} = \mathfrak{u}_1$.

We consider the invariants (cf. §8.2 and §11.2)

$$\pi_{\mathfrak{u}}\colon \mathfrak{u}_{\mathrm{red}} \longrightarrow \mathfrak{b}_{\mathrm{red}} = \mathbb{A}\times\mathbb{A}\times\mathfrak{s}_1$$

given by the formulas (11.1) (in the case of $\mathfrak{s}_{\mathrm{red}}$, but the cases of $\mathfrak{u}_{0,\mathrm{red}}$ and $\mathfrak{u}_{1,\mathrm{red}}$ are the same, comp. (8.8) for $\mathfrak{u}_{1,\mathrm{red}}$ and (11.4) for $\mathfrak{u}_{0,\mathrm{red}}$),

$$\begin{bmatrix} A & \mathbf{b}\\ \mathbf{c} & d\end{bmatrix} \longmapsto (\lambda, u, w).$$

19.1. **Germ expansion around** $x_0 = 0$**.** The germ expansion around $x_0 = 0$ for $\mathfrak{u}_{\mathrm{red}}$ is stated in [34, Th. 2.8]. Since we will not use it directly, let us not repeat it here. We only recall the classification of the nilpotent orbits and their orbital integrals, which are used in our calculations in Part 3.

The nilpotent orbits for $\mathfrak{u}_{0,\mathrm{red}}$ are classified in [34, §2.1]. For our purposes we only need $\{0\}$ and the continuous family

$$n(\beta) := \pi\begin{bmatrix} 0 & \beta\pi & 1\\ 0 & 0 & 0\\ 0 & \pi & 0\end{bmatrix} \in \mathfrak{u}_{0,\mathrm{red}}, \quad \beta\in F_0. \tag{19.1}$$

The stabilizer of $n(\beta)$ is the standard unipotent subgroup N sitting inside $\mathrm{SL}_2 = \mathrm{SU}(J_0^\flat)$ (cf. (11.3)). We define the corresponding nilpotent orbital integrals by

$$\operatorname{Orb}(n(\beta),\phi) = \int_{H/N} \phi\big(h^{-1}n(\beta)h\big)\,d\bar{h}, \tag{19.2}$$

and

$$\operatorname{Orb}(0,\phi) = e_{F/F_0}q^{-1}\zeta_{F_0}(1)\phi(0), \tag{19.3}$$

where e_{F/F_0} is the ramification index of F/F_0. It is easy to see that both expressions converge absolutely. *It is important to note here that the measure on* $H = U(W^\flat)$ *is chosen such that* $\mathrm{vol}(K) = 1$ *for the special parahoric subgroup* K *(the hyperspecial one when* F/F_0 *is unramified).*

We also define the orbital integral for any $x \in \mathfrak{u}_{0,\mathrm{red}}$ with compact stabilizer or any $x \in \mathfrak{u}_{1,\mathrm{red}}$ by

$$\mathrm{Orb}(x, \phi) = \int_H \phi(h^{-1}xh)\, dh,$$

whenever the integral is absolutely convergent (this will always be true in the cases of interest to us).

19.2. **Germ expansion around** $x_0 \neq 0$**.** Let $x_0 \in \mathfrak{b}_{\mathrm{red}} \smallsetminus \mathfrak{b}_{\mathrm{red,rs}}$. We first classify the H-orbits (semi-simple or not) in $\mathfrak{u}_{\mathrm{red}}$ in the fiber $\pi_{\mathfrak{u}}^{-1}(x_0)$. Unlike the case of $\mathfrak{s}_{\mathrm{red}}$, two issues will affect the semi-simple orbits on $\mathfrak{u}$: stability, and whether $H = U(W^\flat)$ is quasi-split or non-split.

Similar to the case $\mathfrak{s}_{\mathrm{red}}$, we distinguish two cases according to the rank r of the space spanned by $\mathbf{b}, A\mathbf{b}$.

• $r = 0$. Then x_0 is of the form $x_0 = (\lambda_0, 0, 0)$ with $\lambda_0 \in F_0^\times$. A semi-simple element in $\mathfrak{u}_{\mathrm{red}}$ mapping to x_0 must be of the form

$$y_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

We need the following lemma concerning the stability issue.

