MODULARITY OF GENERATING SERIES OF DIVISORS ON UNITARY SHIMURA VARIETIES

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Abstract. We form generating series of special divisors, valued in the Chow group and in the arithmetic Chow group, on the compactified integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$, and prove their modularity. The main ingredient of the proof is the calculation of the vertical components appearing in the divisor of a Borcherds product on the integral model.

Contents

1. Introduction

The goal of this paper is to prove the modularity of a generating series of divisors on the integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$.

This generating series is an arithmetic analogue of the classical theta kernel used to lift modular forms from $\mathrm{U}(2)$ and $\mathrm{U}(n)$. In a similar vein, our modular generating series can be used to define a lift from classical cuspidal modular forms of weight n to the codimension one Chow group of the unitary Shimura variety.

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1.1. **Background.** The construction of modular generating series whose coefficients are geometric cycles begins with the work of Hirzebruch-Zagier [HZ76], who considered the cohomology classes of divisors $T(m)$ on compactified Hilbert modular surfaces over $\mathbb{C}$.

An extensive study of the modularity of generating series for cohomology classes of special cycles in Riemannian locally symmetric spaces $S = \Gamma\backslash X$ was undertaken by in a series of papers [KM86, KM87, KM90] by the third author and John Millson. The main technical tool was a family of Siegel type theta series valued in the deRham complex of S, from which modularity was inherited by the image in cohomology.

The special cycles involved are given by an explicit geometric construction, and so, in the cases where S is (the set of complex points of) a Shimura variety, it is natural to ask whether the analogous generating series for special cycle classes in the Chow group is likewise modular. In the case of Shimura varieties of orthogonal type, this question was raised in [Kud97a]. Of course, up to (often non-trivial) issues about compactifications, the modularity of the image of such a series under the cycle class map to cohomology follows from the work of Kudla-Millson. Indeed, in some cases, this already implies the modularity of the Chow group generating series. See, for example, [Zag85, YZZ09].

The generating series for the images of Heegner points in the Jacobian of a modular curve was proved to be modular by Gross-Kohnen-Zagier [GKZ87]. Motivated by their work, Borcherds [Bor99] proved the modularity of the generating series for Chow groups classes of Heegner (=special) divisors on Shimura varieties of orthogonal type. His method depended on the miraculous construction of a family of meromorphic modular forms on such varieties via a regularized theta lift [Bor98], whose explicitly known divisors provide enough relations among such classes to prove modularity.

The passage to cycles on integral models of Shimura varieties and the generating series for their classes in the arithmetic Chow groups was initiated in [Kud97b]. In the case of special divisors on Shimura curves, or, more generally, Shimura varieties of orthogonal type, the required Green functions constructed explicitly there are derived from the KM theta series. Quite complete results on the modularity properties of generating series for special cycles on Shimura curves were obtained in the book [KRY06]. There the case of arithmetic 0-cycles is also treated and the corresponding generating series is shown to coincide with the central derivative of a weight 3/2 Siegel genus 2 incoherent Eisenstein series. We will not include a further discussion of such higher codimension cases here.

In [Bru02], the first author generalized Borcherd's regularized theta lift by allowing harmonic Maass forms as inputs. This provides an alternative construction of Green functions for special divisors on Shimura varieties of orthogonal type, and has the advantage that one can try to use the method of Borcherds to establish modularity of the generating series built with these Green functions. The main issue in doing this is that the divisor of the

Borcherds form is, a priori, only known on the generic fiber and hence more detailed information is needed on its extension to the integral model. In the case of Hilbert modular surfaces, where the arithmetic cycles are integral extensions of the Hirzebruch-Zagier curves $T(m)$, this is carried out in [BBGK07]. Recent advances in our knowledge of the integral models for Hodge type Shimura varieties, [Madb, Mada], in particular the Shimura varieties of orthogonal type over $\mathbb{Q}$, suggest that a general result about modularity of the generating series for classes of special divisors in arithmetic Chow groups is now accessible.

In this paper we deal with unitary Shimura varieties for signature $(n-1,1)$. In this case a definition of arithmetic special cycles was given in [KR11, KR14], and was extended to the toroidal compactification in [How15]. The Bruinier-Borcherds construction of Green functions was carried over to the unitary case in [Hof14, BHY15]. The subject matter of the present paper is a proof of the modularity of the generating series for these classes of arithmetic special divisors.

1.2. **Statement of the main result.** Fix a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. We are concerned with the arithmetic of certain unitary Shimura varieties, whose definition depends on the following initial data: let W_0 and W be $\boldsymbol{k}$-hermitian spaces of signature $(1,0)$ and $(n-1,1)$, respectively, where $n \geqslant 3$. We assume that these spaces each admit an $\mathcal{O}_{\boldsymbol{k}}$-lattice that is self-dual with respect to the hermitian form.

Attached to this data is a reductive algebraic group

$$G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W) \tag{1.2.1}$$

over $\mathbb{Q}$, defined as the subgroup on which the unitary similitude characters are equal, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ depending on the above choice of self-dual lattices. As explained in §2, there is an associated hermitian symmetric domain $\mathcal{D}$, and a stack $\mathrm{Sh}(G,\mathcal{D})$ over $\boldsymbol{k}$ whose complex points are identified with the orbifold quotient

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

This is the unitary Shimura variety of the title.

This stack can be interpreted as a moduli space of pairs (A_0, A) in which A_0 is an elliptic curve with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, and A is a principally polarized abelian scheme of dimension n endowed with an $\mathcal{O}_{\boldsymbol{k}}$-action. The pair (A_0, A) is required to satisfy some additional conditions, which need not concern us in the introduction.

Using the moduli interpretation, one can construct an integral model of $\mathrm{Sh}(G,\mathcal{D})$ over $\mathcal{O}_{\boldsymbol{k}}$. In fact, following work of Pappas and Krämer, we explain in §2.3 that there are two natural integral models related by a morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Each integral model has a canonical toroidal compactification whose boundary is a disjoint union of smooth Cartier divisors, and the above morphism extends uniquely to a morphism

$$\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}} \tag{1.2.2}$$

of compactifications.

Each compactified integral model has its own desirable and undesirable properties. For example, $\mathcal{S}^*_{\mathrm{Kra}}$ is regular, while $\mathcal{S}^*_{\mathrm{Pap}}$ is not. On the other hand, every vertical (*i.e.* supported in nonzero characteristic) Weil divisor on $\mathcal{S}^*_{\mathrm{Pap}}$ has nonempty intersection with the boundary, while $\mathcal{S}^*_{\mathrm{Kra}}$ has certain *exceptional* divisors in characteristics $p \mid D$ that do not meet the boundary. An essential part of our method is to pass back and forth between these two models in order to exploit the best properties of each. For simplicity, we will state our main results in terms of the regular model $\mathcal{S}^*_{\mathrm{Kra}}$.

In §2 we define a distinguished line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$, called the *line bundle of weight one modular forms*, and a family of Cartier divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ indexed by integers $m > 0$. These special divisors were introduced in [KR11, KR14], and studied further in [BHY15, How12, How15]. For the purposes of the introduction, we note only that one should regard the divisors as arising from embeddings of smaller unitary groups into G.

Denote by

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \cong \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the Chow group of rational equivalence classes of divisors with $\mathbb{Q}$ coefficients. Each special divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$ can be extended to a divisor on the toroidal compactification simply by taking its Zariski closure, denoted $\mathcal{Z}^*_{\mathrm{Kra}}(m)$. The *total special divisor* is defined as

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m) \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \tag{1.2.3}$$

where the boundary contribution is defined, as in (5.3.3), by

$$\mathcal{B}_{\mathrm{Kra}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x \rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi).$$

The notation here is the following: The sum is over the equivalence classes of *proper cusp label representatives* Φ as defined in §3.1. These index the connected components $\mathcal{S}^*_{\mathrm{Kra}}(\Phi) \subset \partial \mathcal{S}^*_{\mathrm{Kra}}$ of the boundary[1]. Inside the sum, $(L_0, \langle \cdot, \cdot \rangle)$ is a hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2, 0)$, which depends on Φ.

The line bundle of modular forms $\boldsymbol{\omega}$ admits a canonical extension to the toroidal compactification, denoted the same way. For the sake of notational uniformity, we textend (1.2.3) to $m = 0$ by setting

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \boldsymbol{\omega}^{-1} + \mathrm{Exc} \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

Here Exc is the exceptional divisor of Theorem 2.3.2. It is a reduced effective divisor supported in characteristics $p \mid D$, disjoint from the boundary of the compactification. The following result appears in the text as Theorem 7.1.5.

[1] After base change to $\mathbb{C}$, each $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ decomposes into h connected components, where h is the class number of $\boldsymbol{k}$.

Theorem A. *Let $\chi_{\boldsymbol{k}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ be the Dirichlet character determined by $\boldsymbol{k}/\mathbb{Q}$. The formal generating series*

$$\sum_{m \geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$ in the following sense: for every $\mathbb{Q}$-linear functional $\alpha : \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}$, the series

$$\sum_{m \geqslant 0} \alpha(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)) \cdot q^m \in \mathbb{C}[[q]]$$

is the q-expansion of a classical modular form of the indicated weight, level, and character.

In fact we prove a stronger version of this theorem. Denote by $\widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ the Gillet-Soulé [GS90] arithmetic Chow group of rational equivalence classes of pairs $\widehat{\mathcal{Z}} = (\mathcal{Z}, \mathrm{Gr})$, where $\mathcal{Z}$ is a divisor on $\mathcal{S}^*_{\mathrm{Kra}}$ with rational coefficients, and Gr is a Green function for $\mathcal{Z}$. We allow the Green function to have additional log-log singularities along the boundary, as in the more general theory developed in [BGKK07]. See also [BBGK07, How15].

In §7.3 we use the theory of regularized theta lifts to construct Green functions for the special divisors $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$, and hence obtain arithmetic divisors

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$$

for $m > 0$. We also endow the line bundle $\boldsymbol{\omega}$ with a metric, and the resulting metrized line bundle $\widehat{\boldsymbol{\omega}}$ defines a class

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}),$$

where the vertical divisor Exc has been endowed with the constant Green function $-\log(D)$. The following result is Theorem 7.3.1 in the text.

Theorem B. *The formal generating series*

$$\widehat{\phi}(\tau) = \sum_{m \geqslant 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$, where modularity is understood in the same sense as Theorem A.

Remark 1.2.1. Theorem B implies that the $\mathbb{Q}$-span of the classes $\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ is finite dimensional. See Remark 7.1.2.

Remark 1.2.2. There is a second method of constructing Green functions for the special divisors, based on the methods of [Kud97b], which gives rise to a non-holomorphic variant of $\widehat{\phi}(\tau)$. It is a recent theorem of Ehlen-Sankaran [ES16] that Theorem B implies the modularity of this non-holomorphic generating series. See §7.4.

The motivating desire for the modularity result of Theorem B is that it allows one to construct arithmetic theta lifts. If $g(\tau) \in S_n(\Gamma_0(D), \chi_{\boldsymbol{k}}^n)$ is a classical scalar valued cusp form, we may form the Petersson inner product

$$\widehat{\theta}(g) \stackrel{\text{def}}{=} \langle \widehat{\phi}, g \rangle_{\text{Pet}} \in \widehat{\text{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\text{Kra}})$$

as in [Kud04]. One expects, as in [*loc. cit.*], that the arithmetic intersection pairing of $\widehat{\theta}(g)$ against other cycle classes should be related to derivatives of L-functions, providing generalizations of the Gross-Zagier and Gross-Kohnen-Zagier theorems. Specific instances in which this expectation is fulfilled can be deduced from [BHY15, How12, How15]. This will be explained in the companion paper [BHK$^+$].

As this paper is rather long, we explain in the next two subsections the main ideas that go into the proof of Theorem A. The proof of Theorem B is exactly the same, but one must keep track of Green functions.

1.3. **Sketch of the proof, part I: the generic fiber.** In this subsection we sketch the proof of modularity only in the generic fiber. That is, the modularity of

$$\sum_{m \geqslant 0} \mathcal{Z}^{\text{tot}}_{\text{Kra}}(m)_{/\boldsymbol{k}} \cdot q^m \in \text{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\text{Kra}/\boldsymbol{k}})[[q]]. \tag{1.3.1}$$

The key to the proof is the study of *Borcherds products* [Bor98, Bor99].

A Borcherds product is a meromorphic modular form on an orthogonal Shimura variety, whose construction depends on a choice of weakly holomorphic input form, typically of negative weight. In our case the input form is any

$$f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D, \chi_{\boldsymbol{k}}^{n-2}), \tag{1.3.2}$$

where the superscripts ! and ∞ indicate that the weakly holomorphic form $f(\tau)$ of weight $2-n$ and level $\Gamma_0(D)$ is allowed to have a pole at the cusp ∞, but must be holomorphic at all other cusps. We assume also that all $c(m) \in \mathbb{Z}$.

Our Shimura variety $\text{Sh}(G, \mathcal{D})$ admits a natural map to an orthogonal Shimura variety. Indeed, the $\boldsymbol{k}$-vector space

$$V = \text{Hom}_{\boldsymbol{k}}(W_0, W)$$

admits a natural hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n-1, 1)$, induced by the hermitian forms on W_0 and W. The natural action of G on V determines an exact sequence

$$1 \to \text{Res}_{\boldsymbol{k}/\mathbb{Q}} \mathbb{G}_m \to G \to \text{U}(V) \to 1 \tag{1.3.3}$$

of reductive groups over $\mathbb{Q}$.

We may also view V as a $\mathbb{Q}$-vector space endowed with the quadratic form $Q(x) = \langle x, x \rangle$ of signature $(2n-2, 2)$, and so obtain a homomorphism $G \to \text{SO}(V)$. This induces a map from $\text{Sh}(G, \mathcal{D})$ to the Shimura variety associated with the group $\text{SO}(V)$.

After possibly replacing f by a nonzero integer multiple, Borcherds constructs a meromorphic modular form on the orthogonal Shimura variety, which can be pulled back to a meromorphic modular form on $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. The result is a meromorphic section $\boldsymbol{\psi}(f)$ of $\boldsymbol{\omega}^k$, where the weight

$$k = \sum_{r\mid D} \gamma_r \cdot c_r(0) \in \mathbb{Z}$$

is the integer defined in §5.3. The constant $\gamma_r = \prod_{p\mid r} \gamma_p$ is a 4^{th} root of unity (with $\gamma_1 = 1$) and $c_r(0)$ is the constant term of f at the cusp

$$\infty_r = \frac{r}{D} \in \Gamma_0(D)\backslash\mathbb{P}^1(\mathbb{Q}),$$

in the sense of Definition 4.1.1.

Initially, $\boldsymbol{\psi}(f)$ is characterized by specifying $-\log||\boldsymbol{\psi}(f)||$, where $||\cdot||$ is the Petersson norm on $\boldsymbol{\omega}^k$. In particular, $\boldsymbol{\psi}(f)$ is only defined up to rescaling by a complex number of absolute value 1 on each connected component of $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. We prove that, after a suitable rescaling, $\boldsymbol{\psi}(f)$ is the analytification of a rational section of the line bundle $\boldsymbol{\omega}^k$ on $\mathrm{Sh}(G,\mathcal{D})$. In other words, the Borcherds product is algebraic and defined over the reflex field $\boldsymbol{k}$. This result is not really new, as one could first prove the analogous result on the orthogonal Shimura variety using the explicit q-expansion of Borcherds, as is done in [Hör14], from which the result on $\mathrm{Sh}(G,\mathcal{D})$ follows immediately. In any case, the algebraicity and descent to the reflex field allow us to view $\boldsymbol{\psi}(f)$ as a rational section of $\boldsymbol{\omega}^k$ both on the integral model $\mathcal{S}_{\mathrm{Kra}}$, and on its toroidal compactification.

We compute the divisor of $\boldsymbol{\psi}(f)$ on the generic fiber of the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}}$, and find

$$\mathrm{div}(\boldsymbol{\psi}(f))_{/\boldsymbol{k}} = \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}. \tag{1.3.4}$$

The calculation of the divisor on the interior $\mathcal{S}_{\mathrm{Kra}/\boldsymbol{k}}$ follows immediately from the corresponding calculations of Borcherds on the orthogonal Shimura variety. The multiplicities of the boundary components are computed using the results of [Kud16], which describe the structure of the Fourier-Jacobi expansions of $\boldsymbol{\psi}(f)$ along the various boundary components.

The cusp $\infty_1 = 1/D$ is $\Gamma_0(D)$-equivalent to the usual cusp at ∞, and so $c_1(0) = c(0)$. It follows from this and (1.3.4) that

$$\sum_{\substack{r\mid D\\ r>1}} \gamma_r c_r(0)\cdot\boldsymbol{\omega} = \sum_{m\geqslant 0} c(-m)\cdot\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} \tag{1.3.5}$$

in $\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}})$. In §4.2 we construct for each $r\mid D$ an Eisenstein series

$$E_r(\tau) = \sum_{m\geqslant 0} e_r(m)\cdot q^m \in M_n(D,\chi^n_{\boldsymbol{k}}),$$

which, by a simple residue calculation, satisfies

$$c_r(0) = -\sum_{m>0} c(-m)e_r(m).$$

Substituting this expression into (1.3.5) yields

$$0 = \sum_{m\geqslant 0} c(-m)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} + \sum_{\substack{r\mid D\\ r>1}} \gamma_r e_r(m)\cdot\boldsymbol{\omega}\Big), \tag{1.3.6}$$

where we have also used the relation $e_r(0) = 0$ for $r > 1$.

We now invoke a variant of the modularity criterion of [Bor99], which is our Theorem 4.2.3: if a formal q-expansion

$$\sum_{m\geqslant 0} d(m)q^m \in \mathbb{C}[[q]]$$

satisfies $0 = \sum_{m\geqslant 0} c(-m)d(m)$ for every input form (1.3.2), then it must be the q-expansion of a modular form of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$. It follows immediately from this and (1.3.6) that the formal q-expansion

$$\sum_{m\geqslant 0}\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} + \sum_{\substack{r\mid D\\ r>1}} \gamma_r e_r(m)\cdot\boldsymbol{\omega}\Big)\cdot q^m$$

is modular in the sense of Theorem A. Rewriting this as

$$\sum_{m\geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}\cdot q^m + \sum_{\substack{r\mid D\\ r>1}} \gamma_r E_r(\tau)\cdot\boldsymbol{\omega}$$

and using the modularity of each Eisenstein series $E_r(\tau)$, we deduce that (1.3.1) is modular.

1.4. **Sketch of the proof, part II: vertical components.** In order to extend the arguments of §1.3 to prove Theorem A, it is clear that one should attempt to compute the divisor of $\boldsymbol{\psi}(f)$ on the integral model $\mathcal{S}^*_{\mathrm{Kra}}$ and hope for an expression similar to (1.3.4). Indeed, the bulk of this paper is devoted to precisely this problem.

The subtlety is that both $\mathrm{div}(\boldsymbol{\psi}(f))$ and $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ will turn out to have vertical components supported in characteristics dividing D. Even worse, in these bad characteristics the components of the exceptional divisor $\mathrm{Exc}\subset\mathcal{S}^*_{\mathrm{Kra}}$ do not intersect the boundary, and so the multiplicities of these components in $\mathrm{div}(\boldsymbol{\psi}(f))$ cannot be detected by examining the Fourier-Jacobi expansions of $\boldsymbol{\psi}(f)$.

This is where the second integral model $\mathcal{S}^*_{\mathrm{Pap}}$ plays an essential role. The morphism (1.2.2) sits in a cartesian diagram

$$\begin{tikzcd} \mathrm{Exc} \arrow[r] \arrow[d] & \mathcal{S}^*_{\mathrm{Kra}} \arrow[d] \\ \mathrm{Sing} \arrow[r] & \mathcal{S}^*_{\mathrm{Pap}}, \end{tikzcd}$$

where the *singular locus* $\mathrm{Sing} \subset \mathcal{S}^*_{\mathrm{Pap}}$ is the reduced closed substack of points at which the structure morphism $\mathcal{S}^*_{\mathrm{Pap}} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ is not smooth. It is 0-dimensional and supported in characteristics dividing D. The right vertical arrow restricts to an isomorphism

$$\mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}. \tag{1.4.1}$$

For each connected component $s \in \pi_0(\mathrm{Sing})$ the fiber

$$\mathrm{Exc}_s = \mathrm{Exc} \times_{\mathcal{S}^*_{\mathrm{Pap}}} s$$

is a smooth, irreducible, vertical Cartier divisor on $\mathcal{S}^*_{\mathrm{Kra}}$, and $\mathrm{Exc} = \bigsqcup_s \mathrm{Exc}_s$.

As $\mathcal{S}^*_{\mathrm{Pap}}$ has geometrically normal fibers, every vertical divisor on it meets the boundary. Thus one could hope to use (1.4.1) to view $\boldsymbol{\psi}(f)$ as a rational section on $\mathcal{S}^*_{\mathrm{Pap}}$, compute its divisor there by examining Fourier-Jacobi expansions, and then pull that calculation back to $\mathcal{S}^*_{\mathrm{Kra}}$.

This is precisely what we do, but there is an added complication: The line bundle $\boldsymbol{\omega}$ on (1.4.1) does not extend to $\mathcal{S}^*_{\mathrm{Pap}}$, and similarly the divisor $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ on (1.4.1) cannot be extended across the singular locus to a Cartier divisor on $\mathcal{S}^*_{\mathrm{Pap}}$. However, if you square the line bundle and the divisors, they have much better behavior. This is the content of the following result, which combines Theorems 2.4.3, 2.5.2, and 2.6.3 of the text.

Theorem C. *There is a unique line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$ whose restriction to (1.4.1) is isomorphic to $\boldsymbol{\omega}^2$. Denoting by $\boldsymbol{\Omega}_{\mathrm{Kra}}$ its pullback to $\mathcal{S}^*_{\mathrm{Kra}}$, there is an isomorphism*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

Similarly, there is a unique line bundle Cartier divisor $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)$ on $\mathcal{S}^*_{\mathrm{Pap}}$ whose restriction to (1.4.1) is equal to $2\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$. Its pullback $\mathcal{Y}_{\mathrm{Kra}}(m)$ to $\mathcal{S}_{\mathrm{Kra}}$ satisfies

$$2\mathcal{Z}_{\mathrm{Kra}}(m) = \mathcal{Y}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x \rangle = m\} \cdot \mathrm{Exc}_s.$$

Here L_s is a positive definite self-dual hermitian lattice of rank n associated to the singular point s, and $\langle \cdot\,, \cdot \rangle$ is its hermitian form.

Setting $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}}$, we obtain a formal generating series

$$\sum_{m \geqslant 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]]$$

whose pullback via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ is twice the generating series of Theorem A, up to an error term coming from the exceptional divisors. More precisely, Theorem C shows that the pullback is

$$2 \sum_{m \geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m - \sum_{s \in \pi_0(\mathrm{Sing})} \vartheta_s(\tau) \cdot \mathrm{Exc}_s \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]],$$

where each $\vartheta_s(\tau)$ is a classical theta function whose coefficients count points in the positive definite hermitian lattice L_s.

Over (1.4.1) we have $\boldsymbol{\omega}^{2k} \cong \boldsymbol{\Omega}^k_{\mathrm{Pap}}$, which allows us to view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$. We examine its Fourier-Jacobi expansions along the boundary components and are able to compute its divisor completely (it happens to include nontrivial vertical components). We then pull this calculation back to $\mathcal{S}^*_{\mathrm{Kra}}$, and find that $\boldsymbol{\psi}(f)$, when viewed as a rational section of $\boldsymbol{\omega}^k$, has divisor

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)) = \sum_{m>0} c(-m) \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + \sum_{r|D} \gamma_r c_r(0) \cdot \Big(\frac{\mathrm{Exc}}{2} + \sum_{p|r} \mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}} \Big) \\ - \sum_{m>0} \frac{c(-m)}{2} \sum_{s\in\pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s \\ - k \cdot \mathrm{div}(\delta)\end{aligned}$$

where $\delta \in \mathcal{O}_{\boldsymbol{k}}$ is a square root of $-D$, $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above $p \mid D$, and $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}$ is the mod $\mathfrak{p}$ fiber of $\mathcal{S}^*_{\mathrm{Kra}}$, viewed as a divisor. This is stated in the text as Theorem 5.3.3. Passing to the generic fiber recovers (1.3.4), as it must.

As in the argument leading to (1.3.6), this implies the relation

$$\begin{aligned}0 = \sum_{m\geqslant 0} c(-m) \cdot \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \frac{1}{2} \sum_{s\in\pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s \Big) \\ + \sum_{m\geqslant 0} c(-m) \cdot \sum_{\substack{r|D \\ r>1}} \gamma_r e_r(m) \Big(\boldsymbol{\omega} - \frac{\mathrm{Exc}}{2} - \sum_{p|r} \mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}} \Big)\end{aligned}$$

in the Chow group of $\mathcal{S}^*_{\mathrm{Kra}}$, and the modularity criterion implies that

$$\begin{aligned}\sum_{m\geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m - \frac{1}{2} \sum_{s\in\pi_0(\mathrm{Sing})} \vartheta_s(\tau) \cdot \mathrm{Exc}_s \\ + \sum_{\substack{r|D \\ r>1}} \gamma_r E_r(\tau) \cdot \Big(\boldsymbol{\omega} - \frac{\mathrm{Exc}}{2} - \sum_{p|r} \mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}} \Big)\end{aligned}$$

is a modular form. As the theta series $\vartheta_s(\tau)$ and Eisenstein series $E_r(\tau)$ are all modular, so is $\sum \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) q^m$. This completes the outline of the proof of Theorem A.

Remark 1.4.1. The work of Hörmann [Hör14] includes the calculation of divisors of Borcherds products on integral models of orthogonal Shimura varieties, but with strong restrictions on the level. In particular, there is no analogue in that work of the kind of bad reduction appearing in our Shimura variety at primes dividing D, and all of the Borcherds products considered by Hörmann have divisors with no vertical components. This is in stark contrast to the divisor $\mathrm{div}(\boldsymbol{\psi}(f))$ computed above.

1.5. **The structure of the paper.** We now briefly describe the contents of the various sections of the paper.

In §2 we introduce the unitary Shimura variety associated to the group G of (1.2.1), and explain its realization as a moduli space of pairs (A_0, A) of abelian varieties with extra structure. We then review the integral models constructed by Pappas and Krämer, and the singular and exceptional loci of these models. These are related by a cartesian diagram

$$\begin{tikzcd} \mathrm{Exc} \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Kra}} \arrow[d] \\ \mathrm{Sing} \arrow[r] & \mathcal{S}_{\mathrm{Pap}}, \end{tikzcd}$$

where the vertical arrow on the right is an isomorphism outside of the 0-dimensional singular locus Sing. We also define the line bundle of modular forms $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$. The first main result of §2 is Theorem 2.4.3, which asserts the existence of a bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ restricting to $\boldsymbol{\omega}^2$ over the complement of the singular locus. We then define the Cartier divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$ on $\mathcal{S}_{\mathrm{Kra}}$. The second main result of the section is Theorem 2.5.2, which asserts the existence of a unique Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to the complement of the singular locus coincides with $2\mathcal{Z}_{\mathrm{Kra}}(m)$. The final main result is Theorem 2.6.3, which completes the proof of Theorem C.

In §3 we describe the canonical toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$, and the structure of their formal completions along the boundary. In §3.1 and §3.2 we introduce the cusp labels Φ that index the cusps, and their associated mixed Shimura varieties. In §3.3 we construct smooth integral models $\mathcal{C}_\Phi$ of these mixed Shimura varieties, following the general recipes of the theory of arithmetic toroidal compactification, as moduli space of 1-motives. In §3.4 we give a second moduli interpretation of these integral models. The expressions for the Fourier-Jacobi expansions based on this second interpretation are more easily related to Fourier-Jacobi expansions on orthogonal Shimura varieties. This is one of the key technical steps in our work; see the remarks at the beginning of §3 for further discussion. In §3.5 and §3.6 we describe the line bundle of modular forms and the special divisors on the boundary charts. Theorem 3.7.1 describes the canonical toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ and their properties. In §3.8 we describe the Fourier-Jacobi expansions of sections of $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ in algebraic language, and in §3.9 we explain how to express these Fourier-Jacobi coefficients in classical complex analytic coordinates.

In §4 we review the weakly holomorphic modular forms $f \in M^{!,\infty}_k(D,\chi)$, whose regularized theta lifts are used to define the Borcherds forms that ultimately provide relations in (arithmetic) Chow groups. We also state in Theorem 4.2.3 a variant of the modularity criterion of Borcherds.

In §5 we consider unitary Borcherds forms associated to weakly holomorphic forms

$$f \in M^{!,\infty}_{2-n}(D, \chi_{\boldsymbol{k}}^{n-2}). \tag{1.5.1}$$

Ultimately, the integrality properties of the unitary Borcherds forms will be deduced from an analysis of their Fourier-Jacobi expansions. These expansions involve certain products of Jacobi theta functions, and so, in §5 we review facts about the arithmetic theory of Jacobi forms, viewed as sections of a suitable line bundle $\mathcal{J}_{k,m}$ on the universal elliptic curve $\mathcal{E} \to \mathcal{Y}$ over $\mathbb{Z}$. The key point is to have a precise description of the divisor of the basic section $\Theta^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$ of Proposition 5.1.4. In §5.2 we prove Borcherds quadratic identity, allowing us to relate $\mathcal{J}_{0,1}$ to a certain line bundle determined by a Borcherds form on the boundary component $\mathcal{B}_\Phi$ associated to a cusp label Φ. After these technical preliminaries, we come to the statements of our main results about unitary Borcherds forms. Theorem 5.3.1 asserts that, for each weakly holomorphic form (1.5.1) satisfying integrality conditions on the Fourier coefficients $c(-m)$ with $m \geqslant 0$, there is a rational section $\boldsymbol{\psi}(f)$ of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ with explicit divisor on the generic fiber and prescribed zeros and poles along each boundary component. Moreover, for each cusp label Φ, the leading Fourier-Jacobi coefficient $\boldsymbol{\psi}_0$ of $\boldsymbol{\psi}(f)$ has an expression as a product of three factors, two of which, P^{vert}_Φ and P^{hor}_Φ, are constructed in terms of Θ^{24}. Theorem 5.3.3 gives the precise divisor of $\boldsymbol{\psi}(f)$ on $\mathcal{S}^*_{\mathrm{Kra}}$, and Theorem 5.3.4 gives an analogous formula on $\mathcal{S}^*_{\mathrm{Pap}}$. An essential ingredient in the calculation of these divisors is the calculation of the divisors of the factors P^{vert}_Φ and P^{hor}_Φ, which is done in §5.4.

In §6 we prove the main results stated in §5. In §6.1 we construct a vector valued form $\tilde{f}$ from (1.5.1), and give expressions for its Fourier coefficients in terms of those of f. The vector valued form $\tilde{f}$ defines a Borcherds form $\tilde{\boldsymbol{\psi}}(f)$ on the symmetric space $\tilde{\mathcal{D}}$ for the orthogonal group of the quadratic space (V, Q) and, in §6.2, we obtain the unitary Borcherds form $\boldsymbol{\psi}(f)$ as its pullback to $\mathcal{D}$. In §6.3 we determine the analytic Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ at the cusp Φ by pulling back the product formula for $\tilde{\boldsymbol{\psi}}(f)$ computed in [Kud16] along a one-dimensional boundary component of $\tilde{\mathcal{D}}$. In §6.4 we show that the unitary Borcherds form constructed analytically arises from a rational section of $\boldsymbol{\omega}^k$ and that, after rescaling by a constant of absolute value 1, this section is defined over $\boldsymbol{k}$. This is Proposition 6.4.3. In §6.5 we complete the proofs of Theorems 5.3.1, 5.3.3, and 5.3.4.

In §7 we give the proofs of the modularity results discussed in detail earlier in the introduction.

Finally, in §8, we provide some technical details omitted from or supplementary to the earlier sections.

1.6. **Thanks.** The results of this paper are the outcome of a long term project, begun initially in Bonn in June of 2013, and supported in a crucial way by three weeklong meetings at AIM, in Palo Alto (May of 2014) and San

Jose (November of 2015 and 2016), as part of their AIM SQuaRE's program. The opportunity to spend these periods of intensely focused efforts on the problems involved was essential. We would like to thank the University of Bonn and AIM for their support.

1.7. **Notation.** Throughout the paper, $\boldsymbol{k} \subset \mathbb{C}$ is a quadratic imaginary field of odd discriminant $\operatorname{disc}(\boldsymbol{k}) = -D$. Denote by $\delta = \sqrt{-D} \in \boldsymbol{k}$ the unique choice of square root with $\operatorname{Im}(\delta) > 0$, and by $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ the different of $\mathcal{O}_{\boldsymbol{k}}$.

Fix a $\pi \in \mathcal{O}_{\boldsymbol{k}}$ satisfying $\mathcal{O}_{\boldsymbol{k}} = \mathbb{Z} + \mathbb{Z}\pi$. If S is any $\mathcal{O}_{\boldsymbol{k}}$-scheme, define

$$\begin{aligned} \epsilon_S &= \pi \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \\ \overline{\epsilon}_S &= \overline{\pi} \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S, \end{aligned}$$

where $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$ is the structure map. The ideals generated by these elements are independent of the choice of π, and sit in exact sequences of free $\mathcal{O}_S$-modules

$$0 \to (\overline{\epsilon}_S) \to \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\alpha)x} \mathcal{O}_S \to 0$$

and

$$0 \to (\epsilon_S) \to \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\overline{\alpha})x} \mathcal{O}_S \to 0.$$

It is easy to see that $\epsilon_S \cdot \overline{\epsilon}_S = 0$, and that the images of (ϵ_S) and $(\overline{\epsilon}_S)$ under

$$\begin{aligned} \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S &\xrightarrow{\alpha\otimes x \mapsto i_S(\alpha)x} \mathcal{O}_S \\ \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S &\xrightarrow{\alpha\otimes x \mapsto i_S(\overline{\alpha})x} \mathcal{O}_S, \end{aligned}$$

respectively, are both equal to the sub-sheaf $\mathfrak{d}\mathcal{O}_S$. This defines isomorphisms of $\mathcal{O}_S$-modules

$$(\epsilon_S) \cong \mathfrak{d}\mathcal{O}_S \cong (\overline{\epsilon}_S). \tag{1.7.1}$$

If N is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S$-module then $N/\overline{\epsilon}_S N$ is the maximal quotient of N on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and $N/\epsilon_S N$ is the maximal quotient on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the conjugate of the structure morphism. If $D \in \mathcal{O}_S^\times$ then more is true: there is a decomposition

$$N = \epsilon_S N \oplus \overline{\epsilon}_S N, \tag{1.7.2}$$

and the summands are the maximal submodules on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism and its conjugate, respectively. From this discussion it is clear that one should regard ϵ_S and $\overline{\epsilon}_S$ as integral substitutes for the orthogonal idempotents in $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$. The $\mathcal{O}_{\boldsymbol{k}}$-scheme S will usually be clear from context, and we abbreviate ϵ_S and $\overline{\epsilon}_S$ to ϵ and $\overline{\epsilon}$.

Let $\boldsymbol{k}^{\mathrm{ab}} \subset \mathbb{C}$ be the maximal abelian extension of $\boldsymbol{k}$ in $\mathbb{C}$, and let

$$\mathrm{art} : \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times \to \operatorname{Gal}(\boldsymbol{k}^{\mathrm{ab}}/\boldsymbol{k})$$

be the Artin map of class field theory, normalized as in [Mil05, §11]. As usual, $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is Deligne's torus.

For a prime $p \leqslant \infty$ we write $(a,b)_p$ for the Hilbert symbol of $a, b \in \mathbb{Q}_p^\times$. Recall that the *invariant* of a hermitian space V over $\boldsymbol{k}_p = \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is defined by

$$\mathrm{inv}_p(V) = (\det V, -D)_p, \tag{1.7.3}$$

where $\det V$ is the determinant of the matrix of the hermitian form with respect to a $\boldsymbol{k}_p$-basis. If $p < \infty$ then V is determined up to isomorphism by its $\boldsymbol{k}_p$-rank and invariant. If $p = \infty$ then V is determined up to isomorphism by its signature (r, s), and its invariant satisfies $\mathrm{inv}_\infty(V) = (-1)^s$.

2. Unitary Shimura varieties

In this section we define a unitary Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ over $\boldsymbol{k}$ and describe its moduli interpretation. We then recall the work of Pappas and Krämer, which provides us with two integral models related by a surjection $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. This surjection becomes an isomorphism after restriction to $\mathcal{O}_{\boldsymbol{k}}[1/D]$. We define a line bundle of weight one modular forms $\boldsymbol{\omega}$ and a family of Cartier divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$, $m > 0$, on $\mathcal{S}_{\mathrm{Kra}}$,

The line bundle $\boldsymbol{\omega}$ and the divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ do not descend to $\mathcal{S}_{\mathrm{Pap}}$, and the main original material in §2 is the construction of suitable substitutes on $\mathcal{S}_{\mathrm{Pap}}$. These substitutes consist of a line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ that agrees with $\boldsymbol{\omega}^2$ after restricting to $\mathcal{O}_{\boldsymbol{k}}[1/D]$, and Cartier divisors $\mathcal{Y}_{\mathrm{Pap}}(m)$ that agree with $2\mathcal{Z}_{\mathrm{Kra}}(m)$ after restricting to $\mathcal{O}_{\boldsymbol{k}}[1/D]$.

2.1. The Shimura variety.

Let W_0 and W be $\boldsymbol{k}$-vector spaces endowed with hermitian forms H_0 and H of signatures $(1,0)$ and $(n-1,1)$, respectively. We always assume that $n \geqslant 3$. Abbreviate

$$W(\mathbb{R}) = W \otimes_{\mathbb{Q}} \mathbb{R}, \quad W(\mathbb{C}) = W \otimes_{\mathbb{Q}} \mathbb{C}, \quad W(\mathbb{A}_f) = W \otimes_{\mathbb{Q}} \mathbb{A}_f,$$

and similarly for W_0. We assume the existence of $\mathcal{O}_{\boldsymbol{k}}$-lattices $\mathfrak{a}_0 \subset W_0$ and $\mathfrak{a} \subset W$, self-dual with respect to the hermitian forms H_0 and H. This is equivalent to self-duality with respect to the symplectic forms

$$\psi_0(w, w') = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} H_0(\delta^{-1} w, w'), \quad \psi(w, w') = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} H(\delta^{-1} w, w'). \tag{2.1.1}$$

This data will remain fixed throughout the paper.

As in (1.2.1), let $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ be the subgroup of pairs for which the similitude factors are equal. The common similitude character is denoted $\nu : G \to \mathbb{G}_m$. The three reductive groups in (1.2.1) determine three hermitian symmetric domains: let $\mathcal{D}(W_0) = \{y_0\}$ be a one-point set, let

$$\mathcal{D}(W) = \{\text{negative definite } \boldsymbol{k}\text{-stable } \mathbb{R}\text{-planes } y \subset W(\mathbb{R})\}, \tag{2.1.2}$$

and define

$$\mathcal{D} = \mathcal{D}(W_0) \times \mathcal{D}(W).$$

The lattices $\mathfrak{a}_0$ and $\mathfrak{a}$ determine a maximal compact open subgroup

$$K = \left\{ g \in G(\mathbb{A}_f) : g\widehat{\mathfrak{a}}_0 \subset \widehat{\mathfrak{a}}_0 \text{ and } g\widehat{\mathfrak{a}} \subset \widehat{\mathfrak{a}} \right\} \subset G(\mathbb{A}_f), \tag{2.1.3}$$

and the orbifold quotient

$$\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K$$

is the space of complex points of a smooth $\boldsymbol{k}$-stack of dimension $n-1$, denoted $\mathrm{Sh}(G, \mathcal{D})$.

The symplectic forms (2.1.1) determine a $\boldsymbol{k}$-conjugate-linear isomorphism

$$\mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\boldsymbol{k}}(W, W_0), \tag{2.1.4}$$

characterized by $\psi(xw_0, w) = \psi_0(w_0, x^\vee w)$. The $\boldsymbol{k}$-vector space

$$V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

carries a hermitian form of signature $(n-1, 1)$ defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \mathrm{End}_{\boldsymbol{k}}(W_0) \cong \boldsymbol{k}. \tag{2.1.5}$$

The group G acts on V in a natural way, defining an exact sequence (1.3.3).

