# ON THE SHAPE OF THE CONTRIBUTION OF A FIXED POINT ON THE BOUNDARY: THE CASE OF Q-RANK ONE

MICHAEL RAPOPORT

(WITH AN APPENDIX BY L. SAPER AND M. STERN)

This note has its origin in a letter to R. Kottwitz (October 1985), written shortly after my visit to Seattle where we had discussed at length the problem mentioned in the title. In 1986 in Paris we had occasion to discuss these problems again and a certain conjecture (4.1 below) based on the Comptes Rendus note of A. Borel and W. Casselman [BC], suggested itself (letter to Borel of June 30, 1986). I wrote a preliminary version of this note in 1989, to provide some motivation for this conjecture. Just recently L. Saper and M. Stern were able to prove this conjecture in the rank-1 case, as a consequence of their work on the Zucker conjecture. Their proof appears here as an appendix to this note. This result available, I decided to revise my preliminary version. I was helped in this by a recent preprint of R. Pink [P] in which he proves a general theorem on $\ell$-adic sheaves on Shimura varieties which implies a fact I need (3.2 below) and for which no proof had been included in the preliminary version.

To get an idea of the main result of this note, the reader should glance at *1* and then turn to formula (6.3) which uses notation in *2*, and then glance at the final remarks in *7*, putting this result in a more general context and comparing it to results of M. Goresky, G. Harder, R. Kottwitz, and R. MacPherson.

I wish to thank R. Kottwitz for generously sharing his insights with me. Also, conversations on Lefschetz numbers with G. Harder and J. Rohlfs at the Arbeitsbeitstagung Oberwolfach in 1985 were very useful to me. On a more fundamental level I would like to express my gratitude to G. Harder for introducing me to many ideas and technical tools in this area. I also thank L. Saper and M. Stern for writing up their proof as an appendix, and M. Goresky and R. MacPherson for suggesting

the inclusion of the revised note with its appendix in this volume. I also thank R. Pink for his comments. Various institutions have supported me during this project. I wish to thank the University of Washington, the C.N.R.S., the Volkswagenstiftung, and M.I.T.

**1.** Let $G$ be a semi-simple almost simple algebraic group over $\mathbf{Q}$ such that the associated symmetric space $X$ is a hermitian symmetric domain. Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic group, which we assume neat, and put $X_\Gamma = \Gamma \backslash X$. Let $\overline{X}_\Gamma$ be the Baily-Borel compactification,

$$j : X_\Gamma \hookrightarrow \overline{X}_\Gamma .$$

Let $V$ be a rational representation of $G$. We denote by $\mathbf{V} = \mathbf{V}_\Gamma$ the associated local system (with $\mathbf{Q}$-coefficients) on $X_\Gamma$. We denote $\mathcal{IC}(\mathbf{V})$ the intermediate extension to $\overline{X}_\Gamma$.

It is known from the theory of Shimura varieties that the complex algebraic variety $X_\Gamma$ has a model over a number field and that there is a canonical way (in fact, several canonical ways of which we choose one) to associate to the representation $V$ an $\ell$-adic sheaf on this model. We fix these objects in the sequel and denote by the same letter $X_\Gamma$ such a model, resp. a model over its ring of integers; similarly, we denote by the same symbol $\mathbf{V}$ the $\ell$-adic sheaf resp. an extension of this $\ell$-adic sheaf over this integral model. The choice of integral model will be irrelevant for our purposes since we will be interested in the reduction modulo $p$ only for sufficiently general $p$. By excluding finitely many rational primes $p$ we may assume that $X_\Gamma$ and $\mathbf{V}$ have good reduction and that the same holds with respect to the natural strata of $\overline{X}_\Gamma$ and $\mathcal{IC}(\mathbf{V})$.

Here is now the problem we would like to understand. Let $y \in \overline{X}_\Gamma(\overline{\kappa}_\wp)$ be a point in the reduction modulo a prime ideal $\wp$ which is fixed by a certain power $\Phi$ of the Frobenius at $\wp$. Then $\Phi$ operates on the fibre at $y$ of $\mathcal{IC}(\mathbf{V})$ and, using the customary notation for the alternating sum of traces, we would like to determine

$$\mathrm{Tr}(\Phi;\ \mathcal{IC}(\mathbf{V})_y)\,. \tag{1.1}$$

This is the contribution of $y$ to the Lefschetz fixed point formula for the correspondence $\Phi$ and $H^*(\overline{X}_\Gamma \otimes \overline{\kappa}_\wp, \mathcal{IC}(\mathbf{V}))$. Only the case when $y$ is at the boundary is of interest.