Lemma 19.1. *Let $\lambda_0 \in F_0 \smallsetminus \{0\}$. Then the set*

$$X_{\lambda_0} := \bigl\{\, x \in \mathfrak{su}(W^\flat) \bigm| \det x = \lambda_0 \,\bigr\}$$

forms one orbit under $U(W^\flat)(F_0)$, unless $F' = F_0[X]/(X^2 + \lambda_0)$ is isomorphic to F, in which case X_{λ_0} decomposes into two such orbits.

Proof. Obviously, X_{λ_0} is one *geometric* orbit (i.e., after passing to the algebraic closure $\overline{F}_0$, all elements of X_{λ_0} are conjugate). Let $x_0 \in X_{\lambda_0}$, and let $T = T_{x_0}$ be the stabilizer subgroup of x_0. Then T is a maximal torus in $U(W^\flat)$. By general principles, the number of orbits under $U(W^\flat)(F_0)$ in a geometric orbit is in one-to-one correspondence with

$$\ker\bigl[H^1(F_0, T) \longrightarrow H^1\bigl(F_0, U(W^\flat)\bigr)\bigr]. \tag{19.4}$$

Let $\underline{F}^1$ be the algebraic subtorus of $\underline{F}^\times := \mathrm{Res}_{F/F_0}(\mathbb{G}_m)$ defined by $\mathrm{Nm}_{F/F_0} = 1$. Then $\underline{F}^1$ is the maximal torus quotient of $U(W^\flat)$, and (19.4) is identified with

$$\ker\bigl[H^1(F_0, T) \longrightarrow H^1(F_0, \underline{F}^1)\bigr]. \tag{19.5}$$

Now, for T there are the following possibilities, up to isomorphism.
(1) $T = \underline{F}^\times$, mapping via $a \mapsto a/\overline{a}$ to $\underline{F}^1$.
(2) $T = \underline{F}^1 \times \underline{F}^1$, mapping via multiplication to $\underline{F}^1$.
(3) $T = \underline{K}^1$, mapping via $\mathrm{Nm}_{K/F}$ to $\underline{F}^1$. Here $K = F'.F$ is a bi-quadratic extension of F_0, and $\underline{K}^1$ is the algebraic subtorus of $\underline{K}^\times := \mathrm{Res}_{K/F_0}(\mathbb{G}_m)$, defined by $\mathrm{Nm}_{K/F'} = 1$.
Furthermore, case (1) corresponds to the case when $F' \simeq F_0 \oplus F_0$, case (2) to the case when $F' \simeq F$, and case (3) to the remaining possibilities.

Let $T' = \ker(T \to \underline{F}^1)$. In case (1), $T' = \mathbb{G}_m$ and (19.5) is trivial. In case (2), $T' = \underline{F}^1$, and (19.5) is identified with $H^1(F_0, \underline{F}^1) = F_0^\times / \mathrm{Nm}_{F/F_0}(F^\times) = \mathbb{Z}/2$; in case (3), the map $H^1(F_0, T) \to H^1(F_0, \underline{F}^1)$ is identified with

$$F'^\times / \mathrm{Nm}_{K/F'}(K^\times) \xrightarrow{\mathrm{Nm}_{F'/F_0}} F_0^\times / \mathrm{Nm}_{F/F_0}(F^\times),$$

which is injective, and hence (19.5) is trivial. The lemma is proved. □

Subcase 0i. When F' is not isomorphic to F, then by Lemma 19.1, there is a unique semi-simple orbit with invariants $x_0 = (\lambda_0, 0, 0), \lambda_0 \in F_0^\times$ and we fix a choice of representative $y_0 \in \mathfrak{u}_{\mathrm{red}}$. A non-semi-simple orbit exists only when $W^\flat$ is split, and the quadratic algebra $F' = F_0[X]/(X^2 + \lambda_0)$ is split as $F_0 \times F_0$. We exclude in the sequel the case when F' is isomorphic to $F_0 \times F_0$. The reason is that the closure of $\mathfrak{b}_{\mathrm{red,rs},1}$ in $\mathfrak{b}_{\mathrm{red}}$ does not contain such element and therefore we will not need this case in Part 3.