The hermitian form on V induces a quadratic form $Q(x) = \langle x, x \rangle$, with associated $\mathbb{Q}$-bilinear form

$$[x, y] = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} \langle x, y \rangle. \tag{2.1.6}$$

In particular, we obtain a representation $G \to \mathrm{SO}(V)$.

Proposition 2.1.1. *The stack* $\mathrm{Sh}(G, \mathcal{D})_{/\mathbb{C}}$ *has* $2^{1-o(D)}h^2$ *connected components, where* h *is the class number of* $\boldsymbol{k}$ *and* $o(D)$ *is the number of prime divisors of* D. *Every component is defined over the Hilbert class field* H *of* $\boldsymbol{k}$, *and the set of components is a disjoint union of simply transitive* $\mathrm{Gal}(H/\boldsymbol{k})$-*orbits.*

Proof. Each $g \in G(\mathbb{A}_f)$ determines $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$g\mathfrak{a}_0 = W_0 \cap g\widehat{\mathfrak{a}}_0, \quad g\mathfrak{a} = W \cap g\widehat{\mathfrak{a}}.$$

The hermitian forms H_0 and H need not be $\mathcal{O}_{\boldsymbol{k}}$-valued on these lattices. However, if $\mathrm{rat}(\nu(g))$ denotes the unique positive rational number such that

$$\frac{\nu(g)}{\mathrm{rat}(\nu(g))} \in \widehat{\mathbb{Z}}^\times$$

then the rescaled hermitian forms $\mathrm{rat}(\nu(g))^{-1}H_0$ and $\mathrm{rat}(\nu(g))^{-1}H$ make $g\mathfrak{a}_0$ and $g\mathfrak{a}$ into self-dual hermitian lattices.

As $\mathcal{D}$ is connected, the components of $\mathrm{Sh}(G, \mathcal{D})_{/\mathbb{C}}$ are in bijection with the set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$. The function $g \mapsto (g\mathfrak{a}_0, g\mathfrak{a})$ then establishes a bijection from $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ to the set of isometry classes of pairs of self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices $(\mathfrak{a}_0', \mathfrak{a}')$ of signatures $(1, 0)$ and $(n-1, 1)$, respectively, for which the self-dual hermitian lattice $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0', \mathfrak{a}')$ lies in the same genus as $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \subset V$.

Using the fact that $\mathrm{SU}(V)$ satisfies strong approximation, one can show that there are exactly $2^{1-o(D)}h$ isometry classes in the genus of $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$, and each isometry class arises from exactly h isometry classes of pairs $(\mathfrak{a}_0', \mathfrak{a}')$.

Identifying the connected components with pairs of lattices as above, Deligne's reciprocity law [Mil05, §13] reads as follows: if $\sigma \in \mathrm{Aut}(\mathbb{C}/\boldsymbol{k})$ and $s \in \widehat{\boldsymbol{k}}^\times \subset G(\widehat{\boldsymbol{k}})$ are related by $\mathrm{art}(s) = \sigma|_{\boldsymbol{k}^{ab}}$, then

$$(\mathfrak{a}_0', \mathfrak{a}')^\sigma = (s\mathfrak{a}_0', s\mathfrak{a}').$$

The right hand side obviously only depends on the image of s in $\boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$, and all claims about the Galois action follow easily. □

It will be useful at times to have other interpretations of the hermitian domain $\mathcal{D}$. The following remarks provide alternate points of view. Define isomorphisms of real vector spaces

$$\mathrm{pr}_\epsilon : W(\mathbb{R}) \cong \epsilon W(\mathbb{C}), \quad \mathrm{pr}_{\overline{\epsilon}} : W(\mathbb{R}) \cong \overline{\epsilon} W(\mathbb{C}) \tag{2.1.7}$$

as, respectively, the compositions

$$W(\mathbb{R}) \hookrightarrow W(\mathbb{C}) = \epsilon W(\mathbb{C}) \oplus \overline{\epsilon} W(\mathbb{C}) \xrightarrow{\text{proj.}} \epsilon W(\mathbb{C})$$
$$W(\mathbb{R}) \hookrightarrow W(\mathbb{C}) = \epsilon W(\mathbb{C}) \oplus \overline{\epsilon} W(\mathbb{C}) \xrightarrow{\text{proj.}} \overline{\epsilon} W(\mathbb{C}).$$

Remark 2.1.2. Define a Hodge structure

$$F^1 W_0(\mathbb{C}) = 0, \quad F^0 W_0(\mathbb{C}) = \overline{\epsilon} W_0(\mathbb{C}), \quad F^{-1} W_0(\mathbb{C}) = W_0(\mathbb{C})$$

on $W_0(\mathbb{C})$, and identify the unique point $y_0 \in \mathcal{D}(W_0)$ with the corresponding morphism $\mathbb{S} \to \mathrm{GU}(W_0)_\mathbb{R}$. Every $y \in \mathcal{D}(W)$ defines a Hodge structure

$$F^1 W(\mathbb{C}) = 0, \quad F^0 W(\mathbb{C}) = \mathrm{pr}_\epsilon(y) \oplus \mathrm{pr}_{\overline{\epsilon}}(y^\perp), \quad F^{-1} W(\mathbb{C}) = W(\mathbb{C})$$

on $W(\mathbb{C})$. If we identify $y \in \mathcal{D}(W)$ with the corresponding morphism $\mathbb{S} \to \mathrm{GU}(W)_\mathbb{R}$, then for any point $z = (y_0, y) \in \mathcal{D}$ the product morphism

$$y_0 \times y : \mathbb{S} \to \mathrm{GU}(W_0)_\mathbb{R} \times \mathrm{GU}(W)_\mathbb{R}$$

takes values in $G_\mathbb{R}$. This realizes $\mathcal{D} \subset \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$ as a $G(\mathbb{R})$-conjugacy class.

Remark 2.1.3. Each pair $(y_0, y) \in \mathcal{D}$ determines a line $\mathrm{pr}_\epsilon(y) \subset W(\mathbb{C})$, and hence a line

$$w = \mathrm{Hom}_\mathbb{C}(W_0(\mathbb{C})/\overline{\epsilon} W_0(\mathbb{C}), \mathrm{pr}_\epsilon(y) \subset \epsilon V(\mathbb{C}).$$

This construction identifies

$$\mathcal{D} \cong \big\{ w \in \epsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^\times \subset \mathbb{P}(\epsilon V(\mathbb{C}))$$

as an open subset of projective space.

Remark 2.1.4. In fact, the discussion above shows that $\mathrm{Sh}(G, \mathcal{D})$ admits a map to the Shimura variety defined the group $\mathrm{U}(V)$ together with the homomorphism

$$h_{\mathrm{Gross}} : \mathbb{S} \to \mathrm{U}(V)(\mathbb{R}), \qquad z \mapsto \mathrm{diag}(1, \ldots, 1, \bar{z}/z).$$

Here we have chosen a basis for $V(\mathbb{R})$ for which the hermitian form has matrix $\mathrm{diag}(1_{n-1}, -1)$. Note that, for analogous choices of bases for $W_0(\mathbb{R})$ and $W(\mathbb{R})$, the corresponding map is

$$h : \mathbb{S} \to G(\mathbb{R}), \qquad z \mapsto (z) \times \mathrm{diag}(z, \ldots, z, \bar{z}),$$

which, under composition with the homomorphism $G(\mathbb{R}) \to \mathrm{U}(V)(\mathbb{R})$, gives h_{Gross}. The existence of this map provides an answer to a question posed by Gross: how can one explicitly relate the Shimura variety defined by the unitary group $\mathrm{U}(V)$, as opposed to the Shimura variety defined by the similitude group $\mathrm{GU}(V)$, to a moduli space of abelian varieties? Our answer is that Gross's unitary Shimura variety is a quotient of our $\mathrm{Sh}(G, \mathcal{D})$, whose interpretation as a moduli space is explained in the next section.

2.2. **Moduli interpretation.** We wish to interpret $\mathrm{Sh}(G, \mathcal{D})$ as a moduli space of pairs of abelian varieties with additional structure. First, we recall some generalities on abelian schemes.

For an abelian scheme $\pi : A \to S$ over an arbitrary base S, define the *first relative de Rham cohomology sheaf* $H^1_{\mathrm{dR}}(A) = \mathbb{R}^1\pi_*\Omega^\bullet_{A/S}$ as the relative hypercohomology of the de Rham complex $\Omega^\bullet_{A/S}$. The *relative de Rham homology*

$$H_1^{\mathrm{dR}}(A) = \underline{\mathrm{Hom}}(H^1_{\mathrm{dR}}(A), \mathcal{O}_S)$$

is a locally free $\mathcal{O}_S$-module of rank $2 \cdot \dim(A)$, sitting in an exact sequence

$$0 \to F^0 H_1^{\mathrm{dR}}(A) \to H_1^{\mathrm{dR}}(A) \to \mathrm{Lie}(A) \to 0.$$

Any polarization of A induces an $\mathcal{O}_S$-valued alternating pairing on $H_1^{\mathrm{dR}}(A)$, which in turn induces a pairing

$$F^0 H_1^{\mathrm{dR}}(A) \otimes \mathrm{Lie}(A) \to \mathcal{O}_S. \tag{2.2.1}$$

If the polarization is principal then both pairings are perfect. When $S = \mathrm{Spec}(\mathbb{C})$, Betti homology satisfies $H_1(A, \mathbb{C}) \cong H_1^{\mathrm{dR}}(A)$, and

$$A(\mathbb{C}) \cong H_1(A, \mathbb{Z}) \backslash H_1^{\mathrm{dR}}(A) / F^0 H_1^{\mathrm{dR}}(A).$$

For any pair of nonnegative integers (s, t), define an algebraic stack $M_{(s,t)}$ over $\boldsymbol{k}$ as follows: for any $\boldsymbol{k}$-scheme S let $M_{(s,t)}(S)$ be the groupoid of triples (A, ι, ψ) in which

- $A \to S$ is an abelian scheme of relative dimension $s + t$,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action such that the locally free summands

  $$\mathrm{Lie}(A) = \epsilon \mathrm{Lie}(A) \oplus \overline{\epsilon} \mathrm{Lie}(A)$$

  of (1.7.2) have $\mathcal{O}_S$-ranks s and t, respectively,
- $\psi : A \to A^\vee$ is a principal polarization, such that the induced Rosati involution $\dagger$ on $\mathrm{End}^0(A)$ satisfies $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$.

We usually omit ι and ψ from the notation, and just write $A \in M_{(s,t)}(S)$.

Proposition 2.2.1. *The Shimura variety* $\mathrm{Sh}(G,\mathcal{D})$ *is isomorphic to an open and closed substack*

$$\mathrm{Sh}(G,\mathcal{D}) \subset M_{(1,0)} \times_{\boldsymbol{k}} M_{(n-1,1)}. \tag{2.2.2}$$

More precisely, $\mathrm{Sh}(G,\mathcal{D})(S)$ *classifies, for any* $\boldsymbol{k}$*-scheme* S*, pairs*

$$(A_0, A) \in M_{(1,0)}(S) \times M_{(n-1,1)}(S) \tag{2.2.3}$$

for which there exists, at every geometric point $s \to S$*, an isomorphism of hermitian* $\mathcal{O}_{\boldsymbol{k},\ell}$*-modules*

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(T_\ell A_{0,s}, T_\ell A_s) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \otimes \mathbb{Z}_\ell \tag{2.2.4}$$

for every prime ℓ*. Here the hermitian form on the right hand side of (2.2.4) is the restriction of the hermitian form (2.1.5) on* $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \otimes \mathbb{Q}_\ell$*. The hermitian form on the left hand side is defined similarly, replacing the symplectic forms (2.1.1) on* W_0 *and* W *with the Weil pairings on the Tate modules* $T_\ell A_{0,s}$ *and* $T_\ell A_s$*.*

Proof. As this is routine, we only describe the open and closed immersion on complex points. Fix a point

$$(z, g) \in \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}).$$

The component g determines $\mathcal{O}_{\boldsymbol{k}}$-lattices $g\mathfrak{a}_0 \subset W_0$ and $g\mathfrak{a} \subset W$, which are self-dual with respect to the symplectic forms $\mathrm{rat}(\nu(g))^{-1}\psi_0$ and $\mathrm{rat}(\nu(g))^{-1}\psi$ of (2.1.1), rescaled as in the proof of Proposition 2.1.1.

By Remark 2.1.2 the point $z \in \mathcal{D}$ determines Hodge structures on W_0 and W, and in this way (z, g) determines principally polarized complex abelian varieties

$$A_0(\mathbb{C}) = g\mathfrak{a}_0 \backslash W_0(\mathbb{C})/F^0(W_0)$$
$$A(\mathbb{C}) = g\mathfrak{a} \backslash W(\mathbb{C})/F^0(W)$$

with actions of $\mathcal{O}_{\boldsymbol{k}}$. One can easily check that the pair (A_0, A) determines a complex point of $M_{(1,0)} \times_{\boldsymbol{k}} M_{(n-1,1)}$, and this construction defines (2.2.2) on complex points. ☐

The following lemma will be needed in §2.3 for the construction of integral models for $\mathrm{Sh}(G,\mathcal{D})$.

Lemma 2.2.2. *Fix a* $\boldsymbol{k}$*-scheme* S*, a geometric point* $s \to S$*, a prime* p*, and a point (2.2.3). If the relation (2.2.4) holds for all* $\ell \neq p$*, then it also holds for* $\ell = p$*.*

Proof. As the stack $\mathrm{Sh}(G,\mathcal{D})$ is of finite type over $\boldsymbol{k}$, we may assume that $s = \mathrm{Spec}(\mathbb{C})$. The polarizations on A_0 and A induce symplectic forms on the first homology groups $H_1(A_{0,s}(\mathbb{C}),\mathbb{Z})$ and $H_1(A_s(\mathbb{C}),\mathbb{Z})$, and the construction (2.1.5) makes

$$L_{\mathrm{Be}}(A_{0,s}, A_s) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1(A_{0,s}(\mathbb{C}),\mathbb{Z}), H_1(A_s(\mathbb{C}),\mathbb{Z})\big)$$

into a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1,1)$, satisfying

$$L_{\mathrm{Be}}(A_{0,s}, A_s) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(T_\ell A_{0,s}, T_\ell A_s)$$

for all primes ℓ.

If the relation (2.2.4) holds for all primes $\ell \neq p$, then $L_{\mathrm{Be}}(A_{0,s}, A_s) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ are isomorphic as $\boldsymbol{k}$-hermitian spaces everywhere locally except at p, and so they are isomorphic at p as well. In particular, for every ℓ (including $\ell = p$) both sides of (2.2.4) are isomorphic to self-dual lattices in the hermitian space $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. By the results of Jacobowitz [Jac62] all self-dual lattices in this local hermitian space are isomorphic[2], and so (2.2.4) holds for all ℓ. □

Remark 2.2.3. For any positive integer m define

$$K(m) = \ker\big(K \to \mathrm{Aut}_{\mathcal{O}_{\boldsymbol{k}}}(\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0) \times \mathrm{Aut}_{\mathcal{O}_{\boldsymbol{k}}}(\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}})\big).$$

For a $\boldsymbol{k}$-scheme S, a $K(m)$-*structure* on $(A_0, A) \in \mathrm{Sh}(G, \mathcal{D})(S)$ is a triple $(\alpha_0, \alpha, \zeta)$ in which $\zeta : \underline{\mu_m} \cong \underline{\mathbb{Z}/m\mathbb{Z}}$ is an isomorphism of S-group schemes, and

$$\alpha_0 : A_0[m] \cong \underline{\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0}, \quad \alpha : A[m] \cong \underline{\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}}}$$

are $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphisms identifying the Weil pairings on $A_0[m]$ and $A[m]$ with the $\mathbb{Z}/m\mathbb{Z}$-valued symplectic forms on $\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0$ and $\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}}$ deduced from the pairings (2.1.1). The Shimura variety $G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K(m)$ admits a canonical model over $\boldsymbol{k}$, parametrizing $K(m)$-structures on points of $\mathrm{Sh}(G, \mathcal{D})$.

2.3. **Integral models.** In this subsection we describe two integral models of $\mathrm{Sh}(G, \mathcal{D})$ over $\mathcal{O}_{\boldsymbol{k}}$, related by a morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$.

The first step is to construct an integral model of the moduli space $M_{(1,0)}$. More generally, we will construct an integral model of $M_{(s,0)}$ for any $s > 0$. Define an $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{M}_{(s,0)}$ as the moduli space of triples (A, ι, ψ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S such that

- $A \to S$ is an abelian scheme of relative dimension s,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action such $\overline{\epsilon}\mathrm{Lie}(A) = 0$, or, equivalently, such that that the induced action of $\mathcal{O}_{\boldsymbol{k}}$ on the $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ is through the structure map $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$,
- $\psi : A \to A^\vee$ is a principal polarization whose Rosati involution satisfies $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$.

The stack $\mathcal{M}_{(s,0)}$ is smooth of relative dimension 0 over $\mathcal{O}_{\boldsymbol{k}}$ by [How15, Proposition 2.1.2], and its generic fiber is the stack $M_{(s,0)}$ defined earlier.

The question of integral models for $M_{(n-1,1)}$ is more subtle, but well-understood after work of Pappas and Krämer. The first integral model was defined by Pappas: let

$$\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

[2]This uses our standing hypothesis that D is odd.

be the algebraic stack whose functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of triples (A, ι, ψ) in which

- $A \to S$ is an abelian scheme of relative dimension n,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action satisfying the determinant condition
$$\det(T - \iota(\alpha) \mid \mathrm{Lie}(A)) = (T - \alpha)^{n-1}(T - \overline{\alpha}) \in \mathcal{O}_S[T]$$
for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,
- $\psi : A \to A^\vee$ is a principal polarization whose Rosati involution satisfies $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,
- viewing the elements ϵ_S and $\overline{\epsilon}_S$ of §1.7 as endomorphisms of $\mathrm{Lie}(A)$, the induced endomorphisms
$$\bigwedge^n \epsilon_S : \bigwedge^n \mathrm{Lie}(A) \to \bigwedge^n \mathrm{Lie}(A)$$
$$\bigwedge^2 \overline{\epsilon}_S : \bigwedge^2 \mathrm{Lie}(A) \to \bigwedge^2 \mathrm{Lie}(A)$$
are trivial (*Pappas's wedge condition*).

It is clear that the generic fiber of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is isomorphic to the moduli space $M_{(n-1,1)}$ defined earlier. Denote by
$$\mathrm{Sing}_{(n-1,1)} \subset \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$$
the singular locus: the reduced substack of points at which the structure morphism to $\mathcal{O}_{\boldsymbol{k}}$ is not smooth.

Theorem 2.3.1 (Pappas [Pap00]). *The stack $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is flat over $\mathcal{O}_{\boldsymbol{k}}$ of relative dimension $n-1$, and is Cohen-Macaulay and normal. Moreover:*

1. *For any prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$, the reduction $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)/\mathbb{F}_\mathfrak{p}}$ is Cohen-Macaulay and geometrically normal.*
2. *The singular locus is a 0-dimensional stack, finite over $\mathcal{O}_{\boldsymbol{k}}$ and supported in characteristics dividing D. It is the reduced substack underlying the closed substack defined by $\delta \cdot \mathrm{Lie}(A) = 0$.*
3. *The stack $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ becomes regular after blow-up along the singular locus.*

The blow up of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ along the singular locus is denoted

$$\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \to \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}. \tag{2.3.1}$$

Krämer has shown that its functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of quadruples $(A, \iota, \psi, \mathcal{F}_A)$ in which

- $A \to S$ is an abelian scheme of relative dimension n,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action of $\mathcal{O}_{\boldsymbol{k}}$,
- $\psi : A \to A^\vee$ is a principal polarization satisfying $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,

- $\mathcal{F}_A \subset \mathrm{Lie}(A)$ is an $\mathcal{O}_{\boldsymbol{k}}$-stable $\mathcal{O}_S$-module local direct summand of rank $n-1$ satisfying *Krämer's condition*: $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathcal{F}_A$ via the structure map $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and acts on the line bundle $\mathrm{Lie}(A)/\mathcal{F}_A$ via the complex conjugate of the structure map.

The morphism (2.3.1) simply forgets the subsheaf $\mathcal{F}_A$.

Recalling (2.2.2), we define our first integral model

$$\mathcal{S}_{\mathrm{Pap}} \subset \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$$

as the Zariski closure of $\mathrm{Sh}(G,\mathcal{D})$ in the fiber product on the right, which, like all fiber products below, is taken over over $\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. Using Lemma 2.2.2, one can show that it is characterized asthe open and closed substack whose functor of points assigns to any $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of pairs

$$(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}(S)$$

such that, at any geometric point $s \to S$, the relation (2.2.4) holds for all primes $\ell \neq \mathrm{char}(k(s))$.

Our second integral model of $\mathrm{Sh}(G,\mathcal{D})$ is defined as the cartesian product

$$\begin{tikzcd}
\mathcal{S}_{\mathrm{Kra}} \arrow[r] \arrow[d] & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \arrow[d] \\
\mathcal{S}_{\mathrm{Pap}} \arrow[r] & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}.
\end{tikzcd}$$

The *singular locus* $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ and *exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ are defined by the cartesian squares

$$\begin{tikzcd}
\mathrm{Exc} \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Kra}} \arrow[d] \\
\mathrm{Sing} \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Pap}} \arrow[d] \\
\mathcal{M}_{(1,0)} \times \mathrm{Sing}_{(n-1,1)} \arrow[r] & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}.
\end{tikzcd}$$

Both are proper over $\mathcal{O}_{\boldsymbol{k}}$, and supported in characteristics dividing D.

Theorem 2.3.2 (Krämer [Krä03], Pappas [Pap00])**.** *The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}_{\mathrm{Kra}}$ is regular and flat. The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}_{\mathrm{Pap}}$ is Cohen-Macaulay and normal, with Cohen-Macaulay and geometrically normal fibers. Furthermore:*

(1) *The exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ *is a disjoint union of smooth Cartier divisors. The singular locus* $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ *is a reduced closed stack of dimension* 0*, supported in characteristics dividing* D*.*

(2) *The fiber of* Exc *over a geometric point* $s \to \mathrm{Sing}$ *is isomorphic to the projective space* $\mathbb{P}^{n-1}$ *over* $k(s)$*.*

(3) *The morphism* $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ *is surjective, and restricts to an isomorphism*

$$\mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc} \cong \mathcal{S}_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}. \tag{2.3.2}$$

For an $\mathcal{O}_{\boldsymbol{k}}$*-scheme* S*, the inverse of this isomorphism endows*

$$(A_0, A) \in \big(\mathcal{S}_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}\big)(S)$$

with the subsheaf $\mathcal{F}_A = \ker\big(\overline{\epsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A)\big)$.

Remark 2.3.3. Let (A_0, A) be the universal pair over $\mathcal{S}_{\mathrm{Pap}}$. The vector bundle $H_1^{\mathrm{dR}}(A_0)$ is locally free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$ and, by definition of the moduli problem defining $\mathcal{S}_{\mathrm{Pap}}$, its quotient $\mathrm{Lie}(A_0)$ is annihilated by $\overline{\epsilon}$. From this it is not hard to see that

$$F^0 H_1^{\mathrm{dR}}(A_0) = \overline{\epsilon} H_1^{\mathrm{dR}}(A_0).$$

2.4. **The line bundle of modular forms.** We now construct a line bundle of modular forms $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$, and consider the subtle question of whether or not it descends to $\mathcal{S}_{\mathrm{Pap}}$. The short answer is that doesn't, but a more complete answer can be found in Theorems 2.4.3 and 2.6.3.

By Remark 2.1.2, every point $z \in \mathcal{D}$ determines Hodge structures on W_0 and W of weight -1, and hence a Hodge structure of weight 0 on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. Consider the holomorphic line bundle $\boldsymbol{\omega}^{an}$ on $\mathcal{D}$ whose fiber at z is the complex line $\boldsymbol{\omega}_z^{an} = F^1 V(\mathbb{C})$ determined by this Hodge structure.

Remark 2.4.1. It is useful to interpret $\boldsymbol{\omega}^{an}$ in notation of Remark 2.1.3. The fiber of $\boldsymbol{\omega}^{an}$ at $z = (y_0, y)$ is the line

$$\boldsymbol{\omega}_z^{an} = \mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\overline{\epsilon} W_0(\mathbb{C}), \mathrm{pr}_{\epsilon}(y)) \subset \epsilon V(\mathbb{C}), \tag{2.4.1}$$

and hence $\boldsymbol{\omega}^{an}$ is simply the restriction of the tautological bundle via the inclusion

$$\mathcal{D} \cong \big\{ w \in \epsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^{\times} \subset \mathbb{P}(\epsilon V(\mathbb{C})).$$

There is a natural action of $G(\mathbb{R})$ on the total space of $\boldsymbol{\omega}^{an}$, lifting the natural action on $\mathcal{D}$, and so $\boldsymbol{\omega}^{an}$ descends to a line bundle on the complex orbifold $\mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$. This descent is algebraic, has a canonical model over the reflex field, and extends in a natural way to the integral model $\mathcal{S}_{\mathrm{Kra}}$, as we now explain.

Let (A_0, A) be the universal object over $\mathcal{S}_{\mathrm{Kra}}$, let $\mathcal{F}_A \subset \mathrm{Lie}(A)$ be the universal subsheaf of Krämer's moduli problem, and let

$$\mathcal{F}_A^{\perp} \subset F^0 H_1^{\mathrm{dR}}(A)$$

be the orthogonal to $\mathcal{F}_A$ under the pairing (2.2.1). It is a rank one $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$-module local direct summand on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$. Define the *line bundle of weight one modular forms* on $\mathcal{S}_{\mathrm{Kra}}$ by

$$\boldsymbol{\omega} = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^{\perp}),$$

or, equivalently, $\boldsymbol{\omega}^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}_A$.

Proposition 2.4.2. *The line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ just defined restricts to the already defined $\boldsymbol{\omega}^{an}$ in the complex fiber. Moreover, on the complement of the exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ *we have*

$$\boldsymbol{\omega} = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \epsilon F^0 H_1^{\mathrm{dR}}(A)).$$

Proof. The equality $\mathcal{F}_A^{\perp} = \epsilon F^0 H_1^{\mathrm{dR}}(A)$ on the complement of Exc follows from the characterization

$$\mathcal{F}_A = \ker(\bar{\epsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A))$$

of Theorem 2.3.2, and all of the claims follow easily from this and examination of the proof of Proposition 2.2.1. □

The line bundle $\boldsymbol{\omega}$ does not descend to $\mathcal{S}_{\mathrm{Pap}}$, but it is closely related to another line bundle that does. This is the content of the following theorem, whose proof will occupy the remainder of §2.4. The result will be strengthened in Theorem 2.6.3.

Theorem 2.4.3. *There is a unique line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to the nonsingular locus (2.3.2) is isomorphic to $\boldsymbol{\omega}^2$. We denote by $\boldsymbol{\Omega}_{\mathrm{Kra}}$ its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$.*

Proof. Let (A_0, A) be the universal object over $\mathcal{S}_{\mathrm{Pap}}$, and recall the short exact sequence

$$0 \to F^0 H_1^{\mathrm{dR}}(A) \to H_1^{\mathrm{dR}}(A) \xrightarrow{q} \mathrm{Lie}(A) \to 0$$

of vector bundles on $\mathcal{S}_{\mathrm{Pap}}$. As $H_1^{\mathrm{dR}}(A)$ is a locally free $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$-module of rank n, the quotient $H_1^{\mathrm{dR}}(A)/\bar{\epsilon} H_1^{\mathrm{dR}}(A)$ is a rank n vector bundle.

Define a line bundle

$$\mathcal{P}_{\mathrm{Pap}} = \underline{\mathrm{Hom}}\Big({\bigwedge}^n H_1^{\mathrm{dR}}(A)/\bar{\epsilon} H_1^{\mathrm{dR}}(A), {\bigwedge}^n \mathrm{Lie}(A)\Big)$$

on $\mathcal{S}_{\mathrm{Pap}}$, and denote by $\mathcal{P}_{\mathrm{Kra}}$ its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Let

$$\psi : H_1^{\mathrm{dR}}(A) \otimes H_1^{\mathrm{dR}}(A) \to \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$$

be the alternating pairing induced by the principal polarization on A. If a and b are local sections of $H_1^{\mathrm{dR}}(A)$, define a local section $P_{a\otimes b}$ of $\mathcal{P}_{\mathrm{Pap}}$ by

$$P_{a\otimes b}(e_1 \wedge \cdots \wedge e_n) = \sum_{k=1}^n (-1)^{k+1} \cdot \psi(\bar{\epsilon} a, e_k) \cdot q(\bar{\epsilon} b) \wedge \underbrace{q(e_1) \wedge \cdots \wedge q(e_n)}_{\text{omit } q(e_k)}.$$

If we modify any e_k by a section of $\bar{\epsilon} H_1^{\mathrm{dR}}(A)$, the right hand side is unchanged; this is a consequence of the vanishing of

$${\bigwedge}^2 \bar{\epsilon} : {\bigwedge}^2 \mathrm{Lie}(A) \to {\bigwedge}^2 \mathrm{Lie}(A)$$

imposed in the moduli problem defining $\mathcal{S}_{\mathrm{Pap}}$. We have just constructed a morphism

$$P : H_1^{\mathrm{dR}}(A) \otimes H_1^{\mathrm{dR}}(A) \to \mathcal{P}_{\mathrm{Pap}} \tag{2.4.2}$$

sending $a \otimes b \mapsto P_{a\otimes b}$.

Lemma 2.4.4. *The morphism P factors through a morphism*

$$P : \mathrm{Lie}(A) \otimes \mathrm{Lie}(A) \to \mathcal{P}_{\mathrm{Pap}}.$$

After pullback to $\mathcal{S}_{\mathrm{Kra}}$ there is a further factorization

$$P : \mathrm{Lie}(A)/\mathcal{F}_A \otimes \mathrm{Lie}(A)/\mathcal{F}_A \to \mathcal{P}_{\mathrm{Kra}}, \tag{2.4.3}$$

and this map becomes an isomorphism after restriction to $\mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}$.

Proof. Let a and b be local sections of $H_1^{\mathrm{dR}}(A)$.

Assume first that a is contained in $F^0H_1^{\mathrm{dR}}(A)$. As $F^0H_1^{\mathrm{dR}}(A)$ is isotropic under the pairing ψ, $P_{a\otimes b}$ factors through a map

$$\bigwedge^n \mathrm{Lie}(A)/\overline{\epsilon}\mathrm{Lie}(A) \to \bigwedge^n \mathrm{Lie}(A).$$

In the generic fiber of $\mathcal{S}_{\mathrm{Pap}}$, the sheaf $\mathrm{Lie}(A)/\overline{\epsilon}\mathrm{Lie}(A)$ is a vector bundle of rank $n-1$. This proves that $P_{a\otimes b}$ is trivial over the generic fiber. As $P_{a\otimes b}$ is a morphism of vector bundles on a flat $\mathcal{O}_{\boldsymbol{k}}$-stack, we deduce that $P_{a\otimes b} = 0$ identically on $\mathcal{S}_{\mathrm{Pap}}$.

If instead b is contained in $F^0H_1^{\mathrm{dR}}(A)$ then $q(\overline{\epsilon}b) = 0$, and again $P_{a\otimes b} = 0$. These calculations prove that P factors through $\mathrm{Lie}(A) \otimes \mathrm{Lie}(A)$.

Now pullback to $\mathcal{S}_{\mathrm{Kra}}$. We need to check that $P_{a\otimes b}$ vanishes if either of a or b lies in $\mathcal{F}_A$. Once again it suffices to check this in the generic fiber, where it is clear from

$$\mathcal{F}_A = \ker(\overline{\epsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A)). \tag{2.4.4}$$

Over $\mathcal{S}_{\mathrm{Kra}}$ we now have a factorization (2.4.3), and it only remains to check that its restriction to (2.3.2) is an isomorphism. For this, it suffices to verify that (2.4.3) is surjective on the fiber at any geometric point

$$s = \mathrm{Spec}(\mathbb{F}) \to \mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}.$$

First suppose that $\mathrm{char}(\mathbb{F})$ is prime to D. In this case $\epsilon, \overline{\epsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$ are (up to scaling by $\mathbb{F}^\times$) orthogonal idempotents, $\mathcal{F}_{A_s} = \epsilon\mathrm{Lie}(A_s)$, and we may choose an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$-basis $e_1, \ldots, e_n \in H_1^{\mathrm{dR}}(A_s)$ in such a way that

$$\epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n \in F^0H_1^{\mathrm{dR}}(A_s)$$

and

$$q(\overline{\epsilon} e_1), q(\epsilon e_2), \ldots, q(\epsilon e_n) \in \mathrm{Lie}(A_s)$$

are $\mathbb{F}$-bases. This implies that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \psi(\overline{\epsilon} e_1, \epsilon e_1) \cdot q(\overline{\epsilon} e_1) \wedge q(\epsilon e_2) \wedge \cdots \wedge q(\epsilon e_n) \neq 0,$$

and so

$$P_{e_1\otimes e_1} \in \mathrm{Hom}\big(\bigwedge^n H_1^{\mathrm{dR}}(A_s)/\overline{\epsilon} H_1^{\mathrm{dR}}(A_s), \bigwedge^n \mathrm{Lie}(A_s)\big)$$

is a generator. Thus P is surjective in the fiber at z.

Now suppose that $\mathrm{char}(\mathbb{F})$ divides D. In this case there is an isomorphism

$$\mathbb{F}[x]/(x^2) \xrightarrow{x \mapsto \epsilon = \overline{\epsilon}} \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}.$$

By Theorem 2.3.2 the relation (2.4.4) holds in an étale neighborhood of s, and it follows that we may choose an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$-basis $e_1, \ldots, e_n \in H_1^{\mathrm{dR}}(A_s)$ in such a way that

$$e_2, \epsilon e_2, \epsilon e_3, \ldots, \epsilon e_n \in F^0 H_1^{\mathrm{dR}}(A_s)$$

and

$$q(e_1), q(\epsilon e_1), q(e_3) \ldots, q(e_n) \in \mathrm{Lie}(A_s)$$

are $\mathbb{F}$-bases. This implies that

$$P_{e_1 \otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \psi(\epsilon e_1, e_2) \cdot q(\epsilon e_1) \wedge q(e_1) \wedge q(e_3) \wedge \cdots \wedge q(e_n) \neq 0,$$

and so, as above, P is surjective in the fiber at z. $\square$

We now complete the proof of Theorem 2.4.3. To prove the existence part of the claim, we define $\boldsymbol{\Omega}_{\mathrm{Pap}}$ by

$$\boldsymbol{\Omega}_{\mathrm{Pap}}^{-1} = \mathrm{Lie}(A_0)^{\otimes 2} \otimes \mathcal{P}_{\mathrm{Pap}},$$

and let $\boldsymbol{\Omega}_{\mathrm{Kra}}$ be its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Tensoring both sides of (2.4.3) with $\mathrm{Lie}(A_0)^{\otimes 2}$ defines a morphism

$$\boldsymbol{\omega}^{-2} \to \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1},$$

whose restriction to $\mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}$ is an isomorphism. In particular $\boldsymbol{\omega}^2$ and $\boldsymbol{\Omega}_{\mathrm{Pap}}$ are isomorphic over (2.3.2).

The uniqueness of $\boldsymbol{\Omega}_{\mathrm{Pap}}$ is clear: as $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ is a codimension $\geqslant 2$ closed substack of a normal stack, any line bundle on the complement of Sing admits at most one extension to all of $\mathcal{S}_{\mathrm{Pap}}$.

$\square$

2.5. **Special divisors.** Suppose S is a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme, and

$$(A_0, A) \in \mathcal{S}_{\mathrm{Pap}}(S).$$

Imitating the construction of (2.1.5), there is a positive definite hermitian form on $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A_0) \cong \mathcal{O}_{\boldsymbol{k}}, \tag{2.5.1}$$

where

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A, A_0)$$

is the $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism induced by the principal polarizations on A_0 and A.

For any positive $m \in \mathbb{Z}$, define the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{Z}_{\mathrm{Pap}}(m)$ as the moduli stack assigning to a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of triples (A_0, A, x), where

- $(A_0, A) \in \mathcal{S}_{\mathrm{Pap}}(S)$,
- $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x \rangle = m$.

Define a stack $\mathcal{Z}_{\mathrm{Kra}}(m)$ in exactly the same way, but replacing $\mathcal{S}_{\mathrm{Pap}}$ by $\mathcal{S}_{\mathrm{Kra}}$. Thus we obtain a cartesian diagram

$$\begin{tikzcd} \mathcal{Z}_{\mathrm{Kra}}(m) \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Kra}} \arrow[d] \\ \mathcal{Z}_{\mathrm{Pap}}(m) \arrow[r] & \mathcal{S}_{\mathrm{Pap}}, \end{tikzcd}$$

in which the horizontal arrows are relatively representable, finite, and unramified.

Each $\mathcal{Z}_{\mathrm{Kra}}(m)$ is, étale locally on $\mathcal{S}_{\mathrm{Kra}}$, a disjoint union of Cartier divisors. More precisely, around any geometric point of $\mathcal{S}_{\mathrm{Kra}}$ one can find an étale neighborhood U with the property that the morphism $\mathcal{Z}_{\mathrm{Kra}}(m)_U \to U$ restricts to a closed immersion on every connected component $Z \subset \mathcal{Z}_{\mathrm{Kra}}(m)_U$, and $Z \subset U$ is defined locally by one equation; this is [How15, Proposition 3.2.3]. Summing over all connected components Z allows us to view $\mathcal{Z}_{\mathrm{Kra}}(m)_U$ as a Cartier divisor on U, and glueing as U varies over an étale cover defines a Cartier divisor on $\mathcal{S}_{\mathrm{Kra}}$, which we again denote by $\mathcal{Z}_{\mathrm{Kra}}(m)$.

Remark 2.5.1. It follows from (2.3.2) and the paragraph above that $\mathcal{Z}_{\mathrm{Pap}}(m)$ is locally defined by one equation away from the singular locus. However, this property may fail at points of Sing, and so $\mathcal{Z}_{\mathrm{Pap}}(m)$ does not define a Cartier divisor on all of $\mathcal{S}_{\mathrm{Pap}}$.

The following theorem, whose proof will occupy the remainder of §2.5, shows that $\mathcal{Z}_{\mathrm{Kra}}(m)$ is closely related to another Cartier divisor on $\mathcal{S}_{\mathrm{Kra}}$ that descends to $\mathcal{S}_{\mathrm{Pap}}$. This result will be strengthened in Theorem 2.6.3.

Theorem 2.5.2. *For every $m > 0$ there is a unique Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to (2.3.2) is equal to the square of the Cartier divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$. We denote by $\mathcal{Y}_{\mathrm{Kra}}(m)$ its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$.*

Remark 2.5.3. The Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ is flat over $\mathcal{O}_{\boldsymbol{k}}$, as is the restriction of $\mathcal{Z}_{\mathrm{Kra}}(m)$ to $\mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}$. This will be proved in Corollary 3.7.2, by studying the structure of the divisors near the boundary of a toroidal compactification.

Proof. The map $\mathcal{Z}_{\mathrm{Pap}}(m) \to \mathcal{S}_{\mathrm{Pap}}$ is finite, unramified, and relatively representable. It follows that every geometric point of $\mathcal{S}_{\mathrm{Pap}}$ admits an étale neighborhood $U \to \mathcal{S}_{\mathrm{Pap}}$ such that U is a scheme, and the morphism

$$\mathcal{Z}_{\mathrm{Pap}}(m)_U \to U$$

restricts to a closed immersion on every connected component

$$Z \subset \mathcal{Z}_{\mathrm{Pap}}(m)_U.$$

We will construct a Cartier divisor on any such U, and then glue them together as U varies over an étale cover to obtain the divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$.

Fix Z as above, let $\mathcal{I} \subset \mathcal{O}_U$ be its ideal sheaf, and let Z' be the closed subscheme of U defined by the ideal sheaf $\mathcal{I}^2$. Thus we have closed immersions

$$Z \subset Z' \subset U,$$

the first of which is a square-zero thickening.