**2.** We will also consider the reductive Borel-Serre compactification $\widehat{X}_\Gamma$ (cf. [Z1], 4.1; cf. also [H3], p. 37/38). This is a stratified topological compactification of $X_\Gamma$ which we are going to describe briefly. Let $P \subset G$ be a $\mathbf{Q}$-parabolic. Let $N$ be its unipotent radical and denote by $M = M_P$ the Levi factor $P/N$. If ${}^0M = \cap \ker \alpha^2$

denotes the intersection of the kernels of the squares of all rational characters of $M$ defined over $\mathbf{Q}$, we let $X_M$ be the symmetric space of ${}^0M$ (space of all maximal compact subgroups of ${}^0M(\mathbf{R})$). Starting with the arithmetic group $\Gamma$ we obtain the arithmetic group $\Gamma_M$ of $M$ by taking the image of $\Gamma \cap P$ in $M$. As a point set, the reductive Borel-Serre compactification is given as

$$\widehat{X_\Gamma} = \coprod_P \Gamma_M \backslash X_M. \tag{2.1}$$

Here $P$ ranges over the $\mathbf{Q}$-parabolics, taken modulo $\Gamma$-conjugacy, with Levi factor $M$. This set is finite. The open dense subset $X_\Gamma$ corresponds to the improper parabolic $P = G$. The closure of the stratum corresponding to the parabolic $P$ is the union of all strata corresponding to parabolics $P'$ with $P' \subset P$. The reductive Borel-Serre compactification is defined for any semi-simple algebraic group over $\mathbf{Q}$. If we assume now that $G$ satisfies the hypothesis stated in *1*, there is a continuous proper stratified map to the Baily-Borel compactification, inducing the identity on $X_\Gamma$,

$$\pi : \widehat{X_\Gamma} \to \overline{X}_\Gamma. \tag{2.2}$$

This map was constructed by S. Zucker in [Z2], but the description of it I am going to give seems closer to that of G. Harder in [H3], p. 38. (I intend to give the proofs of these assertions at another occasion.) Let $P$ be a maximal proper $\mathbf{Q}$-parabolic. Let $U = U_P$ be the center of its unipotent radical. Let $H = H_P = \ker(M \to \mathrm{Aut}(U))$ under the adjoint action. Then $H$ is a semi-simple group and there is a reductive subgroup $L = L_P \subset M$ commuting with $H$ and such that $M = L \cdot H$ with finite central intersection. Furthermore the symmetric space $X_H = X_M^h$ is hermitian symmetric, whereas the symmetric space $X_L = X_M^\ell$ associated to ${}^0L$ is the quotient by $\mathbf{R}_+^\times$ of an open self-adjoint cone in $\mathrm{U}(\mathbf{R})$. We obtain a product decomposition

$$X_M = X_M^h \times X_M^\ell. \tag{2.3}$$

By projecting $\Gamma_M$ into $H$ resp. $L$ we obtain arithmetic groups $\Gamma_H$ and $\Gamma_L$ in $H$ and $L$ respectively. The Baily-Borel compactification can be given as

$$\overline{X}_\Gamma = X_\Gamma \sqcup \coprod_P \Gamma_H \backslash X_H. \tag{2.4}$$

Here $P$ ranges over the $\Gamma$-conjugacy classes of maximal proper $\mathbf{Q}$-parabolics with associated subgroup $H = H_P$. The closure relation on the proper strata defines a total order among the maximal proper $\mathbf{Q}$-parabolics containing a fixed $\mathbf{Q}$-parabolic which may be expressed in group-theoretic terms as follows:

$$P' \le P \iff \text{stratum of } P' \text{ contained in the closure of stratum } P \iff U_{P'} \supset U_P.$$

Now let $P$ be any proper $\mathbf{Q}$-parabolic and let $P^+$ be the unique maximal proper $\mathbf{Q}$-parabolic containing P and minimal with respect to the total ordering among maximal proper $\mathbf{Q}$-parabolics containing $P$ introduced above. We obtain as follows a map

$$X_{M_P} \longrightarrow X_{M_{P^+}}. \tag{2.5}$$

Fix a maximal compact subgroup $K$ in $G(\mathbf{R})$ and denote by $\theta_K : G_{\mathbf{R}} \to G_{\mathbf{R}}$ the associated Cartan involution ([BS], 1.6). Then, for any $\mathbf{Q}$-parabolic $P$, there is a unique $\theta_K$-stable lifting of the Levi factor $M_{P\mathbf{R}}$ into a Levi subgroup $\widetilde{M}_{P\mathbf{R}}$ ([BS], 1.9). Applying this to $P$ and $P^+$ we obtain an inclusion $\widetilde{M}_{P\mathbf{R}} \subset \widetilde{M}_{P^+\mathbf{R}}$ which induces an inclusion ${}^0\widetilde{M}_{P\mathbf{R}} \subset {}^0\widetilde{M}_{P^+\mathbf{R}}$ (where these are defined as inverse images of the corresponding subgroups of $M_P$ resp. $M_{P^+}$ under the projection map isomorphism). This induces the desired map (2.5), which when composing with the first projection in (2.3) yields

$$X_{M_P} \to X^h_{M_{P^+}} = X_{H^+}. \tag{2.6}$$

The restriction of the map $\pi$ (2.2) to the stratum corresponding to $P$ is the map induced by (2.6),

$$\Gamma_{M_P}\backslash X_{M_P} \to \Gamma_{H^+}\backslash X_{H^+}.$$

We have obtained a commutative diagram

$$\begin{tikzcd} & \widehat{X}_\Gamma \arrow[d, "\pi"] \\ X_\Gamma \arrow[ur, "\hat{\jmath}"] \arrow[r, "j"'] & \overline{X}_\Gamma \end{tikzcd}$$