Subcase 0ii. When F' is isomorphic to F, by Lemma 19.1, there are two semi-simple orbits mapping to $x_0 = (\lambda_0, 0, 0)$, and we fix the representatives $y_+, y_- \in \mathfrak{u}_{\mathrm{red}}$. We will label the two orbits $y_\pm$ as follows. Consider a regular semi-simple element y in $\mathfrak{u}_{\mathrm{red}}$ near $y_\pm$. Choose a basis such that y may be written in the form

$$y = \begin{bmatrix} A & * \\ * & 0 \end{bmatrix} \text{ with } \textit{diagonal } A = \pi \begin{bmatrix} \alpha & \\ & -\alpha \end{bmatrix}, \ \alpha \in F_0^\times.$$

Then we choose y_+, resp. y_-, such that all regular semi-simple elements near y_+ have the following property: $z_1 := \frac{1}{2}(u + \frac{w}{\pi\alpha})$ is a norm, resp. a non-norm. Here we are using the coordinates z_1, z_2 from Theorem 18.2(b). An easy calculation shows that this is possible and we will choose a small open neighborhood $V_{x_0,\pm}$ of x_0 such that, for all $(\lambda, u, w) \in V_{x_0,\pm} \cap \mathfrak{b}_{\mathrm{red,rs}}$, we have $\eta(z_1) = \pm 1$. Moreover, there are no non-semi-simple orbits in the fiber of such $x_0 \in \mathfrak{b}_{\mathrm{red}}$.

• $r = 1$, then $\mathbf{b} \neq 0$ and $A\mathbf{b}$ is a multiple of $\mathbf{b}$. It is not hard to show that there is a unique orbit (which therefore has to be semi-simple) mapping to x_0. For our calculation in Part 3, we given an explicit representative when $W^\flat$ is split. If $\lambda_0 \neq 0$ we choose

$$y_0 = \begin{bmatrix} 0 & -\lambda_0 & b_1 \\ 1 & 0 & b_2 \\ \cdots & \cdots & 0 \end{bmatrix}, \tag{19.6}$$

where $b_1 = \pi\alpha b_2$ with $\alpha^2 = -\lambda_0/\varpi$ ($\alpha \in F_0^\times$) and $(b_1\bar{b}_2 - b_2\bar{b}_1)/\pi = u_0$. If $\lambda_0 = 0$, we choose any

$$y_0 = \begin{bmatrix} 0 & 0 & b_1 \\ 0 & 0 & 1 \\ \cdots & \cdots & 0 \end{bmatrix}, \quad \mathrm{Im}(b_1) \neq 0. \tag{19.7}$$

In either case, the stabilizer is an anisotropic torus.

Having classified the orbits in $\pi^{-1}(x_0)$, it is easy to prove the following explicit germ expansion.

Theorem 19.2. *Let $x_0 = (\lambda_0, u_0, w_0) \neq (0,0,0) \in \mathfrak{b}_{\mathrm{red}}$, and let $\phi \in C_c^\infty(\mathfrak{u}_{\mathrm{red}})$. For x in a small neighborhood of $x_0 \in \pi_{\mathfrak{u}}(\mathfrak{u}_{\mathrm{red}})$, let $\sigma(x)$ be any element in $\mathfrak{u}_{\mathrm{red}}$ mapping to $x \in \mathfrak{b}_{\mathrm{red}}$.*

(a) *If $F' = F_0[X]/(X^2 + \lambda_0) \neq F, F_0 \times F_0$, then the orbital integral* $\mathrm{Orb}(\sigma(x), \phi)$ *is equal to* $\mathrm{Orb}(y_0, \phi)$, *where $y_0 \in \mathfrak{u}_{\mathrm{red}}$ is any representative of the unique orbit $\pi_{\mathfrak{u}}^{-1}(x_0)$.*

(b) *If $F' = F_0[\sqrt{-\lambda_0}] \simeq F$, then the orbital integral* $\mathrm{Orb}(\sigma(x), \phi)$ *is equal to*

$$\mathrm{Orb}(y_+, \phi)\, \mathbf{1}_{V_{x_0,\mathrm{rs},+}} + \mathrm{Orb}(y_-, \phi)\, \mathbf{1}_{V_{x_0,\mathrm{rs},-}}.$$

Proof. This is proved using the same argument as (and is easier than) the case $x_0 = 0$ in [34]. □