By the very definition of $\mathcal{Z}_{\mathrm{Pap}}(m)$, along Z there is a universal $\mathcal{O}_{\boldsymbol{k}}$-linear map $x : A_{0Z} \to A_Z$. This map does not extend to a map $A_{0Z'} \to A_{Z'}$, however, by deformation theory [Lan13, Chapter 2.1.6] the induced $\mathcal{O}_{\boldsymbol{k}}$-linear morphism of vector bundles

$$x : H_1^{\mathrm{dR}}(A_{0Z}) \to H_1^{\mathrm{dR}}(A_Z)$$

admits a canonical extension to

$$x' : H_1^{\mathrm{dR}}(A_{0Z'}) \to H_1^{\mathrm{dR}}(A_{Z'}). \tag{2.5.2}$$

Recalling the morphism (2.4.2), define $Y \subset Z'$ as the largest closed subscheme over which the composition

$$H_1^{\mathrm{dR}}(A_{0Z'}) \otimes H_1^{\mathrm{dR}}(A_{0Z'}) \xrightarrow{x'\otimes x'} H_1^{\mathrm{dR}}(A_{Z'}) \otimes H_1^{\mathrm{dR}}(A_{Z'}) \xrightarrow{P} \mathcal{P}_{\mathrm{Pap}}|_{Z'} \tag{2.5.3}$$

vanishes.

Lemma 2.5.4. *If $U \to \mathcal{S}_{\mathrm{Pap}}$ factors through $\mathcal{S}_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}$, then $Y = Z'$.*

Proof. Lemma 2.4.4 provides us with a commutative diagram

$$\begin{tikzcd}
H_1^{\mathrm{dR}}(A_{0Z'})^{\otimes 2} \arrow[r, "x'\otimes x'"] \arrow[drr, "(2.5.3)"'] & H_1^{\mathrm{dR}}(A_{Z'})^{\otimes 2} \arrow[r, "q\otimes q"] & \big(\mathrm{Lie}(A_{Z'})/\mathcal{F}_{A_{Z'}}\big)^{\otimes 2} \arrow[d, "\cong"] \\
 & & \mathcal{P}_{\mathrm{Pap}}|_{Z'},
\end{tikzcd}$$

where

$$\mathcal{F}_{A_{Z'}} = \ker(\overline{\epsilon} : \mathrm{Lie}(A_{Z'}) \to \mathrm{Lie}(A_{Z'}))$$

as in Theorem 2.3.2.

By deformation theory, $Z \subset Z'$ is characterized as the largest closed subscheme over which (2.5.2) respects the Hodge filtrations. Using Remark 2.3.3, it is easily seen that $Z \subset Z'$ can also be characterized as the largest closed subscheme over which

$$H_1(A_{0Z'}) \xrightarrow{q\circ x'} \mathrm{Lie}(A_{Z'})/\mathcal{F}_{A_{Z'}}$$

vanishes identically. As $Z \subset Z'$ is a square zero thickening, it follows first that the horizontal composition in the above diagram vanishes identically, and then that (2.5.3) vanishes identically. In other words $Y = Z'$. $\square$

Lemma 2.5.5. *The closed subscheme $Y \subset U$ is defined locally by one equation.*

Proof. Fix a closed point $y \in Y$ of characteristic p, let $\mathcal{O}_{U,y}$ be the local ring of U at y, and let $\mathfrak{m} \subset \mathcal{O}_{U,y}$ be the maximal ideal. For a fixed $k > 0$, let

$$\boldsymbol{U} = \mathrm{Spec}(\mathcal{O}_{U,y}/\mathfrak{m}^k) \subset U$$

be the k^{th}-order infinitesimal neighborhood of y in U. The point of passing to the infinitesimal neighborhood is that p is nilpotent in $\mathcal{O}_{\boldsymbol{U}}$, and so we may apply Grothendieck-Messing deformation theory.

By construction we have closed immersions

$$\begin{tikzcd} & Y \arrow[d] & \\ Z \arrow[r] & Z' \arrow[r] & U. \end{tikzcd}$$

Applying the fiber product $\times_U \boldsymbol{U}$ throughout the diagram, we obtain closed immersions

$$\begin{tikzcd} & \boldsymbol{Y} \arrow[d] & \\ \boldsymbol{Z} \arrow[r] & \boldsymbol{Z}' \arrow[r] & \boldsymbol{U} \end{tikzcd}$$

of Artinian schemes. As k is arbitrary, it suffices to prove that $\boldsymbol{Y} \subset \boldsymbol{U}$ is defined by one equation.

First suppose that $p \nmid D$. In this case $\boldsymbol{U} \to U \to \mathcal{S}_{\mathrm{Pap}}$ factors through the nonsingular locus (2.3.2). It follows from Remark 2.5.1 that $\boldsymbol{Z} \subset \boldsymbol{U}$ is defined by one equation, and $\boldsymbol{Z}'$ is defined by the square of that equation. By Lemma 2.5.4, $\boldsymbol{Y} \subset \boldsymbol{U}$ is also defined by one equation.

For the remainder of the proof we assume that $p \mid D$. In particular $p > 2$. Consider the closed subscheme $Z'' \hookrightarrow U$ with ideal sheaf $\mathcal{I}^3$, so that we have closed immersions $Z \subset Z' \subset Z'' \subset U$. Taking the fiber product with $\boldsymbol{U}$, the above diagram extends to

$$\begin{tikzcd} & \boldsymbol{Y} \arrow[d] & & \\ \boldsymbol{Z} \arrow[r] & \boldsymbol{Z}' \arrow[r] & \boldsymbol{Z}'' \arrow[r] & \boldsymbol{U}. \end{tikzcd}$$

As $p > 2$, the cube zero thickening $\boldsymbol{Z} \subset \boldsymbol{Z}''$ admits divided powers extending the trivial divided powers on $\boldsymbol{Z} \subset \boldsymbol{Z}'$. Therefore, by Grothendieck-Messing theory, the restriction of (2.5.2) to

$$x' : H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}'}) \to H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}'}).$$

admits a canonical extension to

$$x'' : H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \to H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}).$$

Define $\boldsymbol{Y}' \subset \boldsymbol{Z}''$ as the largest closed subscheme over which
(2.5.4)

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \xrightarrow{x'' \otimes x''} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \xrightarrow{P} \mathcal{P}_{\mathrm{Pap}}|_{\boldsymbol{Z}''}$$

vanishes identically, so that there are closed immersions

$$\begin{tikzcd} & \boldsymbol{Y} \arrow[r] \arrow[d] & \boldsymbol{Y}' \arrow[d] & \\ \boldsymbol{Z} \arrow[r] & \boldsymbol{Z}' \arrow[r] & \boldsymbol{Z}'' \arrow[r] & \boldsymbol{U}. \end{tikzcd}$$

We pause the proof of Lemma 2.5.5 for a sub-lemma.

Lemma 2.5.6. *We have $\boldsymbol{Y} = \boldsymbol{Y}'$.*

Proof. As in the proof of Lemma 2.5.4, we may characterize $\boldsymbol{Z} \subset \boldsymbol{Z}''$ as the largest closed subscheme along which x'' respects the Hodge filtrations. Equivalently, by Remark 2.3.3, $\boldsymbol{Z} \subset \boldsymbol{Z}''$ is the largest closed subscheme over which the composition

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \xrightarrow{x''\circ\overline{\epsilon}} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \xrightarrow{q} \mathrm{Lie}(A_{\boldsymbol{Z}''})$$

vanishes identically. This implies that $\boldsymbol{Z}' \subset \boldsymbol{Z}''$ is the largest closed subscheme over which

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{(x''\circ\overline{\epsilon})^{\otimes 2}} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \xrightarrow{q^{\otimes 2}} \mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \tag{2.5.5}$$

vanishes identically.

It follows directly from the definitions that $\boldsymbol{Y} = \boldsymbol{Y}' \cap \boldsymbol{Z}'$, and hence it suffices to show that $\boldsymbol{Y}' \subset \boldsymbol{Z}'$. In other words, it suffices to shows that the vanishing of (2.5.4) implies the vanishing of (2.5.5).

For local sections a and b of $H_1(A_{\boldsymbol{Z}''})$, define

$$Q_{a\otimes b} : F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \bigwedge^{n-1} \mathrm{Lie}(A_{\boldsymbol{Z}''}) \to \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})$$

by

$$Q_{a\otimes b}(e_1 \otimes q(e_2) \wedge \cdots \wedge q(e_n)) = \psi(a, e_1) \cdot q(b) \wedge q(e_2) \wedge \cdots \wedge q(e_n).$$

It is clear that $Q_{a\otimes b}$ depends on the images of a and b in $\mathrm{Lie}(A_{\boldsymbol{Z}''})$, and that this construction defines an isomorphism

$$\mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{Q} \underline{\mathrm{Hom}}\Big(F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \bigwedge^{n-1} \mathrm{Lie}(A_{\boldsymbol{Z}''}), \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})\Big). \tag{2.5.6}$$

It is related to the map

$$\mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{P} \underline{\mathrm{Hom}}\Big(\bigwedge^{n} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''})/\overline{\epsilon} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}), \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})\Big)$$

of Lemma 2.4.4 by

$$P_{a\otimes b}(e_1 \wedge \cdots \wedge e_n) = Q_{\overline{\epsilon}a\otimes\overline{\epsilon}b}(e_1 \otimes q(e_2) \wedge \cdots \wedge q(e_n))$$

for any local section $e_1 \otimes e_2 \otimes \cdots \otimes e_n$ of

$$F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \cdots \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}).$$

Putting everything together, if (2.5.4) vanishes, then $P_{x''(a_0)\otimes x''(b_0)} = 0$ for all local sections a_0 and b_0 of $H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})$. Therefore

$$Q_{x''(\overline{\epsilon}a_0)\otimes x''(\overline{\epsilon}b_0)} = 0$$

for all local sections a_0 and b_0, which implies, as (2.5.6) is an isomorphism, that (2.5.5) vanishes. This proves that $\boldsymbol{Y}' \subset \boldsymbol{Z}'$, and hence $\boldsymbol{Y} = \boldsymbol{Y}'$. ☐

Returning to the proof of Lemma 2.5.5, the map (2.5.4), whose vanishing defines $\boldsymbol{Y}' \subset \boldsymbol{Z}''$, factors through a morphism of line bundles

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})/\epsilon H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})/\epsilon H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \to \mathcal{P}_{\mathrm{Pap}}|_{\boldsymbol{Z}''},$$

and hence $\boldsymbol{Y} = \boldsymbol{Y}'$ is defined inside of $\boldsymbol{Z}''$ locally by one equation. In other words, if we denote by $\boldsymbol{\mathcal{I}} \subset \mathcal{O}_{\boldsymbol{U}}$ and $\boldsymbol{\mathcal{J}} \subset \mathcal{O}_{\boldsymbol{U}}$ the ideal sheaves of $\boldsymbol{Z} \subset \boldsymbol{U}$ and $\boldsymbol{Y} \subset \boldsymbol{U}$, respectively, then $\boldsymbol{\mathcal{I}}^3$ is the ideal sheaf of $\boldsymbol{Z}'' \subset \boldsymbol{U}$, and

$$\boldsymbol{\mathcal{J}} = (f) + \boldsymbol{\mathcal{I}}^3$$

for some $f \in \mathcal{O}_{\boldsymbol{U}}$. But $\boldsymbol{Y} \subset \boldsymbol{Z}'$ implies that $\boldsymbol{\mathcal{I}}^2 \subset \boldsymbol{\mathcal{J}}$, and hence $\boldsymbol{\mathcal{I}}^3 \subset \boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{J}}$. It follows that the image of f under the composition

$$\boldsymbol{\mathcal{J}}/\boldsymbol{\mathcal{I}}^3 \to \boldsymbol{\mathcal{J}}/\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{J}}/\mathfrak{m}\boldsymbol{\mathcal{J}}$$

is an $\mathcal{O}_{\boldsymbol{U}}$-module generator, and $\boldsymbol{\mathcal{J}}$ is principal by Nakayama's lemma. ☐

At last we can complete the proof of Theorem 2.5.2. For each connected component $Z \subset \mathcal{Z}_{\mathrm{Pap}}(m)_U$ we have now defined a closed subscheme $Y \subset Z'$. By Lemma 2.5.5 it is an effective Cartier divisor, and summing these Cartier divisors as Z varies over all connected components yields an effective Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)_U$ on U. Letting U vary over an étale cover and applying étale descent defines a effective Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$.

The Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ just defined agrees with the square of $\mathcal{Z}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}$. This is clear from Lemma 2.5.4 and the definition of $\mathcal{Y}_{\mathrm{Pap}}(m)$. The uniqueness claim follows from the normality of $\mathcal{S}_{\mathrm{Pap}}$, exactly as in the proof of Theorem 2.4.3. ☐

2.6. **Pullbacks of Cartier divisors.** After Theorem 2.4.3 we have two line bundles $\boldsymbol{\Omega}_{\mathrm{Kra}}$ and $\boldsymbol{\omega}^2$ on $\mathcal{S}_{\mathrm{Kra}}$, which agree over the complement of the exceptional locus Exc. We wish to pin down more precisely the relation between them.

Similarly, after Theorem 2.5.2 we have Cartier divisors $\mathcal{Y}_{\mathrm{Kra}}(m)$ and $2\mathcal{Z}_{\mathrm{Kra}}(m)$. These agree on the complement of Exc, and again we wish to pin down more precisely the relation between them.

Denote by $\pi_0(\mathrm{Sing})$ the set of connected components of the singular locus $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$. For each $s \in \pi_0(\mathrm{Sing})$ there is a corresponding irreducible effective Cartier divisor

$$\mathrm{Exc}_s = \mathrm{Exc} \times_{\mathcal{S}_{\mathrm{Pap}}} s \hookrightarrow \mathcal{S}_{\mathrm{Kra}}$$

supported in a single characteristic dividing D. These satisfy

$$\mathrm{Exc} = \bigsqcup_{s \in \pi_0(\mathrm{Sing})} \mathrm{Exc}_s.$$

Remark 2.6.1. As Sing is a reduced 0-dimensional stack of finite type over $\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}$, each $s \in \pi_0(\mathrm{Sing})$ can be realized as the stack quotient

$$s \cong G_s \backslash \mathrm{Spec}(\mathbb{F}_s)$$

for a finite field $\mathbb{F}_s$ of characteristic $p \mid D$ acted on by a finite group G_s.

Fix a geometric point $\mathrm{Spec}(\mathbb{F}) \to s$, and set $p = \mathrm{char}(\mathbb{F})$. By mild abuse of notation this geometric point will again be denoted simply by s. It determines a pair

$$(A_{0,s}, A_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathbb{F}), \tag{2.6.1}$$

and hence a positive definite hermitian $\mathcal{O}_{\boldsymbol{k}}$-module

$$L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0,s}, A_s)$$

as in (2.5.1). This hermitian lattice depends only on $s \in \pi_0(\mathrm{Sing})$, not on the choice of geometric point above it.

Proposition 2.6.2. *For each $s \in \pi_0(\mathrm{Sing})$ the abelian varieties A_{0s} and A_s are supersingular, and there is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism of p-divisible groups*

$$A_s[p^\infty] \cong \underbrace{A_{0s}[p^\infty] \times \cdots \times A_{0s}[p^\infty]}_{n\,\text{times}} \tag{2.6.2}$$

identifying the polarization on the left with the product polarization on the right. Moreover, the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module L_s is self-dual of rank n.

Proof. Certainly A_{0s} is supersingular, as p is ramified in $\mathcal{O}_{\boldsymbol{k}} \subset \mathrm{End}(A_{0s})$.

Denote by $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ be the unique prime above p. Let $W = W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$, and let $\mathrm{Fr} \in \mathrm{Aut}(W)$ be the unique continuous lift of the p-power Frobenius on $\mathbb{F}$. Let $\mathbb{D}(W)$ denote the covariant Dieudonné module of A_s, endowed with its operators F and V satisfying $FV = p = VF$. The Dieudonné module is free of rank n over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$, and the short exact sequence

$$0 \to F^0 H_1^{\mathrm{dR}}(A_s) \to H_1^{\mathrm{dR}}(A_s) \to \mathrm{Lie}(A_s) \to 0$$

of $\mathbb{F}$-modules is identified with

$$0 \to V\mathbb{D}(W)/p\mathbb{D}(W) \to \mathbb{D}(W)/p\mathbb{D}(W) \to \mathbb{D}(W)/V\mathbb{D}(W) \to 0.$$

As D is odd, the element $\delta \in \mathcal{O}_{\boldsymbol{k}}$ fixed in §1.7 satisfies $\mathrm{ord}_{\mathfrak{p}}(\delta) = 1$. This implies that

$$\delta \cdot \mathbb{D}(W) = V\mathbb{D}(W).$$

Indeed, by Theorem 2.3.1 the Lie algebra $\mathrm{Lie}(A_s)$ is annihilated by δ, and hence $\delta \cdot \mathbb{D}(W) \subset V\mathbb{D}(W)$. Equality holds as

$$\dim_{\mathbb{F}}\big(\mathbb{D}(W)/\delta \cdot \mathbb{D}(W)\big) = n = \dim_{\mathbb{F}}\big(\mathbb{D}(W)/V\mathbb{D}(W)\big).$$

Denote by $N \subset \mathbb{D}(W)$ the set of fixed points of the Fr-semilinear bijection

$$V^{-1} \circ \delta : \mathbb{D}(W) \to \mathbb{D}(W).$$

It is a free $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-module of rank n endowed with an isomorphism

$$\mathbb{D}(W) \cong N \otimes_{\mathbb{Z}_p} W$$

identifying $V = \delta \otimes \mathrm{Fr}^{-1}$. Moreover, the alternating form ψ on $\mathbb{D}(W)$ induced by the polarization on A_s has the form

$$\psi(n_1 \otimes w_1, n_2 \otimes w_2) = w_1 w_2 \cdot \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\left(\frac{h(n_1, n_2)}{\delta}\right)$$

for a perfect hermitian pairing $h : N \times N \to \mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$. By diagonalizing this hermitian form, we obtain an orthogonal decomposition of N into rank one hermitian $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-modules, and tensoring this decomposition with W yields a decomoposition of $\mathbb{D}(W)$ as a direct sum of principally polarized Dieudonné modules, each of height 2 and slope 1/2. This corresponds to a decomposition (2.6.2) on the level of p-divisible groups.

In particular, A_s is supersingular, and hence is isogenous to n copies of A_{0s}. Using the Noether-Skolem theorem, this isogeny may be chosen to be $\mathcal{O}_{\boldsymbol{k}}$-linear. It follows first that L_s has $\mathcal{O}_{\boldsymbol{k}}$-rank n, and then that the natural map

$$L_s \otimes_{\mathbb{Z}} \mathbb{Z}_q \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}[q^\infty], A_s[q^\infty])$$

is an isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k},q}$-modules for every rational prime q. It is easy to see, using (2.6.2) when $q = p$, that the hermitian module on the right is self-dual, and hence the same is true of each $L_s \otimes_{\mathbb{Z}} \mathbb{Z}_q$. $\square$

The remainder of §2.6 is devoted to proving the following result.

Theorem 2.6.3. *There is an isomorphism*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc})$$

of line bundles on $\mathcal{S}_{\mathrm{Kra}}$, as well as an equality

$$2\mathcal{Z}_{\mathrm{Kra}}(m) = \mathcal{Y}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s$$

of Cartier divisors.

Proof. Recall from the proof of Theorem 2.4.3 the morphism

$$\begin{array}{ccc} \boldsymbol{\omega}^{-2} & & \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1} \\ \| & & \| \\ \mathrm{Lie}(A_0)^{\otimes 2} \otimes (\mathrm{Lie}(A)/\mathcal{F}_A))^{\otimes 2} & \xrightarrow{(2.4.3)} & \mathrm{Lie}(A_0)^{\otimes 2} \otimes \mathcal{P}_{\mathrm{Kra}}, \end{array}$$

whose restriction to $\mathcal{S}_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}$ is an isomorphism. If we view this morphism as a global section

$$\sigma \in H^0(\mathcal{S}_{\mathrm{Kra}}, \boldsymbol{\omega}^2 \otimes \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1}), \tag{2.6.3}$$

then

$$\mathrm{div}(\sigma) = \sum_{s \in \pi_0(\mathrm{Sing})} \ell_s(0) \cdot \mathrm{Exc}_s \tag{2.6.4}$$

for some integers $\ell_s(0) \geqslant 0$, and hence

$$\boldsymbol{\omega}^2 \otimes \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1} \cong \bigotimes_{s \in \pi_0(\mathrm{Sing})} \mathcal{O}(\mathrm{Exc}_s)^{\otimes \ell_s(0)}. \tag{2.6.5}$$

We must show that each $\ell_s(0) = 1$.

Similarly, suppose $m > 0$. It follows from Theorem 2.5.2 that

$$2\mathcal{Z}_{\mathrm{Kra}}(m) = \mathcal{Y}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \ell_s(m) \cdot \mathrm{Exc}_s \tag{2.6.6}$$

for some integers $\ell_s(m)$. Moreover, it is clear from the construction of $\mathcal{Y}_{\mathrm{Kra}}(m)$ that $2\mathcal{Z}_{\mathrm{Kra}}(m) - \mathcal{Y}_{\mathrm{Kra}}(m)$ is effective, and so $\ell_s(m) \geqslant 0$. We must show that

$$\ell_s(m) = \#\{x \in L_s : \langle x, x\rangle = m\}.$$

Fix $s \in \pi_0(\mathrm{Sing})$, and let $\mathrm{Spec}(\mathbb{F}) \to s$, $p = \mathrm{char}(\mathbb{F})$, and $(A_{0s}, A_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathbb{F})$ be as in (2.6.1). Let $W = W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$, and set $\mathcal{W} = \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$. It is a complete discrete valuation ring of absolute ramification degree 2. Fix a uniformizer $\varpi \in \mathcal{W}$. As p is odd, the quotient map

$$\mathcal{W} \to \mathcal{W}/\varpi\mathcal{W} = \mathbb{F}$$

admits canonical divided powers.

Denote by $\mathbb{D}_0$ and $\mathbb{D}$ the Grothendieck-Messing crystals of A_{0s} and A_s, respectively. Evaluation of the crystals[3] along the divided power thickening $\mathcal{W} \to \mathbb{F}$ yields free $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-modules $\mathbb{D}_0(\mathcal{W})$ and $\mathbb{D}(\mathcal{W})$ endowed with alternating $\mathcal{W}$-bilinear forms ψ_0 and ψ, and $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphisms

$$\mathbb{D}_0(\mathcal{W})/\varpi\mathbb{D}_0(\mathcal{W}) \cong \mathbb{D}_0(\mathbb{F}) \cong H_1^{\mathrm{dR}}(A_{0s}).$$

and

$$\mathbb{D}(\mathcal{W})/\varpi\mathbb{D}(\mathcal{W}) \cong \mathbb{D}(\mathbb{F}) \cong H_1^{\mathrm{dR}}(A_s).$$

The W-modules $\mathbb{D}_0(W)$ and $\mathbb{D}(W)$ are canonically identified with the covariant Dieudonnś modules of A_{0s} and A_s, respectively. The operators F and V on these Dieudonné modules induce operators, denoted the same way, on

$$\mathbb{D}_0(\mathcal{W}) \cong \mathbb{D}_0(W) \otimes_W \mathcal{W}, \quad \mathbb{D}(\mathcal{W}) \cong \mathbb{D}(W) \otimes_W \mathcal{W}.$$

For any elements $y_1, \ldots, y_k$ in an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-module, let $\langle y_1, \ldots, y_k\rangle$ be the $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-submodule generated by them. Recall from §1.7 the elements

$$\epsilon, \bar{\epsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}.$$

Lemma 2.6.4. *There is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-basis $e_0 \in \mathbb{D}_0(\mathcal{W})$ such that*

$$F\,\mathbb{D}_0(\mathcal{W}) \stackrel{\mathrm{def}}{=} \langle \bar{\epsilon} e_0\rangle \subset \mathbb{D}_0(\mathcal{W})$$

is a totally isotropic $\mathcal{W}$-module direct summand lifting the Hodge filtration on $\mathbb{D}_0(\mathbb{F})$, and such that $Ve_0 = \delta e_0$.

[3] If $p = 3$, the divided powers on $\mathcal{W} \to \mathbb{F}$ are not nilpotent, and so we cannot evaluate the usual Grothendieck-Messing crystals on this thickening. However, Proposition 2.6.2 implies that the p-divisible groups of A_{0s} and A_s are formal, and Zink's theory of displays [Zin02] can be used as a substitute.

Similarly, there is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$*-basis* $e_1, \ldots, e_n \in \mathbb{D}(\mathcal{W})$ *such that*

$$F\,\mathbb{D}(\mathcal{W}) \stackrel{\text{def}}{=} \langle \epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n \rangle \subset \mathbb{D}(\mathcal{W})$$

is a totally isotropic $\mathcal{W}$*-module direct summand lifting the Hodge filtration on* $\mathbb{D}(\mathbb{F})$*. This basis may be chosen so that so that* $V e_{k+1} = \delta e_k$*, where the indices are understood in* $\mathbb{Z}/n\mathbb{Z}$*, and also so that*

$$\psi\big(\langle e_i \rangle, \langle e_i \rangle\big) = \begin{cases} \mathcal{W} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Proposition 2.6.2, we may identify

$$\mathbb{D}_0(W) \cong N_0 \otimes_{\mathbb{Z}_p} W$$

for some free $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-module N_0 of rank 1, in such a way that $V = \delta \otimes \mathrm{Fr}^{-1}$, and the alternating form on $\mathbb{D}_0(W)$ arises as the W-bilinear extension of an alternating form ψ_0 on N_0. Any $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-generator $e_0 \in N_0$ determines a generator of the $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}} \otimes_{\mathbb{Z}_p} \mathcal{W}$-module

$$\mathbb{D}_0(\mathcal{W}) \cong N_0 \otimes_{\mathbb{Z}_p} \mathcal{W},$$

which, using Remark 2.3.3 has the desired properties.

Now set $N = N_0 \oplus \cdots \oplus N_0$ (n copies), so that, by Proposition 2.6.2, there is an isomorphism

$$\mathbb{D}(W) \cong N \otimes_{\mathbb{Z}_p} W$$

identifying $V = \delta \otimes \mathrm{Fr}^{-1}$, and the alternating bilinear form on $\mathbb{D}(W)$ arises from an alternating form ψ on N. Let $\mathbb{Z}_{p^n} \subset W$ be the ring of integers in the unique unramified degree n extension of $\mathbb{Q}_p$, and fix an action

$$\iota : \mathbb{Z}_{p^n} \to \mathrm{End}_{\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}}(N)$$

in such a way that $\psi(\iota(\alpha)x, y) = \psi(x, \iota(\alpha)y)$ for all $\alpha \in \mathbb{Z}_{p^n}$.

There is an induced decomposition

$$\mathbb{D}(W) \cong \bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} \mathbb{D}(W)_k,$$

where

$$\mathbb{D}(W)_k = \{e \in \mathbb{D}(W) : \forall \alpha \in \mathbb{Z}_{p^n},\ \iota(\alpha) \cdot e = \mathrm{Fr}^k(\alpha) \cdot e\}$$

is free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$. Now pick any $\mathbb{Z}_{p^n}$-module generator $e \in N$, view it as an element of $\mathbb{D}(W)$, and let $e_k \in \mathbb{D}(W)_k$ be its projection to the k^{th} summand. This gives an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$-basis $e_1, \ldots, e_n \in \mathbb{D}(W)$, which determines an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-basis of $\mathbb{D}(\mathcal{W})$ with the required properties. ∎

By the Serre-Tate theorem and Grothendieck-Messing theory, the lifts of the Hodge filtrations specified in Lemma 2.6.4 determine a lift

$$(\tilde{A}_{0s}, \tilde{A}_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathcal{W}) \tag{2.6.7}$$

of the pair (A_{0s}, A_s). These come with canonical identifications

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s}) \cong \mathbb{D}_0(\mathcal{W}), \quad H_1^{\mathrm{dR}}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})$$

under which the Hodge filtrations correspond to the filtrations chosen in Lemma 2.6.4. In particular, the Lie algebra of $\tilde{A}_s$ is

$$\mathrm{Lie}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/F\,\mathbb{D}(\mathcal{W}) = \langle e_1, e_2, \ldots, e_n\rangle / \langle \epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n\rangle.$$

The $\mathcal{W}$-module direct summand

$$\mathcal{F}_{\tilde{A}_s} = \langle e_2, \ldots, e_n\rangle / \langle \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n\rangle$$

satisfies Krämer's condition (§2.3), and so determines a lift of (2.6.7) to

$$(\tilde{A}_{0s}, \tilde{A}_s) \in \mathcal{S}_{\mathrm{Kra}}(\mathcal{W}).$$

To summarize: starting from a geometric point $\mathrm{Spec}(\mathbb{F}) \to s$, we have used Lemma 2.6.4 to construct a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{F}) \arrow[r] \arrow[d] & \mathrm{Exc}_s \arrow[r] \arrow[d] & s \arrow[d] \\ \mathrm{Spec}(\mathcal{W}) \arrow[r] & \mathcal{S}_{\mathrm{Kra}} \arrow[r] & \mathcal{S}_{\mathrm{Pap}}. \end{tikzcd} \tag{2.6.8}$$

Lemma 2.6.5. *The pullback of the map (2.4.3) via* $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ *vanishes identically along the closed subscheme* $\mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$*, but not along* $\mathrm{Spec}(\mathcal{W}/\varpi^2\mathcal{W})$*.*

Proof. The $\mathcal{W}$-submodule of

$$\mathrm{Lie}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/\langle \epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n\rangle \tag{2.6.9}$$

generated by e_1 is $\mathcal{O}_{\boldsymbol{k}}$-stable. The action of $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$ on this $\mathcal{W}$-line is via

$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W} \xrightarrow{\alpha\otimes x \mapsto i_{\mathcal{W}}(\overline{\alpha})x} \mathcal{W}$$

(where $i_{\mathcal{W}} : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{W}$ is the inclusion), and this map sends $\overline{\epsilon}$ to a uniformizer of $\mathcal{W}$; see §1.7. Thus the quotient map $q : \mathbb{D}(\mathcal{W}) \to \mathrm{Lie}(\tilde{A}_s)$ satisfies $q(\overline{\epsilon} e_1) = \varpi q(e_1)$ up to multiplication by an element of $\mathcal{W}^\times$. It follows that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \varpi \cdot \psi(\overline{\epsilon} e_1, e_1) \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n)$$

up to scaling by $\mathcal{W}^\times$.

We claim that $\psi(\overline{\epsilon} e_1, e_1) \in \mathcal{W}^\times$. Indeed, as $q(e_1)$ generates a $\mathcal{W}$-module direct summand of (2.6.9), there is some

$$x \in F\,\mathbb{D}(\mathcal{W}) = \langle \epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n\rangle \subset \mathbb{D}(\mathcal{W})$$

such that $\psi(x, e_1) \in \mathcal{W}^\times$. We chose our basis in Lemma 2.6.4 in such a way that $\psi(\overline{\epsilon} e_i, e_1) = 0$ for $i > 1$. It follows that $\psi(\epsilon e_1, e_1)$ is a unit, and hence the same is true of $\psi(\overline{\epsilon} e_1, e_1) = \psi(e_1, \epsilon e_1) = -\psi(\epsilon e_1, e_1)$.

We have now proved that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \varpi \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n)$$

up to scaling by $\mathcal{W}^\times$, from which it follows that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) \in \bigwedge\nolimits^n \mathrm{Lie}(\tilde{A}_s)$$

is divisible by ϖ, but not by ϖ^2.

The quotient

$$H_1^{\mathrm{dR}}(\tilde{A}_s)/\bar{\epsilon}H_1^{\mathrm{dR}}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/\langle \bar{\epsilon}e_1, \ldots, \bar{\epsilon}e_n\rangle$$

is generated as a $\mathcal{W}$-module by $e_1, \ldots, e_n$. From the calculation of the previous paragraph, it now follows that $P_{e_1 \otimes e_1} \in \mathcal{P}_{\mathrm{Kra}}|_{\mathrm{Spec}(\mathcal{W})}$ is divisible by ϖ but not by ϖ^2. The quotient

$$\mathrm{Lie}(\tilde{A}_s)/\mathcal{F}_{\tilde{A}_s} \cong \mathbb{D}(\mathcal{W})/\langle \epsilon e_1, e_2, \ldots, e_n\rangle$$

is generated as a $\mathcal{W}$-module by the image of e_1, and we at last deduce that

$$P \in \underline{\mathrm{Hom}}\big((\mathrm{Lie}(A)/\mathcal{F}_A)^{\otimes 2}, \mathcal{P}_{\mathrm{Kra}}\big)|_{\mathrm{Spec}(\mathcal{W})}$$

is divisible by ϖ but not by ϖ^2. □

Recall the global section σ of (2.6.3). It follows immediately from Lemma 2.6.5 that its pullback via $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ has divisor $\mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$, and hence

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{div}(\sigma) = \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}),$$

Comparison with (2.6.4) proves both that $\ell_s(0) = 1$, and that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Exc}_s = \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}). \tag{2.6.10}$$

Recalling (2.6.5), this completes the proof that

$$\boldsymbol{\omega}^2 \otimes \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

It remains to prove the second claim of Theorem 2.6.3. Given any $x \in L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}, A_s)$, denote by $k(x)$ the largest integer such that x lifts to a morphism

$$\tilde{A}_{0s} \otimes_{\mathcal{W}} \mathcal{W}/(\varpi^{k(x)}) \to \tilde{A}_s \otimes_{\mathcal{W}} \mathcal{W}/(\varpi^{k(x)}).$$

Lemma 2.6.6. *As Cartier divisors on* $\mathrm{Spec}(\mathcal{W})$*, we have*

$$\mathcal{Z}_{\mathrm{Kra}}(m) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Spec}(\mathcal{W}) = \sum_{\substack{x \in L_s \\ \langle x, x\rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi^{k(x)}\mathcal{W}).$$

Proof. Each $x \in L_s$ with $\langle x, x\rangle = m$ determines a geometric point

$$(A_{0z}, A_z, x) \in \mathcal{Z}_{\mathrm{Kra}}(m)(\mathbb{F}). \tag{2.6.11}$$

and surjective morphisms

$$\begin{tikzcd}
 & \mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x} \arrow[dl] \arrow[dr] & \\
\mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x} & & \mathcal{W},
\end{tikzcd}$$

where $\mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x}$ is the étale local ring at (2.6.11), $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}$ is the étale local ring at the point below it, and the arrow on the right is induced by the map $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ of (2.6.8). There is an induced isomorphism of $\mathcal{W}$-schemes

$$\mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x} \otimes_{\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}} \mathcal{W} \cong \mathcal{W}/(\varpi^{k(x)}),$$

and the claim follows by summing over x. $\square$

Lemma 2.6.7. *As Cartier divisors on* $\mathrm{Spec}(\mathcal{W})$, *we have*

$$\mathcal{Y}_{\mathrm{Kra}}(m) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Spec}(\mathcal{W}) = \sum_{\substack{x \in L_s \\ \langle x,x\rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi^{2k(x)-1}\mathcal{W}).$$

Proof. Each $x \in L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}, A_s)$ with $\langle x, x\rangle = m$ induces a morphism of crystals $\mathbb{D}_0 \to \mathbb{D}$, and hence a map

$$\mathbb{D}_0(\mathcal{W}) \xrightarrow{x} \mathbb{D}(\mathcal{W})$$

respecting the F and V operators. By Grothendieck-Messing deformation theory, the integer $k(x)$ is characterized as the largest integer such that the composition

$$\begin{array}{ccccccc} F^0 H_1^{\mathrm{dR}}(\tilde{A}_{0s}) & \xrightarrow{\subset} & H_1^{\mathrm{dR}}(\tilde{A}_{0s}) & \xrightarrow{x} & H_1^{\mathrm{dR}}(\tilde{A}_s) & \xrightarrow{q} & \mathrm{Lie}(\tilde{A}_s) \\ \| & & \| & & \| & & \| \\ \overline{\epsilon}\mathbb{D}_0(\mathcal{W}) & \xrightarrow{\subset} & \mathbb{D}_0(\mathcal{W}) & \xrightarrow{x} & \mathbb{D}(\mathcal{W}) & \longrightarrow & \frac{\mathbb{D}(\mathcal{W})}{\langle \epsilon e_1, \overline{\epsilon} e_2, \ldots, \overline{\epsilon} e_n\rangle}. \end{array}$$

vanishes modulo $\varpi^{k(x)}$. In other words the composition

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s}) \xrightarrow{x \circ \overline{\epsilon}} H_1^{\mathrm{dR}}(\tilde{A}_s) \xrightarrow{q} \mathrm{Lie}(\tilde{A}_s)$$

vanishes modulo $\varpi^{k(x)}$, but not modulo $\varpi^{k(x)+1}$.

Using the bases of Lemma 2.6.4, we expand

$$x(e_0) = a_1 e_1 + \cdots + a_n e_n$$

with $a_1, \ldots, a_n \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$. The condition that x respects V implies that $a_1 = \cdots = a_n$. Let us call this common value a, so that

$$q(x(\overline{\epsilon} e_0)) = \overline{\epsilon} \cdot q(a e_1 + \cdots + a e_n) = a\overline{\epsilon} \cdot q(e_1)$$

in $\mathrm{Lie}(\tilde{A}_s)$. By the previous paragraph, this element is divisible by $\varpi^{k(x)}$ but not by $\varpi^{k(x)+1}$, and so

$$q(a\overline{\epsilon} e_1) = \varpi^{k(x)} q(e_1) \tag{2.6.12}$$

up to scaling by $\mathcal{W}^\times$.

On the other hand, the submodule of $\mathrm{Lie}(\tilde{A}_s)$ generated by $q(e_1)$ is isomorphic to $(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \epsilon \rangle \cong \mathcal{W}$, and $\overline{\epsilon}$ acts on this quotient by a uniformizer in $\mathcal{W}$. Thus

$$\overline{\epsilon} q(e_1) = \varpi q(e_1) \tag{2.6.13}$$

up to scaling by $\mathcal{W}^\times$.

Combining (2.6.12) and (2.6.13) shows that, up to scaling by $\mathcal{W}^\times$,

$$a\overline{\epsilon} = \varpi^{k(x)-1}\overline{\epsilon}$$

in the quotient $(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \epsilon \rangle$. By the injectivity of the quotient map $\langle \overline{\epsilon} \rangle \to (\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \epsilon \rangle$, this same equality holds in $\langle \overline{\epsilon} \rangle \subset \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$. Using this and (2.6.12), we compute

$$\begin{aligned} & P_{x(e_0)\otimes x(e_0)}(e_1 \wedge \cdots \wedge e_n) \\ &= \psi(a\overline{\epsilon}e_1, e_1) \cdot q(a\overline{\epsilon}e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n) \\ &= \varpi^{2k(x)-1} \cdot \psi(\overline{\epsilon}e_1, e_1) \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n) \\ &= \varpi^{2k(x)-1} \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n) \end{aligned}$$

up to scaling by $\mathcal{W}^\times$. Here, as in the proof of Lemma 2.6.5, we have used $\psi(\overline{\epsilon}e_1, e_1) \in \mathcal{W}^\times$.

This calculation shows that the composition

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s})^{\otimes 2} \xrightarrow{x \otimes x} H_1^{\mathrm{dR}}(\tilde{A}_s)^{\otimes 2} \xrightarrow{P} \mathcal{P}|_{\mathrm{Spec}(\mathcal{W})}$$

vanishes modulo $\varpi^{2k(x)-1}$, but not modulo $\varpi^{2k(x)}$, and the remainder of the proof is the same as that of Lemma 2.6.6: Comparing with the definition of $\mathcal{Y}_{\mathrm{Kra}}(m)$, see especially (2.5.3), shows that

$$\mathcal{O}_{\mathcal{Y}_{\mathrm{Kra}}(m),x} \otimes_{\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}} \mathcal{W} \cong \mathcal{W}/(\varpi^{2k(x)-1}),$$

and summing over all x proves the claim. □

Combining Lemmas 2.6.6 and 2.6.7 shows that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \big(2\mathcal{Z}_{\mathrm{Kra}}(m) - \mathcal{Y}_{\mathrm{Kra}}(m)\big) = \sum_{\substack{x \in L_s \\ \langle x,x \rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$$

as Cartier divisors on $\mathrm{Spec}(\mathcal{W})$. We know from (2.6.10) that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Exc}_t = \begin{cases} \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}) & \text{if } t = s \\ 0 & \text{if } t \neq s, \end{cases}$$

and comparison with (2.6.6) shows that

$$\ell_s(m) = \#\{x \in L_s : \langle x, x \rangle = m\},$$

completing the proof of Theorem 2.6.3. □

3. Toroidal compactification

In this section we describe canonical toroidal compactifications

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{Kra}} & \longrightarrow & \mathcal{S}^*_{\mathrm{Kra}} \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Pap}} & \longrightarrow & \mathcal{S}^*_{\mathrm{Pap}}, \end{array}$$

and the structure of their formal completions along the boundary. Using this description, we define Fourier-Jacobi expansions of modular forms.

