Non-Archimedean Period Domains

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The best known example of a non-archimedean period domain is the Drinfeld upper half space Ω_E^d of dimension $d-1$ associated to a finite extension E of $\mathbf{Q}_p$ (complement of all E-rational hyperplanes in the projective space $\mathbf{P}^{d-1}$). Drinfeld [D2] interpreted this rigid-analytic space as the generic fibre of a formal scheme over O_E parametrizing certain p-divisible groups. He used this to p-adically uniformize certain Shimura curves (Cherednik's theorem) and to construct highly nontrivial étale coverings of Ω_E^d. This report gives an account of joint work of Zink and myself [RZ] that generalizes the construction of Drinfeld (Sections 1–3). In the last two sections these results are put in a more general framework (Fontaine conjecture) and the problem of the computation of ℓ-adic cohomology is addressed (Kottwitz conjecture). In this report we return to the subject of Grothendieck's talk at the Nice congress [G, esp. Section 5] where he stressed the relation between the local moduli of p-divisible groups and filtered Dieudonné modules.

Throughout this report we fix a prime number p. Denote by k a fixed algebraically closed field of characteristic p. Let $W(k)$ be its ring of Witt vectors and $K_0 = W(k) \otimes_{\mathbf{Z}} \mathbf{Q}$. Let σ be the Frobenius automorphism of K_0. For most results one must assume $p \neq 2$.

1. Formal moduli of p-divisible groups

If O is a complete discrete valuation ring with uniformizer π, we denote by Nilp_O the category of locally noetherian schemes S over $\mathrm{Spec}\, O$ such that the ideal sheaf $\pi \cdot \mathcal{O}_S$ is locally nilpotent. We denote by $\bar{S}$ the closed subscheme defined by $\pi \cdot \mathcal{O}_S$. A locally noetherian formal scheme over $\mathrm{Spf}\, O$ will be identified with the set-valued functor on Nilp_O it defines. A morphism $\mathcal{X} \to \mathcal{Y}$ of formal schemes is called *locally formally of finite type* if the induced morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ between their underlying reduced schemes of definition is locally of finite type.

In what follows, by a *quasi-isogeny between p-divisible groups* X and Y over a scheme $S \in \mathrm{Nilp}_{\mathbf{Z}_p}$ we mean a global section f of the Zariski sheaf $\underline{\mathrm{Hom}}(X,Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ for which there exists locally on S an integer n such that $p^n f$ is an isogeny.

THEOREM 1.1. *Let* $\mathbf{X}$ *be a p-divisible group over* $\mathrm{Spec}\ k$. *We consider the functor* $\mathcal{M}$ *on* $\mathrm{Nilp}_{W(k)}$, *which associates to* $S \in \mathrm{Nilp}_{W(k)}$ *the set of isomorphism classes of pairs* (X, ϱ) *consisting of a p-divisible group* X *over* S *and a quasi-isogeny* $\varrho : \mathbf{X} \times_{\mathrm{Spec}\, k} \bar{S} \to X \times_S \bar{S}$ *of p-divisible groups over* $\bar{S}$. *Then* $\mathcal{M}$ *is representable by a formal scheme locally formally of finite type over* $\mathrm{Spf}\ W(k)$.

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This representability theorem [RZ, Section 2] allows one to show that certain functors of p-divisible groups endowed with endomorphisms and level structures (case (EL)) resp. with polarizations and endomorphisms and level structures (case (PEL)) are also representable. These functors depend on certain "rational" and "integral" data that we now formulate in both cases.

Case (EL): The *rational data* consists of a 4-tuple (B, V, b, μ), where B is a finite-dimensional semi-simple algebra over $\mathbf{Q}_p$ and V a finite left B-module. Let $G = GL_B(V)$ (algebraic group over $\mathbf{Q}_p$). Then b is an element of $G(K_0)$. The final datum μ is a homomorphism $\mathbf{G}_m \to G_K$ defined over a finite extension K of K_0. Let $V \otimes_{\mathbf{Q}_p} K = \bigoplus V_i$ be the corresponding eigenspace decomposition and $V_K^j = \bigoplus_{i \geq j} V_i$ the associated decreasing filtration. We require that the *filtered isocrystal over* K, $(V \otimes_{\mathbf{Q}_p} K_0, b(\mathrm{id} \otimes \sigma), V_K^{\bullet})$, be the filtered isocrystal associated to a p-divisible group over Spec O_K ([G], [Fo1], [Me]). The *integral data* consists of a maximal order O_B in B and an O_B-lattice chain $\mathcal{L}$ in V [RZ, Section 3].

Case (PEL): In this case the *rational data* are given by a 6-tuple $(B, *, V, (\ ,\ ), b, \mu)$. Here B and V are as before. Furthermore, B is endowed with an anti-involution $*$ and V is endowed with a nondegenerate alternating bilinear form $(\ ,\ ) : V \otimes_{\mathbf{Q}_p} V \to \mathbf{Q}_p$ such that

$$(dv, v') = (v, d^* v'), \quad d \in B. \tag{1}$$

The remaining data are as before relative to the algebraic group G over $\mathbf{Q}_p$ whose values in a $\mathbf{Q}_p$-algebra R are

$$G(R) = \{g \in GL_B(V \otimes R);\ \ (gv, gv') = c(g)(v, v'),\ \ c(g) \in R^{\times}\}.$$

We require that the rational data define the filtered isocrystal associated to a p-divisible group over Spec O_K endowed with a polarization (= symmetric isogeny to its dual). The *integral data* are as before. We assume that O_B is stable under $*$ and that $\mathcal{L}$ is self-dual w.r.t. $(\ ,\ )$.