19.3. **Matching orbital integrals around $x_0 = 0$.** As in [34, §4], we let $\mathcal{C}_1(F_0)$ be the space of locally constant functions f on F such that, when $|x|$ is large enough, $f(x)$ is a linear combination of the following functions,

$$\eta(x)|x|^{-1}, \quad \eta(x)\log|x||x|^{-1}.$$

Let $\mathcal{C}_2(F_0)$ be the space similarly defined by requiring that, when $|x|$ is large enough, $f(x)$ is a linear combination of

$$|x|^{-1}, \quad \eta(x)|x|^{-1}.$$

Let $\mathcal{C}(F_0) = \mathcal{C}_1(F_0) \cup \mathcal{C}_2(F_0)$. For $f \in \mathcal{C}(F_0)$, we define the *extended Fourier transform* by [34, §4.1]:

$$\widetilde{f}(v) := \int_{F_0} f(v + x)\eta(x)\,\frac{dx}{|x|},$$

which is understood in the sense of analytic continuation (cf. loc. cit.). We have

$$\widetilde{\widetilde{f}}(v) = \gamma(1, \eta)^2 f(v),$$

where the square of the gamma factor is equal to

$$\gamma(1,\eta)^2 = \begin{cases} \left(\dfrac{L(0,\eta)}{L(1,\eta)}\right)^2 = \dfrac{(1+q^{-1})^2}{4} & \eta \text{ unramified;} \\ \eta(-1)q^{-1}, & \eta \text{ ramified.} \end{cases}$$

Definition 19.3. Let (ϕ_0, ϕ_1) with $\phi_i \in C_c^\infty(\mathfrak{u}_{\mathrm{red},i})$, and let $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$. Then (ϕ_0, ϕ_1) and ϕ' are *local transfers* around $x_0 \in \mathfrak{b}_{\mathrm{red}}$ if there exists a small neighborhood V_{x_0} of x_0 in $\mathfrak{b}_{\mathrm{red}}$ such that

$$\omega(y') \operatorname{Orb}(y', \phi') = \operatorname{Orb}(y, \phi_i)$$

for any $y' \in \mathfrak{s}_{\mathrm{red}}$ with invariants $x \in V_{x_0,\mathrm{rs}}$, and any $y \in \mathfrak{u}_{\mathrm{red},i}$ matching y'.

We will use this definition with the following transfer factor (we are allowed to do so by Remark 12.5, since this transfer factor differs from our original transfer factor by a constant multiple):

$$\omega(y') = \eta\big(\det(\widetilde{y}^i e)_{i=0,1,\dots,n-1}\big), \quad \widetilde{y}' = y'/\pi, \quad y' \in \mathfrak{s}_{\mathrm{red}}. \tag{19.8}$$

This transfer factor is chosen such that $\omega(\sigma(x)) = 1$ for the section $\sigma(x)$ defined by (17.3), which we use frequently.

For $\phi \in C_c^\infty(\mathfrak{u}_{\mathrm{red}})$, resp. $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$, we define

$$\operatorname{Orb}_\phi(\beta) := \operatorname{Orb}\big(n(\beta), \phi\big), \quad \text{resp.} \quad \operatorname{Orb}_{\phi'}(\mu) := \operatorname{Orb}\big(n(\mu), \phi'\big), \quad \beta, \mu \in F_0.$$

Then the functions Orb_ϕ, resp. $\operatorname{Orb}_{\phi'}$ lie in $\mathcal{C}(F_0)$ (cf. loc. cit.). Then the matching conditions around zero are essentially given by the extended Fourier transform between nilpotent orbital integrals. Indeed, set (cf. loc. cit.)

$$\kappa_{F/F_0} = e_{F/F_0} L(1, \eta)^{-1} = \begin{cases} 1 + q^{-1}, & F/F_0 \text{ unramified;} \\ 2, & F/F_0 \text{ ramified.} \end{cases}$$