The existence of toroidal compactifications with reasonable properties is not a new result. In fact the proof of Theorem 3.7.1, which asserts the existence of good compactifications of $\mathcal{S}_{\mathrm{Pap}}$ and $\mathcal{S}_{\mathrm{Kra}}$, simply refers to [How15]. Of course [*loc. cit.*] is itself a very modest addition to the established literature [FC90, Lan13, Lar92, Rap78]. Because of this, the reader is perhaps owed a few words of explanation as to why §3 is so long.

It is well-known that the boundary charts used to construct toroidal compactifications of PEL-type Shimura varieties are themselves moduli spaces of 1-motives (or, what is nearly the same thing, degeneration data in the sense of [FC90]). This moduli interpretation is explained in §3.3.

It is a special feature of our particular Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ that the boundary charts have a second, very different, moduli interpretation. This second moduli interpretation is explained in §3.4. In some sense, the main result of §3 is not Theorem 3.7.1 at all, but rather Proposition 3.4.3, which proves the equivalence of the two moduli problems.

The point is that our goal is to eventually study the integrality and rationality properties of Fourier-Jacobi expansions of Borcherds products on the integral models of $\mathrm{Sh}(G, \mathcal{D})$. A complex analytic description of these Fourier-Jacobi expansions can be deduced from [Kud16], but it is not a priori clear how to deduce integrality and rationality properties from these purely complex analytic formulas.

To do so, we will exploit the fact that the formulas of [Kud16] express the Fourier-Jacobi coefficients in terms of the classical Jacobi theta function. The Jacobi theta function can be viewed as a section of a line bundle on the universal elliptic curve fibered over the modular curve, and when interpreted in this way it has known integrality and rationality properties (this is explained in §5.1).

By converting the moduli interpretation of the boundary charts from 1-motives to an interpretation that makes explicit reference to the universal elliptic curve and the line bundles that live over it, the integrality and rationality properties of the Fourier-Jacobi coefficients can be deduced, ultimately, from those of the classical Jacobi theta function.

3.1. **Cusp label representatives.**

The group G acts on both W_0 and W. If $J \subset W$ is an isotropic $\boldsymbol{k}$-line, its stabilizer $P = \mathrm{Stab}_G(J)$ is a parabolic subgroup of G. This establishes a bijection between isotropic $\boldsymbol{k}$-lines in W and proper parabolic subgroups of G.

Definition 3.1.1. A *cusp label representative* for $(G, \mathcal{D})$ is a pair $\Phi = (P, g)$ in which $g \in G(\mathbb{A}_f)$ and $P \subset G$ is a parabolic subgroup. If $P = \mathrm{Stab}_G(J)$ for an isotropic $\boldsymbol{k}$-line $J \subset W$, we call Φ a *proper cusp label representative*. If $P = G$ we call Φ an *improper cusp label representative*.

For each cusp label representative $\Phi = (P, g)$ there is a distinguished normal subgroup $Q_\Phi \lhd P$. If $P = G$ we simply take $Q_\Phi = G$. If $P = \mathrm{Stab}_G(J)$ for an isotropic $\boldsymbol{k}$-line $J \subset W$ then, following the recipe of [Pin89,

§4.7], we define Q_Φ as the fiber product

$$\begin{tikzcd} Q_\Phi \arrow[r, "\nu_\Phi"] \arrow[d] & \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \arrow[d, "a\mapsto(a,\mathrm{Nm}(a),a,\mathrm{id})"] \\ P \arrow[r] & \mathrm{GU}(W_0)\times\mathrm{GL}(J)\times\mathrm{GU}(J^\perp/J)\times\mathrm{GL}(W/J^\perp). \end{tikzcd} \tag{3.1.1}$$

The morphism $G \to \mathrm{GU}(W)$ restricts to an injection $Q_\Phi \hookrightarrow \mathrm{GU}(W)$, as the action of Q_Φ on $J^\perp/J$ determines its action on W_0.

Let $K \subset G(\mathbb{A}_f)$ be the compact open subgroup (2.1.3). Any cusp label representative $\Phi = (P, g)$ determines compact open subgroups

$$K_\Phi = gKg^{-1} \cap Q_\Phi(\mathbb{A}_f), \quad \tilde{K}_\Phi = gKg^{-1} \cap P(\mathbb{A}_f),$$

and a finite group

$$\Delta_\Phi = \big(P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi\big)/Q_\Phi(\mathbb{Q}). \tag{3.1.2}$$

Definition 3.1.2. Two cusp label representatives $\Phi = (P, g)$ and $\Phi' = (P', g')$ are *K-equivalent* if there exist $\gamma \in G(\mathbb{Q})$, $h \in Q_\Phi(\mathbb{A}_f)$, and $k \in K$ such that

$$(P', g') = (\gamma P\gamma^{-1}, \gamma hgk).$$

One may easily verify that this is an equivalence relation. Obviously, there is a unique K-equivalence class of improper cusp label representatives.

From now through §3.6, we fix a proper cusp label representative $\Phi = (P, g)$, with $P \subset G$ the stabilizer of an isotropic $\boldsymbol{k}$-line $J \subset W$. There is an induced weight filtration $\mathrm{wt}_i W \subset W$ defined by

$$\begin{array}{ccccccc} 0 & \subset & J & \subset & J^\perp & \subset & W \\ \| & & \| & & \| & & \| \\ \mathrm{wt}_{-3}W & \subset & \mathrm{wt}_{-2}W & \subset & \mathrm{wt}_{-1}W & \subset & \mathrm{wt}_0 W, \end{array}$$

and an induced weight filtration on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ defined by

$$\begin{array}{ccccccc} \mathrm{Hom}_{\boldsymbol{k}}(W_0,0) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0,J) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0,J^\perp) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0,W) \\ \| & & \| & & \| & & \| \\ \mathrm{wt}_{-2}V & \subset & \mathrm{wt}_{-1}V & \subset & \mathrm{wt}_0 V & \subset & \mathrm{wt}_1 V, \end{array}$$

It is easy to see that $\mathrm{wt}_{-1}V$ is an isotropic $\boldsymbol{k}$-line, whose orthogonal with respect to (2.1.5) is $\mathrm{wt}_0 V$. Denote by $\mathrm{gr}_i W = \mathrm{wt}_i W/\mathrm{wt}_{i-1}W$ the graded pieces, and similarly for V.

The $\mathcal{O}_{\boldsymbol{k}}$-lattice $g\mathfrak{a} \subset W$ determines an $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$\mathrm{gr}_i(g\mathfrak{a}) = \big(g\mathfrak{a} \cap \mathrm{wt}_i W\big)/\big(g\mathfrak{a} \cap \mathrm{wt}_{i-1}W\big) \subset \mathrm{gr}_i W.$$

The middle graded piece $\mathrm{gr}_{-1}(g\mathfrak{a})$ is endowed with a positive definite self-dual hermitian form, inherited from the self-dual hermitian form on $g\mathfrak{a}$ appearing in the proof of Proposition 2.1.1. The outer graded pieces

$$\mathfrak{m} = \mathrm{gr}_{-2}(g\mathfrak{a}), \quad \mathfrak{n} = \mathrm{gr}_0(g\mathfrak{a}) \tag{3.1.3}$$

are projective rank one $\mathcal{O}_{\boldsymbol{k}}$-modules[4], endowed with a perfect $\mathbb{Z}$-bilinear pairing $\mathfrak{m} \times \mathfrak{n} \to \mathbb{Z}$ inherited by the perfect symplectic form on $g\mathfrak{a}$ appearing in the proof of Proposition 2.2.1.

Remark 3.1.3. The isometry class of $g\mathfrak{a}$ as a hermitian lattice is determined by the isomorphism classes of $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathcal{O}_{\boldsymbol{k}}$-modules, and the isometry class of $\mathrm{gr}_{-1}(g\mathfrak{a})$ as a hermitian lattice. This follows from the proof of [How15, Proposition 2.6.3], which shows that one can find a splitting [5]

$$g\mathfrak{a} \cong \mathrm{gr}_{-2}(g\mathfrak{a}) \oplus \mathrm{gr}_{-1}(g\mathfrak{a}) \oplus \mathrm{gr}_0(g\mathfrak{a}),$$

in such a way that the outer summands are totally isotropic, and each is orthogonal to the middle summand.

Exactly as in (2.1.4), there is a $\boldsymbol{k}$-conjugate linear isomorphism

$$\mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-1}W) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0).$$

If we define

$$\begin{aligned} L_0 &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-1}(g\mathfrak{a})) \\ \Lambda_0 &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_{-1}(g\mathfrak{a}), g\mathfrak{a}_0), \end{aligned} \tag{3.1.4}$$

then $x \mapsto x^\vee$ restricts to an $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism $L_0 \cong \Lambda_0$. We endow L_0 with the positive definite hermitian form

$$\langle x_1, x_2 \rangle = x_1^\vee \circ x_2 \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}) \cong \mathcal{O}_{\boldsymbol{k}}$$

analogous to (2.1.5), and endow Λ_0 with the "dual" hermitian form

$$\langle x_2^\vee, x_1^\vee \rangle = \langle x_1, x_2 \rangle.$$

Lemma 3.1.4. *Two proper cusp label representatives Φ and Φ' are K-equivalent if and only if $\Lambda_0 \cong \Lambda_0'$ as hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules, and $\mathfrak{n} \cong \mathfrak{n}'$ as $\mathcal{O}_{\boldsymbol{k}}$-modules. Moreover, the finite group (3.1.2) satisfies*

$$\Delta_\Phi \cong \mathrm{U}(\Lambda_0) \times \mathrm{GL}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}). \tag{3.1.5}$$

Proof. The first claim is an elementary exercise, left to the reader. For the second claim we only define the isomorphism (3.1.5), and again leave the details to the reader. The group $P(\mathbb{Q})$ acts on both W_0 and W, preserving their weight filtrations, and so acts on both the hermitian space $\mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0)$ and the $\boldsymbol{k}$-vector space $\mathrm{gr}_0 W$. The subgroup $P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$ preserves the lattices

$$\Lambda_0 \subset \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0)$$

[4] In fact, it is not difficult to see that $\mathfrak{m} \cong \mathfrak{n}$ as $\mathcal{O}_{\boldsymbol{k}}$-modules, but identifying them can only lead to confusion.

[5] This uses our standing assumption that $\boldsymbol{k}$ has odd discriminant.

and $\mathfrak{n} \subset \mathrm{gr}_0 W$, inducing (3.1.5). ☐

3.2. **Mixed Shimura varieties.** The subgroup $Q_\Phi(\mathbb{R}) \subset G(\mathbb{R})$ acts on

$$\mathcal{D}_\Phi(W) = \{\boldsymbol{k}\text{-stable } \mathbb{R}\text{-planes } y \subset W(\mathbb{R}) : W(\mathbb{R}) = J^\perp(\mathbb{R}) \oplus y\},$$

and so also acts on

$$\mathcal{D}_\Phi = \mathcal{D}(W_0) \times \mathcal{D}_\Phi(W).$$

The hermitian domain of (2.1.2) satisifies $\mathcal{D}(W) \subset \mathcal{D}_\Phi(W)$, and hence there is a canonical $Q_\Phi(\mathbb{R})$-equivariant inclusion $\mathcal{D} \subset \mathcal{D}_\Phi$.

The mixed Shimura variety

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q}) \backslash \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi \tag{3.2.1}$$

admits a canonical model $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$ over $\boldsymbol{k}$ by the general results of [Pin89]. By rewriting the double quotient as

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \cong Q_\Phi(\mathbb{Q}) \backslash \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi/\tilde{K}_\Phi,$$

we see that (3.2.1) admits an action of the finite group Δ_Φ of (3.1.2), induced by the action of $P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$ on both factors of $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$. This action descends to an action on the canonical model.

Proposition 3.2.1. *The morphism ν_Φ of (3.1.1) induces a surjection*

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \xrightarrow{(z,h) \mapsto \nu_\Phi(h)} \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$$

with connected fibers. This map is Δ_Φ-equivariant, where Δ_Φ acts trivially on the target. In particular, the number of connected components of (3.2.1) is equal to the class number of $\boldsymbol{k}$, and the same is true of its orbifold quotient by the action of Δ_Φ.

Proof. The space $\mathcal{D}_\Phi$ is connected, and the kernel of $\nu_\Phi : Q_\Phi \to \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}} \mathbb{G}_m$ is unipotent (so satisfies strong approximation). Therefore

$$\pi_0\big(\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})\big) \cong Q_\Phi(\mathbb{Q}) \backslash Q_\Phi(\mathbb{A}_f)/K_\Phi \cong \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \nu_\Phi(K_\Phi),$$

and an easy calculation shows that $\nu_\Phi(K_\Phi) = \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$. ☐

It will be useful to have other interpretations of $\mathcal{D}_\Phi$.

Remark 3.2.2. Any point $y \in \mathcal{D}_\Phi(W)$ determines a mixed Hodge structure on W whose weight filtration $\mathrm{wt}_i W \subset W$ was define above, and whose Hodge filtration is defined exactly as in Remark 2.1.2. As in [PS08, p. 64] or [Pin89, Proposition 1.2] there is an induced bigrading $W(\mathbb{C}) = \bigoplus W^{(p,q)}$, and this bigrading is induced by a morphism $\mathbb{S}_\mathbb{C} \to \mathrm{GU}(W)_\mathbb{C}$ taking values in the stabilizer of $J(\mathbb{C})$. The product of this morphism with the morphism $\mathbb{S}_\mathbb{C} \to \mathrm{GU}(W_0)_\mathbb{C}$ of Remark 2.1.2 defines a map $z : \mathbb{S}_\mathbb{C} \to Q_{\Phi\mathbb{C}}$, and this realizes $\mathcal{D}_\Phi \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$.

Remark 3.2.3. Imitating the construction of Remark 2.1.3 identifies

$$\mathcal{D}_\Phi \cong \big\{ w \in \epsilon V(\mathbb{C}) : V(\mathbb{C}) = \mathrm{wt}_0 V(\mathbb{C}) \oplus \mathbb{C}w \oplus \mathbb{C}\overline{w} \big\} / \mathbb{C}^\times \subset \mathbb{P}(\epsilon V(\mathbb{C}))$$

as an open subset of projective space.

3.3. **The first moduli interpretation.** Starting from the pair $(\Lambda_0, \mathfrak{n})$, we now construct a smooth integral model of (3.2.1). Following the general recipes of the theory of arithmetic toroidal compactifications, as in [FC90, How15, Madb, Lan13], this integral model will be defined as the top layer of a tower of morphisms

$$\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}),$$

smooth of relative dimensions 1, $n-2$, and 0, respectively.

Recall from §2.3 the smooth $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,0)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

of relative dimension 0 parametrizing certain pairs (A_0, B) of polarized abelian schemes over S with $\mathcal{O}_{\boldsymbol{k}}$-actions. The étale sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$ on S is locally constant; this is a consequence of [BHY15, Theorem 5.1].

Define $\mathcal{A}_\Phi$ as the moduli space of triples (A_0, B, ϱ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which (A_0, B) is an S-point of $\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,0)}$, and

$$\varrho : \underline{\Lambda}_0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$$

is an isomorphism of étale sheaves of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules.

Define $\mathcal{B}_\Phi$ as the moduli space of quadruples (A_0, B, ϱ, c) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which (A_0, B, ϱ) is an S-point of $\mathcal{A}_\Phi$, and $c : \mathfrak{n} \to B$ is an $\mathcal{O}_{\boldsymbol{k}}$-linear homomorphism of group schemes over S. In other words, if (A_0, B, ϱ) is the universal object over $\mathcal{A}_\Phi$, then

$$\mathcal{B}_\Phi = \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B).$$

Suppose we fix $\mu, \nu \in \mathfrak{n}$. For any scheme U and any morphism $U \to \mathcal{B}_\Phi$, there is a corresponding quadruple (A_0, B, ϱ, c) over U. Evaluating the morphism of U-group schemes $c : \mathfrak{n} \to B$ at μ and ν determines U-points $c(\mu), c(\nu) \in B(U)$, and hence determines a morphism of $\mathcal{B}_\Phi$-schemes

$$U \xrightarrow{c(\mu)\times c(\nu)} B \times B \cong B \times B^\vee.$$

Denote by $\mathcal{L}(\mu,\nu)_U$ the pullback of the Poincaré bundle via this morphism.

As U varies, these line bundles are obtained as the pullback of a single line bundle $\mathcal{L}(\mu,\nu)$ on $\mathcal{B}_\Phi$, which depends, up to canonical isomorphism, only on the image of $\mu \otimes \nu$ in

$$\mathrm{Sym}_\Phi = \mathrm{Sym}^2_{\mathbb{Z}}(\mathfrak{n}) / \big\langle (x\mu) \otimes \nu - \mu \otimes (\overline{x}\nu) : x \in \mathcal{O}_{\boldsymbol{k}},\ \mu, \nu \in \mathfrak{n} \big\rangle.$$

Thus we may associate to every $\chi \in \mathrm{Sym}_\Phi$ a line bundle $\mathcal{L}(\chi)$ on $\mathcal{B}_\Phi$, and there are canonical isomorphisms

$$\mathcal{L}(\chi) \otimes \mathcal{L}(\chi') \cong \mathcal{L}(\chi + \chi').$$

Our assumption that D is odd implies that Sym_Φ is a free $\mathbb{Z}$-module of rank one. Moreover, there is positive cone in $\mathrm{Sym}_\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ uniquely determined by the condition $\mu \otimes \mu \geqslant 0$ for all $\mu \in \mathfrak{n}$. Thus all of the line bundles $\mathcal{L}(\chi)$ are powers of the distinguished line bundle

$$\mathcal{L}_\Phi = \mathcal{L}(\chi_0) \tag{3.3.1}$$

determined by the unique positive generator $\chi_0 \in \mathrm{Sym}_\Phi$.

At last, define $\mathcal{B}_\Phi$-stacks

$$\mathcal{C}_\Phi = \underline{\mathrm{Iso}}(\mathcal{L}_\Phi, \mathcal{O}_{\mathcal{B}_\Phi}), \quad \mathcal{C}_\Phi^* = \underline{\mathrm{Hom}}(\mathcal{L}_\Phi, \mathcal{O}_{\mathcal{B}_\Phi}).$$

In other words, $\mathcal{C}_\Phi^*$ is the total space of the line bundle $\mathcal{L}_\Phi^{-1}$, and $\mathcal{C}_\Phi$ is the complement of the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$. In slightly fancier language,

$$\mathcal{C}_\Phi = \underline{\mathrm{Spec}}_{\mathcal{B}_\Phi}\Big(\bigoplus_{\ell\in\mathbb{Z}} \mathcal{L}_\Phi^\ell\Big), \quad \mathcal{C}_\Phi^* = \underline{\mathrm{Spec}}_{\mathcal{B}_\Phi}\Big(\bigoplus_{\ell\geqslant 0} \mathcal{L}_\Phi^\ell\Big),$$

and the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$ is defined by the ideal sheaf $\bigoplus_{\ell>0} \mathcal{L}_\Phi^\ell$.

Remark 3.3.1. Using the isomorphism of Lemma 3.1.4, the group Δ_Φ acts on $\mathcal{B}_\Phi$ via

$$(u,t) \bullet (A_0, B, \varrho, c) = (A_0, B, \varrho \circ u^{-1}, c \circ t^{-1}),$$

for $(u,t) \in \mathrm{U}(\Lambda_0) \times \mathrm{GL}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n})$. The line bundle $\mathcal{L}_\Phi$ is invariant under Δ_Φ, and hence the action of Δ_Φ lifts to both $\mathcal{C}_\Phi$ and $\mathcal{C}_\Phi^*$.

Proposition 3.3.2. *There is a* Δ_Φ*-equivariant isomorphism*

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi) \cong \mathcal{C}_{\Phi/\boldsymbol{k}}.$$

Proof. This is a special case of the general fact that mixed Shimura varieties appearing at the boundary of PEL Shimura varieties are themselves moduli spaces of 1-motives endowed with polarizations, endomorphisms, and level structure. The core of this is Deligne's theorem [Del74, §10] that the category of 1-motives over $\mathbb{C}$ is equivalent to the category of integral mixed Hodge structures of types $(-1,-1)$, $(-1,0)$, $(0,-1)$, $(0,0)$. See [Madb], where this is explained for Siegel modular varieties, and also [Bry83]. A good introduction to 1-motives is [ABV05].

To make this a bit more explicit in our case, denote by $\mathcal{X}_\Phi$ the $\mathcal{O}_{\boldsymbol{k}}$-stack whose functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid $\mathcal{X}_\Phi(S)$ of principally polarized 1-motives A consisting of diagrams

$$\begin{tikzcd} & & \mathfrak{n} \arrow[d] & & \\ 0 \arrow[r] & \mathfrak{m}\otimes_{\mathbb{Z}} \mathbb{G}_m \arrow[r] & \mathbb{B} \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

in which $B \in \mathcal{M}_{(n-2,0)}(S)$, $\mathbb{B}$ is an extension of B by the rank two torus $\mathfrak{m}\otimes_{\mathbb{Z}} \mathbb{G}_m$ in the category of group schemes with $\mathcal{O}_{\boldsymbol{k}}$-action, and the arrows are morphisms of fppf sheaves of $\mathcal{O}_{\boldsymbol{k}}$-modules.

To explain what it means to have a principal polarization of such a 1-motive A, set $\mathfrak{m}^\vee = \mathrm{Hom}(\mathfrak{m},\mathbb{Z})$ and $\mathfrak{n}^\vee = \mathrm{Hom}(\mathfrak{n},\mathbb{Z})$, and recall from [Del74, §10] that A has a dual 1-motive $A^\vee$ consisting of a diagram

$$\begin{tikzcd} & & \mathfrak{m}^\vee \arrow[d] & & \\ 0 \arrow[r] & \mathfrak{n}^\vee\otimes_{\mathbb{Z}} \mathbb{G}_m \arrow[r] & \mathbb{B}^\vee \arrow[r] & B^\vee \arrow[r] & 0. \end{tikzcd}$$

A principal polarization is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism $\mathbb{B} \cong \mathbb{B}^\vee$ compatible with the given polarization $B \cong B^\vee$, and with the isomorphisms $\mathfrak{m} \cong \mathfrak{n}^\vee$ and $\mathfrak{n} \cong \mathfrak{m}^\vee$ determined by the perfect pairing $\mathfrak{m} \otimes_{\mathbb{Z}} \mathfrak{n} \to \mathbb{Z}$ defined after (3.1.3).

Using the "description plus symétrique" of 1-motives [Del74, (10.2.12)], the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{C}_\Phi$ can be realized as the moduli space whose S-points are triples (A_0, A, ϱ) in which

- $(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{X}_\Phi(S)$,
- $\varrho : \underline{\Lambda}_0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$ is an isomorphism of étale sheaves of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules, where $B \in \mathcal{M}_{(n-2,0)}(S)$ is the abelian scheme part of A.

To verify that $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$ has the same functor of points, one uses Remark 3.2.2 to interpret $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$ as a moduli space of mixed Hodge structures on W_0 and W, and uses the theorem of Deligne cited above to interpret these mixed Hodge structures as 1-motices. This defines an isomorphism $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \cong \mathcal{C}_\Phi(\mathbb{C})$. The proof that it descends to the reflex field is identical to the proof for Siegel mixed Shimura varieties [Madb].

We remark in passing that any triple (A_0, A, ϱ) as above automatically satisfies (2.2.4) for every prime ℓ. Indeed, both sides of (2.2.4) are now endowed with weight filtrations, analogous to the weight filtration on $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ defined in §3.1. The isomorphism ϱ induces an isomorphism (as hermitian $\mathcal{O}_{\boldsymbol{k},\ell}$-lattices) between the gr_0 pieces on either side. The gr_{-1} and gr_1 pieces have no structure other then projective $\mathcal{O}_{\boldsymbol{k},\ell}$-modules of rank 1, so are isomorphic. These isomorphisms of graded pieces imply the existence of an isomorphism (2.2.4), exactly as in Remark 3.1.3. □

3.4. **The second moduli interpretation.** In order to make explicit calculations, it will be useful to interpret the moduli spaces

$$\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

in a different way.

Suppose $E \to S$ is an elliptic curve over any base scheme. If U is any S-scheme and $a, b \in E(U)$, we obtain an $\mathcal{O}_U$-module $\mathcal{P}_E(a,b)$ by pulling back the Poincare bundle via

$$U \xrightarrow{(a,b)} E \times_S E \cong E \times_S E^\vee.$$

The notation is intended to remind the reader of the bilinearity properties of the Poincaré bundle, as expressed by canonical isomorphisms

$$\begin{aligned} \mathcal{P}_E(a+b,c) &\cong \mathcal{P}_E(a,c) \otimes \mathcal{P}_E(b,c) \\ \mathcal{P}_E(a,b+c) &\cong \mathcal{P}_E(a,b) \otimes \mathcal{P}_E(a,c) \\ \mathcal{P}_E(a,b) &\cong \mathcal{P}_E(b,a), \end{aligned} \tag{3.4.1}$$

along with $\mathcal{P}_E(e,b) \cong \mathcal{O}_U \cong \mathcal{P}_E(a,e)$. Here $e \in E(U)$ is the zero section.

Let $E \to \mathcal{M}_{(1,0)}$ be the universal elliptic curve with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$. Its Poincaré bundle satisfies, for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$, the additional relation $\mathcal{P}_E(\alpha a, b) \cong \mathcal{P}_E(a, \overline{\alpha} b)$. Abbreviate

$$E \otimes L_0 = E \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$$

for Serre's tensor construction, and denote by $\mathcal{P}_{E\otimes L_0}$ the line bundle on $(E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0)$ whose pullback via a pair of U-valued points

$$c = \sum s_i \otimes x_i \in E(U) \otimes L_0, \quad c' = \sum s'_j \otimes x'_j \in E(U) \otimes L_0$$

is the $\mathcal{O}_U$-module

$$\mathcal{P}_{E\otimes L_0}(c, c') = \bigotimes_{i,j} \mathcal{P}_E(\langle x_i, x'_j \rangle s_i, s'_j).$$

Define $\mathcal{Q}_{E\otimes L_0}$ to be the line bundle on $E \otimes L_0$ whose restriction to the U-valued point $c = \sum s_i \otimes x_i$ is

$$\mathcal{Q}_{E\otimes L_0}(c) = \bigotimes_{i<j} \mathcal{P}_E(\langle x_i, x_j \rangle s_i, s_j) \otimes \bigotimes_{i} \mathcal{P}_E(\gamma \langle x_i, x_i \rangle s_i, s_i), \tag{3.4.2}$$

where

$$\gamma = \frac{1+\delta}{2} \in \mathcal{O}_{\boldsymbol{k}}.$$

It is related to $\mathcal{P}_{E\otimes L_0}$ by canonical isomorphisms

$$\begin{aligned} \mathcal{P}_{E\otimes L_0}(a, b) &\cong \mathcal{Q}_{E\otimes L_0}(a+b) \otimes \mathcal{Q}_{E\otimes L_0}(a)^{-1} \otimes \mathcal{Q}_{E\otimes L_0}(b)^{-1} \\ \mathcal{P}_{E\otimes L_0}(a, a) &\cong \mathcal{Q}_{E\otimes L_0}(a)^{\otimes 2}. \end{aligned} \tag{3.4.3}$$

for all U-valued points $a, b \in E(U) \otimes L_0$.

Remark 3.4.1. The line bundle $\mathcal{P}_{E\otimes L_0}(\delta a, a)$ is canonically trivial. This follows by comparing

$$\mathcal{P}_{E\otimes L_0}(\gamma a, a)^{\otimes 2} \cong \mathcal{P}_{E\otimes L_0}(a, a) \otimes \mathcal{P}_{E\otimes L_0}(\delta a, a)$$

with

$$\mathcal{P}_{E\otimes L_0}(\gamma a, a)^{\otimes 2} \cong \mathcal{P}_{E\otimes L_0}(\gamma a, a) \otimes \mathcal{P}_{E\otimes L_0}(\overline{\gamma} a, a) \cong \mathcal{P}_{E\otimes L_0}(a, a).$$

Remark 3.4.2. One should compare (3.4.3) with [Lan13, Construction 1.3.2.7] or [MFK94, Definition 6.2]. The line bundle $\mathcal{Q}_{E\otimes L_0}$ determines a polarization of $E \otimes L_0$, and $\mathcal{P}_{E\otimes L_0}$ is the pullback of the Poincaré bundle via

$$(E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0) \to (E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0)^\vee.$$

Proposition 3.4.3. *As above, let $E \to \mathcal{M}_{(1,0)}$ be the universal object. There are canonical isomorphisms*

$$\begin{tikzcd} \mathcal{C}_\Phi \arrow[r] \arrow[d, "\cong"] & \mathcal{B}_\Phi \arrow[r] \arrow[d, "\cong"] & \mathcal{A}_\Phi \arrow[d, "\cong"] \\ \underline{\mathrm{Iso}}(\mathcal{Q}_{E\otimes L_0}, \mathcal{O}_{E\otimes L_0}) \arrow[r] & E \otimes L_0 \arrow[r] & \mathcal{M}_{(1,0)}, \end{tikzcd}$$

and the middle vertical arrow identifies $\mathcal{L}_\Phi \cong \mathcal{Q}_{E\otimes L_0}$.

Proof. Define a morphism $\mathcal{A}_\Phi \to \mathcal{M}_{(1,0)}$ by sending a triple (A_0, B, ϱ) to the CM elliptic curve

$$E = \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0). \tag{3.4.4}$$

To show that this map is an isomorphism we will construct the inverse.

If S is any $\mathcal{O}_{\boldsymbol{k}}$-scheme and $E \in \mathcal{M}_{(1,0)}(S)$, we may define $(A_0, B, \varrho) \in \mathcal{A}_\Phi(S)$ by setting

$$A_0 = E \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{n}, \quad B = \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0),$$

and taking for $\varrho : \underline{\Lambda}_0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$ the tautological isomorphism. It remains to endow B with a principal polarization. Let U be an S-scheme, and suppose we are given points

$$b, b' \in B(U) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_{0U})$$

of the form $b = \langle \cdot, \lambda \rangle a$ and $b' = \langle \cdot, \lambda' \rangle a'$ with $\lambda, \lambda' \in \Lambda_0$ and $a, a' \in A_0(U)$. Note that all points of $B(U)$ are sums of points of this form. Define a line bundle

$$\mathcal{P}_B(b, b') = \mathcal{P}_{A_0}(a, \langle \lambda, \lambda' \rangle a')$$

on U, and extend the definition $\mathbb{Z}$-bilinearly, in the sense of (3.4.1), to arbitrary points of $B(U)$. There is a unique line bundle $\mathcal{P}_B$ on $B \times_S B$ whose restriction to any U-point $(b, b') \in B(U) \times B(U)$ is $\mathcal{P}_B(b, b')$, and a unique principal polarization on B such that $\mathcal{P}_B$ is the pullback of the Poincaré bundle via $B \times_S B \to B \times_S B^\vee$. The construction $E \mapsto (A_0, B, \varrho)$ is inverse to the above morphism $\mathcal{A}_\Phi \to \mathcal{M}_{(1,0)}$.

Now identify $\mathcal{A}_\Phi \cong \mathcal{M}_{(1,0)}$ using the above isomorphism, and denote by (A_0, B, ϱ) and E the universal objects on the source and target. They are related by canonical isomorphisms

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \Lambda_0, A_0) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathcal{B}_\Phi = \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B) & & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E).
\end{tikzcd}$$

Combining this with the $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism

$$E \otimes L_0 \xrightarrow{a \otimes x \mapsto \langle \cdot, x^\vee \rangle a} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E)$$

defines $\mathcal{B}_\Phi \cong E \otimes L_0$. All that remains is to prove that this isomorphism identifies $\mathcal{L}_\Phi$ with $\mathcal{Q}_{E \otimes L_0}$.

Any fractional ideal $\mathfrak{b} \subset \boldsymbol{k}$ admits a unique positive definite self-dual hermitian form, given explicitly by $\langle b_1, b_2 \rangle = b_1 \overline{b}_2 / \mathrm{N}(\mathfrak{b})$. It follows that any rank one projective $\mathcal{O}_{\boldsymbol{k}}$-module admits a unique positive definite self-dual hermitian form. For the $\mathcal{O}_{\boldsymbol{k}}$-module $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, \mathcal{O}_{\boldsymbol{k}})$, this hermitian form is

$$\langle \ell_1, \ell_2 \rangle = \ell_1(\mu)\overline{\ell_2(\nu)} + \ell_1(\nu)\overline{\ell_2(\mu)},$$

where $\mu \otimes \nu = \chi_0 \in \mathrm{Sym}_\Phi$ is the positive generator appearing in (3.3.1).

The relation (3.4.4) implies a relation between the line bundles $\mathcal{P}_E$ and $\mathcal{P}_{A_0}$. If U is any $\mathcal{A}_\Phi$-scheme and we are given points

$$s, s' \in E(U) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_{0U})$$

of the form $s = \ell(\cdot)a$ and $s' = \ell'(\cdot)a'$ with $\ell, \ell' \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, \mathcal{O}_{\boldsymbol{k}})$ and $a, a' \in A_0(U)$, then

$$\begin{aligned} \mathcal{P}_E(s, s') &\cong \mathcal{P}_{A_0}\big(\langle \ell, \ell' \rangle a, a'\big) \\ \mathcal{P}_E(\gamma s, s) &\cong \mathcal{P}_{A_0}\big(\ell(\mu)a, \ell(\nu)a\big). \end{aligned}$$

Fix any point $c \in \mathcal{B}_\Phi(U)$. Using the isomorphisms in the diagram above, c admits three different interpretations. In one of them, c has the form

$$c = \sum \ell_i(\cdot)\langle \cdot, \lambda_i \rangle a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \Lambda, A_{0U}).$$

By setting

$$\begin{aligned} b_i &= \langle \cdot, \lambda_i \rangle a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_{0U}) = B(U) \\ s_i &= \ell_i(\cdot) a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_{0U}) = E(U), \end{aligned}$$

we find the other two interpretations

$$\begin{aligned} c &= \sum \ell_i(\cdot) b_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B_U) \\ c &= \sum \langle \cdot, \lambda_i \rangle s_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E_U). \end{aligned}$$

The above relations between $\mathcal{P}_B$, $\mathcal{P}_E$, and $\mathcal{P}_{A_0}$ imply

$$\begin{aligned} &\mathcal{P}_B(c(\mu), c(\nu)) \\ &\quad\cong \bigotimes_{i,j} \mathcal{P}_B(\ell_i(\mu) b_i, \ell_j(\nu) b_j) \\ &\quad\cong \bigotimes_{i,j} \mathcal{P}_{A_0}(\ell_i(\mu) a_i, \langle \lambda_i, \lambda_j \rangle \ell_j(\nu) a_j) \\ &\quad\cong \bigotimes_{i<j} \mathcal{P}_{A_0}(\langle \ell_i, \ell_j \rangle a_i, \langle \lambda_i, \lambda_j \rangle a_j) \otimes \bigotimes_{i} \mathcal{P}_{A_0}(\ell_i(\mu) a_i, \ell_i(\nu) \langle \lambda_i, \lambda_i \rangle a_i) \\ &\quad\cong \bigotimes_{i<j} \mathcal{P}_E(s_i, \langle \lambda_i, \lambda_j \rangle s_j) \otimes \bigotimes_{i} \mathcal{P}_E(\gamma s_i, \langle \lambda_i, \lambda_i \rangle s_i). \end{aligned}$$

Now write $\lambda_i = x_i^\vee$ with $x_i \in L_0$, and use the relation

$$\mathcal{P}_E(s_i, \langle \lambda_i, \lambda_j \rangle s_j) = \mathcal{P}_E(\langle \lambda_j, \lambda_i \rangle s_i, s_j) = \mathcal{P}_E(\langle x_i, x_j \rangle s_i, s_j)$$

to obtain an isomorphism $\mathcal{P}_B(c(\mu), c(\nu)) \cong \mathcal{Q}_{E \otimes L_0}(c)$. The line bundle on the left is precisely the pullback of $\mathcal{L}_\Phi$ via c, and letting c vary we obtain an isomorphism $\mathcal{L}_\Phi \cong \mathcal{Q}_{E \otimes L_0}$. $\square$

3.5. **The line bundle of modular forms.** We now define a line bundle of weight one modular forms on our mixed Shimura variety, analogous to the one on the pure Shimura variety defined in §2.4.

The holomorphic line bundle $\boldsymbol{\omega}^{an}$ on $\mathcal{D}$ defined in §2.4 admits a canonical extension to $\mathcal{D}_\Phi$, denoted $\boldsymbol{\omega}^{an}_\Phi$, whose fiber at a point $z = (y_0, y)$ is again defined by (2.4.1). If we embed $\mathcal{D}_\Phi$ into projective space over $\epsilon V(\mathbb{C})$ as in Remark 3.2.3, this is simply the restriction of the tautological bundle. There is an obvious action of $Q_\Phi(\mathbb{R})$ on the total space of $\boldsymbol{\omega}^{an}_\Phi$, lifting the natural action on $\mathcal{D}_\Phi$, and so $\boldsymbol{\omega}^{an}_\Phi$ descends to a holomorphic line bundle on the complex orbifold $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$.

As in §2.4, the holomorphic line bundle $\boldsymbol{\omega}^{an}_\Phi$ is algebraic and descends to the canonical model $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$. In fact, it admits a canonical extension to the integral model $\mathcal{C}_\Phi$, as we now explain.

Define rank two vector bundles on $\mathcal{A}_\Phi$ by

$$\mathfrak{M} = \mathfrak{m} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}, \quad \mathfrak{N} = \mathfrak{n} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}.$$

Each is free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$, and the perfect pairing between $\mathfrak{m}$ and $\mathfrak{n}$ defined after (3.1.3) induces a perfect pairing $\mathfrak{M} \times \mathfrak{N} \to \mathcal{O}_{\mathcal{A}_\Phi}$. Using the almost idempotents $\epsilon, \overline{\epsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$ of §1.7, there is an induced perfect pairing of line bundles

$$(\mathfrak{M}/\epsilon\mathfrak{M}) \otimes (\epsilon\mathfrak{N}) \to \mathcal{O}_{\mathcal{A}_\Phi}.$$

Recalling that $\mathcal{A}_\Phi$ carries over it a universal triple (A_0, B, ϱ), in which A_0 is an elliptic curve with $\mathcal{O}_{\boldsymbol{k}}$-action, we now define a line bundle on $\mathcal{A}_\Phi$ by

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \epsilon\mathfrak{N}),$$

or, equivalently,

$$\boldsymbol{\omega}_\Phi^{-1} = \mathrm{Lie}(A_0) \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \mathfrak{M}/\epsilon\mathfrak{M}.$$

Denote in the same way its pullback to any step in the tower

$$\mathcal{C}^*_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi.$$

The above definition of $\boldsymbol{\omega}_\Phi$ is a bit unmotivated, and so we explain why $\boldsymbol{\omega}_\Phi$ is analogous to the line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ defined in §2.4. Recall from the proof of Proposition 3.3.2 that $\mathcal{C}_\Phi$ carries over it a universal 1-motive A. This 1-motive has a de Rham realization $H_1^{\mathrm{dR}}(A)$, defined as the Lie algebra of the universal vector extension of A, as in [Del74, (10.1.7)]. It is a rank $2n$-vector bundle on $\mathcal{C}_\Phi$, locally free of rank n over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{C}_\Phi}$, and sits in a diagram of vector bundles

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & F^0H_1^{\mathrm{dR}}(B) \arrow[d] & & \mathfrak{M} \arrow[d] & \\
0 \arrow[r] & F^0H_1^{\mathrm{dR}}(A) \arrow[r] \arrow[d] & H_1^{\mathrm{dR}}(A) \arrow[r] & \mathrm{Lie}(A) \arrow[r] \arrow[d] & 0 \\
 & \mathfrak{N} \arrow[d] & & \mathrm{Lie}(B) \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}$$

with exact rows and columns. The polarization on A induces a perfect symplectic form on $H_1^{\mathrm{dR}}(A)$. This induces a perfect pairing

$$F^0H_1^{\mathrm{dR}}(A) \otimes \mathrm{Lie}(A) \to \mathcal{O}_{\mathcal{C}_\Phi} \tag{3.5.1}$$

as in (2.2.1), which is compatible (in the obvious sense) with the pairings

$$F^0H_1^{\mathrm{dR}}(B) \otimes \mathrm{Lie}(B) \to \mathcal{O}_{\mathcal{C}_\Phi}$$

and $\mathfrak{N} \otimes \mathfrak{M} \to \mathcal{O}_{\mathcal{C}_\Phi}$ that we already have.