**3.** Fix a maximal proper $\mathbf{Q}$-parabolic $P$ and denote by $\Gamma_H\backslash X_H$ the corresponding stratum of the Baily-Borel compactification, cf. (2.4). There is then the following expression for the restriction to this stratum of the higher direct image sheaves under the open immersion $j$ of the local system $\mathbf{V}$,

$$R^k j_* \mathbf{V}_{|\Gamma_H\backslash X_H} \simeq \bigoplus_{i+j=k} H^i(\Gamma_L, H^j(\mathbf{n}, V)). \tag{3.1}$$

Here $\mathbf{n}$ denotes the Lie algebra of the unipotent radical $N$ of $P$, and the action of $\Gamma_L$ on the Lie algebra cohomology is induced from the action of $L$ simultaneously on $\mathbf{n}$ and on $V$. Each summand on the right hand side defines a representation of $\Gamma_H$ via its representation on $H^j(\mathbf{n}, V)$ and the fact that $H$ centralizes $L$. The isomorphism in (3.1) is to be interpreted as asserting that the sheaf appearing on the left hand side is locally constant and isomorphic to the local system defined by the representation of $\Gamma_H$ given by the right hand side. A proof of (3.1) can be given either by using the map $\pi : \widehat{X_\Gamma} \to \overline{X}_\Gamma$ introduced in (2.2) (cf. Harder [H2], Satz 1, p. 54, [H3], p. 40; for the rank-1 case comp. [H1],Theorem 2.8) or by using the toroidal compactifications of $X_\Gamma$ (cf. Pink [P], Theorem 4.2.1). Implicit here is in particular Harder's result on the degeneracy of the Hochschild-Serre spectral sequence for the normal subgroup $\Gamma \cap N \subset \Gamma \cap (NL)$ and a van Est-type argument. The approach of Pink has the additional bonus of yielding a description of the $\ell$-adic sheaf associated to $R^k j_* \mathbf{V}$ (cf. [P], Theorem 5.3.1). He works in the context of Shimura varieties and the following statement is a coarsening of his result. (In the preliminary version of this note I stated this only in the rank-1 case). Recall from *1* that we fixed a model of $\overline{X}_\Gamma$ and $Rj_*\mathbf{V}$ over the ring of integers in a number field and that we exclude a finite number of prime ideals; in particular, all terms appearing below have a meaning as objects "modulo $\wp$".

**Theorem 3.2.** (G. Harder, R. Pink) *Let $y \in \overline{X}_\Gamma(\bar{\kappa}_\wp)$ be a point in the reduction modulo $\wp$ for a sufficiently general prime ideal $\wp$ and assume that $y$ lies in the stratum corresponding to the proper maximal $\mathbf{Q}$-parabolic $P$. For a sufficiently divisible power $\Phi$ of the Frobenius at $\wp$ which fixes $y$ there is on $H^j(\mathbf{n}, V)$ an operation of $\Phi$ commuting with the action of $L$ and such that the isomorphism*

$$(R^k j_* \mathbf{V})_y \simeq \bigoplus_{i+j=k} H^i(\Gamma_L, H^j(\mathbf{n}, V))$$

*is $\Phi$-equivariant.* (The action of $\Phi$ on the left hand side is induced from its action on $Rj_*\mathbf{V}$.) $\square$

**4.** We now return to the map $\pi : \widehat{X_\Gamma} \to \overline{X}_\Gamma$ from the reductive Borel-Serre compactification to the Baily-Borel compactification, cf. (2.2). Since $\widehat{X}_\Gamma$ is a stratified topological space we may apply the construction of Deligne-Goresky-MacPherson [GM1],[BBD] of the intermediate extension of $\mathbf{V}$ to $\widehat{X_\Gamma}$ through successive extension and truncation. Since the strata are not necessarily of even dimension we have the upper and the lower intermediate extension ([GM1]),

$${}^+\widehat{\mathcal{IC}}(\mathbf{V}) \quad \text{and} \quad {}^-\widehat{\mathcal{IC}}(\mathbf{V}).$$

(Examples with odd-dimensional strata are given by the symplectic group $Sp_6$ or by the Hilbert-Blumenthal groups $R_{F/\mathbf{Q}}SL_2$, with $F$ a totally real field of even degree.)