Theorem 19.4. *The functions $(\phi_0, \phi_1), \phi_i \in C_c^\infty(\mathfrak{u}_{\mathrm{red},i})$ and $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ are local transfers around zero if and only if*

$$\operatorname{Orb}_{\phi_0} = 2\eta(-1)|\varpi|^{-1} \kappa_{F/F_0}^{-1} \widetilde{\operatorname{Orb}}_{\phi'},$$

and

$$\begin{aligned} -\operatorname{Orb}(0, \phi_0) &= \eta(-1) \operatorname{Orb}(n_{0,+}, \phi') + \operatorname{Orb}(n_{0,-}, \phi'); \\ \operatorname{Orb}(0, \phi_1) &= \eta(-1) \operatorname{Orb}(n_{0,+}, \phi') - \operatorname{Orb}(n_{0,-}, \phi'). \end{aligned}$$

Proof. The statement is easily reduced to the corresponding one for $\pi\mathfrak{s}_{\mathrm{red}} = \mathfrak{sl}_{\mathrm{red}}$ and $\pi\mathfrak{u}_{\mathrm{red}}$ which are given by [34, Prop. 4.4, 4.7], by comparing the germ expansions on $\mathfrak{u}_{\mathrm{red}}$ and $\mathfrak{s}_{\mathrm{red}}$ (and note $\eta(\Delta(x)/\varpi) = (-1)^i$ if $x \in \mathfrak{b}_{\mathrm{red,rs},i}$, cf. Proposition 10.2). Here we note that there is an error in the germ expansion [34, Th. 2.8(1)(i)]: τ should be τ^{-1} (τ being ϖ in the current notation). This leads to the correction factor $|\varpi|^{-1}$ in the statement above. ☐

19.4. Matching orbital integrals around $x_0 \neq 0$.

Theorem 19.5. *Let $x_0 = (\lambda_0, u_0, w_0) \in \mathfrak{b}_{\mathrm{red}}$, where $x_0 \neq 0$. The functions (ϕ_0, ϕ_1), $\phi_i \in C_c^\infty(\mathfrak{u}_{\mathrm{red},i})$, and $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ are local transfers around x_0 if and only if the following identities hold:*

(a) *In case (0i), and when $F' \neq F_0 \times F_0$,*

$$\begin{aligned} \operatorname{Orb}(y_0, \phi_0) &= \operatorname{Orb}(y_+, \phi') + \eta(-\lambda) \operatorname{Orb}(y_-, \phi'), \\ \operatorname{Orb}(y_0, \phi_1) &= \operatorname{Orb}(y_+, \phi') - \eta(-\lambda) \operatorname{Orb}(y_-, \phi'). \end{aligned}$$

Here $y_0 \in \mathfrak{u}_{i,\mathrm{red}}$ is any representative of the unique orbit in $\pi_{\mathfrak{u}_i}^{-1}(x_0)$, and $y_\pm \in \mathfrak{s}_{\mathrm{red}}$ are the representatives given by (18.2) *of the two non-semi-simple orbits in $\pi_{\mathfrak{s}}^{-1}(x_0)$.*

(b) *In case (0ii),*

$$\begin{aligned} \eta(-\alpha) \operatorname{Orb}(y_+, \phi_0) &= \operatorname{Orb}(y_{++}, \phi') + \operatorname{Orb}(y_{-+}, \phi') + \eta(-1) \operatorname{Orb}(y_{+-}, \phi') + \eta(-1) \operatorname{Orb}(y_{--}, \phi'), \\ \eta(-\alpha) \operatorname{Orb}(y_-, \phi_0) &= \operatorname{Orb}(y_{++}, \phi') - \operatorname{Orb}(y_{-+}, \phi') - \eta(-1) \operatorname{Orb}(y_{+-}, \phi') + \eta(-1) \operatorname{Orb}(y_{--}, \phi'), \end{aligned}$$

and

$$\begin{aligned} \eta(-\alpha) \operatorname{Orb}(y_+, \phi_1) &= \operatorname{Orb}(y_{++}, \phi') + \operatorname{Orb}(y_{-+}, \phi') - \eta(-1) \operatorname{Orb}(y_{+-}, \phi') - \eta(-1) \operatorname{Orb}(y_{--}, \phi'), \\ \eta(-\alpha) \operatorname{Orb}(y_-, \phi_1) &= \operatorname{Orb}(y_{++}, \phi') - \operatorname{Orb}(y_{-+}, \phi') + \eta(-1) \operatorname{Orb}(y_{+-}, \phi') - \eta(-1) \operatorname{Orb}(y_{--}, \phi'). \end{aligned}$$