In either case let E be the field of definition of the conjugacy class of μ, a finite extension of $\mathbf{Q}_p$ contained in K. Let $\breve{E} = E.K_0$, with ring of integers $O_{\breve{E}}$.

THEOREM 1.2. *We fix data of type (EL) or (PEL). Let* $\mathbf{X}$ *be a* p*-divisible group with action of* O_B *over* Spec k *with associated isocrystal isomorphic to* $(V \otimes_{\mathbf{Q}_p} K_0, b(\mathrm{id} \otimes \sigma))$. *In the case (PEL) we endow* $\mathbf{X}$ *with an* O_B*-polarization defined by the alternating form on* $V \otimes_{\mathbf{Q}_p} K_0$. *We consider the functor* $\breve{\mathcal{M}}$ *on* $\mathrm{Nilp}_{O_{\breve{E}}}$, *which associates to* S *the isomorphism classes of the following data.*

(1) *A* p*-divisible group* X_Λ *over* S *with* O_B*-action, for each* $\Lambda \in \mathcal{L}$.
(2) *An* O_B*-quasi-isogeny* $\varrho_\Lambda : \mathbf{X} \times_{\mathrm{Spec}\, k} \bar{S} \to X_\Lambda \times_S \bar{S}$, *for each* $\Lambda \in \mathcal{L}$.

Among the conditions these data are required to satisfy we mention only the following.

(i) *For each* $\Lambda \in \mathcal{L}$ *we have* $\det_{\mathcal{O}_S}(d; \mathrm{Lie}\, X_\Lambda) = \det_K(d; V_K^0/V_K^1)$, $d \in B$ *(Kottwitz condition* [Ko3]*).*

(ii) *Let* $M(X_\Lambda)$ *be the Lie algebra of the universal extension of* X_Λ. *Then locally on* S *there is an* O_B*-isomorphism* $M(X_\Lambda) \simeq \Lambda \otimes_{\mathbf{Z}} \mathcal{O}_S$. *If* $\Lambda \subset \Lambda'$, *the quasi-isogeny* $\varrho_{\Lambda'} \circ \varrho_\Lambda^{-1}$ *lifts to an isogeny* $X_\Lambda \to X_{\Lambda'}$ *of height* $\log_p |\Lambda'/\Lambda|$.

The functor $\breve{\mathcal{M}}$ is representable by a formal scheme locally formally of finite type over Spf $O_{\breve{E}}$.

REMARKS 1.3. (i) To the pair (G, b) there is associated the algebraic group J over $\mathbf{Q}_p$ with points in a $\mathbf{Q}_p$-algebra R

$$J(R) = \{g \in G(R \otimes_{\mathbf{Q}_p} K_0);\ \sigma(g) = b^{-1}gb\}.$$

The group $J(\mathbf{Q}_p)$ of quasi-isogenies of $\mathbf{X}$ acts on the left of $\breve{\mathcal{M}}$, via

$$g \cdot (X_\Lambda, \varrho_\Lambda) = (X_\Lambda, \varrho_\Lambda \circ g^{-1}).$$

Let Δ be the abelian group dual to the group of $\mathbf{Q}_p$-rational characters of G. The group $J(\mathbf{Q}_p)$ acts on Δ by translations. There is a canonical $J(\mathbf{Q}_p)$-equivariant map $\kappa : \breve{\mathcal{M}} \to \Delta$ [RZ, Section 3].

(ii) We conjecture that $\breve{\mathcal{M}}$ is flat over Spf $O_{\breve{E}}$. This conjecture can be reduced to a similar statement on the *local model*, an explicit closed subscheme of a finite product of Grassmannian varieties over Spec O_E associated to the moduli problem [RZ, Section 3]. In the numerous special cases where this conjecture is proved, the singularities of $\breve{\mathcal{M}}$ have turned out to be roughly of a "determinantal nature", comp. [CN], [dJ]. For a moduli problem of type (EL) the scheme $\breve{\mathcal{M}}_{\text{red}}$ turns out in all known cases to be "elementary". For instance, the zeta function of a model of $\breve{\mathcal{M}}_{\text{red}}$ over a finite field is given by an elementary expression. On the other hand, for type (PEL) there are simple examples where the irreducible components of $\breve{\mathcal{M}}_{\text{red}}$ fibre over nonrational curves [KO]. Laumon has pointed out the similarity with the behaviour of the varieties connected with the local harmonic analysis of G [H].

(iii) The formal scheme $\breve{\mathcal{M}}$ is equipped with a canonical Weil descent datum from Spf $O_{\breve{E}}$ to Spf O_E [RZ, Section 3]. Even though this is not effective, a suitable completion of $\breve{\mathcal{M}}$ can be written in a canonical way as $\mathcal{M} \times_{\text{Spf } O_E} \text{Spf } O_{\breve{E}}$ for a (pro-)formal scheme $\mathcal{M}$ over Spf O_E [RZ, Section 3].

EXAMPLES 1.4. (i) Let B be a division algebra with invariant $1/d$ over its center F, with maximal order O_B. Set $E = F$ and $\breve{E} = E.K_0$. Drinfeld [D2] considers the moduli problem classifying quasi-isogenies of *special formal O_B-modules* (X, ϱ) over schemes $S \in \text{Nilp}_{O_{\breve{E}}}$ (it can be identified with a moduli problem of type (EL)). He exhibits in this case an isomorphism

$$\breve{\mathcal{M}} = \coprod_{n \in \mathbf{Z}} \hat{\Omega}^d_E \times_{\text{Spf } O_E} \text{Spf } O_{\breve{E}}. \tag{2}$$

Here $\hat{\Omega}^d_E$ is the formal scheme over Spf O_E associated by Deligne to the local field E and the integer d [D2]. The disjoint sum decomposition is induced by the function $(X, \varrho) \mapsto \text{height}\,(\varrho)$.