The signature condition on B implies that $\epsilon F^0H_1^{\mathrm{dR}}(B) = 0$ and $\overline{\epsilon}\mathrm{Lie}(B) = 0$. Using this, and arguing as in [How15, Lemma 2.3.6], it is not difficult to see that

$$\mathcal{F}_A = \ker(\overline{\epsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A))$$

is the unique codimension one local direct summand of $\mathrm{Lie}(A)$ satisfying Kramer's condition as in §2.3, and that its orthogonal under the pairing (3.5.1) is $\mathcal{F}_A^\perp = \epsilon F^0H_1^{\mathrm{dR}}(A)$. Moreover, the natural maps

$$\mathfrak{M}/\epsilon\mathfrak{M} \to \mathrm{Lie}(A)/\mathcal{F}_A, \quad \mathcal{F}_A^\perp \to \epsilon\mathfrak{N}$$

are isomorphisms. These latter isomorphisms allow us to identify

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^\perp), \quad \boldsymbol{\omega}_\Phi^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}_A$$

in perfect analogy with §2.4.

Proposition 3.5.1. *The isomorphism*

$$\mathcal{C}_\Phi(\mathbb{C}) \cong \mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

of Proposition 3.3.2 identifies the analytification of $\boldsymbol{\omega}_\Phi$ with the already defined $\boldsymbol{\omega}_\Phi^{an}$. Moreover, the isomorphism $\mathcal{A}_\Phi \cong \mathcal{M}_{(1,0)}$ of Proposition 3.4.3 identifies

$$\boldsymbol{\omega}_\Phi \cong \mathfrak{d} \cdot \mathrm{Lie}(E)^{-1} \subset \mathrm{Lie}(E)^{-1}$$

where $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ is the different of $\mathcal{O}_{\boldsymbol{k}}$, and $E \to \mathcal{M}_{(1,0)}$ is the universal elliptic curve with CM by $\mathcal{O}_{\boldsymbol{k}}$.

Proof. Any point $z = (y_0, y) \in \mathcal{D}_\Phi$ determines, by Remarks 2.1.2 and 3.2.2, a pure Hodge structure on W_0 and a mixed Hodge structure on W, these induce a mixed Hodge structure on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$, and the fiber of $\boldsymbol{\omega}^{an}_\Phi$ at z is

$$\boldsymbol{\omega}^{an}_{\Phi,z} = F^1 V(\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\overline{\epsilon} W_0(\mathbb{C}), \epsilon F^0 W(\mathbb{C})).$$

On the other hand, we have just seen that

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^\perp) = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \epsilon F^0 H_1^{\mathrm{dR}}(A)).$$

With these identifications, the proof of the first claim amounts to carefully tracing through the construction of the isomorphism of Proposition 3.3.2.

For the second claim, the isomorphism $A_0 \cong E \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{n}$ induces a canonical isomorphism

$$\mathrm{Lie}(A_0) \cong \mathrm{Lie}(E) \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{n} \cong \mathrm{Lie}(E) \otimes \mathfrak{N}/\overline{\epsilon}\mathfrak{N},$$

where we have used the fact that $\mathfrak{n} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{O}_{\mathcal{A}_\Phi} = \mathfrak{N}/\overline{\epsilon}\mathfrak{N}$ is the largest quotient of $\mathfrak{N}$ on which $\mathcal{O}_{\boldsymbol{k}}$ acts via the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{A}_\Phi}$. Thus

$$\begin{aligned}\boldsymbol{\omega}_\Phi &= \underline{\mathrm{Hom}}(\mathrm{Lie}(A), \epsilon\mathfrak{N}) \\ &\cong \underline{\mathrm{Hom}}(\mathrm{Lie}(E) \otimes \mathfrak{N}/\overline{\epsilon}\mathfrak{N}, \epsilon\mathfrak{N}) \\ &\cong \mathrm{Lie}(E)^{-1} \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \underline{\mathrm{Hom}}(\mathfrak{N}/\overline{\epsilon}\mathfrak{N}, \epsilon\mathfrak{N}).\end{aligned}$$

Now recall the ideal sheaf $(\epsilon) \subset \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$ of §1.7. There are canonical isomorphisms of line bundles

$$\mathfrak{d}\mathcal{O}_{\mathcal{A}_\Phi} \cong (\epsilon) \cong \underline{\mathrm{Hom}}(\mathfrak{N}/\overline{\epsilon}\mathfrak{N}, \epsilon\mathfrak{N}),$$

where the first is (1.7.1) and the second is the tautological isomorphism sending ϵ to the multiplication-by-ϵ map $\mathfrak{N}/\overline{\epsilon}\mathfrak{N} \to \epsilon\mathfrak{N}$. These constructions determine the desired isomorphism

$$\boldsymbol{\omega}_\Phi \cong \mathrm{Lie}(E)^{-1} \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \mathfrak{d}\mathcal{O}_{\mathcal{A}_\Phi}.$$

□

3.6. **Special divisors.** Let $\mathcal{Y}_0(D)$ be the moduli stack over $\mathcal{O}_{\boldsymbol{k}}$ parametrizing cyclic D-isogenies of elliptic curves over $\mathcal{O}_{\boldsymbol{k}}$-schemes, and let $\mathcal{E} \to \mathcal{E}'$ be the universal object.

Let (A_0, B, ϱ, c) be the universal object over $\mathcal{B}_\Phi$. Recalling the $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism $L_0 \cong \Lambda_0$ defined after (3.1.4), each $x \in L_0$ defines a morphism

$$\mathfrak{n} \xrightarrow{c} B \xrightarrow{\varrho(x^\vee)} A_0$$

of sheaves of $\mathcal{O}_{\boldsymbol{k}}$-modules on $\mathcal{B}_\Phi$. Define $\mathcal{Z}_\Phi(x) \subset \mathcal{B}_\Phi$ as the largest closed substack over which this morphism is trivial. We will see in a moment that this closed substack is defined locally by one equation. For any $m > 0$ define stack over $\mathcal{B}_\Phi$ by

$$\mathcal{Z}_\Phi(m) = \bigsqcup_{\substack{x \in L_0 \\ \langle x, x\rangle = m}} \mathcal{Z}_\Phi(x). \tag{3.6.1}$$

We also view $\mathcal{Z}_\Phi(m)$ as a divisor on $\mathcal{B}_\Phi$, and denote in the same way the pullback of this divisor via $\mathcal{C}_\Phi^* \to \mathcal{B}_\Phi$.

We will now reformulate the definition of $\mathcal{Z}_\Phi(x)$ in terms of the moduli problem of §3.4. Recalling the isomorphisms of Proposition 3.4.3, every $x \in L_0$ determines a commutative diagram

$$\begin{tikzcd}
\mathcal{B}_\Phi \arrow[r, "\cong"] \arrow[d] & E \otimes L_0 \arrow[r, "\langle \cdot, x\rangle"] \arrow[d] & E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{A}_\Phi \arrow[r, "\cong"] & \mathcal{M}_{(1,0)} \arrow[r, equal] & \mathcal{M}_{(1,0)} \arrow[r] & \mathcal{Y}_0(D),
\end{tikzcd}$$

where $\mathcal{M}_{(1,0)} \to \mathcal{Y}_0(D)$ sends E to the cyclic D-isogeny

$$E \to E \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{d}^{-1},$$

and the rightmost square is cartesian. The upper and lower horizontal compositions are denoted j_x and j, giving the diagram

$$\begin{tikzcd}
\mathcal{B}_\Phi \arrow[r, "j_x"] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{A}_\Phi \arrow[r, "j"] & \mathcal{Y}_0(D).
\end{tikzcd} \tag{3.6.2}$$

Proposition 3.6.1. *For any nonzero $x \in L_0$, the closed substack $\mathcal{Z}_\Phi(x) \subset \mathcal{B}_\Phi$ is equal to the pullback of the zero section along j_x. It is an effective Cartier divisor, flat over $\mathcal{A}_\Phi$. In particular, as $\mathcal{A}_\Phi$ is flat over $\mathcal{O}_{\boldsymbol{k}}$, so is each divisor $\mathcal{Z}_\Phi(x)$.*

Proof. Recall the isomorphisms

$$E \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0), \quad B \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0)$$

from the proof of Proposition 3.4.3. If we identify $A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \cong B$ using

$$A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \xrightarrow{a \otimes x \mapsto \langle \cdot, x^\vee \rangle a} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0) \cong B,$$

we obtain a commutative diagram of $\mathcal{A}_\Phi$-stacks

$$\begin{tikzcd}
E \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \arrow[r] \arrow[d, "\langle \cdot, x\rangle"'] & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0) \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B) = \mathcal{B}_\Phi \arrow[d, "\varrho(x^\vee)"] \\
E \arrow[rr] & & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0),
\end{tikzcd}$$

in which all horizontal arrows are isomorphisms. The first claim follows immediately.

The remaining claims now follow from the cartesian diagram

$$\begin{tikzcd}
\mathcal{Z}_\Phi(x) \arrow[rr] \arrow[d] & & \mathcal{M}_{(1,0)} \arrow[d, "e"] \\
\mathcal{B}_\Phi \arrow[r, "\cong"] & E \otimes L_0 \arrow[r, "\langle \cdot, x \rangle"] & E.
\end{tikzcd}$$

The zero section $e : \mathcal{M}_{(1,0)} \hookrightarrow E$ is locally defined by one equation [KM85, Lemma 1.2.2], and so the same is true of its pullback $\mathcal{Z}_\Phi(x) \hookrightarrow \mathcal{B}_\Phi$. Composition along the bottom row is flat by [MFK94, Lemma 6.12], and hence so is the top horizontal arrow. □

Remark 3.6.2. For those who prefer the language of 1-motives: As in the proof of Proposition 3.3.2, there is a universal triple (A_0, A, ϱ) over $\mathcal{C}_\Phi$ in which A_0 is an elliptic curve with $\mathcal{O}_{\boldsymbol{k}}$-action and A is a principally polarized 1-motive with $\mathcal{O}_{\boldsymbol{k}}$-action. The functor of points of $\mathcal{Z}_\Phi(m)$ assigns to any scheme $S \to \mathcal{C}_\Phi$ the set

$$\mathcal{Z}_\Phi(m)(S) = \{x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0,S}, A_S) : \langle x, x \rangle = m\},$$

where the positive definite hermitian form $\langle \cdot, \cdot \rangle$ is defined as in (2.5.1). When expressed this way, our special divisors are obviously analogous to the special divisors on $\mathcal{S}_{\mathrm{Kra}}$ defined in §2.5.

3.7. **The toroidal compactification.** We describe the canonical toroidal compactification of the integral model $\mathcal{S}_{\mathrm{Kra}}$ from §2.3.

Theorem 3.7.1. *Let $\mathcal{S}_\square$ denote either $\mathcal{S}_{\mathrm{Kra}}$ or $\mathcal{S}_{\mathrm{Pap}}$. There is a canonical toroidal compactification $\mathcal{S}_\square \hookrightarrow \mathcal{S}^*_\square$, flat over $\mathcal{O}_{\boldsymbol{k}}$ of relative dimension $n-1$. It admits a stratification*

$$\mathcal{S}^*_\square = \bigsqcup_{\Phi} \mathcal{S}^*_\square(\Phi)$$

as a disjoint union of locally closed substacks, indexed by the K-equivalence classes of cusp label representatives.

1. *The stack $\mathcal{S}^*_{\mathrm{Kra}}$ is regular.*
2. *The stack $\mathcal{S}^*_{\mathrm{Pap}}$ is Cohen-Macaulay and normal, with Cohen-Macaulay and geometrically normal fibers.*
3. *The open dense substack $\mathcal{S}_\square \subset \mathcal{S}^*_\square$ is the stratum indexed by the unique equivalence class of improper cusp label representatives. Its complement*

$$\partial \mathcal{S}^*_\square = \bigsqcup_{\Phi \text{ proper}} \mathcal{S}^*_\square(\Phi)$$

is a smooth divisor, flat over $\mathcal{O}_{\boldsymbol{k}}$.

(4) *For each proper Φ the stratum $\mathcal{S}^*_\Box(\Phi)$ is closed. All components of $\mathcal{S}^*_\Box(\Phi)_{/\mathbb{C}}$ are defined over the Hilbert class field H, and they are permuted simply transitively by* $\mathrm{Gal}(H/\boldsymbol{k})$. *Moreover, there is a canonical identification of $\mathcal{O}_{\boldsymbol{k}}$-stacks*

$$\begin{tikzcd} \Delta_\Phi\backslash\mathcal{B}_\Phi \arrow[r, equal] \arrow[d] & \mathcal{S}^*_\Box(\Phi) \arrow[d] \\ \Delta_\Phi\backslash\mathcal{C}^*_\Phi & \mathcal{S}^*_\Box \end{tikzcd}$$

such that the two stacks in the bottom row become isomorphic after completion along their common closed substack in the top row. In other words, there is a canonical isomorphism of formal stacks

$$\Delta_\Phi\backslash(\mathcal{C}^*_\Phi)^\wedge_{\mathcal{B}_\Phi} \cong (\mathcal{S}^*_\Box)^\wedge_{\mathcal{S}^*_\Box(\Phi)}. \tag{3.7.1}$$

(5) *The morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ extends uniquely to a stratum preserving morphism of toroidal compactifications. This extension restricts to an isomorphism*

$$\mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}, \tag{3.7.2}$$

compatible with (3.7.1) for any proper Φ.

(6) *The line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ defined in §2.4 admits a unique extension (denoted the same way) to the toroidal compactification in such a way that (3.7.1) identifies it with the line bundle $\boldsymbol{\omega}_\Phi$ on $\mathcal{C}^*_\Phi$. A similar statement holds for $\boldsymbol{\Omega}_{\mathrm{Kra}}$, and these two extensions are related by*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

(7) *The line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ defined in §2.4 admits a unique extension (denoted the same way) to the toroidal compactification, in such a way that (3.7.1) identifies it with $\boldsymbol{\omega}^2_\Phi$.*

(8) *For any $m > 0$, define $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ as the Zariski closure of $\mathcal{Z}_{\mathrm{Kra}}(m)$ in $\mathcal{S}^*_{\mathrm{Kra}}$. The isomorphism (3.7.1) identifies it with the Cartier divisor $\mathcal{Z}_\Phi(m)$ on $\mathcal{C}^*_\Phi$.*

(9) *For any $m > 0$, define $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ as the Zariski closure of $\mathcal{Y}_{\mathrm{Pap}}(m)$ in $\mathcal{S}^*_{\mathrm{Pap}}$. The isomorphism (3.7.1) identifies it with the square of the Cartier divisor $\mathcal{Z}_\Phi(m)$. Moreover, the pullback of $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ to $\mathcal{S}^*_{\mathrm{Kra}}$, denoted $\mathcal{Y}^*_{\mathrm{Kra}}(m)$, satisfies*

$$2 \cdot \mathcal{Z}^*_{\mathrm{Kra}}(m) = \mathcal{Y}^*_{\mathrm{Kra}}(m) + \sum_{s\in\pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s.$$

Proof. Briefly, in [How15, §2] one finds the construction a canonical toroidal compactification

$$\mathcal{M}^\Box_{(n-1,1)} \hookrightarrow \mathcal{M}^{\Box,*}_{(n-1,1)}.$$

Using the open and closed immersion

$$\mathcal{S}_\Box \hookrightarrow \mathcal{M}_{(1,0)} \times \mathcal{M}^\Box_{(n-1,1)},$$

the toroidal compactification $\mathcal{S}^*_\Box$ is defined as the Zariski closure of $\mathcal{S}_\Box$ in $\mathcal{M}_{(1,0)} \times \mathcal{M}^{\Box,*}_{(n-1,1)}$. All of the claims follow by examination of the construction of the compactification, along with Theorem 2.6.3. $\Box$

Corollary 3.7.2. *The Cartier divisor $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ on $\mathcal{S}^*_{\mathrm{Pap}}$ is $\mathcal{O}_{\boldsymbol{k}}$-flat, as is the restriction of $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ to $\mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}$.*

Proof. Fix a prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$. In general, if X is a Noetherian, proper, flat $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-scheme with geometrically normal fibers, one can find a finite unramified extension $F/\boldsymbol{k}_{\mathfrak{p}}$ such that every connected component of $X_{/\mathcal{O}_F}$ has geometrically irreducible fibers. This follows from [FGI+05, Proposition 8.5.16].

Choose a finite extension $F/\boldsymbol{k}_{\mathfrak{p}}$ so that every connected component of $\mathcal{S}^*_{\mathrm{Pap}/\mathcal{O}_F}$ has geometrically irreducible fibers. Proposition 3.6.1 implies that $\mathcal{Z}_\Phi(m)$ is $\mathcal{O}_{\boldsymbol{k}}$-flat for every proper cusp label representative Φ. Theorem 3.7.1 now implies that $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ is $\mathcal{O}_{\boldsymbol{k}}$-flat when restricted to some étale neighborhood U of the boundary of $\mathcal{S}^*_{\mathrm{Pap}/\mathcal{O}_F}$.

The boundary, and hence the étale neighborhood U, meets every connected component of $\mathcal{S}^*_{\mathrm{Pap}/F}$ (this can be checked in the complex fiber, where it follows from the fact that a $\boldsymbol{k}$-hermitian space of signature $(n-1,1)$ with $n \geqslant 3$ admits an isotropic $\boldsymbol{k}$-line). It follows that U meets every connected component of $\mathcal{S}^*_{\mathrm{Pap}/\mathcal{O}_F}$, and hence meets every irreducible component of the special fiber. We deduce that the support of the Cartier divisor $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ contains no irreducible components of the special fiber, so is $\mathcal{O}_{\boldsymbol{k}}$-flat.

As the isomorphism (3.7.2) identifies $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ with $2\mathcal{Z}^*_{\mathrm{Kra}}(m)$, it follows that the restriction of $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ to the complement of Exc is also flat. $\Box$

3.8. **Fourier-Jacobi expansions.** We now define Fourier-Jacobi expansions of sections of the line bundle $\boldsymbol{\omega}^k$ of weight k modular forms on $\mathcal{S}^*_{\mathrm{Kra}}$.

Fix a proper cusp label representative $\Phi = (P, g)$. Suppose ψ is a rational function on $\mathcal{S}^*_{\mathrm{Kra}}$, regular on an open neighborhood of the closed stratum $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$. Using the isomorphism (3.7.1) we obtain a formal function, again denoted ψ, on the formal completion

$$(\mathcal{C}^*_\Phi)^\wedge_{\mathcal{B}_\Phi} = \underline{\mathrm{Spf}}_{\mathcal{B}_\Phi}\Big(\prod_{\ell \geqslant 0} \mathcal{L}^\ell_\Phi\Big).$$

Tautologically, there is a formal Fourier-Jacobi expansion

$$\psi = \sum_{\ell \geqslant 0} \mathrm{FJ}_\ell(\psi) \cdot q^\ell \tag{3.8.1}$$

with coefficients $\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \mathcal{L}^\ell_\Phi)$. In the same way, any rational section ψ of $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$, regular on an open neighborhood of $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$, admits a Fourier-Jacobi expansion (3.8.1), but now with coefficients

$$\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \boldsymbol{\omega}^k_\Phi \otimes \mathcal{L}^\ell_\Phi).$$

Remark 3.8.1. Let $\pi : \mathcal{C}^*_\Phi \to \mathcal{B}_\Phi$ be the natural map. The formal symbol q can be understood as follows. As $\mathcal{C}^*_\Phi$ is the total space of the line bundle $\mathcal{L}_\Phi^{-1}$, there is a tautological section

$$q \in H^0(\mathcal{C}^*_\Phi, \pi^*\mathcal{L}_\Phi^{-1})$$

whose divisor is the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}^*_\Phi$. Any $\mathrm{FJ}_\ell \in H^0(\mathcal{B}_\Phi, \mathcal{L}_\Phi^\ell)$ pulls back to a section of $\pi^*\mathcal{L}_\Phi^\ell$, and so defines a function $\mathrm{FJ}_\ell \cdot q^\ell$ on $\mathcal{C}^*_\Phi$.

3.9. **Explicit coordinates.** Once again, let $\Phi = (P, g)$ be a proper cusp label representative. The algebraic theory of §3.8 realizes the Fourier-Jacobi coefficients of

$$\psi \in H^0(\mathcal{S}^*_{\mathrm{Kra}}, \boldsymbol{\omega}^k) \tag{3.9.1}$$

as sections of line bundles on the stack

$$\mathcal{B}_\Phi \cong E \otimes L_0.$$

Here $E \to \mathcal{M}_{(1,0)}$ is the universal CM elliptic curve, the tensor product is over $\mathcal{O}_{\boldsymbol{k}}$, and we are using the isomorphism of Proposition 3.4.3. Our goal is to relate this to the classical analytic theory of Fourier-Jacobi expansions by choosing explicit complex coordinates, so as to identify each coefficient $\mathrm{FJ}_\ell(\psi)$ with a holomorphic function on a complex vector space satisfying a particular transformation law.

The point of this discussion is to allow us, eventually, to show that the Fourier-Jacobi coefficients of Borcherds products, expressed in the classical way as holomorphic functions satisfying certain transformation laws, have algebraic meaning. More precisely, the following discussion will be used to deduce the algebraic statement of Proposition 6.4.1 from the analytic statement of Proposition 6.3.1.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \arrow[r, "\cong"] \arrow[d] & \mathcal{C}_\Phi(\mathbb{C}) \arrow[r] & \mathcal{B}_\Phi(\mathbb{C}) \arrow[r] & \mathcal{A}_\Phi(\mathbb{C}) \arrow[d, "\cong"] \\
\boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times \arrow[rrr, "a \mapsto E^{(a)}"'] & & & \mathcal{M}_{(1,0)}(\mathbb{C}).
\end{tikzcd}$$

Here the isomorphisms are those of Propositions 3.3.2 and 3.4.3, and the vertical arrow on the left is the surjection of Proposition 3.2.1. The bottom horizontal arrow is defined as the unique function making the diagram commute. It is a bijection, and is given explicitly by the following recipe: Each $a \in \widehat{\boldsymbol{k}}^\times$ determines a projective $\mathcal{O}_{\boldsymbol{k}}$-module

$$\mathfrak{b} = a \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, g\mathfrak{a}_0)$$

of rank one, and the elliptic curve $E^{(a)}$ has complex points

$$E^{(a)}(\mathbb{C}) = \mathfrak{b} \backslash (\mathfrak{b} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathbb{C}). \tag{3.9.2}$$

For each $a \in \widehat{\boldsymbol{k}}^\times$ there is a cartesian diagram

$$\begin{tikzcd} E^{(a)} \otimes L_0 \arrow[r] \arrow[d] & E \otimes L_0 \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "E^{(a)}"] & \mathcal{M}_{(1,0)}. \end{tikzcd}$$

Now suppose we have a section ψ as in (3.9.1). Using the isomorphisms $\mathcal{B}_\Phi \cong E \otimes L_0$ and $\boldsymbol{\omega}_{\boldsymbol{\Phi}} \cong \mathfrak{d} \cdot \mathrm{Lie}(E)^{-1}$ of Propositions 3.4.3 and 3.5.1, we view its Fourier-Jacobi coefficients

$$\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \boldsymbol{\omega}_\Phi^k \otimes \mathcal{L}_\Phi^\ell)$$

as sections

$$\mathrm{FJ}_\ell(\psi) \in H^0\big(E \otimes L_0, \mathfrak{d}^k \cdot \mathrm{Lie}(E)^{-k} \otimes \mathcal{Q}^\ell_{E\otimes L_0}\big),$$

which we pull back along the top map in the above diagram to obtain a section

$$\mathrm{FJ}^{(a)}_\ell(\psi) \in H^0\big(E^{(a)} \otimes L_0, \mathrm{Lie}(E^{(a)})^{-k} \otimes \mathcal{Q}^\ell_{E^{(a)}\otimes L_0}\big). \tag{3.9.3}$$

Remark 3.9.1. Recalling that $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ is the different of $\boldsymbol{k}$, we are using the inclusion $\mathfrak{d}^k \subset \boldsymbol{k} \subset \mathbb{C}$ to identify $\mathfrak{d}^k \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathbb{C} \cong \mathbb{C}$, and hence

$$\mathfrak{d}^k \cdot \mathrm{Lie}(E^{(a)})^{-k} \cong \mathrm{Lie}(E^{(a)})^{-k}.$$

In particular, this isomorphism is *not* multiplication by δ^{-k}.

The explicit coordinates we will use to express (3.9.3) as a holomorphic function arise from a choice of Witt decomposition of the hermitian space $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. The following lemma will allow us to choose this decomposition in a particularly nice way.

Lemma 3.9.2. *The homomorphism ν_Φ of (3.1.1) admits a section*

$$\begin{tikzcd} Q_\Phi \arrow[r, "\nu_\Phi"'] & \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m. \arrow[l, bend right, "s"'] \end{tikzcd}$$

This section may be chosen so that $s(\widehat{\mathcal{O}}^\times_{\boldsymbol{k}}) \subset K_\Phi$, and such a choice determines a decomposition

$$\bigsqcup_{a\in \boldsymbol{k}^\times\backslash\widehat{\boldsymbol{k}}^\times/\widehat{\mathcal{O}}^\times_{\boldsymbol{k}}} (Q_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1})\backslash\mathcal{D}_\Phi \cong \mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}), \tag{3.9.4}$$

where the isomorphism is $z \mapsto (z, s(a))$ on the copy of $\mathcal{D}_\Phi$ indexed by a.

Proof. Fix an isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules

$$g\mathfrak{a}_0 \oplus g\mathfrak{a} \cong g\mathfrak{a}_0 \oplus \mathrm{gr}_{-2}(g a) \oplus \mathrm{gr}_{-1}(g a) \oplus \mathrm{gr}_0(g a)$$

as in Remark 3.1.3. After tensoring with $\mathbb{Q}$, we let $\boldsymbol{k}^\times$ act on the right hand side by $a \mapsto (a, \mathrm{Nm}(a), a, 1)$. This defines a morphism $\boldsymbol{k}^\times \to G(\mathbb{Q})$, which, using (3.1.1), is easily seen to take values in the subgroup $Q_\Phi(\mathbb{Q})$. This defines the desired section s, and the decomposition (3.9.4) is immediate from Proposition 3.2.1. $\square$

Fix a section s as in Lemma 3.9.2. Recall from §3.1 the weight filtration $\mathrm{wt}_i V \subset V$ whose graded pieces

$$\begin{aligned}
\mathrm{gr}_{-1}V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-2}W) \\
\mathrm{gr}_{0}V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-1}W) \\
\mathrm{gr}_{1}V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{0}W)
\end{aligned}$$

have $\boldsymbol{k}$-dimensions 1, $n-2$, and 1, respectively. Recalling (3.1.1), which describes the action of Q_Φ on the graded pieces of V, the section s determines a splitting $V = V_{-1} \oplus V_0 \oplus V_1$ of the weight filtration by

$$\begin{aligned}
V_{-1} &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = \overline{a}v\} \\
V_{0} &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = v\} \\
V_{1} &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = a^{-1}v\}.
\end{aligned}$$

The summands V_{-1} and V_1 are isotropic $\boldsymbol{k}$-lines, and V_0 is the orthogonal complement of $V_{-1} + V_1$ with respect to the hermitian form on V. In particular, the restriction of the hermitian form to $V_0 \subset V$ is positive definite.

Fix an $a \in \widehat{\boldsymbol{k}}^\times$ and define an $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(s(a)g\mathfrak{a}_0, s(a)g\mathfrak{a}) \subset V.$$

Using the assumption $s(\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times) \subset K_\Phi$, we obtain a decomposition

$$L = L_{-1} \oplus L_0 \oplus L_1$$

with $L_i = L \cap V_i$. The images of the lattices L_i in the graded pieces $\mathrm{gr}_i V$ are given by

$$\begin{aligned}
L_{-1} &= \overline{a} \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-2}(g\mathfrak{a})) \\
L_{0} &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-1}(g\mathfrak{a})) \\
L_{1} &= a^{-1} \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{0}(g\mathfrak{a})).
\end{aligned}$$

In particular, L_0 is independent of a and agrees with (3.1.4).

Choose a $\mathbb{Z}$-basis $\mathrm{e}_{-1}, \mathrm{f}_{-1} \in L_{-1}$, and let $\mathrm{e}_1, \mathrm{f}_1 \in \mathfrak{d}^{-1}L_1$ be the dual basis with respect to the (perfect) $\mathbb{Z}$-bilinear pairing

$$[\,\cdot\,,\,\cdot\,] : L_{-1} \times \mathfrak{d}^{-1}L_1 \to \mathbb{Z}$$

obtained by restricting (2.1.6). This basis may be chosen so that

$$\begin{aligned}
L_{-1} &= \mathbb{Z}\mathrm{e}_{-1} + \mathbb{Z}\mathrm{f}_{-1} & \mathfrak{d}^{-1}L_{-1} &= \mathbb{Z}\mathrm{e}_{-1} + D^{-1}\mathbb{Z}\mathrm{f}_{-1} \\
L_{1} &= \mathbb{Z}\mathrm{e}_{1} + D\mathbb{Z}\mathrm{f}_{1} & \mathfrak{d}^{-1}L_{1} &= \mathbb{Z}\mathrm{e}_{1} + \mathbb{Z}\mathrm{f}_{1}.
\end{aligned} \tag{3.9.5}$$

As $\epsilon V_1(\mathbb{C}) \subset V_1(\mathbb{C})$ is a line, there is a unique $\tau \in \mathbb{C}$ satisfying

$$\tau\mathrm{e}_1 + \mathrm{f}_1 \in \epsilon V_1(\mathbb{C}). \tag{3.9.6}$$

After possibly replacing both e_1 and e_{-1} by their negatives, we may assume that $\mathrm{Im}(\tau) > 0$.

Proposition 3.9.3. *The $\mathbb{Z}$-lattice $\mathfrak{b} = \mathbb{Z}\tau + \mathbb{Z}$ is contained in $\boldsymbol{k}$, and is a fractional $\mathcal{O}_{\boldsymbol{k}}$-ideal. The elliptic curve*

$$E^{(a)}(\mathbb{C}) = \mathfrak{b}\backslash\mathbb{C} \tag{3.9.7}$$

is isomorphic to (3.9.2), and there is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism of complex abelian varieties

$$E^{(a)}(\mathbb{C}) \otimes L_0 \cong \mathfrak{b}L_0\backslash V_0(\mathbb{R}). \tag{3.9.8}$$

Under this isomorphism the inverse of line bundle (3.4.2) has the form

$$\mathcal{Q}^{-1}_{E^{(a)}(\mathbb{C})\otimes L_0} \cong \mathfrak{b}L_0\backslash(V_0(\mathbb{R}) \times \mathbb{C}), \tag{3.9.9}$$

where the action of $y_0 \in \mathfrak{b}L_0$ on $V_0(\mathbb{R}) \times \mathbb{C}$ is

$$y_0 \cdot (w_0, q) = \big(w_0 + \epsilon y_0, q \cdot e^{\pi i \frac{\langle y_0, y_0\rangle}{\mathrm{N}(\mathfrak{b})}} e^{-\pi \frac{\langle w_0, y_0\rangle}{\mathrm{Im}(\tau)} - \pi \frac{\langle y_0, y_0\rangle}{2\mathrm{Im}(\tau)}}\big).$$

Proof. Consider the $\mathbb{Q}$-linear map

$$V_{-1} \xrightarrow{\alpha \mathrm{e}_{-1} + \beta \mathrm{f}_{-1} \mapsto \alpha\tau + \beta} \mathbb{C}. \tag{3.9.10}$$

Its $\mathbb{C}$-linear extension $V_{-1}(\mathbb{C}) \to \mathbb{C}$ kills the vector $\mathrm{e}_{-1} - \tau \mathrm{f}_{-1} \in \epsilon V_{-1}(\mathbb{C})$, and hence factors through an isomorphism $V_{-1}(\mathbb{C})/\epsilon V_{-1}(\mathbb{C}) \cong \mathbb{C}$. This implies that (3.9.10) is $\boldsymbol{k}$-conjugate linear. As this map identifies $L_{-1} \cong \mathfrak{b}$, we find that the $\mathbb{Z}$-lattice $\mathfrak{b} \subset \mathbb{C}$ is $\mathcal{O}_{\boldsymbol{k}}$-stable. From $1 \in \mathfrak{b}$ we then deduce that $\mathfrak{b} \subset \boldsymbol{k}$, and is a fractional $\mathcal{O}_{\boldsymbol{k}}$-ideal. Moreover, we have just shown that

$$L_{-1} \xrightarrow{\alpha \mathrm{e}_{-1} + \beta \mathrm{f}_{-1} \mapsto \alpha\tau + \beta} \mathfrak{b}. \tag{3.9.11}$$

is an $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism.

Exactly as in (2.1.4), the self-dual hermitian forms on $g\mathfrak{a}_0$ and $g\mathfrak{a}$ induce an $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-2}(g\mathfrak{a})) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_0(g\mathfrak{a}), g\mathfrak{a}_0),$$

and hence determine an $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism

$$\begin{aligned} L_{-1} &= \overline{a} \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-2}(g\mathfrak{a})) \\ &\cong a \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_0(g\mathfrak{a}), g\mathfrak{a}_0) \\ &= a \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, g\mathfrak{a}_0). \end{aligned}$$

The composition

$$a \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, g\mathfrak{a}_0) \cong L_{-1} \xrightarrow{(3.9.11)} \mathfrak{b}$$

is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism, which identifies the fractional ideal $\mathfrak{b}$ with the projective $\mathcal{O}_{\boldsymbol{k}}$-module used in the definition of (3.9.2). In particular it identifies the elliptic curves (3.9.2) and (3.9.7), and also identifies

$$E^{(a)}(\mathbb{C}) \otimes L_0 = (\mathfrak{b}\backslash\mathbb{C}) \otimes L_0 \cong (\mathfrak{b} \otimes L_0)\backslash(\mathbb{C} \otimes L_0).$$

Here, and throughout the remainder of the proof, all tensor products are over $\mathcal{O}_{\boldsymbol{k}}$. Identifying $\mathbb{C} \otimes L_0 \cong V_0(\mathbb{R})$ proves (3.9.8).

It remains to explain the isomorphism (3.9.9). First consider the Poincaré bundle on the product

$$E^{(a)}(\mathbb{C}) \times E^{(a)}(\mathbb{C}) \cong (\mathfrak{b} \times \mathfrak{b}) \backslash (\mathbb{C} \times \mathbb{C}).$$

Using classical formulas, the space of this line bundle can be identified with the quotient

$$\mathcal{P}_{E^{(a)}(\mathbb{C})} = (\mathfrak{b} \times \mathfrak{b}) \backslash (\mathbb{C} \times \mathbb{C} \times \mathbb{C}),$$

where the action is given by

$$(b_1, b_2) \cdot (z_1, z_2, q) = \left(z_1 + b_1, z_2 + b_2, q \cdot e^{\pi H_\tau(z_1, b_2) + \pi H_\tau(z_2, b_1) + \pi H_\tau(b_1, b_2)} \right),$$

and we have set $H_\tau(w, z) = w\overline{z}/\mathrm{Im}(\tau)$ for complex numbers w and z.

Directly from the definition, the line bundle (3.4.2) on

$$E^{(a)}(\mathbb{C}) \otimes L_0 \cong (\mathfrak{b} \otimes L_0) \backslash (\mathbb{C} \otimes L_0)$$

is given by

$$\mathcal{Q}_{E^{(a)}(\mathbb{C}) \otimes L_0} \cong (\mathfrak{b} \otimes L_0) \backslash \big((\mathbb{C} \otimes L_0) \times \mathbb{C} \big),$$

where the action of $\mathfrak{b} \otimes L_0$ on $(\mathbb{C} \otimes L_0) \times \mathbb{C}$ is given as follows: Choose any set $x_1, \ldots, x_n \in L_0$ of $\mathcal{O}_{\boldsymbol{k}}$-module generators, and the extend the $\mathcal{O}_{\boldsymbol{k}}$-hermitian form on L_0 to a $\mathbb{C}$-hermitian form on $\mathbb{C} \otimes L_0$. If

$$y_0 = \sum_i b_i \otimes x_i \in \mathfrak{b} \otimes L_0$$

and

$$w_0 = \sum_i z_i \otimes x_i \in \mathbb{C} \otimes L_0$$

then

$$y_0 \cdot (w_0, q) = (w_0 + y_0, q \cdot e^{\pi X + \pi Y}),$$

where the factors X and Y are

$$\begin{aligned} X &= \sum_{i<j} \Big(H_\tau(\langle x_i, x_j \rangle z_i, b_j) + H_\tau(z_j, \langle x_i, x_j \rangle b_i) + H_\tau(\langle x_i, x_j \rangle b_i, b_j) \Big) \\ &= \frac{1}{\mathrm{Im}(\tau)} \sum_{i \neq j} \langle z_i \otimes x_i, b_j \otimes x_j \rangle + \frac{1}{\mathrm{Im}(\tau)} \sum_{i<j} \langle b_i \otimes x_i, b_j \otimes x_j \rangle \end{aligned}$$

and, recalling $\gamma = (1 + \delta)/2$,

$$\begin{aligned} Y &= \sum_i \Big(H_\tau(\gamma \langle x_i, x_i \rangle z_i, b_i) + H_\tau(z_i, \gamma \langle x_i, x_i \rangle b_i) + H_\tau(\gamma \langle x_i, x_i \rangle b_i, b_i) \Big) \\ &= \frac{1}{\mathrm{Im}(\tau)} \sum_i \langle z_i \otimes x_i, b_i \otimes x_i \rangle + \frac{1}{\mathrm{Im}(\tau)} \sum_i \gamma \langle b_i \otimes x_i, b_i \otimes x_i \rangle. \end{aligned}$$

For elements $y_1, y_2 \in \mathfrak{b} \otimes L_0$, we abbreviate

$$\alpha(y_1, y_1) = \frac{\langle y_1, y_2 \rangle}{\delta \mathrm{N}(\mathfrak{b})} - \frac{\langle y_2, y_1 \rangle}{\delta \mathrm{N}(\mathfrak{b})} \in \mathbb{Z}.$$

Using $2i\mathrm{Im}(\tau) = \delta\mathrm{N}(\mathfrak{b})$, some elementary calculations show that

$$\begin{aligned}
&\pi X + \pi Y - \frac{\pi\langle w_0, y_0\rangle}{\mathrm{Im}(\tau)} \\
&= \frac{2\pi i}{\delta\mathrm{N}(\mathfrak{b})}\sum_{i<j}\langle b_i\otimes x_i, b_j\otimes x_j\rangle + \frac{2\pi i}{\delta\mathrm{N}(\mathfrak{b})}\sum_i\langle \gamma b_i\otimes x_i, b_i\otimes x_i\rangle \\
&= \frac{\pi}{2\mathrm{Im}(\tau)}\sum_{i,j}\langle b_i\otimes x_i, b_j\otimes x_j\rangle - \frac{\pi i}{\mathrm{N}(\mathfrak{b})}\sum_{i,j}\langle b_i\otimes x_i, b_j\otimes x_j\rangle \\
&\quad + 2\pi i\sum_{i<j}\alpha(\gamma b_i\otimes x_i, b_j\otimes x_j) + \frac{2\pi i}{\mathrm{N}(\mathfrak{b})}\sum_i\langle b_i\otimes x_i, b_i\otimes x_i\rangle.
\end{aligned}$$

All terms in the final line lie in $2\pi i\mathbb{Z}$, and so

$$e^{\pi X+\pi Y} = e^{\frac{\pi\langle w_0,y_0\rangle}{\mathrm{Im}(\tau)}} e^{\frac{\pi\langle y_0,y_0\rangle}{2\mathrm{Im}(\tau)}} e^{-\frac{\pi i\langle y_0,y_0\rangle}{\mathrm{N}(\mathfrak{b})}}.$$

The relation (3.9.9) follows immediately. □

Proposition 3.9.3 allows us to express Fourier-Jacobi coefficients explicitly as functions on $V_0(\mathbb{R})$ satisfying certain transformation laws. Suppose we start with a global section

$$\psi \in H^0\big(\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}, \boldsymbol{\omega}^k\big). \tag{3.9.12}$$

For each $a \in \widehat{\boldsymbol{k}}^\times$ and $\ell \geqslant 0$ we have the algebraically defined Fourier-Jacobi coefficient

$$\mathrm{FJ}^{(a)}_\ell(\psi) \in H^0\big(E^{(a)}\otimes L_0, \mathcal{Q}^\ell_{E^{(a)}\otimes L_0}\big), \tag{3.9.13}$$

of (3.9.3), where we have trivialized $\mathrm{Lie}(E^{(a)})$ using (3.9.7). The isomorphism (3.9.9) now identifies (3.9.13) with a function on $V_0(\mathbb{R})$ satisfying the transformation law

$$\mathrm{FJ}^{(a)}_\ell(\psi)(w_0+y_0) = \mathrm{FJ}^{(a)}_\ell(\psi)(w_0)\cdot e^{i\pi\ell\frac{\langle y_0,y_0\rangle}{\mathrm{N}(\mathfrak{b})}} e^{\pi\ell\frac{\langle w_0,y_0\rangle}{\mathrm{Im}(\tau)}+\pi\ell\frac{\langle y_0,y_0\rangle}{2\mathrm{Im}(\tau)}} \tag{3.9.14}$$

for all $y_0 \in \mathfrak{b}L_0$.