Here $y_+, y_- \in \mathfrak{u}_{i,\mathrm{red}}$ are any representatives of the two semi-simple orbits in $\pi_{\mathfrak{u}_i}^{-1}(x_0)$ labeled in §19.2, and $y_{\pm\pm} \in \mathfrak{s}_{\mathrm{red}}$ are the representatives given by (18.3) *and* (18.4) *of the four non-semi-simple orbits in $\pi_{\mathfrak{s}}^{-1}(x_0)$.*

(c) *In case* (1),

$$\mathrm{Orb}(y_0, \phi_0) = \mathrm{Orb}(y_+, \phi') + \mathrm{Orb}(y_-, \phi'),$$
$$\mathrm{Orb}(y_0, \phi_1) = \mathrm{Orb}(y_+, \phi') - \mathrm{Orb}(y_-, \phi').$$

Here $y_0 \in \mathfrak{u}_{i,\mathrm{red}}$ *is any representative of the unique orbit in* $\pi_{\mathfrak{u}_i}^{-1}(x_0)$, *and* $y_\pm \in \mathfrak{s}_{\mathrm{red}}$ *are the representatives given by* (18.5) *of the two non-semi-simple orbits in* $\pi_{\mathfrak{s}}^{-1}(x_0)$.

Proof. This follows by comparing Theorem 18.2 (specialized to $s = 0$) and Theorem 19.2. Note that $x \in \mathfrak{b}_{\mathrm{red,rs},0}$ if and only if $\eta(\Delta/\varpi) = \eta(-\Delta) = 1$ (cf. Proposition 10.2). In case $(0ii)$, we note the following facts about the section σ defined by (18.6):

- $\Delta = -4\alpha^2\varpi z_1 z_2$ and hence $\eta(\Delta) = \eta(z_1 z_2)$.
- The choice of y_+ is such that $\eta(z_1) = 1$ for $x \in V_{x_0,\mathrm{rs},+}$.
- The transfer factor (19.8) is given by $\omega(\sigma(x)) = \eta(-\alpha)$. □

20. Proofs of Theorems 16.2 and 16.5

20.1. **Proof of Theorem 16.2.** To show the *"only if"* part, by Theorem 17.1 and 18.2, it suffices to show that the function $\mathrm{Orb}_{\phi'}$ lies in $\mathcal{C}_1(F_0)$. This is proved in [34, Lem. 2.3].

To show the *"if"* part, by [35, Prop. 3.8], it suffices to show that φ is a local orbital integral function around every $x_0 \in \mathfrak{b}_{\mathrm{red}}$, comp. Theorem 10.11. This amounts to showing the following two lemmas.

Lemma 20.1. *For each* x_0, *and each* discrete $n_0 \in \pi^{-1}(x_0)$ *with nonzero germ function value* $\Gamma_{n_0}(x, 0)$, *there exists a function* $\phi' \in C_c^\infty(\mathfrak{s}_{\mathrm{red}})$ *such that*

$$\mathrm{Orb}(n, \phi') = \begin{cases} 1, & n = n_0; \\ 0, & n \textit{ is not in the same orbit as } n_0. \end{cases}$$

Proof. Though not stated explicitly in [34], this can be proved in the same way as [34, Lem. 2.1, 2.3]. □

Lemma 20.2. *Every function in* $\mathcal{C}_1(F_0)$ *arises as* $\mathrm{Orb}_{\phi'}$ *for some* ϕ', *and such* ϕ' *can be chosen so that* $\mathrm{Orb}(n, \phi') = 0$ *for the two regular nilpotents* $n = n_{0,\pm}$.

Proof. This is proved in the same way as [34, Lem. 2.3]. □

20.2. **Proof of Theorem 16.5.** We need to verify the hypotheses of Theorem 16.2.

The case $x_0 = 0$. First assume that $i = 0$. Then for the one-dimensional family $n(\mu)$, by Corollary 17.2 the germ function $\Gamma_{n(\mu)}(x, 0)$ vanishes identically. By Theorem 17.1, we have for $x \in \mathfrak{b}_{\mathrm{red,rs}}$ around $x_0 = 0$,

$$\varphi(x) = \partial\mathrm{Orb}(n_{0,+}, \phi, 0)\, \Gamma_{n_{0,+}}(x, 0) + \partial\mathrm{Orb}(n_{0,-}, \phi, 0)\, \Gamma_{n_{0,-}}(x, 0).$$

Clearly $\partial\mathrm{Orb}(n_{0\pm}, \phi, 0)$ are constants. Therefore the function satisfies the hypotheses of Theorem 16.2 concerning the summands for these two elements. This proves the case $i = 0$.