This example (and trivial variants of it) is the only one we know where the formal scheme $\breve{\mathcal{M}}$ is p-adic, i.e., $p \cdot O_{\breve{\mathcal{M}}}$ is an ideal of definition.

(ii) Let F be a finite extension of $\mathbf{Q}_p$, set $E = F$ and $\breve{E} = E.K_0$. Let $\breve{\mathcal{M}}$ be the moduli problem (of type (EL)) over $\text{Nilp}_{O_{\breve{E}}}$ classifying quasi-isogenies (X, ϱ) of formal O_F-modules of dimension 1 and height d. In this case $\breve{\mathcal{M}}_{\text{red}}$ is a discrete set indexed by height $(\varrho) \in \mathbf{Z}$. The infinitesimal deformation theory of Lubin and Tate yields a (noncanonical) isomorphism

$$\breve{\mathcal{M}} = \coprod_{n\in\mathbf{Z}} \mathrm{Spf}\ O_{\breve{E}}[[T_1, \dots, T_{d-1}]].$$

(iii) Consider the moduli problem $\breve{\mathcal{M}}$ over Spf $W(k)$ (of type (EL)) classifying quasi-isogenies (X, ϱ) of *ordinary* p-divisible groups of height $2n$ and dimension n (i.e. deformations of $\mathbf{X} = \hat{\mathbf{G}}_m^n \times (\mathbf{Q}_p/\mathbf{Z}_p)^n$). There is an isomorphism (Serre-Tate canonical coordinates [Ka])

$$\breve{\mathcal{M}} = \coprod \mathrm{Spf}\ W(k)[[T_{11}, T_{12}, \dots, T_{nn}]].$$

The index set is equal to $(GL_n(\mathbf{Q}_p)/GL_n(\mathbf{Z}_p))^2$.

2. Non-archimedean uniformization of Shimura varieties

In this section we use slightly different notation. Let B be a finite-dimensional algebra over $\mathbf{Q}$ equipped with a *positive* anti-involution $*$. Let V be a finite B-module with a nondegenerate alternating bilinear form $(\ ,\ )$ with values in $\mathbf{Q}$ satisfying the identity (1). We define the algebraic group G over $\mathbf{Q}$ in complete analogy with the case (PEL) in Section 1. Let $h : \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\, \mathbf{G}_m \to G_{\mathbf{R}}$ be such that (G, h) satisfies the axioms of Deligne defining a Shimura variety over the Shimura field $E \subset \mathbf{C}$. We fix an order O_B of B such that $O_B \otimes \mathbf{Z}_p$ is a maximal order of $B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ stable under $*$, and a self-dual $O_B \otimes_{\mathbf{Z}} \mathbf{Z}_p$-lattice chain $\mathcal{L}$ in $V \otimes_{\mathbf{Q}} \mathbf{Q}_p$. We fix an open compact subgroup $K^p \subset G(\mathbf{A}_f^p)$. Let $\bar{\mathbf{Q}}$ be the field of algebraic numbers and fix an embedding $\bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p$. Let ν be the corresponding place of E above p and E_ν the completion of E in ν.

These data define a moduli problem of PEL-type parametrizing triples $(A, \bar{\lambda}, \bar{\eta}^p)$ consisting of an $\mathcal{L}$-chain of O_B-abelian varieties, a $\mathbf{Q}$-homogeneous O_B-polarization and a K^p-level structure that is representable by a quasi-projective scheme $\mathcal{A}_{K^p}$ over Spec O_{E_ν} (cf. [RZ, Section 6] for details). The generic fibre of $\mathcal{A}_{K^p}$ contains the Shimura variety of (G, h) as an open and closed subscheme.

We take for k the algebraic closure of the residue field of O_{E_ν}. We fix a point $(A_0, \bar{\lambda}_0, \bar{\eta}_0^p)$ of $\mathcal{A}_{K^p}(k)$. Let N_0 be the isocrystal associated to A_0. We fix an isomorphism $N_0 \simeq V \otimes_{\mathbf{Q}_p} K_0$ that respects the actions of $B \otimes K_0$ and the alternating bilinear forms on both sides. This allows us to write the Frobenius operator on N_0 as $b(\mathrm{id} \otimes \sigma)$, with $b \in G(K_0)$. Let $\mathcal{M}$ be the (pro-)formal scheme over Spf O_{E_ν} associated to the data of type (PEL), $(B \otimes \mathbf{Q}_p, *, V \otimes \mathbf{Q}_p, (\ ,\ ), b, \mu, O_B \otimes \mathbf{Z}_p, \mathcal{L})$, cf. Remark 1.3, (iii). It is acted on by the group $J(\mathbf{Q}_p)$, Remark 1.3, (i). Here μ denotes a 1-parameter subgroup of G defined over a finite extension K of K_0 in the conjugacy class defined by h.