Remark 3.9.4. If we use the isomorphism $\mathrm{pr}_\epsilon : V_0(\mathbb{R}) \cong \epsilon V_0(\mathbb{C})$ of (2.1.7) to view (3.9.13) as a function of $w_0 \in \epsilon V_0(\mathbb{C})$, the transformation law can be expressed in terms of the $\mathbb{C}$-bilinear form $[\cdot,\cdot]$ as

$$\mathrm{FJ}^{(a)}_\ell(\psi)(w_0+\mathrm{pr}_\epsilon(y_0)) = \mathrm{FJ}^{(a)}_\ell(\psi)(w_0)\cdot e^{i\pi\ell\frac{Q(y_0)}{\mathrm{N}(\mathfrak{b})}} e^{\pi\ell\frac{[w_0,y_0]}{\mathrm{Im}(\tau)}+\pi\ell\frac{Q(y_0)}{2\mathrm{Im}(\tau)}}$$

for all $y_0 \in \mathfrak{b}L_0$. This uses the (slightly confusing) commutativity of

$$\begin{tikzcd}
V_0(\mathbb{R}) \arrow[r, "\mathrm{pr}_\epsilon"] \arrow[d, "\langle\cdot,y_0\rangle"'] & \epsilon V_0(\mathbb{C}) \arrow[r, "\subset"] & V_0(\mathbb{C}) \arrow[d, "{[\cdot,y_0]}"] \\
\boldsymbol{k}\otimes_{\mathbb{Q}}\mathbb{R} \arrow[rr, equal] & & \mathbb{C}.
\end{tikzcd}$$

In order to give another interpretation of our explicit coordinates, let $N_\Phi \subset Q_\Phi$ be the unipotent radical, and let $U_\Phi \subset N_\Phi$ be its center. The unipotent radical may be characterized as the kernel of the morphism ν_Φ of (3.1.1), or, equivalently, as the largest subgroup acting trivially on all graded pieces $\mathrm{gr}_i V$.

Proposition 3.9.5. *There is a commutative diagram*

$$\begin{tikzcd} (U_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1})\backslash \mathcal{D}_\Phi \arrow[r, "z\mapsto(w_0,q)"] \arrow[d] & \epsilon V_0(\mathbb{C}) \times \mathbb{C}^\times \arrow[d] \\ (N_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1})\backslash \mathcal{D}_\Phi \arrow[r] & \mathfrak{b}L_0\backslash(\epsilon V_0(\mathbb{C}) \times \mathbb{C}^\times) \end{tikzcd} \tag{3.9.15}$$

in which the horizontal arrows are holomorphic isomorphisms, and the action of $\mathfrak{b}L_0$ *on*

$$\epsilon V_0(\mathbb{C}) \times \mathbb{C}^\times \cong V_0(\mathbb{R}) \times \mathbb{C}^\times$$

is the same as in Proposition 3.9.3

Proof. Recall from Remark 3.2.3 the isomorphism

$$\mathcal{D}_\Phi \cong \big\{ w \in \epsilon V(\mathbb{C}) : \epsilon V(\mathbb{C}) = \epsilon V_{-1}(\mathbb{C}) \oplus \epsilon V_0(\mathbb{C}) \oplus \mathbb{C}w \big\}/\mathbb{C}^\times.$$

As $\epsilon V(\mathbb{C})$ is totally isotropic with respect to $[\cdot,\cdot]$, a simple calculation shows that every line $w \in \mathcal{D}_\Phi$ has a unique representative of the form

$$-\xi(\mathrm{e}_{-1} - \tau \mathrm{f}_{-1}) + w_0 + (\tau \mathrm{e}_1 + \mathrm{f}_1) \in \epsilon V_{-1}(\mathbb{C}) \oplus \epsilon V_0(\mathbb{C}) \oplus \epsilon V_1(\mathbb{C})$$

with $\xi \in \mathbb{C}$ and $w_0 \in \epsilon V_0(\mathbb{C}) = V_0(\mathbb{R})$. These coordinates define an isomorphism of complex manifolds

$$\mathcal{D}_\Phi \xrightarrow{w\mapsto(w_0,\xi)} \epsilon V_0(\mathbb{C}) \times \mathbb{C}. \tag{3.9.16}$$

The action of G on V restricts to a faithful action of N_Φ, allowing us to express elements of $N_\Phi(\mathbb{Q})$ as matrices

$$n(\phi,\phi^*,u) = \begin{pmatrix} 1 & \phi^* & u + \frac{1}{2}\phi^*\circ\phi \\ & 1 & \phi \\ & & 1 \end{pmatrix} \in N_\Phi(\mathbb{Q})$$

for maps

$$\phi \in \mathrm{Hom}_{\boldsymbol{k}}(V_1, V_0), \quad \phi^* \in \mathrm{Hom}_{\boldsymbol{k}}(V_0, V_{-1}), \quad u \in \mathrm{Hom}_{\boldsymbol{k}}(V_1, V_{-1})$$

satisfying the relations

$$0 = \langle \phi(x_1), y_0\rangle + \langle x_1, \phi^*(y_0)\rangle$$
$$0 = \langle u(x_1), y_1\rangle + \langle x_1, u(y_1)\rangle$$

for $x_i, y_i \in V_i$. The subgroup $U_\Phi(\mathbb{Q})$ is defined by $\phi = 0 = \phi^*$.

The group $U_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1}$ is cyclic, and generated by the element $n(0,0,u)$ defined by

$$u(x_1) = \frac{\langle x_1, a\rangle}{[L_{-1} : \mathcal{O}_{\boldsymbol{k}} a]} \cdot \delta a$$

for any $a \in L_{-1}$. In terms of the bilinear form, this can be rewritten as

$$u(x_1) = -[x_1, \mathrm{f}_{-1}]\mathrm{e}_{-1} + [x_1, \mathrm{e}_{-1}]\mathrm{f}_{-1}.$$

In the coordinates of (3.9.16), the action of $n(0,0,u)$ on $\mathcal{D}_\Phi$ becomes

$$(w_0, \xi) \mapsto (w_0, \xi + 1),$$

and setting $q = e^{2\pi i \xi}$ defines the top horizontal isomorphism in (3.9.15).

Let $\overline{V}_{-1} = V_{-1}$ with its conjugate action of $\boldsymbol{k}$. There are group isomorphisms

$$N_\Phi(\mathbb{Q})/U_\Phi(\mathbb{Q}) \cong \overline{V}_{-1} \otimes_{\boldsymbol{k}} V_0 \cong V_0. \tag{3.9.17}$$

The first sends

$$n(\phi, \phi^*, u) \mapsto y_{-1} \otimes y_0,$$

where y_{-1} and y_0 are defined by the relation $\phi(x_1) = \langle x_1, y_{-1} \rangle \cdot y_0$, and the second sends

$$(\alpha \mathrm{e}_{-1} + \beta \mathrm{f}_{-1}) \otimes y_0 \mapsto (\alpha\tau + \beta) y_0.$$

Compare with (3.9.11).

A slightly tedious calculation shows that (3.9.17) identifies

$$(N_\Phi(\mathbb{Q}) \cap s(a) K_\Phi s(a)^{-1})/(U_\Phi(\mathbb{Q}) \cap s(a) K_\Phi s(a)^{-1}) \cong \mathfrak{b} L_0,$$

defining the bottom horizontal arrow in (3.9.15), and that the resulting action of $\mathfrak{b}L_0$ on $\epsilon V_0(\mathbb{C}) \times \mathbb{C}^\times$ agrees with the one defined in Proposition 3.9.3. We leave this to the reader. $\square$

Any section (3.9.12) may now be pulled back via

$$(N_\Phi(\mathbb{Q}) \cap s(a) K_\Phi s(a)^{-1}) \backslash \mathcal{D} \xrightarrow{z \mapsto (z, s(a) g)} \mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$$

to define a holomorphic section of $(\boldsymbol{\omega}^{an})^k$, the k^{th} power of the tautological bundle on

$$\mathcal{D} \cong \big\{ w \in \epsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^\times.$$

The tautological bundle admits a natural $N_\Phi(\mathbb{R})$-equivariant trivialization: any line $w \in \mathcal{D}$ must satisfy $[w, \mathrm{f}_{-1}] \neq 0$, yielding an isomorphism

$$[\,\cdot\,, \mathrm{f}_{-1}] : \boldsymbol{\omega}^{an} \cong \mathcal{O}_{\mathcal{D}}.$$

This trivialization allows us to identify ψ with a holomorphic function on $\mathcal{D} \subset \mathcal{D}_\Phi$, which then has an *analytic* Fourier-Jacobi expansion

$$\psi = \sum_{\ell} \mathrm{FJ}_\ell^{(a)}(\psi)(w_0) \cdot q^\ell \tag{3.9.18}$$

defined using the coordinates of Proposition 3.9.5. The fact that the coefficients here agree with (3.9.13) is a special case of the main results of [Lan12], which compare algebraic and analytic Fourier-Jacobi coefficients on general PEL-type Shimura varieties.

4. Classical modular forms

Throughout §4 we let D be any odd squarefree positive integer, and abbreviate $\Gamma = \Gamma_0(D)$. Let k be any positive integer.

4.1. Weakly holomorphic forms. The positive divisors of D are bijection with the cusps of the complex modular curve $X_0(D)(\mathbb{C})$, by sending $r \mid D$ to

$$\infty_r = \frac{r}{D} \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q}).$$

Note that $r = 1$ corresponds to the usual cusp at infinity, and so we sometimes abbreviate $\infty = \infty_1$.

Fix a positive divisor $r \mid D$, set $s = D/r$ and choose

$$R_r = \begin{pmatrix} \alpha & \beta \\ s\gamma & r\delta \end{pmatrix} \in \Gamma_0(s)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. The corresponding Aktin-Lehner operator is defined by the matrix

$$W_r = \begin{pmatrix} r\alpha & \beta \\ D\gamma & r\delta \end{pmatrix} = R_r \begin{pmatrix} r & \\ & 1 \end{pmatrix}.$$

The matrix W_r normalizes Γ, and so acts on the cusps of $X_0(D)(\mathbb{C})$. This action satisfies $W_r \cdot \infty = \infty_r$.

Let χ be a quadratic Dirichlet character modulo D, and let

$$\chi = \chi_r \cdot \chi_s$$

be the unique factorization as a product of quadratic Dirichlet characters χ_r and χ_s modulo r and s, respectively. Write

$$M_k(D, \chi) \subset M_k^!(D, \chi)$$

for the spaces of holomorphic modular forms and weakly holomorphic modular forms of weight k, level Γ, and character χ. We assume that $\chi(-1) = (-1)^k$, since otherwise $M_k^!(D, \chi) = 0$.

Denote by $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ the subgroup of elements with positive determinant. It acts on functions on the upper half plane by the usual weight k slash operator

$$(f \mid_k \gamma)(\tau) = \det(\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

and $f \mapsto f \mid_k W_r$ defines an endomorphism of $M_k^!(D, \chi)$ satisfying

$$f \mid_k W_r^2 = \chi_r(-1)\chi_s(r) \cdot f.$$

In particular, W_r is an involution when χ is trivial.

Any weakly holomorphic modular form

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M_k^!(D, \chi)$$

determines another weakly holomorphic modular form

$$\chi_r(\beta)\chi_s(\alpha) \cdot (f \mid_k W_r) \in M_k^!(D, \chi),$$

which is easily seen to be independent of the choice of parameters $\alpha, \beta, \gamma, \delta$ in the definition of W_r. This second modular form has a q-expansion at ∞, denoted

$$\chi_r(\beta)\chi_s(\alpha) \cdot (f \mid_k W_r) = \sum_{m \gg -\infty} c_r(m) \cdot q^m. \tag{4.1.1}$$

Definition 4.1.1. We call (4.1.1) the *q-expansion of f at ∞_r*. Of special interest is $c_r(0)$, the *constant term of f at ∞_r*.

Remark 4.1.2. If χ is nontrivial, the coefficients of (4.1.1) need not lie in the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f.

4.2. **Eisenstein series and the modularity criterion.** If $k > 2$ we may define an Eisenstein series

$$E = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi(d) \cdot (1 \mid_k \gamma) \in M_k(D, \chi).$$

Here $\Gamma_\infty \subset \Gamma$ is the stabilizer of $\infty \in \mathbb{P}^1(\mathbb{Q})$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Define the (normalized) Eisenstein series for the cusp ∞_r by

$$E_r = \chi_r(-\beta)\chi_s(\alpha r) \cdot (E \mid_k W_r) \in M_k(D, \chi).$$

It is independent of the choice of the parameters in W_r, and we denote by

$$E_r(\tau) = \sum_{m \geqslant 0} e_r(m) \cdot q^m$$

its q-expansion at ∞.

Remark 4.2.1. Our notation for the q-expansion of E_r is slightly at odds with (4.1.1), as the q-expansion of E at ∞_r is not $\sum e_r(m) q^m$. Instead, the q-expansion of E at ∞_r is $\chi_r(-1)\chi_s(r) \sum e_r(m) q^m$, while the q-expansion of E_r at ∞_r is $\sum e_1(m) q^m$. In any case, what matters most is that

$$\text{constant term of } E_r \text{ at } \infty_s = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{otherwise.} \end{cases}$$

Denote by

$$M^{!,\infty}_{2-k}(D, \chi) \subset M^!_{2-k}(D, \chi)$$

the subspace of weakly holomorphic forms that are holomorphic outside the cusp ∞, and by

$$M^\infty_k(D, \chi) \subset M_k(D, \chi)$$

the subspace of holomorphic modular forms that vanish at all cusps different from ∞. If $k > 2$ then

$$M^\infty_k(D, \chi) = \mathbb{C}E \oplus S_k(D, \chi)$$

for the weight k Eisenstein series E of §4.2. The constant terms of weakly holomorphic modular forms in $M^{!,\infty}_{2-k}(D, \chi)$ can be computed using the above Eisenstein series.

Proposition 4.2.2. *Assume $k > 2$. Suppose $r \mid D$ and*

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M^{!,\infty}_{2-k}(D, \chi).$$

The constant term of f at the cusp ∞_r, in the sense of Definition 4.1.1, satisfies

$$c_r(0) + \sum_{m>0} c(-m) e_r(m) = 0.$$

Proof. The meromorphic differential form $f(\tau)E_r(\tau)\,d\tau$ on $X_0(D)(\mathbb{C})$ is holomorphic away from the cusps ∞ and ∞_r. Summing its residues at these cusps gives the desired equality. □

Theorem 4.2.3 (Modularity criterion)**.** *Suppose $k \geqslant 2$. For a formal power series*

$$\sum_{m \geqslant 0} d(m) q^m \in \mathbb{C}[[q]], \tag{4.2.1}$$

the following are equivalent:

(1) *The relation $\sum_{m \geqslant 0} c(-m)d(m) = 0$ holds for every weakly holomorphic form*
$$\sum_{m \gg -\infty} c(m) \cdot q^m \in M^{!,\infty}_{2-k}(D, \chi).$$
(2) *The formal power series (4.2.1) is the q-expansion of a modular form in $M^{\infty}_k(D, \chi)$.*

Proof. As we assume $k \geqslant 2$, that the map sending a weakly holomorphic modular form f to its principal part at ∞ identifies

$$M^{!,\infty}_{2-k}(D, \chi) \subset \mathbb{C}[q^{-1}].$$

On the other hand, the map sending a holomorphic modular form to its q-expansion identifies

$$M^{\infty}_k(D, \chi) \subset \mathbb{C}[[q]].$$

A slight variant of the modularity criterion of [Bor99, Theorem 3.1] shows that each subspace is the exact annihilator of the other under the bilinear pairing $\mathbb{C}[q^{-1}] \otimes \mathbb{C}[[q]] \longrightarrow \mathbb{C}$ sending $P \otimes g$ to the constant term of $P \cdot g$. The claim follows. □

5. Unitary Borcherds products

The goal of §5 is to state Theorems 5.3.1, 5.3.3, and 5.3.4, which asserts the existence of Borcherds products on $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ having prescribed divisors and prescribed leading Fourier-Jacobi coefficients. These theorems are the technical core of this work, and their proofs will occupy all of §6.

5.1. **Jacobi forms.** In this section we recall some of the rudiments of the arithmetic theory of Jacobi forms. A more systematic treatment can be found in the work of Kramer [Kra91, Kra95].

Let $\mathcal{Y}$ be the moduli stack over $\mathbb{Z}$ classifying elliptic curves, and let $\pi : \mathcal{E} \to \mathcal{Y}$ be the universal elliptic curve. Abbreviate $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $\mathfrak{H}$ be the complex upper half-plane. The groups Γ and $\mathbb{Z}^2$ each act on $\mathfrak{H} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right), \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot (\tau, z) = (\tau, z + \alpha\tau + \beta),$$

and this defines an action of the semi-direct product $\Gamma^* = \Gamma \ltimes \mathbb{Z}^2$. We identify the commutative diagrams

$$\begin{tikzcd} \Gamma\backslash(\mathfrak{H}\times\mathbb{C}) \arrow[d] \arrow[dr] & \\ \Gamma^*\backslash(\mathfrak{H}\times\mathbb{C}) \arrow[r] & \Gamma\backslash\mathfrak{H} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Lie}(\mathcal{E}(\mathbb{C})) \arrow[d, "\exp"'] \arrow[dr] & \\ \mathcal{E}(\mathbb{C}) \arrow[r] & \mathcal{Y}(\mathbb{C}) \end{tikzcd} \tag{5.1.1}$$

by sending $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$ to the vector z in the Lie algebra of $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

Define a line bundle $\mathcal{O}(e)$ on $\mathcal{E}$ as the inverse ideal sheaf of the zero section $e : \mathcal{Y} \to \mathcal{E}$. The Lie algebra $\mathrm{Lie}(\mathcal{E})$ is (by definition) $e^*\mathcal{O}(e)$, and $\boldsymbol{\omega}_{\mathcal{Y}} = \mathrm{Lie}(\mathcal{E})^{-1}$ is the usual line bundle of weight one modular forms on $\mathcal{Y}$ (see Remark 5.1.3 below). In particular, the line bundle

$$\mathcal{Q} = \mathcal{O}(e) \otimes \pi^*\boldsymbol{\omega}_{\mathcal{Y}}$$

on $\mathcal{E}$ is canonically trivialized along the zero section. For a scheme U and a point $a \in \mathcal{E}(U)$, denote by $\mathcal{Q}(a)$ the pullback of $\mathcal{Q}$ via $a : U \to \mathcal{E}$.

Denote by $\mathcal{P}$ the pullback of the Poincaré bundle via the canonical isomorphism $\mathcal{E} \times_{\mathcal{Y}} \mathcal{E} \cong \mathcal{E} \times_{\mathcal{Y}} \mathcal{E}^\vee$. If U is any scheme and $a, b \in \mathcal{E}(U)$, we obtain a line bundle $\mathcal{P}(a, b)$ on U exactly as in (3.4.1). There are canonical isomorphisms

$$\mathcal{P}(a, b) \cong \mathcal{Q}(a + b) \otimes \mathcal{Q}(a)^{-1} \otimes \mathcal{Q}(b)^{-1}$$

and $\mathcal{P}(a, a) \cong \mathcal{Q}(a) \otimes \mathcal{Q}(a)$.

Definition 5.1.1. The diagonal restriction

$$\mathcal{J}_{0,1} = (\mathrm{diag})^*\mathcal{P} \cong \mathcal{Q}^2$$

is the line bundle of *Jacobi forms of weight* 0 *and index* 1 on $\mathcal{E}$. More generally,

$$\mathcal{J}_{k,m} = \mathcal{J}_{0,1}^m \otimes \pi^*\boldsymbol{\omega}_{\mathcal{Y}}^k$$

is the line bundle of *Jacobi forms of weight* k *and index* m on $\mathcal{E}$.

The isomorphism of the following proposition is presumably well-known. We include the proof in order to make explicit the normalization of the isomorphism (see Remark 5.1.3 below, for example).

Proposition 5.1.2. *Let $p : \mathfrak{H} \times \mathbb{C} \to \mathcal{E}(\mathbb{C})$ be the quotient map. The holomorphic line bundle $\mathcal{J}^{an}_{k,m}$ on $\mathcal{E}(\mathbb{C})$ is isomorphic to the holomorphic line bundle whose sections over an open set $\mathscr{U} \subset \mathcal{E}(\mathbb{C})$ are holomorphic functions $F(\tau, z)$ on $p^{-1}(\mathscr{U})$ satisfying the transformation laws*

$$F\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = F(\tau, z) \cdot (c\tau+d)^k \cdot e^{2\pi imcz^2/(c\tau+d)}$$

and

$$F(\tau, z+\alpha\tau+\beta) = F(\tau, z) \cdot e^{-2\pi im(\alpha^2\tau+2\alpha z)}. \tag{5.1.2}$$

Proof. Let $J_{k,m}$ be the holomorphic line bundle on $\mathcal{E}(\mathbb{C})$ defined by the above transformation laws.

By identifying the diagrams (5.1.1), a function f, defined on a Γ-invariant open subset of $\mathfrak{H}$ and satisfying the transformation law

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \cdot (c\tau+d)^{-1}$$

of a weight -1 modular form, defines a section $\tau \mapsto (\tau, f(\tau))$ of the line bundle

$$\Gamma\backslash(\mathfrak{H} \times \mathbb{C}) \cong \mathrm{Lie}(\mathcal{E}(\mathbb{C})) \cong (\boldsymbol{\omega}^{an}_{\mathcal{Y}})^{-1}$$

on $\Gamma\backslash\mathfrak{H}$. This determines an isomorphism $J_{1,0} \cong \mathcal{J}^{an}_{1,0}$. It now suffices to construct an isomorphism $J_{0,1} \cong \mathcal{J}^{an}_{0,1}$, and then take tensor products.

Fix $\tau \in \mathfrak{H}$, set $E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, and restrict both $\mathcal{J}^{an}_{0,1}$ and $J_{0,1}$ to line bundles on $E_\tau \subset \mathcal{E}(\mathbb{C})$. The imaginary part of the hermitian form

$$H_\tau(z_1, z_2) = \frac{z_1\overline{z_2}}{\mathrm{Im}(\tau)}$$

on $\mathbb{C}$ restricts to a Riemann form on $\mathbb{Z}\tau + \mathbb{Z}$. By classical formulas for the Poincaré bundle on complex abelian varieties, the restriction of $\mathcal{J}^{an}_{0,1}$ to the fiber E_τ is isomorphic to the holomorphic line bundle determined by the Appell-Humbert data $2H_\tau$ and the trivial character $\mathbb{Z}\tau + \mathbb{Z} \to \mathbb{C}^\times$. The sections of this holomorphic line bundle are, by definition, holomorphic functions g_τ on $\mathbb{C}$ satisfying the transformation law

$$g_\tau(z+\ell) = g_\tau(z) \cdot e^{2\pi H_\tau(z,\ell)+\pi H_\tau(\ell,\ell)}$$

for all $\ell \in \mathbb{Z}\tau + \mathbb{Z}$. If we set

$$F(\tau, z) = g_\tau(z) \cdot e^{-\pi H_\tau(z,\overline{z})},$$

this transformation law becomes (5.1.2).

The above shows that $\mathcal{J}^{an}_{0,1}$ and $J_{0,1}$ are isomorphic when restricted to the fiber over any point of $\mathcal{Y}(\mathbb{C})$, but such an isomorphism is only determined up to scaling by $\mathbb{C}^\times$. To pin down the scalars, and to get an isomorphism over the total space, use the fact that both $\mathcal{J}^{an}_{0,1}$ and $J_{0,1}$ come (by construction) with canonical trivializations along the zero section. By the Seesaw Theorem [BL04, Appendix A], there is a unique isomorphism $\mathcal{J}^{an}_{0,1} \cong J_{0,1}$ compatible with these trivializations. $\square$

Remark 5.1.3. The proof of Proposition 5.1.2 identifies a classical modular form $f(\tau) = \sum c(m)q^m$ of weight k and level Γ with a holomorphic section of $(\boldsymbol{\omega}_{\mathcal{Y}}^{an})^k$, again denoted f, satisfying an additional growth condition at the cusp. Under our identification, the q-expansion principle takes the following form: if $R \subset \mathbb{C}$ is any subring, then f is the analytification of a global section $f \in H^0(\mathcal{Y}_{/R}, \boldsymbol{\omega}^k_{\mathcal{Y}/R})$ if and only if $c(m) \in (2\pi i)^k R$ for all m.

For $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}$, we denote by

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau + 2\pi i \left(n + \frac{1}{2}\right)\left(z - \frac{1}{2}\right)}$$

the classical Jacobi theta function, and by

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2n\pi i \tau})$$

Dedekind's eta function. Set

$$\Theta(\tau, z) \stackrel{\text{def}}{=} i \frac{\vartheta_1(\tau, z)}{\eta(\tau)} = q^{1/12}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty} (1 - \zeta q^n)(1 - \zeta^{-1} q^n)$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$.

Proposition 5.1.4. *The Jacobi form* Θ^{24} *defines a global section*

$$\Theta^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$$

with divisor $24e$*, while* $(2\pi i \eta^2)^{12}$ *determines a nowhere vanishing section*

$$(2\pi i \eta^2)^{12} \in H^0(\mathcal{Y}, \boldsymbol{\omega}_{\mathcal{Y}}^{12}).$$

Proof. It is a classical fact that $(2\pi i \eta^2)^{12}$ is a nowhere vanishing modular form of weight 12. Noting Remark 5.1.3, the q-expansion principle shows that it descends to a section on $\mathcal{Y}_{/\mathbb{Q}}$, and thus may be viewed as a rational section on $\mathcal{Y}$. Another application of the q-expansion principle shows that its divisor has no vertical components. Thus its divisor is trivial.

Classical formulas show that Θ^{24} defines a holomorphic section of $\mathcal{J}_{0,12}^{an}$ with divisor $24e$, and so the problem is to show that Θ^{24} is defined over $\mathbb{Q}$, and extends to a section on the integral model with the stated divisors. One could presumably deduce this from the q-expansion principle for Jacobi forms as in [Kra91, Kra95]. We instead borrow an argument from [Sch98, §1.2], which requires only the more elementary q-expansion principle for *functions* on $\mathcal{E}$.

Let d be any positive integer. The bilinear relations (3.4.1) imply that the line bundle $\mathcal{J}_{0,1}^{d^2} \otimes [d]^* \mathcal{J}_{0,1}^{-1}$ on $\mathcal{E}$ is canonically trivial, and so

$$\theta_d^{24} = \Theta^{24d^2} \otimes [d]^* \Theta^{-24}$$

defines a meromorphic function on $\mathcal{E}(\mathbb{C})$. The crucial point is that θ_d^{24} is actually a rational function defined over $\mathbb{Q}$, and extends to a rational function on the integral model $\mathcal{E}$ with divisor

$$\operatorname{div}(\theta_d^{24}) = 24\big(d^2\mathcal{E}[1] - \mathcal{E}[d]\big). \tag{5.1.3}$$

As in [Sch98, p. 387], this follows by computing the divisor first in the complex fiber, then using the explicit formula

$$\theta_d^{24}(\tau, z) = q^{2(d^2-1)}\zeta^{-12d(d-1)}\left(\prod_{n\geqslant 0}\frac{(1-q^n\zeta)^{d^2}}{1-q^n\zeta^d}\prod_{n>0}\frac{(1-q^n\zeta^{-1})^{d^2}}{1-q^n\zeta^{-d}}\right)^{24}$$

and the q-expansion principle on $\mathcal{E}$ to see that the divisor has no vertical components.

The line bundle $\boldsymbol{\omega}_{\mathcal{Y}}^{12}$ is trivial, and hence there are isomorphisms

$$\mathcal{J}_{0,12} \cong \mathcal{Q}^{24} \cong \mathcal{O}(e)^{24} \otimes \pi^*\boldsymbol{\omega}_{\mathcal{Y}}^{12} \cong \mathcal{O}(e)^{24}.$$

Thus there is *some* $\tilde{\Theta}^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$ with divisor $24e$, and the rational function

$$\tilde{\theta}_d^{24} = \tilde{\Theta}^{24d^2} \otimes [d]^*\tilde{\Theta}^{-24}$$

on $\mathcal{E}$ also has divisor (5.1.3).

Consider the meromorphic function $\rho = \Theta^{24}/\tilde{\Theta}^{24}$ on $\mathcal{E}(\mathbb{C})$. By computing the divisor in the complex fiber, we see that ρ is a nowhere vanishing holomorphic function, and hence is constant. But this implies that

$$\rho^{d^2-1} = \theta_d^{24}/\tilde{\theta}_d^{24}.$$

By what was said above, the right hand side is (the analytification of) a nowhere vanishing function on $\mathcal{E}$. This implies that $\rho^{d^2-1} = \pm 1$, and the only way this can hold for all $d > 1$ is if $\rho = \pm 1$. $\square$

Now consider the tower of modular curves

$$\mathcal{Y}_1(D) \to \mathcal{Y}_0(D) \to \mathcal{Y}$$

over $\operatorname{Spec}(\mathbb{Z})$ parametrizing elliptic curves with Drinfeld $\Gamma_1(D)$-level structure, $\Gamma_0(D)$-level structure, and no level structure, respectively. We denote by $\mathcal{E}$ the universal elliptic curve over any one of these bases, and view the line bundle of Jacobi forms $\mathcal{J}_{0,12}$ as a line bundle on any one of the three universal elliptic curves. Similarly, we view the Jacobi forms Θ^{24} and $(2\pi i\eta^2)^{12}$ of Proposition 5.1.4 as being defined over any one of these bases.

The following lemma will be needed in §5.3.

Lemma 5.1.5. *Let $Q : \mathcal{Y}_1(D) \to \mathcal{E}$ be the universal D-torsion point. For any $r \mid D$ the line bundle*

$$\bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} (bQ)^*\mathcal{J}_{0,12} \tag{5.1.4}$$

on $\mathcal{Y}_1(D)$ is canonically trivial, and its section

$$F_r^{24} = \bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z} \\ b\neq 0 \\ rb=0}} (bQ)^*\Theta^{24}$$

admits a canonical descent, denoted the same way, to a section of the trivial bundle on $\mathcal{Y}_0(D)$.

Proof. If $x_1, \ldots, x_r$ are integers representing the r-torsion subgroup of $\mathbb{Z}/D\mathbb{Z}$, then $6\sum x_i^2 \equiv 0 \pmod{D}$. The bilinear relations (3.4.1) therefore provide a canonical isomorphism

$$\bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z} \\ b\neq 0 \\ rb=0}} \mathcal{P}(bQ, bQ)^{\otimes 12} \cong \bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z} \\ b\neq 0 \\ rb=0}} \mathcal{P}(Q, 12b^2Q) \cong \mathcal{P}(Q, e) \cong \mathcal{O}_{\mathcal{Y}_1(D)}$$

of line bundles on $\mathcal{Y}_1(D)$. This is the desired trivialization of (5.1.4). The section F_r^{24} is obviously invariant under the action of the diamond operators on $\mathcal{Y}_1(D)$, and so descends to $\mathcal{Y}_0(D)$. $\square$

5.2. **Borcherds' quadratic identity.** For the remainder of §5 we denote by $\chi_{\boldsymbol{k}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ the Dirichlet character determined by the extension $\boldsymbol{k}/\mathbb{Q}$, abbreviate

$$\chi = \chi_{\boldsymbol{k}}^{n-2}, \tag{5.2.1}$$

and fix a weakly holomorphic form

$$f(\tau) = \sum_{m \gg -\infty} c(m)q^m \in M^{!,\infty}_{2-n}(D, \chi) \tag{5.2.2}$$

with $c(m) \in \mathbb{Z}$ for all $m \leqslant 0$.

For a proper cusp label representative Φ as in Definition 3.1.1, recall the self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice L_0 of signature $(n-2, 0)$ defined by (3.1.4). The hermitian form on L_0 determines a quadratic form $Q(x) = \langle x, x\rangle$, with associated $\mathbb{Z}$-bilinear form $[x_1, x_2] = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle x_1, x_2\rangle$ of signature $(2n-4, 0)$.

The modularity criterion of Theorem 4.2.3 implies the following identity of quadratic forms on $L_0 \otimes \mathbb{R}$.

Proposition 5.2.1 (Borcherds' quadratic identity)**.** *For all $u \in L_0 \otimes \mathbb{R}$,*

$$\sum_{x\in L_0} c(-Q(x)) \cdot [u, x]^2 = \frac{[u, u]}{2n-4} \sum_{x\in L_0} c(-Q(x)) \cdot [x, x].$$

Proof. The homogeneous polynomial

$$P(u, v) = [u, v]^2 - \frac{[u, u] \cdot [v, v]}{2n-4}$$

on $L_0 \otimes \mathbb{R}$ is harmonic in both variables u and v. For any fixed $u \in L_0 \otimes \mathbb{R}$ there is a corresponding theta series

$$\theta(\tau, u, P) = \sum_{x\in L_0} P(u, x) \cdot q^{Q(x)} \in S_n(D, \chi).$$

The modularity criterion of Theorem 4.2.3 therefore shows that

$$\sum_{m>0} c(-m) \sum_{\substack{x\in L_0 \\ Q(x)=m}} \left([u,x]^2 - \frac{[u,u]\cdot[x,x]}{2n-4} \right) = 0$$

for all $u \in L_0 \otimes \mathbb{R}$. This implies the assertion. $\square$

Recall from (3.6.2) that every $x \in L_0$ determines a diagram

$$\begin{tikzcd} \mathcal{B}_\Phi \arrow[r, "j_x"] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{A}_\Phi \arrow[r, "j"] & \mathcal{Y}_0(D), \end{tikzcd} \tag{5.2.3}$$

where, changing notation slightly from §5.1, $\mathcal{Y}_0(D)$ is now the open modular curve over $\mathcal{O}_{\boldsymbol{k}}$. Recall also that $\mathcal{B}_\Phi$ carries a distinguished line bundle $\mathcal{L}_\Phi$ defined by (3.3.1), used to define the Fourier-Jacobi expansions of (3.8.1). We will use Borcherds' quadratic identity to relate the line bundle $\mathcal{L}_\Phi$ to the line bundle $\mathcal{J}_{0,1}$ of Jacobi forms on $\mathcal{E}$.

Proposition 5.2.2. *The rational number*

$$\mathrm{mult}_\Phi(f) = \sum_{m>0} \frac{m \cdot c(-m)}{n-2} \cdot \#\{x \in L_0 : Q(x) = m\} \tag{5.2.4}$$

lies in $\mathbb{Z}$, and there is a canonical isomorphism

$$\mathcal{L}_\Phi^{2\cdot \mathrm{mult}_\Phi(f)} \cong \bigotimes_{m>0} \bigotimes_{\substack{x\in L_0 \\ Q(x)=m}} j_x^* \mathcal{J}_{0,1}^{c(-m)}$$

of line bundles on $\mathcal{B}_\Phi$.

Proof. Proposition 5.2.1 implies the equality of hermitian forms

$$\begin{aligned} \sum_{x\in L_0} c(-Q(x)) \cdot \langle u,x\rangle \cdot \langle x,v\rangle &= \frac{\langle u,v\rangle}{2n-4} \sum_{x\in L_0} c(-Q(x))\cdot [x,x] \\ &= \langle u,v\rangle \cdot \mathrm{mult}_\Phi(f) \end{aligned}$$

for all $u, v \in L_0$. As L_0 is self-dual, we may choose u and v so that $\langle u,v\rangle = 1$, and the integrality of $\mathrm{mult}_\Phi(f)$ follows from the integrality of $c(-m)$.

Set $E = \mathcal{E} \times_{\mathcal{Y}_0(D)} \mathcal{A}_\Phi$, and use Proposition 3.4.3 to identify $\mathcal{B}_\Phi \cong E \otimes L_0$. The pullback of the line bundle

$$\bigotimes_{m>0} \bigotimes_{\substack{x\in L_0 \\ Q(x)=m}} j_x^* \mathcal{J}_{0,1}^{\otimes c(-m)} \cong \bigotimes_{x\in L_0} j_x^* \mathcal{J}_{0,1}^{\otimes c(-Q(x))}$$

via any T-valued point $a = \sum t_i \otimes y_i \in E(T) \otimes L_0$ is, in the notation of §3.4,

$$\begin{aligned}
&\bigotimes_{x \in L_0} \mathcal{P}_E\Big(\sum_i \langle y_i, x\rangle t_i, \sum_j \langle y_j, x\rangle t_j\Big)^{\otimes c(-Q(x))} \\
&\cong \bigotimes_{i,j} \bigotimes_{x \in L_0} \mathcal{P}_E\big(c(-Q(x)) \cdot \langle y_i, x\rangle \cdot \langle x, y_j\rangle \cdot t_i, t_j\big) \\
&\cong \bigotimes_{i,j} \mathcal{P}_E\big(\langle y_i, y_j\rangle \cdot t_i, t_j\big)^{\otimes \mathrm{mult}_\Phi(f)} \\
&\cong \mathcal{P}_{E \otimes L_0}(a, a)^{\otimes \mathrm{mult}_\Phi(f)} \\
&\cong \mathcal{Q}_{E \otimes L_0}(a)^{\otimes 2 \cdot \mathrm{mult}_\Phi(f)}.
\end{aligned}$$

This, along with the isomorphism $\mathcal{Q}_{E \otimes L_0} \cong \mathcal{L}_\Phi$ of Proposition 3.4.3, proves that

$$\mathcal{L}_\Phi^{2 \cdot \mathrm{mult}_\Phi(f)} \cong \mathcal{Q}_{E \otimes L_0}^{2 \cdot \mathrm{mult}_\Phi(f)} \cong \bigotimes_{m > 0} \bigotimes_{\substack{x \in L_0 \\ Q(x) = m}} j_x^* \mathcal{J}_{0,1}^{c(-m)}.$$

□

5.3. **The unitary Borcherds product.** For a prime p dividing D define

$$\gamma_p = \epsilon_p^{-n} \cdot (D, p)_p^n \cdot \mathrm{inv}_p(V_p) \in \{\pm 1, \pm i\}, \tag{5.3.1}$$

where $\mathrm{inv}_p(V_p)$ is the invariant of $V_p = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ in the sense of (1.7.3), and

$$\epsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

It is equal to the local Weil index of the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $S_{L_p} \subset S(V_p)$, where V_p is viewed as a quadratic space as in (2.1.6). This is explained in more detail in §8.1. For any r dividing D we define

$$\gamma_r = \prod_{p \mid r} \gamma_p. \tag{5.3.2}$$

Let $c_r(0)$ denote the constant term of f at the cusp ∞_r, as in Definition 4.1.1, and define

$$k = \sum_{r \mid D} \gamma_r \cdot c_r(0).$$

We will see later in Corollary 6.1.4 that all $\gamma_r \cdot c_r(0) \in \mathbb{Q}$.