**Conjecture 4.1.** $R\pi_*({}^+\widehat{\mathcal{IC}}(\mathbf{V})) = R\pi_*({}^-\widehat{\mathcal{IC}}(\mathbf{V})) = \mathcal{IC}(\mathbf{V})$.

This conjecture has been rediscovered independently by M. Goresky and R. MacPherson (cf. [GM2], conjecture 1) who also announce in loc. cit. a proof for $G = Sp_4$, $Sp_6$, and for $G = Sp_8$ where in the last case $\mathbf{V}$ is assumed to be the constant sheaf associated to the trivial representation. Earlier, S. Zucker had checked the case $G = Sp_4$, $\mathbf{V} = \mathbf{Q}$.

**5.** *From now on we assume that* $rk_{\mathbf{Q}} G = 1$. Then we have the following expression for the intermediate extension

$$\mathcal{IC}(\mathbf{V}) = \tau_{<c} Rj_* \mathbf{V}. \tag{5.1}$$

Here $c$ denotes the codimension of $\overline{X}_\Gamma - X_\Gamma$ (as algebraic variety). We shall reformulate the conjecture above as a vanishing condition in Lie algebra cohomology. We fix a maximal proper $\mathbf{Q}$-parabolic $P$ and introduce as before the corresponding groups $N$, $M$, $H$, $L$, etc. We note for the codimensions of the corresponding strata in $\widehat{X_\Gamma}$ resp. $\overline{X}_\Gamma$:

$$\operatorname{codim}_{\mathbf{R}} X_M = \dim N + 1,$$
$$\operatorname{codim}_{\mathbf{R}} X_H = \dim N + \dim U.$$

Using the characterization of $\mathcal{IC}(\mathbf{V})$ ([GM1]) the assertion of the conjecture is equivalent (always in the case of rank 1) to the conjunction of the following two statements.

$$\mathcal{H}^k(R\pi_*{}^+\widehat{\mathcal{IC}}(\mathbf{V})) = 0, \qquad k \geq \frac{1}{2}\operatorname{codim}_{\mathbf{R}} X_H \tag{a}$$

$$\mathcal{H}^k(D\,R\pi_*{}^+\widehat{\mathcal{IC}}(\mathbf{V})) = 0, \qquad k \geq \frac{1}{2}\operatorname{codim}_{\mathbf{R}} X_H \tag{b}$$

We have left out from the notation the fact that we restrict the above cohomology sheaves to the stratum corresponding to $P$. To be precise, the statements (a) and (b) are equivalent to the equality between the two extreme terms in conjecture 4.1. The second equality sign in 4.1 follows from this, if we let $V$ vary, for

$$D\,R\pi_*({}^+\widehat{\mathcal{IC}}(\mathbf{V})) = R\pi_*(D\,{}^+\widehat{\mathcal{IC}}(\mathbf{V}))$$

and

$$D\,{}^+\widehat{\mathcal{IC}}(\mathbf{V}) = {}^-\widehat{\mathcal{IC}}(\mathbf{V}^*).$$

Here $D$ denotes the *shifted* Verdier dualizer which to the local system $\mathbf{V}$ on the smooth manifold $X_\Gamma$ associates the dual local system $\mathbf{V}^*$. This also implies that both statements (a) and (b) (for all local systems $\mathbf{V}$) are equivalent to statement (a) in general. However, as with (5.1),

$$^{+}\widehat{\mathcal{IC}}(\mathbf{V})|_{\Gamma_M \backslash X_M} \simeq \tau_{<\frac{1}{2}\operatorname{codim} X_M} H^*(\mathbf{n}, V),$$

where on the right we mean the local systems (put in varying degree between 0 and $\frac{1}{2}\operatorname{codim} X_M$) associated to the natural representation of $M$ on $H^*(\mathbf{n}, V)$, with trivial differential. It follows easily from the shape of the map $\pi$ that $\mathcal{H}^k(R\pi_*{}^{+}\widehat{\mathcal{IC}}(\mathbf{V}))$ when restricted to the stratum corresponding to $P$ is a local system with typical fibre

$$\bigoplus_{\substack{i+j=k \\ j<\frac{1}{2}\operatorname{codim} X_M}} H^i(\Gamma_L, H^j(\mathbf{n}, V)).$$

Therefore conjecture 4.1 is equivalent in the rank-1 case to the following vanishing "theorem".