THEOREM 2.1. *Assume that $(A_0, \bar{\lambda}_0, \bar{\eta}_0^p)$ is basic, i.e. the corresponding element $b \in G(K_0)$ is basic* [Kol]. *Then*

(i) *The set of points $(A, \bar{\lambda}, \bar{\eta}^p)$ of $\mathcal{A}_{K^p}(k)$ such that $(A, \bar{\lambda})$ is isogenous to $(A_0, \bar{\lambda}_0)$ is a closed subset Z of $\mathcal{A}_{K^p}$.*

(ii) *Let $\mathcal{A}_{K^p}|Z$ denote the formal completion of $\mathcal{A}_{K^p}$ along Z. There is an isomorphism of formal schemes over* Spf O_{E_ν},

$$I(\mathbf{Q}) \setminus [\mathcal{M} \times G(\mathbf{A}_f^p)/K^p] \xrightarrow{\sim} \mathcal{A}_{K^p}|Z.$$

Here I is an inner form of G such that $I(\mathbf{Q})$ is the group of quasi-isogenies of $(A_0, \bar{\lambda}_0)$ that acts diagonally through suitable natural embeddings of groups,

$$I(\mathbf{Q}) \longrightarrow J(\mathbf{Q}_p), \ \ I(\mathbf{Q}) \longrightarrow G(\mathbf{A}_f^p).$$

The source of this isomorphism is a finite disjoint sum of formal schemes of the form $\Gamma \setminus \mathcal{M}$, where $\Gamma \subset J(\mathbf{Q}_p)$ is a discrete subgroup that is cocompact modulo center.

REMARKS 2.2. (i) In the Siegel case (principally polarized abelian varieties with level structure prime to p) the subscheme Z is the *supersingular locus*. In general it may be conjectured that there always exist basic points $(A_0, \bar{\lambda}_0, \bar{\eta}_0^p) \in \mathcal{A}_{K^p}(k)$.

(ii) It is not understood when it happens that the subscheme Z is open in the special fibre (existence of p-adic uniformization), as in Cherednik's theorem [D2]. This is a very subtle property that occurs only rarely (cf. Example 1.4, (i)). However, there are examples [RZ] uniformized by the disjoint sum of unramified forms of formal schemes of the form $\Gamma \setminus \hat{\Omega}_E^d$, where E is any finite extension of $\mathbf{Q}_p$ and d any integer.

(iii) There also exists a version of Theorem 2.1 for nonbasic isogeny classes. Because these do not in general form a closed subset the formulation becomes more technical.

3. The period morphism

In this section we return to the notation of Section 1. Let $\breve{\mathcal{M}}$ be the formal scheme over Spf $O_{\breve{E}}$ given by Theorem 1.2 and let $(X_\Lambda, \varrho_\Lambda;\ \Lambda \in \mathcal{L})$ be the universal object over $\breve{\mathcal{M}}$.

Let $\breve{\mathcal{M}}^{\text{rig}}$ be the rigid-analytic space over $\breve{E}$ associated to $\breve{\mathcal{M}}$ (the generic fibre of $\breve{\mathcal{M}}$ in the sense of Raynaud-Berthelot [RZ, Section 5]). By Grothendieck's rigidity theorem [G] the quasi-isogenies ϱ_Λ yield canonical isomorphisms $V \otimes_{\mathbf{Q}_p} \mathcal{O}_{\breve{\mathcal{M}}^{\text{rig}}} = M(X_\Lambda) \otimes_{\mathcal{O}_{\breve{\mathcal{M}}}} \mathcal{O}_{\breve{\mathcal{M}}^{\text{rig}}}$. The resulting surjective homomorphism of locally free $\mathcal{O}_{\breve{\mathcal{M}}^{\text{rig}}}$-modules

$$V \otimes_{\mathbf{Q}_p} \mathcal{O}_{\breve{\mathcal{M}}^{\text{rig}}} \longrightarrow \text{Lie}(X_\Lambda) \otimes_{\mathcal{O}_{\breve{\mathcal{M}}}} \mathcal{O}_{\breve{\mathcal{M}}^{\text{rig}}} \tag{3}$$

is independent of Λ.

Let $\mathcal{F}$ be the projective algebraic variety over Spec E parametrizing the subspaces of V in the G-conjugacy class of V_K^1. Set $\breve{\mathcal{F}} = \mathcal{F} \times_{\text{Spec } E} \text{Spec } \breve{E}$. Then (3) corresponds to a rigid-analytic morphism $\breve{\pi}^1 : \breve{\mathcal{M}}^{\text{rig}} \to \breve{\mathcal{F}}^{\text{rig}}$. Let $\breve{\pi}^2 : \breve{\mathcal{M}}^{\text{rig}} \to \Delta$ be the rigid-analytic morphism to the discrete rigid-analytic space associated to the map κ, cf. Remark 1.3, (i). The product morphism

$$\breve{\pi} = \breve{\pi}^1 \times \breve{\pi}^2 : \breve{\mathcal{M}}^{\text{rig}} \longrightarrow \breve{\mathcal{F}}^{\text{rig}} \times \Delta$$

is called the *period morphism associated to the moduli problem* $\breve{\mathcal{M}}$. We list some properties of the period morphism [RZ, Section 5].

PROPERTY 3.1. *The morphism $\breve{\pi}$ is étale and $J(\mathbf{Q}_p)$-equivariant.*

Here $J(\mathbf{Q}_p)$ acts diagonally on $\breve{\mathcal{F}} \times \Delta$, via its embedding in $G(\breve{E})$ on $\breve{\mathcal{F}}$ and via translations on Δ. In the rest of this section we shall assume that the algebraic group G is connected. We also make the assumption that $\breve{\mathcal{M}}^{\mathrm{rig}}$ is nonempty (this would certainly follow if the flatness conjecture 1.3, (ii) were true).