For every $m > 0$ define a divisor

$$\mathcal{B}_{\mathrm{Kra}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x\rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi) \tag{5.3.3}$$

with rational coefficients on $\mathcal{S}^*_{\mathrm{Kra}}$. Here the sum is over all K-equivalence classes of proper cusp label representatives Φ in the sense of §3.2, L_0 is the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2, 0)$ defined by (3.1.4), and $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$

is the boundary divisor of Theorem 3.7.1. It follows immediately from the definition (5.2.4) that

$$\sum_{m>0} c(-m)\cdot \mathcal{B}_{\mathrm{Kra}}(m) = \sum_{\Phi} \mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi).$$

For $m>0$ define the *total special divisor*

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m),$$

where $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ is the special divisor defined on the open Shimura variety in §2.5, and extended to the toroidal compactification in Theorem 3.7.1.

The following theorems assert the existence of Borcherds products on $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ having prescribed divisors and prescribed leading Fourier-Jacobi coefficients. Their proofs will occupy all of §6.

Theorem 5.3.1. *After possibly replacing the form f of (5.2.2) by a positive integer multiple, there is a rational section $\boldsymbol{\psi}(f)$ of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ with the following properties.*

(1) *In the generic fiber, the divisor of $\boldsymbol{\psi}(f)$ is*

$$\mathrm{div}(\boldsymbol{\psi}(f))_{/\boldsymbol{k}} = \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}.$$

(2) *For every proper cusp label representative Φ, the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$, in the sense of (3.8.1), along the boundary divisor*

$$\Delta_\Phi\backslash\mathcal{B}_\Phi \cong \mathcal{S}^*_{\mathrm{Kra}}(\Phi)$$

has the form

$$\boldsymbol{\psi}(f) = q^{\mathrm{mult}_\Phi(f)} \sum_{\ell\geqslant 0} \boldsymbol{\psi}_\ell \cdot q^\ell,$$

where $\boldsymbol{\psi}_\ell$ is a rational section of $\boldsymbol{\omega}^k_\Phi \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)+\ell}$ over $\mathcal{B}_\Phi$.

(3) *For any Φ as above, the leading coefficient $\boldsymbol{\psi}_0$ admits a factorization*

$$\boldsymbol{\psi}_0 = P^\eta_\Phi \otimes P^{hor}_\Phi \otimes P^{vert}_\Phi,$$

where the three terms on the right are defined as follows.

(a) *Proposition 3.5.1 provides us with an isomorphism*

$$\mathfrak{d}^{-1}\boldsymbol{\omega}_\Phi \cong j^*\boldsymbol{\omega}_{\mathcal{Y}}$$

of line bundles on $\mathcal{A}_\Phi$, where $j:\mathcal{A}_\Phi\to\mathcal{Y}_0(D)$ is the morphism of (5.2.3), and $\boldsymbol{\omega}_{\mathcal{Y}} = \mathrm{Lie}(\mathcal{E})^{-1}$ is the pullback via $\mathcal{Y}_0(D)\to\mathcal{Y}$ of the line bundle of weight one modular forms. Pulling back the modular form $(2\pi i\eta^2)^{12}$ of Proposition 5.1.4 defines a nowhere vanishing section

$$j^*(2\pi i \eta^2)^k \in H^0(\mathcal{A}_\Phi, \mathfrak{d}^{-k}\boldsymbol{\omega}^k_\Phi).$$

Using the canonical inclusion $\boldsymbol{\omega}_\Phi \subset \mathfrak{d}^{-1}\boldsymbol{\omega}_\Phi$, define

$$P^\eta_\Phi = j^*(2\pi i\eta^2)^k,$$

but viewed as a rational section of $\boldsymbol{\omega}_\Phi^k$ over $\mathcal{A}_\Phi$. Denote in the same way its pullback to $\mathcal{B}_\Phi$.

(b) *Recalling the function*

$$F_r^{24} = \bigotimes_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0 \\ rb = 0}} (bQ)^* \Theta^{24}$$

on $\mathcal{Y}_0(D)$ of Lemma 5.1.5, define a rational function

$$P_\Phi^{vert} = \bigotimes_{\substack{r \mid D \\ r > 1}} j^* F_r^{\gamma_r c_r(0)}$$

on $\mathcal{A}_\Phi$, and again pull back to $\mathcal{B}_\Phi$.

(c) *Using Proposition 5.2.2, define a rational section*

$$P_\Phi^{hor} = \bigotimes_{m>0} \bigotimes_{\substack{x \in L_0 \\ \langle x, x \rangle = m}} j_x^* \Theta^{c(-m)}$$

of the line bundle $\mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on $\mathcal{B}_\Phi$.

These properties determine $\boldsymbol{\psi}(f)$ uniquely.

Remark 5.3.2. In replacing f by a positive integer multiple, we are tacitly assuming that the constants $\gamma_r c_r(0)$ and $c(-m)$ are integer multiples of 24 for all $r \mid D$ and all $m > 0$. This is necessary in order to guarantee $k \in 12\mathbb{Z}$, and to make sense of the three factors $(2\pi i \eta_\Phi^2)^k$, P_Φ^{hor}, and P_Φ^{vert}.

In fact, we can strengthen Theorem 5.3.1 by computing precisely the divisor of $\boldsymbol{\psi}(f)$ on the integral model $\mathcal{S}_{\mathrm{Kra}}^*$.

Theorem 5.3.3. *The rational section $\boldsymbol{\psi}(f)$ of $\boldsymbol{\omega}^k$ has divisor*

$$\begin{aligned} \mathrm{div}(\boldsymbol{\psi}(f)) = & \sum_{m>0} c(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}^{\mathrm{tot}}(m) \\ & + k \cdot \left(\frac{\mathrm{Exc}}{2} - \mathrm{div}(\delta) \right) + \sum_{r \mid D} \gamma_r c_r(0) \sum_{p \mid r} \mathcal{S}_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}^* \\ & - \sum_{m>0} \frac{c(-m)}{2} \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x \rangle = m\} \cdot \mathrm{Exc}_s, \end{aligned}$$

where $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above p, L_s is the self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice defined in §2.6, and $\mathrm{Exc}_s \subset \mathrm{Exc}$ is the fiber over the component $s \in \pi_0(\mathrm{Sing})$.

It is possible to give a statement analogous to Theorem 5.3.3 for the integral model $\mathcal{S}_{\mathrm{Pap}}^*$. To do this we first define, exactly as in (5.3.3), a Cartier divisor

$$\mathcal{Y}_{\mathrm{Pap}}^{\mathrm{tot}}(m) = \mathcal{Y}_{\mathrm{Pap}}^*(m) + 2\mathcal{B}_{\mathrm{Pap}}(m)$$

with rational coefficients on $\mathcal{S}^*_{\mathrm{Pap}}$. Here $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ is the Cartier divisor of Theorem §3.7.1, and

$$\mathcal{B}_{\mathrm{Pap}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x \rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Pap}}(\Phi).$$

It is clear from Theorem 3.7.1 that

$$2 \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x \rangle = m\} \cdot \mathrm{Exc}_s, \tag{5.3.4}$$

where $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ denotes the pullback of $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)$ via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$.

The isomorphism

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc})$$

of Theorem 3.7.1 identifies $\boldsymbol{\omega}^{2k} \cong \boldsymbol{\Omega}^k_{\mathrm{Kra}}$ in the generic fiber of $\mathcal{S}^*_{\mathrm{Kra}}$, allowing us to view $\boldsymbol{\psi}(f)^2$ as a rational section of $\boldsymbol{\Omega}^k_{\mathrm{Kra}}$. As $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ is an isomorphism in the generic fiber, this section descends to a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$.

Theorem 5.3.4. *When viewed as a rational section of $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$, the Borcherds product $\boldsymbol{\psi}(f)^2$ has divisor*

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) = &\sum_{m>0} c(-m) \cdot \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \\ &- 2k \cdot \mathrm{div}(\delta) + 2 \sum_{r \mid D} \gamma_r c_r(0) \sum_{p \mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}}.\end{aligned}$$

These three theorems will be proved simultaneously in §6. Briefly, we will map our unitary Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ to an orthogonal Shimura variety, where a meromorphic Borcherds product is already known to exist. If we pull back this Borcherds product to $\mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$, the leading coefficient in its analytic Fourier-Jacobi expansion is known from [Kud16], up to multiplication by some unknown constants of absolute value 1.

By converting this analytic Fourier-Jacobi expansion into algebraic language, we will deduce the existence of a Borcherds product $\boldsymbol{\psi}(f)$ satisfying all of the properties stated in Theorem 5.3.1, up to some unknown constants in the leading Fourier-Jacobi coefficient. These unknown constants are the κ_Φ's appearing in Proposition 6.4.1. We then rescale the Borcherds product to make many $\kappa_\Phi = 1$ simultaneously.

After such a rescaling, the divisor of $\boldsymbol{\psi}(f)^2$ on $\mathcal{S}^*_{\mathrm{Pap}}$ can be computed from the Fourier-Jacobi expansions, and agrees with the divisor written in Theorem 5.3.4. Pulling back that divisor calculation via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$, and using Theorem 2.6.3, yields the divisor of Theorem 5.3.3.

Using the above divisor calculations, we prove that all κ_Φ are roots of unity. Thus, after replacing f by a multiple, which replaces $\boldsymbol{\psi}(f)$ by a power, we can force all $\kappa_\Phi = 1$, completing the proofs.

5.4. **A divisor calculation at the boundary.** Let Φ be a proper cusp label representative. The following proposition is a key ingredient in the proofs of Theorems 5.3.1, 5.3.3, and 5.3.4.

Proposition 5.4.1. *The rational sections P^{η}_{Φ}, P^{hor}_{Φ}, and P^{vert}_{Φ} of the line bundles $\boldsymbol{\omega}^k_{\Phi}$, $\mathcal{L}^{\mathrm{mult}_{\Phi}(f)}_{\Phi}$, and $\mathcal{O}_{\mathcal{B}_{\Phi}}$, respectively, have divisors*

$$\begin{aligned}\mathrm{div}(P^{\eta}_{\Phi}) &= -k\cdot \mathrm{div}(\delta)\\ \mathrm{div}(P^{hor}_{\Phi}) &= \sum_{m>0} c(-m)\mathcal{Z}_{\Phi}(m)\\ \mathrm{div}(P^{vert}_{\Phi}) &= \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{B}_{\Phi/\mathbb{F}_{\mathfrak{p}}}.\end{aligned}$$

In particular, the divisor of P^{hor}_{Φ} is purely horizontal (Proposition 3.6.1), while the divisors of P^{η}_{Φ} and P^{vert}_{Φ} are purely vertical.

Proof. By Proposition 5.1.4 the section

$$j^*(2\pi i\eta^2)^k \in H^0(\mathcal{A}_{\Phi}, \mathfrak{d}^{-k}\boldsymbol{\omega}^k_{\Phi}) \cong H^0(\mathcal{Y}_0(D), \boldsymbol{\omega}^k_{\mathcal{Y}})$$

has trivial divisor. When we use the inclusion $\boldsymbol{\omega}_{\Phi} \subset \mathfrak{d}^{-1}\boldsymbol{\omega}_{\Phi}$ to view it instead as a rational section P^{η}_{Φ} of $\boldsymbol{\omega}^k_{\Phi}$, its divisor becomes $\mathrm{div}(\delta^{-k})$. This proves the first equality.

To prove the remaining two equalities, let $\mathcal{E} \to \mathcal{Y}_0(D)$ be the universal elliptic curve, and denote by $e : \mathcal{Y}_0(D) \to \mathcal{E}$ the 0-section. It is an effective Cartier divisor on $\mathcal{E}$.

Directly from the definition of P^{hor}_{Φ} we have the equality

$$\mathrm{div}(P^{hor}_{\Phi}) = \sum_{m>0}\frac{c(-m)}{24}\sum_{\substack{x\in L_0\\ \langle x,x\rangle = m}}\mathrm{div}(j^*_x\Theta^{24}).$$

Combining Proposition 5.1.4 with (3.6.1) shows that

$$\sum_{\substack{x\in L_0\\ \langle x,x\rangle = m}}\mathrm{div}(j^*_x\Theta^{24}) = \sum_{\substack{x\in L_0\\ \langle x,x\rangle = m}} 24 j^*_x(e) = \sum_{\substack{x\in L_0\\ \langle x,x\rangle = m}} 24\mathcal{Z}_{\Phi}(x) = 24\mathcal{Z}_{\Phi}(m),$$

and the first equality follows immediately.

Recall the morphism $j : \mathcal{A}_{\Phi} \to \mathcal{Y}_0(D)$ of §3.6. For the second equality it suffices to prove that the function F^{24}_r on $\mathcal{Y}_0(D)$ defined in Lemma 5.1.5 satisfies

$$\mathrm{div}(j^*F^{24}_r) = 24\sum_{p\mid r}\mathcal{A}_{\Phi/\mathbb{F}_{\mathfrak{p}}}. \tag{5.4.1}$$

Let $C \subset \mathcal{E}$ be the universal cyclic subgroup scheme of order D. For each $s \mid D$ denote by $C[s] \subset C$ the s-torsion subgroup, and by $C[s]^{\times} \subset C[s]$ the closed *subscheme of generators.* More precisely, noting that

$$C[s] = \prod_{p\mid s} C[p],$$

we may use the Oort-Tate theory of group schemes of prime order (see [HR12] for a summary). Define

$$C[s]^\times = \prod_{p\mid s} C[p]^\times,$$

where $C[p]^\times$ denotes the closed subscheme of generators of $C[p]$ as in [HR12, §3.3]. Note that $C[p]^\times$ coincides with the subscheme of points of exact order p. See [HR12, Rem. 3.3.2].

There is an equality of Cartier divisors

$$\frac{1}{24}\mathrm{div}(F_r^{24}) = \big(C[r]-e\big)\times_{\mathcal{E},e}\mathcal{Y}_0(D) = \sum_{\substack{s\mid r\\ s\neq 1}}\big(C[s]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big)$$

on $\mathcal{Y}_0(D)$. Indeed, one can check this after pullback to $\mathcal{Y}_1(D)$, where it is clear from Proposition 5.1.4, which asserts that the divisor of the section Θ^{24} appearing in the definition of F_r^{24} is equal to $24e$. If s is divisible by two distinct primes then

$$\big(C[s]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big) = 0,$$

and hence

$$\mathrm{div}(F_r^{24}) = 24\sum_{p\mid r}\big(C[p]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big).$$

Now pull back this equality of Cartier divisors by j. Recall that j is defined as the composition

$$\mathcal{A}_\Phi \cong \mathcal{M}_{(1,0)} \xrightarrow{i} \mathcal{Y}_0(D),$$

where the isomorphism is the one provided by Proposition 3.4.3, and the arrow labeled i endows the universal CM elliptic curve $E\to\mathcal{M}_{(1,0)}$ with its cyclic subgroup scheme $E[\delta]$. Thus

$$i^*\mathrm{div}(F_r^{24}) = 24\sum_{p\mid r}\big(E[\mathfrak{p}]^\times\times_{E,e}\mathcal{M}_{(1,0)}\big), \tag{5.4.2}$$

where $\mathfrak{p}$ denotes the unique prime ideal in $\mathcal{O}_{\boldsymbol{k}}$ over p.

Fix a geometric point $z : \mathrm{Spec}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}) \to \mathcal{M}_{(1,0)}$, and view z also as a geometric point of E or $\mathcal{E}$ using

$$\mathcal{M}_{(1,0)} \xrightarrow{e} E \xrightarrow{i} \mathcal{E}.$$

Let $\mathcal{O}_{E,z}$ and $\mathcal{O}_{\mathcal{E},z}$ denote the completed étale local rings of E and $\mathcal{E}$ at z.

By Oort-Tate theory there is an isomorphism

$$\mathcal{O}_{\mathcal{E},z} \cong W[[X,Y,Z]]/(XY-w_p)$$

for some uniformizer w_p in the Witt ring $W = W(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$. Compare with [HR12, Theorem 3.3.1]. Under this isomorphism the 0-section of $\mathcal{E}$ is defined by the equation $Z=0$, and the divisor $C[p]^\times$ is defined by $Z^{p-1}-X=0$. Moreover, noting that the completed étale local ring of $\mathcal{M}_{(1,0)}$ at z can be identified with $\mathcal{O}_{\boldsymbol{k}} \otimes W$, the natural map $\mathcal{O}_{\mathcal{E},z} \to \mathcal{O}_{E,z}$ is identified with the quotient map

$$W[[X,Y,Z]]/(XY - w_p) \to W[[X,Y,Z]]/(XY - w_p, X - uY)$$

for some $u \in W^\times$.

Under these identifications, the closed immersion

$$E[\mathfrak{p}]^\times \times_{E,e} \mathcal{M}_{(1,0)} \hookrightarrow \mathcal{M}_{(1,0)}$$

corresponds, on the level of completed local rings, to the quotient map

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}_{(1,0)},z} & = & W[[X,Y,Z]]/(XY - w_p, X - uY, Z) \\ & & \downarrow \\ \mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}} & = & W[[X,Y,Z]]/(XY - w_p, X - uY, Z, Z^{p-1} - X). \end{array}$$

This implies that

$$E[\mathfrak{p}]^\times \times_{E,e} \mathcal{M}_{(1,0)} = \mathcal{M}_{(1,0)/\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}}.$$

The equality (5.4.1) is clear from this and (5.4.2). □

6. Calculation of the Borcherds product divisor

In this section we prove Theorems 5.3.1, 5.3.3, and 5.3.4. Throughout §6 we keep f as in (5.2.2), and again assume that $c(-m) \in \mathbb{Z}$ for all $m \geqslant 0$.

Recall that $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ is endowed with the hermitian form $\langle x, y\rangle$ of (2.1.5), as well as the $\mathbb{Q}$-bilinear form $[x, y]$ of (2.1.6). The associated quadratic form is

$$Q(x) = \langle x, x\rangle = \frac{[x,x]}{2}.$$

6.1. **Vector-valued modular forms.** Let $L \subset V$ be any $\mathcal{O}_{\boldsymbol{k}}$-lattice, self-dual with respect to the hermitian form. The dual lattice of L with respect to the bilinear form $[\cdot,\cdot]$ is $L' = \mathfrak{d}^{-1}L$.

Let ω be the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of the Weil representation of $\mathrm{SL}_2(\widehat{\mathbb{Q}})$ (associated with the standard additive character of $\mathbb{A}/\mathbb{Q}$) on the Schwartz-Bruhat functions on $L \otimes_{\mathbb{Z}} \mathbb{A}_f$. The restriction of ω to $\mathrm{SL}_2(\mathbb{Z})$ preserves the subspace $S_L = \mathbb{C}[L'/L]$ of Schwartz-Bruhat functions that are supported on $\widehat{L}'$ and invariant under translations by $\widehat{L}$. We obtain a representation

$$\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(S_L).$$

For $\mu \in L'/L$, we denote by $\phi_\mu \in S_L$ the characteristic function of μ.

Remark 6.1.1. The conjugate representation $\overline{\omega}_L$ on S_L, defined by

$$\overline{\omega}_L(\gamma)(\phi) = \overline{\omega_L(\gamma)(\overline{\phi})}$$

for $\phi \in S_L$, is the representation denoted ρ_L in [Bor98, Bru02, BF04].

Recall the scalar valued modular form

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M^{!,\infty}_{2-n}(D, \chi)$$

of (5.2.2), and continue to assume that $c(m) \in \mathbb{Z}$ for all $m \leqslant 0$. We will convert f into a $\mathbb{C}[L'/L]$-valued modular form $\tilde{f}$, to be used as input for Borcherds' construction of meromorphic modular forms on orthogonal Shimura varieties. The restriction of ω_L to $\Gamma_0(D)$ acts on the line $\mathbb{C} \cdot \phi_0$ via the character χ, and hence the induced function

$$\tilde{f}(\tau) = \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (f \mid_{2-n} \gamma)(\tau) \cdot \omega_L(\gamma)^{-1} \phi_0 \tag{6.1.1}$$

is an S_L-valued weakly holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2-n$ with representation ω_L. Its Fourier expansion is denoted

$$\tilde{f}(\tau) = \sum_{m \gg -\infty} \tilde{c}(m) \cdot q^m, \tag{6.1.2}$$

and we denote by $\tilde{c}(m, \mu)$ the value of $\tilde{c}(m) \in S_L$ at a coset $\mu \in L'/L$.

For any $r \mid D$ let $\gamma_r \in \{\pm 1, \pm i\}$ be as in (5.3.2), and let $c_r(m)$ be the m^{th} Fourier coefficient of f at the cusp ∞_r as in (4.1.1). For any $\mu \in L'/L$ define $r_\mu \mid D$ by

$$r_\mu = \prod_{\mu_p \neq 0} p, \tag{6.1.3}$$

where $\mu_p \in L'_p/L_p$ is the p-component of μ.

Proposition 6.1.2. *For all $m \in \mathbb{Q}$ the coefficients $\tilde{c}(m) \in S_L$ satisfy*

$$\tilde{c}(m, \mu) = \begin{cases} \sum_{r_\mu | r | D} \gamma_r \cdot c_r(mr) & \text{if } m \equiv -Q(\mu) \pmod{\mathbb{Z}}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $m < 0$ we have

$$\tilde{c}(m, \mu) = \begin{cases} c(m) & \text{if } \mu = 0, \\ 0 & \text{if } \mu \neq 0, \end{cases}$$

and the constant term of $\tilde{f}$ is given by

$$\tilde{c}(0, \mu) = \sum_{r_\mu | r | D} \gamma_r \cdot c_r(0).$$

Proof. The first formula is a special case of results of Scheithauer [Sch09, Section 5]. For the reader's benefit we provide a direct proof in §8.2.

The formula for the $m = 0$ coefficient is immediate from the general formula. So is the formula for $m < 0$, using the fact that the singularities of f are supported at the cusp at ∞. □

Remark 6.1.3. The first formula of Proposition 6.1.2 actually also holds for f in the larger space $M^!_{2-n}(D, \chi)$.

Corollary 6.1.4. *The coefficients $c(m)$ and $\tilde{c}(m)$ satisfy the following:*

(1) *The $c(m)$ are rational for all m.*
(2) *The $\tilde{c}(m,\mu)$ are rational for all m and μ, and are integral if $m<0$.*
(3) *For all $r \mid D$ we have $\gamma_r \cdot c_r(0) \in \mathbb{Q}$. In particular*

$$\tilde{c}(0,0) = \sum_{r \mid D} \gamma_r \cdot c_r(0) \in \mathbb{Q}.$$

Proof. For the first claim, fix any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. The form $f^\sigma - f \in M^{!,\infty}_{2-n}$ is holomorphic at all cusps other than ∞, and vanishes at the cusp ∞ by the assumption that as $c(m) \in \mathbb{Z}$ for $m \leqslant 0$. Hence $f^\sigma - f$ is a holomorphic modular form of negative weight $2-n$, and therefore vanishes identically. It follows that $c(m) \in \mathbb{Q}$ for all m.

Now consider the second claim. In view of the Proposition 6.1.2 the coefficients $\tilde{c}(m,\mu)$ of $\tilde{f}$ with $m<0$ are integers. Hence, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the function $\tilde{f}^\sigma - \tilde{f}$ is a holomorphic modular form of weight $2-n<0$, which is therefore identically 0. Therefore $\tilde{f}$ has rational Fourier coefficients.

The third claim follows from the second claim and the formula for the constant term of $\tilde{f}$ given in Proposition 6.1.2. □

6.2. **Construction of the Borcherds product.** We now construct the Borcherds product $\boldsymbol{\psi}(f)$ of Theorem 5.3.1 as the pullback of a Borcherds product on the orthogonal Shimura variety defined by the quadratic space (V,Q). Useful references here include [Bor98, Bru02, Kud03, Hof14].

After Corollary 6.1.4 we may replace f by a positive integer multiple in order to assume that $c(-m) \in 24\mathbb{Z}$ for all $m \geqslant 0$, and that $\gamma_r c_r(0) \in 24\mathbb{Z}$ for all $r \mid D$. In particular the rational number

$$k = \tilde{c}(0,0)$$

of Corollary 6.1.4 is an integer. Compare with Remark 5.3.2.

Define a hermitian domain

$$\tilde{\mathcal{D}} = \{ w \in V(\mathbb{C}) : [w,w]=0,\ [w,\overline{w}]<0 \}/\mathbb{C}^\times. \tag{6.2.1}$$

Let $\tilde{\boldsymbol{\omega}}^{an}$ be the tautological bundle on $\tilde{\mathcal{D}}$, whose fiber at w is the line $\mathbb{C}w \subset V(\mathbb{C})$. The group of real points of $\mathrm{SO}(V)$ acts on (6.2.1), and this action lifts to an action on $\tilde{\boldsymbol{\omega}}^{an}$.

As in Remark 2.1.3, any point $z \in \mathcal{D}$ determines a line $\mathbb{C}w \subset \epsilon V(\mathbb{C})$. This construction defines a closed immersion

$$\mathcal{D} \hookrightarrow \tilde{\mathcal{D}}, \tag{6.2.2}$$

under which $\tilde{\boldsymbol{\omega}}^{an}$ pulls back to the line bundle $\boldsymbol{\omega}^{an}$ of §2.4. The hermitian domain $\tilde{\mathcal{D}}$ has two connected components. Let $\tilde{\mathcal{D}}^+ \subset \tilde{\mathcal{D}}$ be the connected component containing $\mathcal{D}$.

For a fixed $g \in G(\mathbb{A})$, we apply the constructions of §6.1 to the input form f and the self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, g\mathfrak{a}) \subset V.$$

The result is a vector-valued modular form $\tilde{f}$ of weight $2-n$ and representation $\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to S_L$. The form $\tilde{f}$ determines a Borcherds product $\Psi(\tilde{f})$ on $\tilde{\mathcal{D}}^+$; see [Bor98, Theorem 13.3] and Theorem 7.2.4. For us it is more convenient to use the rescaled Borcherds product

$$\tilde{\boldsymbol{\psi}}_g(f) = (2\pi i)^{\tilde{c}(0,0)} \Psi(2\tilde{f}) \tag{6.2.3}$$

determined by $2\tilde{f}$. It is a meromorphic section of $(\tilde{\boldsymbol{\omega}}^{an})^k$.

The subgroup $\mathrm{SO}(L)^+ \subset \mathrm{SO}(L)$ of elements preserving the component $\tilde{\mathcal{D}}^+$ acts on $\tilde{\boldsymbol{\psi}}_g(f)$ through a finite order character [Bor00]. Replacing f by mf has the effect of replacing $\tilde{\boldsymbol{\psi}}_g(f)$ by $\tilde{\boldsymbol{\psi}}_g(f)^m$, and so after replacing f by a multiple we assume that $\tilde{\boldsymbol{\psi}}_g(f)$ is invariant under this action.

Denote by $\boldsymbol{\psi}_g(f)$ the pullback of $\tilde{\boldsymbol{\psi}}_g(f)$ via the map

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash\mathcal{D} \to \mathrm{SO}(L)^+\backslash\tilde{\mathcal{D}}^+$$

induced by (6.2.2). It is a meromorphic section of $(\boldsymbol{\omega}^{an})^k$ on the connected component

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash\mathcal{D} \xrightarrow{z\mapsto(z,g)} \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}).$$

By repeating the construction for all $g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, we obtain a meromorphic section $\boldsymbol{\psi}(f)$ of the line bundle $(\boldsymbol{\omega}^{an})^k$ on all of $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$.

After rescaling on every connected component by a complex constant of absolute value 1, this will be the section whose existence is asserted in Theorem 5.3.1.

6.3. **Analytic Fourier-Jacobi coefficients.** We return to the notation of §3.9. Thus $\Phi = (P, g)$ is a proper cusp label representative, we have chosen

$$s : \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \to Q_\Phi$$

as in Lemma 3.9.2, and have fixed $a \in \widehat{\boldsymbol{k}}^\times$. This data determines a lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(s(a)g\mathfrak{a}_0, s(a)g\mathfrak{a}),$$

and Witt decompositions

$$V = V_{-1} \oplus V_0 \oplus V_1, \quad L = L_{-1} \oplus L_0 \oplus L_1.$$

Choose bases $\mathrm{e}_{-1}, \mathrm{f}_{-1} \in L_{-1}$ and $\mathrm{e}_1, \mathrm{f}_1 \in L_1$ as in §3.9.

Imitating the construction of (3.9.16) yields a commutative diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "(6.2.2)"] \arrow[d, "w\mapsto(w_0,\xi)"'] & \tilde{\mathcal{D}}^+ \arrow[d, "w\mapsto(\tau,w_0,\xi)"] \\
\epsilon V_0(\mathbb{C}) \times \mathbb{C} \arrow[r] & \mathfrak{H} \times V_0(\mathbb{C}) \times \mathbb{C}
\end{tikzcd}$$

in which the vertical arrows are open immersions, and the horizontal arrows are closed immersions. The vertical arrow on the right is defined as follows:

Any $w \in \tilde{\mathcal{D}}$ pairs nontrivially with the isotropic vector f_{-1}, and so may be scaled so that $[w, \mathrm{f}_{-1}] = 1$. This allows us to identify

$$\tilde{\mathcal{D}} = \{w \in V(\mathbb{C}) : [w, w] = 0,\ [w, \overline{w}] < 0,\ [w, \mathrm{f}_{-1}] = 1\}.$$

Using this model, any $w \in \tilde{\mathcal{D}}^+$ has the form

$$w = -\xi \mathrm{e}_{-1} + (\tau\xi - Q(w_0))\mathrm{f}_{-1} + w_0 + \tau \mathrm{e}_1 + \mathrm{f}_1$$

with $\tau \in \mathfrak{H}$, $w_0 \in V_0(\mathbb{C})$, and $\xi \in \mathbb{C}$. The bottom horizontal arrow is $(w_0, \xi) \mapsto (\tau, w_0, \xi)$, where τ is determined by the relation (3.9.6).

The construction above singles out a nowhere vanishing section of $\tilde{\boldsymbol{\omega}}^{an}$, whose value at an isotropic line $\mathbb{C}w$ is the unique vector in that line with $[w, \mathrm{f}_{-1}] = 1$. As in the discussion leading to (3.9.18), we obtain a trivialization

$$[\,\cdot\,, \mathrm{f}_{-1}] : \tilde{\boldsymbol{\omega}}^{an} \cong \mathcal{O}_{\tilde{\mathcal{D}}^+}.$$

Now consider the Borcherds product $\tilde{\boldsymbol{\psi}}_{s(a)g}(f)$ on $\tilde{\mathcal{D}}^+$ determined by the lattice L above (that is, replace g by $s(a)g$ throughout §6.2). It is a meromorphic section of $(\tilde{\boldsymbol{\omega}}^{an})^k$, and we use the trivialization above to identify it with a meromorphic function. In a neighborhood of the rational boundary component associated to the isotropic plane $V_{-1} \subset V$, this meromorphic function has a product expansion.

Proposition 6.3.1 ([Kud16]). *There are positive constants A and B with the following property: For all points $w \in \tilde{\mathcal{D}}^+$ satisfying*

$$\mathrm{Im}(\xi) - \frac{Q(\mathrm{Im}(w_0))}{\mathrm{Im}(\tau)} > A\,\mathrm{Im}(\tau) + \frac{B}{\mathrm{Im}(\tau)},$$

there is a factorization

$$\tilde{\boldsymbol{\psi}}_{s(a)g}(f) = \kappa \cdot (2\pi i)^k \cdot \eta^{2k}(\tau) \cdot e^{2\pi i I \xi} \cdot P_0(\tau) \cdot P_1(\tau, w_0) \cdot P_2(\tau, w_0, \xi)$$

in which $\kappa \in \mathbb{C}^\times$ has absolute value 1, η is the Dedekind η-function, and

$$I = \frac{1}{12} \sum_{b \in \mathbb{Z}/D\mathbb{Z}} \tilde{c}\left(0, -\frac{b}{D}\mathrm{f}_{-1}\right) - 2 \sum_{m>0} \sum_{x \in L_0} c(-m) \cdot \sigma_1(m - Q(x)).$$

The factors P_0 and P_1 are defined by

$$P_0(\tau) = \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0}} \Theta\left(\tau, \frac{b}{D}\right)^{\tilde{c}(0, \frac{b}{D}\mathrm{f}_{-1})}$$

and

$$P_1(\tau, w_0) = \prod_{m>0} \prod_{\substack{x \in L_0 \\ Q(x)=m}} \Theta\big(\tau, [w_0, x]\big)^{c(-m)}.$$

The remaining factor is

$$P_2(\tau, w_0, \xi) = \prod_{\substack{x \in \delta^{-1}L_0 \\ a \in \mathbb{Z} \\ b \in \mathbb{Z}/D\mathbb{Z} \\ c \in \mathbb{Z}_{>0}}} \left(1 - e^{2\pi i c\xi} e^{2\pi i a\tau} e^{2\pi i b/D} e^{-2\pi i [x, w_0]}\right)^{2\cdot\tilde{c}(ac - Q(x), \mu)},$$

where $\mu = -a\mathrm{e}_{-1} - \frac{b}{D}\mathrm{f}_{-1} + x + c\mathrm{e}_1 \in \delta^{-1}L/L$.

Proof. This is just a restatement of [Kud16, Corollary 2.3], with some simplifications arising from the fact that the vector-valued form $\tilde{f}$ used to define the Borcherds product is induced from a scalar-valued form via (6.1.1).

A more detailed description of how these expressions arise from the general formulas in [Kud16] is given in the appendix. □

If we pull back the formula for the Borcherds product $\tilde{\boldsymbol{\psi}}_{s(a)g}(f)$ found in Proposition 6.3.1 via the closed immersion (6.2.2), we obtain a formula for the Borcherds product $\boldsymbol{\psi}_{s(a)g}(f)$ on the connected component

$$(G(\mathbb{Q}) \cap s(a)gKg^{-1}s(a)^{-1})\backslash\mathcal{D} \xrightarrow{z \mapsto (z, s(a)g)} \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}),$$

from which we can read off the leading analytic Fourier-Jacobi coefficient.

Corollary 6.3.2. *The analytic Fourier-Jacobi expansion of* $\boldsymbol{\psi}(f)$*, in the sense of (3.9.18), has the form*

$$\boldsymbol{\psi}_{s(a)g}(f) = \sum_{\ell \geqslant I} \mathrm{FJ}_\ell^{(a)}(\boldsymbol{\psi}(f))(w_0) \cdot q^\ell.$$

The leading coefficient $\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))$*, viewed as a function on* $V_0(\mathbb{R})$ *as in the discussion leading to (3.9.14), is given by*

$$\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))(w_0) = \kappa \cdot (2\pi i)^k \cdot \eta(\tau)^{2k} \cdot P_0(\tau) \cdot P_1(\tau, w_0), \tag{6.3.1}$$

where $\tau \in \mathfrak{H}$ *is determined by (3.9.6),*

$$P_0(\tau) = \prod_{r|D} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0 \\ rb = 0}} \Theta\left(\tau, \frac{b}{D}\right)^{\gamma_r c_r(0)}$$

and

$$P_1(\tau, w_0) = \prod_{m > 0} \prod_{\substack{x \in L_0 \\ Q(x) = m}} \Theta\big(\tau, \langle w_0, x\rangle\big)^{c(-m)}.$$

The constant $\kappa \in \mathbb{C}$*, which depends on both* Φ *and* a*, has absolute value* 1*.*

Proof. Using Proposition 6.3.1, the pullback of $\tilde{\boldsymbol{\psi}}_{s(a)g}(f)$ via (6.2.2) factors as a product

$$\boldsymbol{\psi}_{s(a)g}(f) = \kappa \cdot (2\pi i)^k \cdot \eta^{2k}(\tau) \cdot e^{2\pi i\xi I} \cdot P_0(\tau)P_1(\tau, w_0)P_2(\tau, w_0, \xi),$$

where $\xi \in \mathbb{C}^\times$ and $w_0 \in V(\mathbb{R}) \cong \epsilon V(\mathbb{C})$. The parameter $\tau \in \mathfrak{H}$ is now fixed, determined by (3.9.6). The equality

$$\prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0}} \Theta\left(\tau, \frac{b}{D}\right)^{\tilde{c}(0, \frac{b}{D}\mathfrak{f}_{-1})} = \prod_{r|D} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0 \\ rb=0}} \Theta\left(\tau, \frac{b}{D}\right)^{\gamma_r c_r(0)}$$

follows from Proposition 6.1.2, and allows us to rewrite P_0 in the stated form. To rewrite the factor P_1 in terms of $\langle \cdot, \cdot \rangle$ instead of $[\cdot, \cdot]$, use the commutative diagram of Remark 3.9.4. Finally, as $\mathrm{Im}(\xi) \to \infty$, so $q = e^{2\pi i \xi} \to 0$, the factor P_2 converges to 1. This P_2 does not contribute to the leading Fourier-Jacobi coefficient. ☐

Proposition 6.3.3. *The integer I defined in Proposition 6.3.1 is equal to the integer $\mathrm{mult}_\Phi(f)$ defined by (5.2.4), and the product (6.3.1) satisfies the transformation law (3.9.14) with $\ell = \mathrm{mult}_\Phi(f)$.*

Proof. The Fourier-Jacobi coefficient $\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))$ is, by definition, a section of the line bundle $\mathcal{Q}^I_{E^{(a)} \otimes L}$ on $E^{(a)} \otimes L$. When viewed as a function of the variable $w_0 \in V_0(\mathbb{R})$ using our explicit coordinates, it therefore satisfies the transformation law (3.9.14) with $\ell = I$.

Now consider the right hand side of (6.3.1), and recall that τ is fixed, determined by (3.9.6). In our explicit coordinates the function $\Theta(\tau, \langle w_0, x \rangle)^{24}$ of $w_0 \in V_0(\mathbb{R})$ is identified with a section of the line bundle $j_x^* \mathcal{J}_{0,12}$ on $E^{(a)} \otimes L$; this is clear from the definition of j_x in (3.6.2), and Proposition 5.1.4. Thus $P_1(\tau, w_0)$, and hence the entire right hand side of (6.3.1), defines a section of the line bundle

$$\bigotimes_{m>0} \bigotimes_{\substack{x \in L_0 \\ Q(x)=m}} j_x^* \mathcal{J}_{0,1}^{c(-m)/2} \cong \mathcal{L}_\Phi^{2 \cdot \mathrm{mult}_\Phi(f/2)} \cong \mathcal{Q}^{\mathrm{mult}_\Phi(f)}_{E^{(a)} \otimes L},$$

where the isomorphisms are those of Proposition 5.2.2 and Proposition 3.4.3. This implies that the right hand side of (6.3.1) satisfies the transformation law (3.9.14) with $\ell = \mathrm{mult}_\Phi(f)$.

The claim now follows from (6.3.1) and the discussion above. For a more direct proof of the proposition, see §8.4. ☐

6.4. **Algebraization and descent.** The following weak form of Theorem 5.3.1 shows that $\boldsymbol{\psi}(f)$ is algebraic, and provides an algebraic interpretation of its leading Fourier-Jacobi coefficients.

Proposition 6.4.1. *The meromorphic section $\boldsymbol{\psi}(f)$ is the analytification of a rational section of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}_{\mathrm{Kra}/\mathbb{C}}$. This rational section satisfies the following properties:*

(1) *When viewed as a rational section over the toroidal compactification,*

$$\mathrm{div}(\boldsymbol{\psi}(f)) = \sum_{m>0} c(-m) \cdot \mathcal{Z}^*_{\mathrm{Kra}}(m)_{/\mathbb{C}} + \sum_\Phi \mathrm{mult}_\Phi(f) \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}}.$$

(2) *For every proper cusp label representative Φ, the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ along $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}}$, in the sense of §3.8, has the form*

$$\boldsymbol{\psi}(f) = q^{\mathrm{mult}_\Phi(f)} \sum_{\ell \geqslant 0} \boldsymbol{\psi}_\ell \cdot q^\ell.$$

(3) *The leading coefficient $\boldsymbol{\psi}_0$, a rational section of $\boldsymbol{\omega}_\Phi^k \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ over $\mathcal{B}_{\Phi/\mathbb{C}}$, factors as*

$$\boldsymbol{\psi}_0 = \kappa_\Phi \otimes P_\Phi^\eta \otimes P_\Phi^{hor} \otimes P_\Phi^{vert}$$

for a unique section

$$\kappa_\Phi \in H^0(\mathcal{A}_{\Phi/\mathbb{C}}, \mathcal{O}^\times_{\mathcal{A}_{\Phi/\mathbb{C}}}).$$

This section satisfies $|\kappa_\Phi(z)| = 1$ at every complex point $z \in \mathcal{A}_\Phi(\mathbb{C})$. (The other factors appearing on the right hand side were defined in Theorem 5.3.1.)