Now assume that $i = 1$. To show that the function in Theorem 16.5 satisfies the hypotheses of Theorem 16.2 around $x_0 = 0$, it suffices to show that the function

$$\partial\mathrm{Orb}_{\phi'}(\mu) := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}\big(n(\mu), \phi', s\big), \quad \mu \in F_0,$$

lies in $\mathcal{C}_1(F_0)$.

Since we are assuming that ϕ' transfers to the zero function on $\mathfrak{u}_{0,\mathrm{red}}$, by Theorem 19.4 we have $\mathrm{Orb}_{\phi'} = 0$ identically as a function on F_0. The claim follows from the next lemma.

Lemma 20.3. *If* $\mathrm{Orb}_{\phi'} = 0$, *then* $\partial\mathrm{Orb}_{\phi'} \in \mathcal{C}_1(F_0)$.

Proof. We have

$$\mathrm{Orb}(n(\mu), \phi', s) = \int \phi'_K \left(\pi \begin{bmatrix} 0 & \mu ab & b \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \right) \eta(ab) |a^{-1}b|^s \, da\, db.$$

When $|\mu|$ is small, this function at $s = 0$ is locally constant in $\mu \in F_0$, hence has the desired property. We now assume that $|\mu| \gg 0$. The same idea as in the proof of Theorem 18.2 shows that this integral is a sum of two terms,

$$\int \phi'_K \left(\pi \begin{bmatrix} 0 & \mu ab & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \right) \eta(ab)|a^{-1}b|^s \, da \, db,$$

and

$$\int \phi'_K \left(\pi \begin{bmatrix} 0 & \mu ab & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \eta(ab)|a^{-1}b|^s \, da \, db.$$

These may be rewritten as

$$\eta(\mu)|\mu|^{-1-s} \int \phi'_K \left(\pi \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \right) \eta(b)|a^{-2}b|^s \frac{da \, db}{|a|},$$

and

$$\eta(\mu)|\mu|^{-1+s} \int \phi'_K \left(\pi \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \eta(a)|a^{-1}b^2|^s \frac{da \, db}{|b|},$$

respectively. For simplicity we write the sum as

$$\eta(\mu)|\mu|^{-1}\bigl(|\mu|^{-s}A(s) + |\mu|^s B(s)\bigr),$$

where $A(s)$ and $B(s)$ both have a simple pole at $s = 0$ with opposite residues. Write the Laurent expansion as

$$A(s) = \frac{A_{-1}}{s} + A_0 + A_1 s + \cdots \quad \text{and} \quad B(s) = \frac{B_{-1}}{s} + B_0 + B_1 s + \cdots,$$

where $A_{-1} + B_{-1} = 0$. Then the constant term of the Laurent expansion of $|\mu|^{-s}A(s) + |\mu|^s B(s)$ is given by

$$(A_0 + B_0) + \log|\mu|(-A_{-1} + B_{-1}).$$

Since $\mathrm{Orb}_{\phi'} = 0$ by assumption, we have $A_0 + B_0 = 0$ and $A_{-1} = B_{-1} = 0$. This implies that the degree one term in the Laurent expansion of $|\mu|^{-s}A(s) + |\mu|^s B(s)$ is given by

$$(A_1 + B_1)s + \log|\mu|(-A_0 + B_0)s.$$

We conclude that when $|\mu| \gg 0$

$$\partial\mathrm{Orb}_{\phi'}(\mu) = \eta(\mu)|\mu|^{-1}\bigl((A_1 + B_1) + \log|\mu|(-A_0 + B_0)\bigr),$$

and hence $\partial\mathrm{Orb}_{\phi'}$ belongs to $\mathcal{C}_1(F_0)$, as desired. □

The case $x_0 \neq 0$. This follows easily from the explicit germ expansion Theorem 18.2, 19.2, and 19.5, with a similar argument as in the case $x_0 = 0$. We omit the details.