PROPERTY 3.2. Assume the validity of the conjecture of Fontaine [Fo2] that a weakly admissible filtered isocrystal over a finite extension K of K_0, with filtration steps contained in [0,1] comes from a p-divisible group over Spec O_K. Let $\breve{\mathcal{F}}^{\mathrm{wa}}$ be the admissible open subset of $\breve{\mathcal{F}}^{\mathrm{rig}}$ formed by weakly admissible filtrations, cf. Proposition 4.1 below. *Then the image of $\breve{\pi}$ is of the form $\breve{\mathcal{F}}^{\mathrm{wa}} \times \Delta'$, where Δ' is a union of cosets of a subgroup of finite index in Δ.*

PROPERTY 3.3. *For all $\Lambda \in \mathcal{L}$ the rational p-adic Tate module $V_p(X_\Lambda)$ yields a constant $\mathbf{Q}_p$-sheaf on $\breve{\mathcal{M}}^{\mathrm{rig}}$ with typical fibre V.* Let $K_{\mathcal{L}}$ be the subgroup of $G(\mathbf{Q}_p)$ that fixes the lattice chain $\mathcal{L}$. For any subgroup K of finite index in $K_{\mathcal{L}}$ we introduce the finite étale covering $\breve{\mathbf{M}}_K$ of $\breve{\mathcal{M}}^{\mathrm{rig}}$ that classifies the trivializations $\alpha : \{T_p(X_\Lambda)\} \to \mathcal{L}$ mod K of the chain of p-adic Tate modules $(T_p(X_\Lambda);\ \Lambda \in \mathcal{L})$. On the tower of étale coverings

$$\{\breve{\mathbf{M}}_K;\ K \subset K_{\mathcal{L}} \text{ of finite index }\}$$

of $\breve{\mathcal{M}}^{\mathrm{rig}} = \breve{\mathbf{M}}_{K_{\mathcal{L}}}$ the group $G(\mathbf{Q}_p)$ acts as a group of Hecke correspondences. Let $\breve{\pi}_K : \breve{\mathbf{M}}_K \to \breve{\mathcal{F}}^{\mathrm{wa}} \times \Delta$ be the resulting morphism. *The fibre of $\breve{\pi}_K$ through a point may be identified with $G(\mathbf{Q}_p)^1/K$.* Here $G(\mathbf{Q}_p)^1$ is the set of points of $G(\mathbf{Q}_p)$ where the values of all $\mathbf{Q}_p$-rational characters of G are units.

EXAMPLES 3.4. (i) Historically the first such period morphism was defined by Dwork for Example 1.4, (iii), cf. [Ka].

(ii) The period morphism for Example 1.4, (ii) induces on each connected component of $\breve{\mathcal{M}}^{\mathrm{rig}}$ a *surjective* morphism of the open unit polydisc of dimension $d-1$ to $\mathbf{P}_E^{d-1}$. The period morphism in this case is due to Gross and Hopkins [HG] (their construction is slightly different). Their paper is at the origin of the results of this section. The passage to the rigid category is the essential novelty compared to Grothendieck [G, Section 5].

(iii) In Drinfeld's Example 1.4, (i) the period morphism coincides with the composition of the isomorphism (2) with the natural inclusion $(\hat{\Omega}_E^d)^{\mathrm{rig}} \times \mathbf{Z} \subset (\mathbf{P}_E^{d-1})^{\mathrm{rig}} \times \mathbf{Z}$ induced by the identification $(\hat{\Omega}_E^d)^{\mathrm{rig}} = \Omega_E^d$ (Faltings). This example (and trivial variants of it) is the only known one where the period morphism is quasi-compact. There are examples where the period morphism has finite fibres but is not quasi-compact.

4. Non-archimedean period domains

Let $\mathcal{FI}(K)^{\mathrm{wa}}$ denote the $\mathbf{Q}_p$-tensor category of weakly admissible filtered isocrystals over the finite extension K of K_0 [Fo2], [Fa]. Assuming the validity of his conjecture (weakly admissible $\Rightarrow$ admissible [Fo2]) Fontaine constructs an exact fully faithful functor from this category to the category of p-adic Galois representations of $\mathrm{Gal}(\bar{K}/K)$. Composing this functor with the natural fibre functor on the latter (forgetting the Galois action) we obtain a fibre functor ω of $\mathcal{FI}(K)^{\mathrm{wa}}$ over $\mathbf{Q}_p$.

Let now G be a linear algebraic group over $\mathbf{Q}_p$ and fix $b \in G(K_0)$. Let $\mathcal{REP}(G)$ be the tensor category of $\mathbf{Q}_p$-rational representations of G. Let $\mu : \mathbf{G}_m \to G_K$. Then to any $V \in \mathcal{REP}(G)$ we have an associated filtered isocrystal $\mathcal{I}(V) = \mathcal{I}_\mu(V) = (V \otimes_{\mathbf{Q}_p} K_0, b(\mathrm{id} \otimes \sigma), V_K^\bullet)$ over K, where $V_K^\bullet$ is the filtration associated to μ, cf. Section 1. We call μ *weakly admissible (w.r.t. b)* if $\mathcal{I}(V)$ is weakly admissible for all $V \in \mathcal{REP}(G)$. Let μ be weakly admissible. Then, assuming Fontaine's conjecture and composing the functor $\mathcal{I}$ with ω we obtain a fibre functor of $\mathcal{REP}(G)$ over $\mathbf{Q}_p$. Let ω_0 be the natural fibre functor of $\mathcal{REP}(G)$ over $\mathbf{Q}_p$. Then the right G-torsor $\mathrm{Hom}(\omega_0, \omega \circ \mathcal{I})$ defines a cohomology class

$$\mathrm{cls}(b, \mu) \in H^1(\mathbf{Q}_p, G). \tag{4}$$

When G is a connected reductive group with simply connected derived group there is an explicit expression for this class [RZ, Section 1].