**Conjecture 5.2.** *$H^i(\Gamma_L, H^j(\mathbf{n}, V)) = 0$ if $j < \frac{1}{2}\operatorname{codim} X_M$ and $i + j \geq \frac{1}{2}\operatorname{codim}_{\mathbf{R}} X_H$.*

This conjecture has been proved by L. Saper and M. Stern. Their proof is given in the appendix to this note. Here I wish to compare this conjecture with the earlier results of A. Borel and W. Casselman ([BC], Theorem 4; [B], Theorem 2.1). Namely, what they prove in loc. cit. is the vanishing of the cohomology groups in question under the conditions $j < \frac{1}{2}\dim N$ and $i + j \geq \frac{1}{2}\operatorname{codim}_{\mathbf{R}} X_H$. (To deduce this from their results one has to express the cohomology of $\Gamma_L$ in terms of the relative Lie algebra cohomology of $\mathfrak{l}, K_L$ with values in a certain unitary representation, comp. [BW].) Therefore, we see that they prove the conjecture in case the stratum $X_M$ has even dimension. If the stratum has odd dimension, i.e., $\dim N$ is even, what remained to be shown was

$$H^i(\Gamma_L, H^{\frac{1}{2}\dim N}(\mathbf{n}, V)) = 0, \qquad \text{for} \quad i \geq \frac{1}{2}\dim U$$

(note that in this case $\dim U$ is even).

**6.** We shall now assume the validity of conjecture (5.2) and draw consequences from it (we continue to assume that the $\mathbf{Q}$-rank of $G$ is equal to 1). We obtain the following expression for the $k$-th cohomology sheaf of the restriction of $\mathcal{IC}(\mathbf{V})$ to the stratum corresponding to $P$,

$$\mathcal{H}^k(\mathcal{IC}(\mathbf{V}))|_{\Gamma_H \backslash X_H} \cong \bigoplus_{\substack{i+j=k \\ j<\frac{1}{2}\dim N}} H^i(\Gamma_L, H^j(\mathbf{n}, V)),$$

(to be interpreted as (3.1)). We now return to the setup of Theorem 3.2. We therefore obtain for the alternating trace of the power $\Phi$ of the Frobenius at $\wp$ on the fibre at $y$ of $\mathcal{IC}(\mathbf{V})$,

$$\mathrm{Tr}(\Phi; \mathcal{IC}(\mathbf{V})_y) = \sum_{j<\frac{1}{2}\dim N} (-1)^j \,\mathrm{Tr}(\Phi; H^*(\Gamma_L, H^j(\mathbf{n}, V))). \tag{6.1}$$

For the calculation of the alternating traces appearing on the right hand side we use the following simple lemma.

**Lemma 6.2.** *Let $\Gamma$ be an abstract group of type $FL$, e.g., an arithmetic group and let $M$ be an arbitrary $k[\Gamma]$-module of finite dimension over the field $k$ and equipped with an endomorphism $\Phi$ commuting with $\Gamma$. Then for the alternating sum of traces of $\Phi$,*

$$\mathrm{Tr}(\Phi; H^*(\Gamma, M)) = \chi(\Gamma) \cdot \mathrm{Tr}(\Phi; M).$$

Here $\chi(\Gamma)$ denotes the Euler-Poincaré characteristic of $\Gamma$.

*Proof.* (cf. [S], Prop. 4, §1) By hypothesis, the constant $k[\Gamma]$-module $k$ has a finite free resolution $L_\bullet$ and the cohomology modules $H^*(\Gamma, M)$ are the cohomology groups of the complex $L_\bullet^\vee \otimes_\Gamma M$, where $L_i^\vee$ is the dual of the free $k[\Gamma]$-module $L_i$. Each $k[\Phi]$-module $L_i^\vee \otimes_\Gamma M$ is isomorphic to $M^{rg(L_i)}$. The assertion therefore follows by the usual Euler-Poincaré principle, since $\chi(\Gamma) = \sum(-1)^i rg(L_i)$. □

We therefore obtain the following expression. We keep the notation used in (6.1).

**Theorem 6.3.** *There is the following expression for the alternating trace of the power $\Phi$ of the Frobenius*

$$\mathrm{Tr}(\Phi; \mathcal{IC}(\mathbf{V})_y) = \chi(\Gamma_L) \cdot \sum_{0\le j<\frac{1}{2}\dim N} (-1)^j \,\mathrm{Tr}(\Phi; H^j(\mathbf{n}, V)).$$

*In particular, this expression vanishes if, denoting by $A$ the 1-dimensional split torus in the center of $L$, the group $L/A$ has no anisotropic maximal torus over* $\mathbf{R}$.

*Proof.* Indeed, in this last case the Euler-Poincaré characteristic appearing above vanishes (comp. e.g. [S], prop. 23, §3.2). □

Examples where the second assertion may be applied are provided by the Hilbert-Blumenthal cases where $G = R_{F/Q}SL_2$, with $F$ a totally real field of degree $> 1$. In this case $L \simeq R_{F/Q}\mathbf{G}_m$. Similarly, $G = R_{F/Q}SU$ are examples. Here $SU$ is a special unitary group of $F$-rank one, and $F$ is again a totally real field of degree $> 1$. (The assertion for the first class of examples appears as cor. 2.2.3 in [BL]; for the second class of examples it appears as theorem 2.4.6 in [BL].)