With this Theorem 16.5 is proved.

References

[1] ARGOS *seminar on intersections of modular correspondences*, Astérisque **312** (2007).

[2] K. Arzdorf, *On local models with special parahoric level structure*, Michgan Math. J. **58** (2009), no. 3, 683–710.

[3] B. Conrad, *Gross–Zagier revisited.* In *Heegner points and Rankin L-series*, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 67–163.

[4] W. T. Gan, B. Gross, and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109.

[5] J. Gordon, *Transfer to characteristic zero*, appendix to [30].

[6] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320.

[7] R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. **84** (1962), 441–465.

[8] H. Jacquet and S. Rallis, *On the Gross-Prasad conjecture for unitary groups.* In *On certain L-functions*, Clay Math. Proc., 13, Amer. Math. Soc., Providence, RI, 2011, pp. 205–264.

[9] S. Kudla and M. Rapoport, *Special cycles on unitary Shimura varieties I. Unramified local theory*, Invent. Math. **184** (2011), no. 3, 629–682.

[10] ———, *Special cycles on unitary Shimura varieties II: Global theory*, J. Reine Angew. Math. **697** (2014), 91–157.
[11] ———, *An alternative description of the Drinfeld p-adic half-plane*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 3, 1203–1228.
[12] A. Mihatsch, *On the arithmetic fundamental lemma conjecture through Lie algebras*, preprint, 2015, `arXiv:1502.02855 [math.AG]`.
[13] G. Pappas, *On the arithmetic moduli schemes of PEL Shimura varieties*, J. Algebraic Geom. **9** (2000), no. 3, 577–605.
[14] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198.
[15] ———, *Local models in the ramified case. III. Unitary groups*, J. Inst. Math. Jussieu **8** (2009), no. 3, 507–564.
[16] G. Pappas, M. Rapoport, and B. Smithling, *Local models of Shimura varieties, I. Geometry and combinatorics.* In *Handbook of Moduli. Vol. III*, Adv. Lect. Math. (ALM), vol. 26, Int. Press, Somerville, MA, 2013, pp. 135–217.
[17] S. Rallis and G. Schiffmann, *Multiplicity one conjectures*, preprint, 2007, `arXiv:0705.2168 [math.NT]`.
[18] M. Rapoport, *Deformations of isogenies of formal groups*, in [1], pp. 139–169.
[19] M. Rapoport, B. Smithling, and W. Zhang, *On arithmetic transfer: conjectures*, in preparation.
[20] M. Rapoport, U. Terstiege, and S. Wilson, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, n-1)$ *over a ramified prime*, Math. Z. **276** (2014), no. 3–4, 1165–1188.
[21] M. Rapoport, U. Terstiege, and W. Zhang, *On the arithmetic fundamental lemma in the minuscule case*, Compositio Math. **149** (2013), no. 10, 1631–1666.
[22] M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), 273–326.
[23] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.
[24] B. Smithling, *On the moduli description of local models for ramified unitary groups*, to appear in Int. Math. Res. Not.
[25] U. Terstiege, *Intersections of arithmetic Hirzebruch–Zagier cycles*, Math. Ann. **349** (2011), no. 1, 161–213.
[26] I. Vollaard, *Endomorphisms of quasi-canonical lifts*, in [1], pp. 105–112.
[27] ———, *The supersingular locus of the Shimura variety for GU(1,s)*, Can. J. Math. **62** (2010), 668–720.
[28] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, n-1)$ *II*, Invent. Math. **184** (2011), no. 3, 591–627.
[29] S. Wewers, *Canonical and quasi-canonical liftings*, in [1], pp. 67–86.
[30] Z. Yun, *The fundamental lemma of Jacquet–Rallis in positive characteristics*, Duke Math. J. **156** (2011), no. 2, 167–228.
[31] ———, *An arithmetic fundamental lemma for function fields*, in preparation.
[32] W. Zhang, *Relative trace formula and arithmetic Gross–Prasad conjecture*, unpublished manuscript, 2009.
[33] ———, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), no. 1, 197–252.
[34] ———, *On the smooth transfer conjecture of Jacquet–Rallis for* $n = 3$, Ramanujan J. **29** (2012), no. 1–3, 225–256.
[35] ———, *Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049.

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