From now on we fix an algebraic closure $\bar{K}_0$ of K_0 and take K to be any finite extension of K_0 inside $\bar{K}_0$. Two 1-parameter subgroups $\mu, \mu' : \mathbf{G}_m \to G_K$ will be called equivalent if they induce the same filtrations on each $V \in \mathcal{REP}(G)$. We fix a conjugacy class $\{\mu\}$ of 1-parameter subgroups of G over $\bar{K}_0$ and denote by $E \subset \bar{K}_0$ its field of definition and $\breve{E} = E.K_0$. Then the equivalence classes of 1-parameter subgroups in $\{\mu\}$ form a projective algebraic variety $\mathcal{F}(G, \{\mu\})$ defined over E that is homogeneous under G_E. We write $\mathcal{F}$ for $\mathcal{F}(G, \{\mu\})$ if this is unambiguous and $\breve{\mathcal{F}} = \mathcal{F} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E}$.

PROPOSITION 4.1. *The weakly admissible points form an admissible open subset $\breve{\mathcal{F}}^{\mathrm{wa}}$ of $\breve{\mathcal{F}}^{\mathrm{rig}}$ stable under the action of $J(\mathbf{Q}_p)$.*

Here J is the algebraic group associated to (G, b), cf. Remark 1.3, (i). We call $\breve{\mathcal{F}}^{\mathrm{wa}}$ the *non-archimedean period domain associated to* $(G, b, \{\mu\})$. From now on we assume that G and hence J are connected reductive groups over $\mathbf{Q}_p$. We also assume that $\breve{\mathcal{F}}^{\mathrm{wa}} \neq \emptyset$. The fundamental open question in this context is the following. We introduce the free abelian group Δ as in Section 3. Let G' be the inner form of G defined by the image of $cl(b, \mu)$ in $H^1(\mathbf{Q}_p, G_{ad})$, cf. (4) (the class $cl(b, \mu)$ should only depend on $(b, \{\mu\})$).

HOPE 4.2. *There exists a canonical tower of rigid-analytic spaces*

$$\{\breve{\mathbf{M}}_{K'};\ K' \subset G'(\mathbf{Q}_p) \text{ open compact }\} \tag{5}$$

each of which is equipped with an action of $J(\mathbf{Q}_p)$ and an equivariant étale morphism $\breve{\pi}_{K'} : \breve{\mathbf{M}}_{K'} \to \breve{\mathcal{F}}^{\mathrm{wa}} \times \Delta$ with image of the form $\breve{\mathcal{F}}^{\mathrm{wa}} \times \Delta'$, where Δ' is a union of cosets of a subgroup of finite index in Δ. We furthermore want $G'(\mathbf{Q}_p)$ to act on (5) as a group of Hecke correspondences covering the action on $\breve{\mathcal{F}}^{\mathrm{wa}} \times \Delta$, which is trivial on the first factor and by translations on the second factor.

Heuristically speaking, the tower (5) should be given by the "K'-level structures on the local system on $\breve{\mathcal{F}}^{\mathrm{wa}}$ defined by the fibre functor $\omega \circ \mathcal{I}_x$, as x varies in $\breve{\mathcal{F}}^{\mathrm{wa}}$", but it is not clear how to make sense of this. The tower of Property 3.3 is a typical candidate (in this case we have $G = G'$). In Examples 1.4, (i)–(iii) this tower does indeed exist.

REMARKS 4.3. (i) Assume b is basic. In the few known cases the fibres of $\breve{\pi}^2_{K'}$ over a point in Δ have turned out to be connected. We do not know whether to expect this in general when the derived group is simply connected.

(ii) Assume b basic and that $\mathrm{cls}(b, \mu)$ is trivial. Then the triple (J, b^{-1}, μ^{-1}) satisfies the same assumptions as (G, b, μ) and the group associated to (J, b^{-1}) is G. One might wonder whether there exists a rigid-analytic space X with an action of $G(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$ such that the towers associated to (G, b, μ) resp. (J, b^{-1}, μ^{-1}) are obtained by taking the quotients of X by open compact subgroups of $G(\mathbf{Q}_p)$ resp. $J(\mathbf{Q}_p)$. The pair formed by Example 1.4, (i) and the moduli problem of formal O_F-modules of dimension $d-1$ and height d (dual in some sense to Example 1.4, (ii)) are in this kind of duality and the question was raised in this case by Gross.

One can characterize $\breve{\mathcal{F}}^{\mathrm{wa}}$ by geometric invariant theory as follows. To $\{\mu\}$ there corresponds an essentially unique ample line bundle $\mathcal{L}$ on $\breve{\mathcal{F}}$ that is homogeneous under the derived group $G_{der_{\breve{E}}}$. For any maximal $\mathbf{Q}_p$-split torus $T \subset J$ let $\breve{\mathcal{F}}(T)^{ss}$ be the Zariski-open subset of $\breve{\mathcal{F}}$ formed by the points that are semi-stable w.r.t. the action of $T_{\breve{E}} \cap G_{der_{\breve{E}}}$ on $(\breve{\mathcal{F}}, \mathcal{L})$.

THEOREM 4.4. (Totaro): *We have*

$$\breve{\mathcal{F}}^{\mathrm{wa}} = \bigcap_{T \subset J} \breve{\mathcal{F}}(T)^{ss}.$$

The admissible open subsets of $\breve{\mathcal{F}}^{\mathrm{rig}}$ appearing on the right-hand side have been considered by van der Put and Voskuil [PV].