An example where the contribution does not vanish is given by an inner twisting of $Sp_4$ of rank 1 over $\mathbf{Q}$. In this case $L$ is the multiplicative group of a quaternion division algebra over $\mathbf{Q}$.

**7.** We conclude this note with some general remarks. In their contribution to this volume [GM3] M. Goresky and R. MacPherson work with the (lower and upper) "middle weighted cohomology complexes" on $\widehat{X_\Gamma}$. They give a formula for the local contribution of a fixed point on the reductive Borel-Serre compactification of a Hecke correspondence. Furthermore, they announce as a theorem (joint work with G. Harder) the analogue of conjecture 4.1 above, where the intersection complexes on $\widehat{X_\Gamma}$ are replaced by the middle weighted cohomology complexes. Therefore, by summing over the fixed points in the reductive Borel-Serre compactification mapping under the map $\pi$ to a fixed point on the Baily-Borel compactification, a formula for the local contribution of a fixed point on the Baily-Borel compactification is obtained. To compare the resulting formula with the one appearing here in the rank-1 case one invokes Kostant's formula for the highest weights of the irreducible representations of $M$ appearing in $H^*(\mathbf{n}, V)$. Similarly, Kostant's formula would have to be invoked to compare our formula (6.3) above with the one obtained by bypassing completely conjecture (5.2) above and using instead the theorem of Harder, Goresky, and MacPherson mentioned above.

Assume that, in the notation of (6.3), $L/A$ has an anisotropic maximal torus over $\mathbf{R}$ and therefore also has a discrete series. Then the factor $\chi(\Gamma_L)$ may be expressed through the Selberg trace formula on $L$ for a suitable function on $L(\mathbf{R})$.

Furthermore, assume that "there are no phenomena of $L$-indistinguishability". Then it seems reasonable to expect that a fixed point $y$ under a power $\Phi$ of the Frobenius at $\wp$ yields a well-determined conjugacy class $\gamma(y)$ in $H_P(\mathbf{Q}).A(\mathbf{Q})$ and that the trace of the operator on $H^*(\mathbf{n}, V)$ appearing in the statement of (6.3) only depends on $\gamma(y)$. Therefore, summing the contributions of the fixed points of $\Phi$ at the boundary by first summing over $\gamma(y)$ the expression given above and then counting the number of fixed points $y$ yielding $\gamma(y)$ we would obtain, taking into account the product decomposition $M = L \cdot H$, a sum of terms parametrized by the conjugacy classes in $M(\mathbf{Q})$. This would then look like the contribution of $M$ to the Arthur-Selberg trace formula for the trace of a Hecke operator on $L^2$-cohomology (cf. [A]). Indeed, the expression

$$tr(\gamma; \tau_{<\frac{1}{2}\dim N} H^*(\mathbf{n}, V))$$

is closely related to the value at $\gamma$ of the stabilized discrete series representation corresponding to $V$. In [GM3] there is an allusion to a similar formula due to R. Kottwitz, M. Goresky, and R. MacPherson for the case of a Hecke operator in the case of arbitrary rank. Unfortunately, $L$-indistinguishability makes for a more complicated picture for the Frobenius than the one sketched in the preceding remarks.

## References


[A] Arthur J., *The $L^2$-Lefschetz numbers of Hecke operators*, Inv. Math. **97** (1989), 257–290.

[BBD] Beilinson A., Bernstein J., and Deligne P., *Faisceaux pervers*, Astérisque **100** (1982).

[B] Borel, A., *A vanishing theorem in relative Lie algebra cohomology*, Algebraic groups Utrecht 1986, vol. 1271, Springer Lecture Notes, 1987, pp. 1–16.

[BC] Borel A., and Casselman, W., *Cohomologie d'intersection et $L^2$-cohomologie de variétés arithmétiques de rang rationnel 2*, C.R. Acad. Sci. Paris **301** (1985), 369–373.

[BS] Borel, A., and Serre, J.-P., *Corners and arithmetic groups*, Comm. Math. Helv. **48** (1973), 436–491.

[BW] Borel, A. and Wallach, N., *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. Math. Stud. **94** (1980), Princeton University Press.

[BL] Brylinski, J.-L. and Labesse, J.-P., *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura*, Ann. Scient. ENS **17** (1984), 361–412.

[GM1] Goresky, M. and MacPherson, R., *Intersection homology* II, Inv. Math. **72** (1983), 77–129.

[GM2] ———, *Weighted cohomology of Satake compactifications*, preliminary announcement, unpublished, (1988) 21p.

[GM3] ———, *Lefschetz numbers of Hecke correspondences (preliminary notes August 1990)*, this volume, 465–478.

[H1] Harder, G., *On the cohomology of discrete arithmetically defined groups*, Proc. of the Int. Coll. on discrete subgroups of Lie groups and applications to moduli Bombay 1973, Oxford University Press, 1975, pp. 129–160.

[H2] ———, *Eisensteinkohomologie arithmetischer Gruppen. Allgemeine Aspekte*, preprint (1985), 70p, Bonn.

[H3] ———, *Arithmetische Eigenschaften von Eisensteinklassen, die modulare Konstruktion von gemischten Motiven und von Erweiterungen endlicher Galoismoduln*, preprint (1989), 103p, Bonn.

[P] Pink, R., *On $\ell$-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, preprint (1990), 32p, Bonn.

[S] Serre, J.-P., *Cohomologie des groupes discrets*, Prospects in Mathematics; vol. 70, Princeton University Press, 1971, pp. 77–169.

[Z1] Zucker, S., *$L^2$-cohomology of warped products and arithmetic groups*, Inv. Math. **70** (1982), 169–218.

[Z2] ———, *Satake compactifications*, Comm. Math. Helv. **58** (1983), 312–343.

# APPENDIX

Leslie Saper
Mark A. Stern

In this appendix, we prove the following theorem conjectured by M. Rapoport (5.2). We follow the notation of the preceding article. We fix a maximal proper $\mathbf{Q}$-parabolic $P$ with associated groups $M = M_P$, $H = H_P$, $L = L_P$, etc. and symmetric spaces $X_M$, $X_L$, $X_H$, cf. **2**.

**Theorem A.1.** *Suppose that $\Gamma_L \backslash X_L$ is compact. Then*

$$H^i(\Gamma_L, H^j(\mathbf{n}, V)) = 0,$$

*if $j < 1/2 \operatorname{codim} X_M$ and $i + j \geq 1/2 \operatorname{codim}_{\mathbf{R}} X_H$.*

**Remark A.2.** The condition that $\Gamma_L \backslash X_L$ is compact can be replaced by the weaker condition that $H^i(\Gamma_L, H^j(\mathbf{n}, V))$ is representable by square integrable forms.

The proof of the theorem follows readily from the results of [SS], as we will now show.

For $\Gamma_L \backslash X_L$ compact, $H^i(\Gamma_L, H^j(\mathbf{n}, V))$ can be realized as the $L_2$-cohomology $H^i_{(2)}(\Gamma_L, H^j(\mathbf{n}, V))$. Let $L^i_2(\Gamma_L \backslash X_L, H^j(\mathbf{n}, V))$ denote the space of square integrable $i$-forms with coefficients in the homogeneous bundle associated to the local system $H^j(\mathbf{n}, V)$. The vanishing of $H^i_{(2)}(\Gamma_L \backslash X_L, H^j(\mathbf{n}, V))$ is equivalent to the existence of a positive constant $c$ such that

$$\|df\|^2 + \|d^* f\|^2 \geq c\|f\|^2, \tag{1}$$

for all smooth forms $f \in L^i_2(\Gamma_L \backslash X_L, H^j(\mathbf{n}, V))$. Here norms and adjoints are computed with respect to admissible invariant metrics.

Let $\Delta_{\circ_\rho}$ be the nonnegative, self-adjoint invariant bounded operator defined in [SS, Proposition 9.4]. The proof of that proposition contains the proof of the following lemma; one merely deletes extraneous variables.

**Lemma A.3.**

$$\|df\|^2 + \|d^*f\|^2 \geq (\Delta_{\circ_\rho} f, f).$$

The spectrum of $\Delta_{\circ_\rho}$ is discrete. Hence, in order to prove Theorem A.1, it suffices to prove the following proposition.

**Proposition A.4.** *If*

(a) $j < 1/2 \ \mathrm{codim}\, X_M$, *and*
(b) $i + j \geq 1/2 \ \mathrm{codim}\, X_H$,

*then* $\Delta_{\circ_\rho}$ *is positive definite on* $L_2^i(\Gamma_L \backslash X_L, H^j(\mathbf{n}, V))$.

*Proof.* Fix a fundamental Cartan subalgebra $\mathbf{h}$ of $\mathbf{m}$, stable under the Cartan involution. Let $\mathbf{h} = \mathbf{h}_k + \mathbf{h}_p$ denote the Cartan decomposition of $\mathbf{h}$. Further decompose $\mathbf{h}_p$ as $\mathbf{h}_p = \mathbf{h}_{p,0} \oplus \mathbf{a}$ where $\mathbf{a}$ is the Lie algebra of the one-dimensional maximal $\mathbf{Q}$-split torus in the center of $L$. Set $\delta_P = 1/2 \sum_{\alpha \in \Phi(\mathbf{n},\mathbf{a})} \alpha$. Let $F_\beta$ be an $\mathbf{m}$-irreducible component of $H^j(\mathbf{n}, V)$ with highest weight $\beta - \delta_P$. (Here we have chosen a positive system $\Phi^+(\mathbf{m}_\mathbf{C}, \mathbf{h}_\mathbf{C})$ for $\Phi(\mathbf{m}_\mathbf{C}, \mathbf{h}_\mathbf{C})$ as in [SS, Proposition 10.2].) Let $W$ denote the Weyl group of $\Phi(\mathbf{g}_\mathbf{C}, \mathbf{h}_\mathbf{C})$. Define