REMARK 4.5. The period domain $\breve{\Omega}^d_E$ (= complement of all E-rational hyperplanes in $\mathbf{P}^{d-1}_{\breve{E}}$) has the following properties.

(i) It is a Stein space [SS].

(ii) The quotient by any discrete co-compact subgroup of $PGL_d(E)$ exists and is a proper rigid-analytic space over $\breve{E}$. In fact, it is a projective algebraic variety [Mu].

(iii) Let G_x be the stabilizer in $PGL_d(E)$ of a point $x \in \mathbf{P}^{d-1}_{\breve{E}}$. Then $x \in \breve{\Omega}^d_E$ iff G_x is compact. In fact, there is an equivariant map from $\breve{\Omega}^d_E$ to the Bruhat-Tits building of $GL_d(E)$ [D1].

None of these statements continue to hold for general period domains. This raises interesting questions (cohomology of coherent sheaves, stratification by the amount of noncompactness of stabilizers, etc).

5. ℓ-adic cohomology

If X is a rigid-analytic space over the local field E we denote by $H^i_c(X)$ the ith ℓ-adic cohomology group with compact supports, for a fixed prime number $\ell \neq p$

and a fixed algebraic closure $\bar{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$,

$$H_c^i(X) = H_c^i(X \otimes_E \bar{E};\ \bar{\mathbf{Q}}_\ell).$$

We continue with the set-up of the previous section but assume in addition that b is *basic*, i.e. J is an inner form of G. The tower $\{\breve{\mathbf{M}}_{K'};\ K' \subset G'(\mathbf{Q}_p)\}$ (with its Weil descent datum from $\breve{E}$ to E, cf. Remark 1.3, (iii)) — in the cases where it exists, cf. Hope 4.2 — defines the ℓ-adic representation

$$H_c^i((G, b, \{\mu\})) = \varinjlim_{K'} H_c^i(\breve{\mathbf{M}}_{K'})$$

of the product group $G'(\mathbf{Q}_p) \times J(\mathbf{Q}_p) \times W_E$. Here W_E denotes the Weil group of E. Drinfeld [D2] has conjectured in his Example 1.4, (i) that these modules give a geometric realization of the Langlands correspondence on the supercuspidal spectrum of $GL_d(E)$, comp. [C]. Partial results in this direction are due to Carayol, Faltings, Genestier, and Harris. We now describe a conjecture of Kottwitz in the general case, which describes the contribution of the *discrete Langlands parameters* to the Euler-Poincaré characteristic in the appropriate Grothendieck group,

$$H_c^*((G, b, \{\mu\})) = \sum (-1)^i H_c^i((G, b, \{\mu\})). \tag{6}$$

Let $\varphi : W_{\mathbf{Q}_p} \to {}^L G = \hat{G} \rtimes W_{\mathbf{Q}_p}$ be a discrete L-parameter, i.e. the connected component S_φ^0 of the centralizer group $S_\varphi = \mathrm{Cent}_{\hat{G}}(\varphi)$ lies in $Z(\hat{G})^\Gamma$. We assume that G is obtained from a quasi-split inner form G^* by twisting with a basic element $b^* \in G^*(K_0)$, i.e. G is the inner form associated by Remark 1.3, (i) to (G^*, b^*). (This is automatic if the center of G is connected.) We use the maps [Ko2]

$$G(K_0) \xrightarrow{\lambda_G} X^*(Z(\hat{G})^\Gamma),\ H^1(\mathbf{Q}_p, G) \longrightarrow X^*(Z(\hat{G})^\Gamma).$$

These maps define elements

$$\lambda_{b^*} = \lambda_{G^*}(b^*),\ \lambda_b = \lambda_G(b),\ \text{and } \mathrm{cls}(b, \mu) \in X^*(Z(\hat{G})^\Gamma),\ \text{cf. } (4)\ .$$

According to Kottwitz, generalizing the notion of *strong inner forms* of Vogan, there should be bijections $\pi' \mapsto \tau_{\pi'}$ resp. $\pi \mapsto \tau_\pi$ that yield identifications of the L-packets on G' resp. J corresponding to φ,

$$\begin{aligned} \Pi_\varphi(G') &= \{\text{irreducible repns } \tau \text{ of } S_\varphi;\ \tau | Z(\hat{G})^\Gamma = \lambda_{b^*} + \mathrm{cls}(b, \mu)\}, \\ \Pi_\varphi(J) &= \{\text{irreducible repns } \tau \text{ of } S_\varphi;\ \tau | Z(\hat{G})^\Gamma = \lambda_b + \lambda_{b^*}\}. \end{aligned}$$

Even though these identifications depend on auxiliary choices, the function $(\pi', \pi) \mapsto \check{\tau}_{\pi'} \otimes \tau_\pi$ should be well defined and associate to (π', π) a representation of S_φ. Here and elsewhere $\check{}$ denotes the contragredient representation. Let $r_{\{\mu\}}$ be the finite-dimensional representation of $\hat{G} \rtimes W_E$ defined by $\{\mu\}$ [Ko2]. If φ_E denotes the restriction of φ to W_E, then $r_{\{\mu\}} \circ \varphi_E$ is in a natural way a representation of $S_\varphi \times W_E$, via $r_{\{\mu\}} \circ \varphi_E(s, w) = r_{\{\mu\}}(s \cdot \varphi_E(w))$.