$$W^1 = \{w \in W : \Phi_w \subset \Phi(\mathbf{n}_\mathbf{C}, \mathbf{h}_\mathbf{C})\},$$

where $\Phi_w = \Phi^+ \cap w(-\Phi^+)$ and $\Phi^+ = \Phi^+(\mathbf{m}_\mathbf{C}, \mathbf{h}_\mathbf{C}) \cup \Phi(\mathbf{n}_\mathbf{C}, \mathbf{h}_\mathbf{C})$. By Kostant's theorem there exists $w \in W^1$ with $|\Phi_w| = j$ such that

$$\beta = w(\delta + \lambda) - \delta + \delta_P,$$

where $\lambda$ is the highest weight of $V$ and $\delta = 1/2 \sum_{\alpha \in \Phi^+} \alpha$. The operator $\Delta_{\circ_\rho}$ commutes with the projection onto $L_2^i(\Gamma_L \backslash X_L, F_\beta)$; hence, it suffices to show that $\Delta_{\circ_\rho}$ is positive definite on $L_2^i(\Gamma_L \backslash X_L, F_\beta)$. By [SS, Proposition 10.2], if this fails then

$$\beta|_{\mathbf{h}_{p,0}} = 0. \tag{2}$$

Fix a $\beta$ for which $\Delta_{\circ_\rho}$ is not positive definite. We shall show that $i$ and $j$ lie outside the given range.

There are two cases to consider. Assume first that for $\alpha \in \Phi(\mathbf{n}, \mathbf{a})$,

$$(\beta|_\mathbf{a}, \alpha) \geq 0. \tag{3}$$

Then we may apply [SS, Propositions 10.2 and 11.1 and (12.1)] to deduce

$$i + j \leq 1/2(\dim \mathbf{n} + \dim X_L - 1) < 1/2 \ \mathrm{codim}\, X_H.$$

This contradicts (b).

Suppose now that for some (and hence all) $\alpha \in \Phi(\mathbf{n}, \mathbf{a})$,

$$(\beta|_{\mathbf{a}}, \alpha) < 0. \tag{4}$$

We will show that this cannot occur when $j < 1/2(1 + \dim \mathbf{n}) = 1/2 \ \operatorname{codim} X_M$.

There exists a basis $S = \{\gamma_1, \cdots, \gamma_r\}$ of $\mathbf{h}_p^*$ consisting of strongly orthogonal positive real roots satisfying $S \subset \Phi(\mathbf{u}_{\mathbf{C}}, \mathbf{h}_{\mathbf{C}})$ (see [SS, Lemma 11.6]). Recall that $\mathbf{u}$ is the center of $\mathbf{n}$ and is a weight space for $\mathbf{a}$. The conditions (2) and (4) on $\beta$ imply that for all $k$,

$$(\beta, \gamma_k) = (\beta, \gamma_1) < 0. \tag{5}$$

From [SS, (11.14)], we may write

$$\Phi_w = \{\mu \in \Phi(\mathbf{n}_{\mathbf{C}}, \mathbf{h}_{\mathbf{C}}) | (\mu, \beta + \delta - \delta_P) < 0\}.$$

Thus, since $(\delta - \delta_P)|_{\mathbf{h}_p} = 0$, (5) implies $S \subset \Phi_w$. Similarly, if $\mu \in \Phi(\mathbf{n}_{\mathbf{C}}, \mathbf{h}_{\mathbf{C}})$, then $(w(\delta + \lambda), \mu + \bar{\mu}) < 0$. Hence either $\mu \in \Phi_w$ or $\bar{\mu} \in \Phi_w$. From this it is immediate that

$$j = |\Phi_w| \geq 1/2(\dim \mathbf{n} + \dim \mathbf{h}_p) = 1/2(\operatorname{codim} X_M + \dim \mathbf{h}_{p,0}), \tag{6}$$

contradicting (a). $\square$

One can recast the above argument to show that if $H^i_{(2)}(\Gamma_L \backslash X_L, H^j(\mathbf{n}, V)) \neq 0$, with $j < 1/2(\dim \mathbf{n} + \dim \mathbf{h}_p)$, then in fact

$$j \leq 1/2(\dim \mathbf{n} - \dim \mathbf{h}_p). \tag{7}$$

## Reference

[SS] Saper, L. and Stern, M., $L^2$*-cohomology of arithmetic varieties*, Ann. Math. **132** (1990), 1–69.