CONJECTURE 5.1. (Kottwitz): *Let $\pi' \otimes \check{\pi} \otimes \varrho$ be an irreducible representation of $G'(\mathbf{Q}_p) \times J(\mathbf{Q}_p) \times W_E$ that contributes in a nontrivial way to* (6). *Then π' lies in an L-packet corresponding to a discrete L-parameter iff π does and then these*

L-packets correspond to the same L-parameter up to equivalence. The total contribution of all (equivalence classes of) discrete L-parameters is given up to sign *by the following expression:*

$$\sum_{\varphi\ \mathrm{discrete}} \quad \sum_{(\pi',\pi)\in\Pi_\varphi(G')\times\Pi_\varphi(J)} \pi' \otimes \check{\pi} \otimes \mathrm{Hom}_{S_\varphi}(\check{\tau}_{\pi'} \otimes \tau_\pi,\ r_{\{\mu\}} \circ \varphi_E).$$

REMARKS 5.2. (i) Kottwitz has the prudence to assume in his conjecture that $\{\mu\}$ is minuscule, as in the examples of Section 3. Based on some heuristical principles he has checked that the above conjecture is compatible (in the sense of Theorem 2.1) with the corresponding global conjecture on Shimura varieties [Ko2].

(ii) Let b be the basic element, unique up to σ-conjugacy such that λ_b coincides with the element of $X^*(Z(\hat{G})^\Gamma)$ defined by $\{\mu\}$. Up to an obvious equivalence relation the representation $H_c^i((G,\{\mu\})) = H_c^i((G,b,\{\mu\}))$ is independent of the choice of b. If the derived group of G is simply connected the above conjectural formula for $H_c^*((G,\{\mu\}))$ simplifies because then $\mathrm{cls}(b,\mu) = 0$ and G' is isomorphic to G.

(iii) To extend this conjecture to include certain nondiscrete L-parameters, one might be tempted to replace cohomology with compact supports by some kind of "middle intersection" cohomology (?), as is done in the global case of Shimura varieties.

A problem independent of the determination of (6) is the calculation of the cohomology of the non-archimedean period domains themselves. We describe a result for $H_c^*(\breve{\mathcal{F}}^{\mathrm{wa}}) = H_c^*(\breve{\mathcal{F}}^{\mathrm{wa}} \otimes_{\breve{E}} \bar{E}, \bar{\mathbf{Q}}_\ell)$ as a virtual representation of $J(\mathbf{Q}_p) \times \mathrm{Gal}(\bar{E}/\breve{E})$. Let P_0 be a minimal parabolic subgroup of J, M_0 a Levi subgroup of P_0, and A_0 the maximal split torus contained in the center of M_0. Let Δ be the set of simple roots of A_0 in the unipotent radical of P_0. For a parabolic subgroup P containing P_0, with Levi subgroup M containing M_0, let A_P be the maximal split torus contained in the center of M, and set $a_P^J = \dim A_P - \dim A_J$. For $x \in X_*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$ let

$$\Delta_x = \{\alpha \in \Delta;\ \langle x, \omega_\alpha\rangle > 0\},$$

where ω_α is the fundamental weight corresponding to α, and let P_x be the unique parabolic subgroup containing P_0 such that Δ_x is the set of simple roots occurring in its unipotent radical. Any element $\mu \in \{\mu\}$ factoring through M_0 defines a unique element $\bar{\mu} \in X_*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$ such that $\langle \bar{\mu}, \chi\rangle = \chi \circ \mu$ for all $\mathbf{Q}_p$-rational characters χ of M_0. Let $\{\mu\}_0 \subset X_*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$ be the finite subset of points obtained in this way. Then $\mathrm{Gal}(\bar{E}/\breve{E})$ acts on $\{\mu\}_0$.

THEOREM 5.3. (Kottwitz, Rapoport): *The representation of $J(\mathbf{Q}_p) \times \mathrm{Gal}(\bar{E}/\breve{E})$ on $H_c^i(\breve{\mathcal{F}}^{\mathrm{wa}})$ is admissible for each i. In the Grothendieck group of admissible representations we have*

$$H_c^*(\breve{\mathcal{F}}^{\mathrm{wa}}) = \sum_{\bar{\mu}\in\{\mu\}_0} (-1)^{a^J_{P_{\bar{\mu}}}} v^J_{P_{\bar{\mu}}},$$

where v_P^J denotes the unique irreducible quotient of the representation of $J(\mathbf{Q}_p)$ on $C^\infty(J(\mathbf{Q}_p)/P(\mathbf{Q}_p))$. The action of $\mathrm{Gal}(\bar{E}/\breve{E})$ *is through permutation of the indices.*

REMARKS 5.4. (i) The theorem has been proved with the help of Huber supposing that certain foundational questions concerning the ℓ-adic cohomology of rigid spaces can be resolved (the case of torsion coefficients was developed by Berkovich and by Huber). The proof is modeled on the approach of [AB] to the calculation of the cohomology of the space of semi-stable vector bundles on a Riemann surface.

(ii) All available evidence seems to indicate that the contribution of $\bar{\mu}$ is in degree $a^J_{P_{\bar{\mu}}} + 2\ell(\bar{\mu})$, where $\ell(\bar{\mu})$ is the number of root hyperplanes separating $\bar{\mu}$ from the positive Weyl chamber corresponding to P_0. This is indeed proved by Schneider and Stuhler [SS] in the case of $\breve{\Omega}^d_E$. Their paper is at the origin of the above theorem.

(iii) As we have in general little control over the morphisms $\breve{\pi}_{K'}$ in the tower (5) (cf. Example 3.4, (iii)), the above result gives almost no information on the nature of (6).

In conclusion, I express my strong belief that there exists a theory in the equal characteristic case that closely parallels the one outlined here.

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