Proof. From Corollary 6.3.2 one can see that $\boldsymbol{\psi}(f)$ extends to a meromorphic section of $\boldsymbol{\omega}^k$ over the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, vanishing to order $I = \mathrm{mult}_\Phi(f)$ along the closed stratum $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}} \subset \mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$ indexed by a proper cusp label representative Φ. The calculation of the divisor of $\boldsymbol{\psi}(f)$ over the open Shimura variety $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ follows from the explicit calculation of the divisor of $\tilde{\boldsymbol{\psi}}(f)$ on the orthogonal Shimura variety due to Borcherds, and the explicit complex uniformization of the divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ found in [KR14] or [BHY15].

The algebraicity claim now follows from GAGA (using the fact that the divisor is already known to be algebraic), proving all parts of the first claim. The second and third claims are just a translation of Corollary 6.3.2 into the algebraic language of Theorem 5.3.1, using the explicit coordinates of §3.9 and the change of notation $(2\pi i \eta^2)^k = P_\Phi^\eta$, $P_0 = P_\Phi^{vert}$ and $P_1 = P_\Phi^{hor}$. □

We now prove that $\boldsymbol{\psi}(f)$, after minor rescaling, descends to $\boldsymbol{k}$.

Recall from Proposition 2.1.1 that the geometric components of $\mathrm{Sh}(G, \mathcal{D})$ are defined over the Hilbert class field $H \subset \mathbb{C}$ of $\boldsymbol{k}$, and that each such component has trivial stabilizer in $\mathrm{Gal}(H/\boldsymbol{k})$. This allows us to choose connected components $X_i \subset \mathrm{Sh}(G, \mathcal{D})_{/H}$ in such a way that

$$\mathrm{Sh}(G, \mathcal{D})_{/H} = \bigsqcup_i \bigsqcup_{\sigma \in \mathrm{Gal}(H/\boldsymbol{k})} \sigma(X_i).$$

For each index i, pick $g_i \in G(\mathbb{A}_f)$ in such a way that $X_i(\mathbb{C})$ is equal to the image of

$$(G(\mathbb{Q}) \cap g_i K g_i^{-1}) \backslash \mathcal{D} \xrightarrow{z \mapsto (z, g_i)} \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}).$$

Choose an isotropic $\boldsymbol{k}$-line $J \subset W$, let $P \subset G$ be its stabilizer, and define a proper cusp label representative $\Phi_i = (P, g_i)$. The above choices pick out one boundary component on every component of the toroidal compactification, as the following lemma demonstrates.

Lemma 6.4.2. *The natural maps*

$$\begin{tikzcd}
 & & \bigsqcup_i \mathcal{S}^*_{\mathrm{Kra}}(\Phi_i) \arrow[r] \arrow[dd, "\cong"] & \mathcal{S}^*_{\mathrm{Kra}} \arrow[dd] \\
\bigsqcup_i \mathcal{A}_{\Phi_i} & \bigsqcup_i \mathcal{B}_{\Phi_i} \arrow[l] \arrow[ur] \arrow[dr] & & \\
 & & \bigsqcup_i \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i) \arrow[r] & \mathcal{S}^*_{\mathrm{Pap}}
\end{tikzcd}$$

induce bijections on connected components. The same is true after base change to $\boldsymbol{k}$ or $\mathbb{C}$.

Proof. Let $X_i^* \subset \mathcal{S}^*_{\mathrm{Pap}}(\mathbb{C})$ be the closure of X_i. By examining the complex analytic construction of the toroidal compactification [How15, Lan12, Pin89], one sees that some component of the divisor $\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$ lies on X_i^*. Now recall from Theorem 3.7.1 that the components of $\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$ are defined over H, and that the action of $\mathrm{Gal}(H/\boldsymbol{k})$ is simply transitive. It follows immediately that

$$\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C}) \subset \bigsqcup_{\sigma \in \mathrm{Gal}(H/\boldsymbol{k})} \sigma(X_i^*),$$

and the inclusion induces a bijection on components. By Proposition 3.2.1 and the isomorphism of Proposition 3.3.2, the quotient map

$$\mathcal{C}_\Phi(\mathbb{C}) \to \Delta_{\Phi_i} \backslash \mathcal{C}_{\Phi_i}(\mathbb{C})$$

induces a bijection on connected components, and both maps $\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi$ have geometrically connected fibers (the first is a $\mathbb{G}_m$-torsor, and the second is an abelian scheme). We deduce that all maps in

$$\mathcal{A}_{\Phi_i}(\mathbb{C}) \leftarrow \mathcal{B}_{\Phi_i}(\mathbb{C}) \to \Delta_{\Phi_i} \backslash \mathcal{B}_{\Phi_i}(\mathbb{C}) \cong \mathcal{S}^*_{\mathrm{Kra}}(\Phi_i)(\mathbb{C}) \cong \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$$

induce bijections on connected components.

The above proves the claim over $\mathbb{C}$, and the claim over $\boldsymbol{k}$ follows formally from this. The claim for integral models follows from the claim in the generic fiber, using the fact that all integral models in question are normal and flat over $\mathcal{O}_{\boldsymbol{k}}$. □

Proposition 6.4.3. *After possibly rescaling by a constant of absolute value 1 on every connected component of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$, the Borcherds product $\boldsymbol{\psi}(f)$ is defined over $\boldsymbol{k}$, and the sections of Proposition 6.4.1 satisfy*

$$\kappa_\Phi \in H^0(\mathcal{A}_{\Phi/\boldsymbol{k}}, \mathcal{O}^\times_{\mathcal{A}_\Phi/\boldsymbol{k}})$$

for all proper cusp label representatives Φ. Furthermore, we may arrange that $\kappa_{\Phi_i} = 1$ for all i.

Proof. Lemma 6.4.2 establishes a bijection between the connected components of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ and the finite set $\bigsqcup_i \mathcal{A}_{\Phi_i}(\mathbb{C})$. On the component indexed by $z \in \mathcal{A}_{\Phi_i}(\mathbb{C})$, rescale $\boldsymbol{\psi}(f)$ by $\kappa_{\Phi_i}(z)^{-1}$. For this rescaled $\boldsymbol{\psi}(f)$ we have $\kappa_{\Phi_i} = 1$ for all i.

Suppose $\sigma \in \mathrm{Aut}(\mathbb{C}/\boldsymbol{k})$. The second claim of Proposition 6.4.1 implies that the divisor of $\boldsymbol{\psi}(f)$, when computed on the compactification $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$, is defined over $\boldsymbol{k}$. Therefore $\sigma(\boldsymbol{\psi}(f))/\boldsymbol{\psi}(f)$ has trivial divisor, and so is constant on every connected component.

By Proposition 6.4.1, the leading coefficient in the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ along the boundary stratum $\mathcal{S}^*_{\mathrm{Kra}}(\Phi_i)$ is

$$\boldsymbol{\psi}_0 = P^{\eta}_{\Phi_i} \otimes P^{hor}_{\Phi_i} \otimes P^{vert}_{\Phi_i},$$

which is defined over $\boldsymbol{k}$. From this it follows that $\sigma(\boldsymbol{\psi}(f))/\boldsymbol{\psi}(f)$ is identically equal to 1 on every connected component of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$ meeting this boundary stratum. Varying i and using Lemma 6.4.2 shows that $\sigma(\boldsymbol{\psi}(f)) = \boldsymbol{\psi}(f)$.

This proves that $\boldsymbol{\psi}(f)$ is defined over $\boldsymbol{k}$, hence so are all of its Fourier-Jacobi coefficients along *all* boundary strata $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$. Appealing again to the calculation of the leading Fourier-Jacobi coefficient of Proposition 6.4.1, we deduce finally that κ_Φ is defined over $\boldsymbol{k}$ for all Φ. □

6.5. **Calculation of the divisor, and completion of the proof.** The Borcherds product $\boldsymbol{\psi}(f)$ on $\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}}$ of Proposition 6.4.3 may now be viewed as a rational section of $\boldsymbol{\omega}^k$ on the integral model $\mathcal{S}^*_{\mathrm{Kra}}$.

Let Φ be any proper cusp label representative. Combining Propositions 6.4.1 and 6.4.3 shows that the leading Fourier-Jacobi coefficient of $\boldsymbol{\psi}(f)$ along the boundary divisor $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is

$$\boldsymbol{\psi}_0 = \kappa_\Phi \otimes P^{\eta}_{\Phi} \otimes P^{hor}_{\Phi} \otimes P^{vert}_{\Phi}. \tag{6.5.1}$$

Recall that this is a rational section of $\boldsymbol{\omega}^k_\Phi \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on $\mathcal{B}_\Phi$. Here, by mild abuse of notation, we are viewing κ_Φ as a rational function on $\mathcal{A}_\Phi$, and denoting in the same way its pullback to any step in the tower

$$\mathcal{C}^*_\Phi \xrightarrow{\pi} \mathcal{B}_\Phi \to \mathcal{A}_\Phi.$$

Lemma 6.5.1. *Recall that π has a canonical section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}^*_\Phi$, realizing $\mathcal{B}_\Phi$ as a divisor on $\mathcal{C}^*_\Phi$. If we use the isomorphism (3.7.1) to view $\boldsymbol{\psi}(f)$ as a rational section of the line bundle $\boldsymbol{\omega}^k_\Phi$ on the formal completion $(\mathcal{C}^*_\Phi)^\wedge_{\mathcal{B}_\Phi}$, its divisor satisfies*

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)) = {} & \mathrm{div}(\delta^{-k}\kappa_\Phi) + \mathrm{mult}_\Phi(f) \cdot \mathcal{B}_\Phi \\ & + \sum_{m>0} c(-m)\mathcal{Z}_\Phi(m) + \sum_{r \mid D} \gamma_r c_r(0) \sum_{p \mid r} \pi^*(\mathcal{B}_{\Phi/\mathbb{F}_p}).\end{aligned}$$

Proof. First we prove

$$\mathrm{div}(\boldsymbol{\psi}(f)) = \pi^*\mathrm{div}(\boldsymbol{\psi}_0) + \mathrm{mult}_\Phi(f) \cdot \mathcal{B}_\Phi. \tag{6.5.2}$$

Recalling the tautological section q with divisor $\mathcal{B}_\Phi$ from Remark 3.8.1, consider the rational section

$$R = q^{-\mathrm{mult}_\Phi(f)} \cdot \boldsymbol{\psi}(f) = \sum_{i \geqslant 0} \boldsymbol{\psi}_i \cdot q^i$$

of $\boldsymbol{\omega}_\Phi^k \otimes \pi^* \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on the formal completion $(\mathcal{C}_\Phi^*)^\wedge_{\mathcal{B}_\Phi}$.

We claim that $\mathrm{div}(R) = \pi^*\Delta$ for *some* divisor Δ on $\mathcal{B}_\Phi$. Indeed, whatever $\mathrm{div}(R)$ is, it may decomposed as a sum of horizontal and vertical components. We know from Theorem 3.7.1 and Proposition 6.4.1 that the horizontal part is a linear combination of the divisors $\mathcal{Z}_\Phi(m)$ on $\mathcal{C}_\Phi^*$ defined by (3.6.1); these divisors are, by definition, pullbacks of divisors on $\mathcal{B}_\Phi$. On the other hand, the morphism $\mathcal{C}_\Phi^* \to \mathcal{B}_\Phi$ is the total space of a line bundle, and hence is smooth with connected fibers. Thus *every* vertical divisor on $\mathcal{C}_\Phi^*$, and in particular the vertical part of $\mathrm{div}(R)$, is the pullback of some divisor on $\mathcal{B}_\Phi$.

Denoting by $i : \mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$ the zero section, we compute

$$\Delta = i^*\pi^*\Delta = i^*\mathrm{div}(R) = \mathrm{div}(i^*R) = \mathrm{div}(\boldsymbol{\psi}_0).$$

Pulling back by π proves that $\mathrm{div}(R) = \pi^*\mathrm{div}(\boldsymbol{\psi}_0)$, and (6.5.2) follows.

We now compute the divisor of $\boldsymbol{\psi}_0$ on $\mathcal{B}_\Phi$ using (6.5.1). The divisors of P_Φ^η, P_Φ^{hor}, and P_Φ^{vert} were computed in Proposition 5.4.1, which shows that on $\mathcal{B}_\Phi$ we have the equality

$$\mathrm{div}(\boldsymbol{\psi}_0) = \mathrm{div}(\delta^{-k}\kappa_\Phi) + \sum_{m>0} c(-m)\mathcal{Z}_\Phi(m) + \sum_{r|D} \gamma_r c_r(0) \sum_{\mathfrak{p}|r} \mathcal{B}_{\Phi/\mathbb{F}_\mathfrak{p}}.$$

Combining this with (6.5.2) completes the proof. □

Proposition 6.5.2. *When viewed as a rational section of $\boldsymbol{\omega}^k$ on $\mathcal{S}_{\mathrm{Kra}}^*$, the Borcherds product $\boldsymbol{\psi}(f)$ has divisor*

$$\begin{aligned} \mathrm{div}(\boldsymbol{\psi}(f)) = & \sum_{m>0} c(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}^*(m) + \sum_{\Phi} \mathrm{mult}_\Phi(f) \cdot \mathcal{S}_{\mathrm{Kra}}^*(\Phi) \\ & + \mathrm{div}(\delta^{-k}) + \sum_{r|D} \gamma_r c_r(0) \sum_{\mathfrak{p}|r} \mathcal{S}_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}^* \end{aligned} \tag{6.5.3}$$

up to a linear combination of irreducible components of the exceptional divisor $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}^*$. *Moreover, each section κ_Φ of Proposition 6.4.3 has finite multiplicative order, and extends to a section $\kappa_\Phi \in H^0(\mathcal{A}_\Phi, \mathcal{O}_{\mathcal{A}_\Phi}^\times)$.*

Proof. Arguing as in the proof of Corollary 3.7.2, we may find a finite extension $F/\boldsymbol{k}_\mathfrak{p}$ such that the maps

$$\begin{array}{ccccc} \bigsqcup_i \mathcal{B}_{\Phi_i/\mathcal{O}_F} & \longrightarrow & \bigsqcup_i \mathcal{S}_{\mathrm{Pap}}^*(\Phi)_{/\mathcal{O}_F} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}/\mathcal{O}_F}^* \\ \downarrow & & & & \\ \bigsqcup_i \mathcal{A}_{\Phi_i/\mathcal{O}_F} & & & & \end{array}$$

of Lemma 6.4.2 induce bijections on connected components, as well as on connected (=irreducible) components of the generic and special fibers. The same property then holds if we replace $\mathcal{S}^*_{\mathrm{Pap}}$ by its dense open substack

$$\mathcal{X} \stackrel{\mathrm{def}}{=} \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing} \cong \mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc}.$$

Suppose Φ is any proper cusp label representative, and let $\mathcal{U}_\Phi \subset \mathcal{X}_{/\mathcal{O}_F}$ be the union of all connected components that meet $\mathcal{S}^*_{\mathrm{Pap}}(\Phi)_{/\mathcal{O}_F}$. If we interpret $\mathrm{div}(\kappa_\Phi)_{/\mathcal{O}_F}$ as a divisor on $\mathcal{X}_{/\mathcal{O}_F}$ using the isomorphism

$$\{\text{vertical divisors on } \mathcal{A}_{\Phi/\mathcal{O}_F}\} \cong \{\text{vertical divisors on } \mathcal{X}_{/\mathcal{O}_F}\},$$

then the equality of divisors (6.5.3) holds after pullback to $\mathcal{U}_\Phi$, up to the error term $\mathrm{div}(\kappa_\Phi)_{/\mathcal{O}_F}$. Indeed, this equality holds in the generic fiber of $\mathcal{U}_\Phi$ by Proposition 6.4.1, and it holds over an open neighborhood of $\mathcal{S}^*_{\mathrm{Pap}}(\Phi)_{/\mathcal{O}_F}$ by Lemma 6.5.1 and the isomorphism of formal completions (3.7.1). As the union of the generic fiber with this open neighborhood is an open substack whose complement has codimension $\geqslant 2$, the stated equality holds over all of $\mathcal{U}_\Phi$.

Letting Φ vary over the Φ_i and using $\kappa_{\Phi_i} = 1$, we see from the paragraph above that (6.5.3) holds after base change to $\bigsqcup_i \mathcal{U}_{\Phi_i} = \mathcal{X}_{/\mathcal{O}_F}$. Using faithfully flat descent for divisors and allowing $\mathfrak{p}$ to vary proves (6.5.3). Now that the equality (6.5.3) is known, we may reverse the argument above to see that the error term $\mathrm{div}(\kappa_\Phi)$ vanishes for every Φ. It follows that κ_Φ extends to a global section of $\mathcal{O}^\times_{\mathcal{A}_\Phi}$.

It only remains to show that each κ_Φ has finite order. Choose a finite extension $L/\boldsymbol{k}$ large enough that every elliptic curve over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$ admits a model over L with everywhere good reduction. Choosing such models determines a faithfully flat morphism

$$\bigsqcup \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{M}_{(1,0)} \cong \mathcal{A}_\Phi,$$

and the pullback of κ_Φ is represented by a tuple of units $(x_\ell) \in \prod \mathcal{O}_L^\times$. Each x_ℓ has absolute value 1 at every complex embedding of L (this follows from the final claim of Proposition 6.4.1), and is therefore a root of unity. This implies that κ_Φ has finite order. □

Proof of Theorem 5.3.1. Start with a weakly holomorphic form (5.2.2). As in §6.2, after possibly replacing f by a positive integer multiple, we obtain a Borcherds product $\boldsymbol{\psi}(f)$. This is a meromorphic section of $(\boldsymbol{\omega}^{an})^k$. By Proposition 6.4.1 it is algebraic, and by Proposition 6.4.3 it may be rescaled by a constant of absolute value 1 on each connected component in such a way that it descends to $\boldsymbol{k}$.

Now view $\boldsymbol{\psi}(f)$ as a rational section of $\boldsymbol{\omega}^k$ over $\mathcal{S}^*_{\mathrm{Kra}}$. By Proposition 6.5.2 we may replace f by a further positive integer multiple, and replace $\boldsymbol{\psi}(f)$ by a corresponding tensor power, in order to make all $\kappa_\Phi = 1$. Having trivialized the κ_Φ, the existence part of Theorem 5.3.1 now follows from Proposition 6.4.1. For uniqueness, suppose $\boldsymbol{\psi}'(f)$ also satisfies the conditions of that theorem. The quotient of the two Borcherds products is a rational function with trivial divisor, which is therefore constant on every connected component of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$. As the leading Fourier-Jacobi coefficients of $\boldsymbol{\psi}'(f)$ and $\boldsymbol{\psi}(f)$ are equal along every boundary stratum, those constants are all equal to 1. $\square$

Proof of Theorem 5.3.4. As in the statement of the theorem, we now view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$. Combining Proposition 6.5.2 with the isomorphism

$$\mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing},$$

of (3.7.2), and recalling from Theorem 3.7.1 that this isomorphism identifies

$$\boldsymbol{\omega}^{2k} \cong \boldsymbol{\Omega}^k_{\mathrm{Kra}} \cong \boldsymbol{\Omega}^k_{\mathrm{Pap}},$$

we deduce the equality

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) = &\sum_{m>0} c(-m)\cdot \mathcal{Y}^*_{\mathrm{Pap}}(m) + 2\sum_{\Phi} \mathrm{mult}_{\Phi}(f)\cdot \mathcal{S}^*_{\mathrm{Pap}}(\Phi)\\ &+ \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}}\end{aligned} \tag{6.5.4}$$

of Cartier divisors on $\mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}$. As $\mathcal{S}^*_{\mathrm{Pap}}$ is normal and Sing lies in codimension $\geqslant 2$, this same equality must hold on the entirety of $\mathcal{S}^*_{\mathrm{Pap}}$. $\square$

Proof of Theorem 5.3.3. If we pull back via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ and view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle

$$\boldsymbol{\Omega}^k_{\mathrm{Kra}} \cong \boldsymbol{\omega}^{2k} \otimes \mathcal{O}(\mathrm{Exc})^{-k},$$

the equality (6.5.4) on $\mathcal{S}^*_{\mathrm{Pap}}$ pulls back to

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) = &\sum_{m>0} c(-m)\cdot \mathcal{Y}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi} \mathrm{mult}_{\Phi}(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ &+ \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_{\mathfrak{p}}}.\end{aligned}$$

Theorem 2.6.3 allows us to rewrite this as

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) = 2&\sum_{m>0} c(-m)\cdot \mathcal{Z}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi} \mathrm{mult}_{\Phi}(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ &+ \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_{\mathfrak{p}}}\\ &- \sum_{m>0} c(-m) \sum_{s\in\pi_0(\mathrm{Sing})} \#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s.\end{aligned}$$

If we instead view $\psi(f)^2$ as a rational section of $\boldsymbol{\omega}^{2k}$, this becomes

$$\begin{aligned}\operatorname{div}(\psi(f)^2) = 2\sum_{m>0} c(-m)\cdot \mathcal{Z}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi}\mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ + \operatorname{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\\ - \sum_{m>0} c(-m)\sum_{s\in\pi_0(\mathrm{Sing})}\#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s\\ + k\cdot \mathrm{Exc}.\end{aligned}$$

as desired. □

7. Modularity of the generating series

Now armed with the modularity criterion of Theorem 4.2.3 and the arithmetic theory of Borcherds products provided by Theorems 5.3.1, 5.3.3, and 5.3.4, we prove our main results: the modularity of generating series of divisors on the integral models $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ of the unitary Shimura variety $\mathrm{Sh}(G,\mathcal{D})$.

The strategy follows that of [Bor99], which proves modularity of the generating series of divisors on the complex fiber of an orthogonal Shimura variety.

7.1. **The modularity theorems.**

Denote by

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \cong \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}})\otimes_{\mathbb{Z}}\mathbb{Q}$$

the Chow group of rational equivalence classes of Cartier divisors on $\mathcal{S}^*_{\mathrm{Kra}}$ with $\mathbb{Q}$ coefficients, and similarly for $\mathcal{S}^*_{\mathrm{Pap}}$. There is a natural pullback map

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}) \to \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

Let $\chi = \chi^n_{\boldsymbol{k}}$ be the quadratic Dirichlet character (5.2.1).

Definition 7.1.1. If V is any $\mathbb{Q}$-vector space, a formal q-expansion

$$\sum_{m\geqslant 0} d(m)\cdot q^m \in V[[q]] \tag{7.1.1}$$

is said to *modular of level D, weight n, and character χ* if for any $\mathbb{Q}$-linear map $\alpha : V\to\mathbb{C}$ the q-expansion

$$\sum_{m\geqslant 0}\alpha(d(m))\cdot q^m \in \mathbb{C}[[q]]$$

is the q-expansion of an element of $M_n(D,\chi)$.

Remark 7.1.2. If (7.1.1) is modular then its coefficients $d(m)$ span a subspace of V of dimension $\leqslant \dim M_n(D,\chi)$. We leave the proof as an exercise for the reader.

We also define the notion of the constant term of (7.1.1) at a cusp ∞_r, generalizing Definition 4.1.1.

Definition 7.1.3. Suppose a formal q-expansion $g \in V[[q]]$ is modular of level D, weight n, and character χ. For any $r \mid D$, a vector $v \in V(\mathbb{C})$ is said to be the *constant term of g at the cusp* ∞_r if, for every linear functional $\alpha : V(\mathbb{C}) \to \mathbb{C}$, $\alpha(v)$ is the constant term of $\alpha(g)$ at the cusp ∞_r in the sense of Definition 4.1.1.

For $m > 0$ we have defined in §5.3 effective Cartier divisors

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \hookrightarrow \mathcal{S}^*_{\mathrm{Pap}}, \quad \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \hookrightarrow \mathcal{S}^*_{\mathrm{Kra}}$$

related by (5.3.4). We have defined in §3.7 line bundles

$$\boldsymbol{\Omega}_{\mathrm{Pap}} \in \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Pap}}), \quad \boldsymbol{\omega} \in \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}})$$

extending the line bundles on the open integral models defined in §2.4. For notational uniformity, we define

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}}, \quad \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \boldsymbol{\omega}^{-1} \otimes \mathcal{O}(\mathrm{Exc}).$$

Theorem 7.1.4. *The formal q-expansion*

$$\sum_{m \geqslant 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]],$$

is a modular form of level D, weight n, and character χ. For any $r \mid D$, its constant term at the cusp ∞_r is

$$\gamma_r \cdot \Big(\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2 \sum_{p \mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}} \Big) \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Here $\gamma_r \in \{\pm 1, \pm i\}$ is defined by (5.3.2), $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above $p \mid r$, and $\mathbb{F}_{\mathfrak{p}}$ is its residue field.

Proof. Let f be a weakly holomorphic form as in (5.2.2), and assume again that $c(m) \in \mathbb{Z}$ for all $m \leqslant 0$. The space $M^{!,\infty}_{2-n}(D,\chi)$ is spanned by such forms. The Borcherds product $\boldsymbol{\psi}(f)$ of Theorem 5.3.1 is a rational section of the line bundle

$$\boldsymbol{\omega}^k = \bigotimes_{r \mid D} \boldsymbol{\omega}^{\gamma_r c_r(0)},$$

on $\mathcal{S}^*_{\mathrm{Kra}}$. If we view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle

$$\boldsymbol{\Omega}^k_{\mathrm{Pap}} \cong \bigotimes_{r \mid D} \boldsymbol{\Omega}^{\gamma_r c_r(0)}_{\mathrm{Pap}}$$

on $\mathcal{S}^*_{\mathrm{Pap}}$, exactly as in Theorem 5.3.4, then

$$\mathrm{div}(\boldsymbol{\psi}(f)^2) = -\sum_{r \mid D} \gamma_r c_r(0) \cdot \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0)$$

holds in the Chow group of $\mathcal{S}^*_{\mathrm{Pap}}$. Comparing this with the calculation of the divisor of $\boldsymbol{\psi}(f)^2$ found in Theorem 5.3.4 shows that

$$0 = \sum_{m \geqslant 0} c(-m) \cdot \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) + \sum_{\substack{r \mid D \\ r > 1}} \gamma_r c_r(0) \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r), \tag{7.1.2}$$

where we abbreviate $\mathcal{V}_r = \sum_{p|r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_\mathfrak{p}}$.

For each $r \mid D$ we have defined in §4.2 an Eisenstein series

$$E_r(\tau) = \sum_{m \geqslant 0} e_r(m) \cdot q^m \in M_n(D, \chi),$$

and Proposition 4.2.2 allows us to rewrite the above equality as

$$0 = \sum_{m \geqslant 0} c(-m) \cdot \Big[\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - \sum_{\substack{r|D \\ r>1}} \gamma_r e_r(m) \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r) \Big].$$

Note that we have used $e_r(0) = 0$ for $r > 1$, a consequence of Remark 4.2.1.

The modularity criterion of Theorem 4.2.3 now shows that

$$\sum_{m \geqslant 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m - \sum_{\substack{r|D \\ r>1}} \gamma_r E_r \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r)$$

is a modular form of level D, weight n, and character χ, whose constant term vanishes at every cusp different from ∞.

The theorem now follows from the modularity of each E_r, together with the description of their constant terms found in Remark 4.2.1. □

Theorem 7.1.5. *The formal q-expansion*

$$\sum_{m \geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]],$$

is a modular form of level D, weight n, and character χ.

Proof. Recall from Theorems 2.6.3 and 3.7.1 that pullback via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ sends

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \mapsto 2 \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x \rangle = m\} \cdot \mathrm{Exc}_s$$

for all $m > 0$. This relation also holds for $m = 0$, as those same theorems show that

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}} \mapsto \boldsymbol{\omega}^{-2} \otimes \mathcal{O}(\mathrm{Exc}) = 2 \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \mathrm{Exc}.$$

Pulling back the relation (7.1.2) shows that

$$\begin{aligned} 0 = \sum_{m \geqslant 0} c(-m) \cdot \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s \in \pi_0(\mathrm{Sing})} \frac{\#\{x \in L_s : \langle x, x \rangle = m\}}{2} \cdot \mathrm{Exc}_s \Big) \\ + \sum_{\substack{r|D \\ r>1}} \gamma_r c_r(0) \cdot \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2} \cdot \mathrm{Exc} + \mathcal{V}_r \Big) \end{aligned}$$

in $\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ for any input form (5.2.2), where we now abbreviate

$$\mathcal{V}_r = \sum_{p|r} \mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}.$$

Using Proposition 4.2.2 we rewrite this as

$$0 = \sum_{m \geqslant 0} c(-m) \cdot \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s \in \pi_0(\mathrm{Sing})} \frac{\#\{x \in L_s : \langle x, x \rangle = m\}}{2} \cdot \mathrm{Exc}_s \Big)$$
$$- \sum_{m \geqslant 0} c(-m) \sum_{\substack{r \mid D \\ r > 1}} \gamma_r e_r(m) \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2} \cdot \mathrm{Exc} + \mathcal{V}_r \Big),$$

where we have again used the fact that $e_r(0) = 0$ for $r > 1$.

The modularity criterion of Theorem 4.2.3 now implies the modularity of

$$\sum_{m \geqslant 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m - \frac{1}{2} \sum_{s \in \pi_0(\mathrm{Sing})} \vartheta_s(\tau) \cdot \mathrm{Exc}_s$$
$$- \sum_{\substack{r \mid D \\ r > 1}} \gamma_r E_r(\tau) \cdot \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2} \cdot \mathrm{Exc} + \mathcal{V}_r \Big).$$

The theorem follows from the modularity of the Eisenstein series $E_r(\tau)$ and the theta series

$$\vartheta_s(\tau) = \sum_{x \in L_s} q^{\langle x, x \rangle} \in M_n(D, \chi).$$

□

7.2. **Green functions.** Here we construct Green functions for special divisors on $\mathcal{S}^*_{\mathrm{Kra}}$ as regularized theta lifts of harmonic Maass forms.

Recall from Section 2 the isomorphism of complex orbifolds

$$\mathcal{S}_{\mathrm{Kra}}(\mathbb{C}) \cong \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K.$$

We use the uniformization on the right hand side and the regularized theta lift to construct Green functions for the special divisors

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m)$$

on $\mathcal{S}^*_{\mathrm{Kra}}$. The construction is a variant of the ones in [BF04] and [BHY15], adapted to our situation.

We now recall some of the basic notions of the theory of harmonic Maass forms, as in [BF04, Section 3]. Let $H^{\infty}_{2-n}(D, \chi)$ denote the space of harmonic Maass forms f of weight $2 - n$ for $\Gamma_0(D)$ with character χ such that

- f is bounded at all cusps of $\Gamma_0(D)$ different from the cusp ∞,
- f has polynomial growth at ∞, in sense that there is a
$$P_f = \sum_{m < 0} c^+(m) q^m \in \mathbb{C}[q^{-1}]$$
such that $f - P_f$ is bounded as q goes to 0.

A harmonic Maass form $f \in H^{\infty}_{2-n}(D, \chi)$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Z} \\ m \gg -\infty}} c^+(m) q^m + \sum_{\substack{m \in \mathbb{Z} \\ m < 0}} c^-(m) \cdot \Gamma\big(n - 1, 4\pi |m| \operatorname{Im}(\tau)\big) \cdot q^m, \tag{7.2.1}$$

where

$$\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1} dt$$

is the incomplete gamma function. The first summand on the right hand side of (7.2.1) is denoted by f^+ and is called the *holomorphic part* of f, the second summand is denoted by f^- and is called the *non-holomorphic part*.

If $f \in H^\infty_{2-n}(D,\chi)$ then (6.1.1) defines an S_L-valued harmonic Maass form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2-n$ with representation ω_L. Proposition 6.1.2 extends to such lifts of harmonic Maass forms, giving the same formulas for the coefficients $\tilde{c}^+(m,\mu)$ of the holomorphic part $\tilde{f}^+$ of $\tilde{f}$. In particular, if $m<0$ we have

$$\tilde{c}^+(m,\mu) = \begin{cases} c^+(m) & \text{if } \mu = 0, \\ 0 & \text{if } \mu \neq 0, \end{cases} \tag{7.2.2}$$

and the constant term of $\tilde{f}$ is given by

$$\tilde{c}^+(0,\mu) = \sum_{r_\mu | r | D} \gamma_r \cdot c_r^+(0).$$

The formula of Proposition 4.2.2 for the contant terms $c_r^+(0)$ of f at the other cusps also extends.

As before, we consider the hermitian self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice $L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$ in $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. The dual lattice of L with respect to the bilinear form $[\cdot,\cdot]$ is $L' = \mathfrak{d}^{-1}L$. Let

$$S_L \subset S(V(\mathbb{A}_f))$$

be the space of Schwartz-Bruhat functions that are supported on $\widehat{L}'$ and invariant under translations by $\widehat{L}$.

Recall from Remark 2.1.3 that we may identify

$$\mathcal{D} \cong \{w \in \epsilon V(\mathbb{C}) : [w, \overline{w}] < 0\}/\mathbb{C}^\times,$$

and also

$$\mathcal{D} \cong \{\text{negative definite } \boldsymbol{k}\text{-stable } \mathbb{R}\text{-planes } z \subset V(\mathbb{R})\}.$$

For any $x \in V$ and $z \in \mathcal{D}$, let x_z be the orthogonal projection of x to the plane $z \subset V(\mathbb{R})$, and let $x_{z^\perp}$ be the orthogonal projection to $z^\perp$.

For $(\tau, z, g) \in \mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f)$ and $\varphi \in S_L$, we define a theta function

$$\theta(\tau, z, g, \varphi) = \sum_{x \in V} \varphi(g^{-1}x) \cdot \varphi_\infty(\tau, z, x),$$

where the Schwartz function at ∞,

$$\varphi_\infty(\tau, z, x) = v \cdot e^{2\pi i Q(x_{z^\perp})\tau + 2\pi i Q(x_z)\bar{\tau}},$$

is the usual Gaussian involving the majorant associated to z. We may view θ as a function $\mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to S_L^\vee$. As a function in (z,g) it is invariant under the left action of $G(\mathbb{Q})$. Under the right action of K it satisfies the transformation law

$$\theta(\tau, z, gk, \varphi) = \theta(\tau, z, g, \omega_L(k)\varphi), \quad k \in K,$$

where ω_L denotes the action of K on S_L by the Weil representation and $v = \mathrm{Im}(\tau)$. In the variable $\tau \in \mathfrak{H}$ it transforms as a $S_L^\vee$-valued modular form of weight $n-2$ for $\mathrm{SL}_2(\mathbb{Z})$.

Fix an $f \in H_{2-n}^\infty(D, \chi)$ with Fourier expansion as in (7.2.1), and assume that $c^+(m) \in \mathbb{Z}$ for $m \leqslant 0$. We associate to f the divisors

$$\mathcal{Z}_{\mathrm{Kra}}(f) = \sum_{m>0} c^+(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}(m)$$
$$\mathcal{Z}_{\mathrm{Kra}}^{\mathrm{tot}}(f) = \sum_{m>0} c^+(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}^{\mathrm{tot}}(m)$$

on $\mathcal{S}_{\mathrm{Kra}}$ and $\mathcal{S}_{\mathrm{Kra}}^*$, respectively. As the actions of $\mathrm{SL}_2(\mathbb{Z})$ and K via the Weil representation commute, the associated S_L-valued harmonic Maass form $\tilde{f}$ is invariant under K. Hence the natural pairing $S_L \times S_L^\vee \to \mathbb{C}$ gives rise to a scalar valued function $(\tilde{f}(\tau), \theta(\tau, z, g))$ in the variables $(\tau, z, g) \in \mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f)$, which is invariant under the right action of K and the left action of $G(\mathbb{Q})$. Hence it descends to a function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \times \mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$.

We define the *regularized theta lift of f* as

$$\Theta^{\mathrm{reg}}(z, g, f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{\mathrm{reg}} \big(\tilde{f}(\tau), \theta(\tau, z, g)\big) \frac{du\, dv}{v^2}.$$

Here the regularization of the integral is defined as in [Bor98, BF04, BHY15]. We extend the incomplete Gamma function

$$\Gamma(0, t) = \int_t^\infty e^{-v} \frac{dv}{v}$$

to a function on $\mathbb{R}_{\geqslant 0}$ by setting

$$\tilde{\Gamma}(0, t) = \begin{cases} \Gamma(0, t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Theorem 7.2.1. *The regularized theta lift $\Theta^{\mathrm{reg}}(z, g, f)$ defines a smooth function on $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C}) \smallsetminus \mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$. For $g \in G(\mathbb{A}_f)$ and $z_0 \in \mathcal{D}$, there exists a neighborhood $U \subset \mathcal{D}$ of z_0 such that*

$$\Theta^{\mathrm{reg}}(z, g, f) - \sum_{\substack{x \in gL \\ x \perp z_0}} c^+(-\langle x, x\rangle) \cdot \tilde{\Gamma}\big(0, 4\pi|\langle x_z, x_z\rangle|\big)$$

is a smooth function on U.

Proof. Note that the sum over $x \in gL \cap z_0^\perp$ is finite. Since $\mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$ decomposes into a finite disjoint union of connected components of the form

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash\mathcal{D},$$

where $g \in G(\mathbb{A}_f)$, it suffices to consider the restriction of $\Theta^{\mathrm{reg}}(f)$ to these components.

On such a component, $\Theta^{\mathrm{reg}}(z, g, f)$ is a regularized theta lift considered in [BHY15, Section 4] of the vector valued form $\tilde{f}$ for the lattice

$$gL = g\widehat{L} \cap V = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, g\mathfrak{a}) \subset V,$$

and hence the assertion follows from (7.2.2) and [BHY15, Theorem 4.1]. □

Remark 7.2.2. Let $\Delta_{\mathcal{D}}$ denote the $\mathrm{U}(V)(\mathbb{R})$-invariant Laplacian on $\mathcal{D}$. There exists a non-zero real constant c (which only depends on the normalization of $\Delta_{\mathcal{D}}$ and which is independent of f), such that

$$\Delta_{\mathcal{D}} \Theta^{\mathrm{reg}}(z, g, f) = c \cdot \deg \mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$$

on the complement of the divisor $\mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$.

Using the fact that

$$\Gamma(0, t) = -\log(t) + \Gamma'(1) + o(t)$$

as $t \to 0$, Theorem 7.2.1 implies that $\Theta^{\mathrm{reg}}(f)$ is a (sub-harmonic) logarithmic Green function for the divisor $\mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$ on the non-compactified Shimura variety $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$. These properties, together with an integrability condition, characterize it uniquely up to addition of a locally constant function [BHY15, Theorem 4.6]. The following result describes the behavior of $\Theta^{\mathrm{reg}}(f)$ on the toroidal compactification.

Theorem 7.2.3. *On $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, the function $\Theta^{\mathrm{reg}}(f)$ is a logarithmic Green function for the divisor $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(f)(\mathbb{C})$ with possible additional log-log singularities along the boundary in the sense of* [BGKK07].

Proof. As in the proof of Theorem 7.2.1 we reduce this to showing that $\Theta^{\mathrm{reg}}(f)$ has the correct growth along the boundary of the connected components of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$. Then it is a direct consequence of [BHY15, Theorem 4.10] and [BHY15, Corollary 4.12]. □

Recall that $\boldsymbol{\omega}^{an}$ is the tautological bundle on

$$\mathcal{D} \cong \big\{ w \in \epsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^\times.$$

We define the Petersson metric $\| \cdot \|$ on $\boldsymbol{\omega}^{an}$ by

$$\|w\|^2 = -\frac{[w, \overline{w}]}{4\pi e^\gamma},$$

where $\gamma = -\Gamma'(1)$ denotes Euler's constant. This choice of metric on $\boldsymbol{\omega}^{an}$ induces a metric on the line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ defined in §2.4, which extends to a metric over $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ with log-log singularities along the boundary [BHY15, Proposition 6.3]. We obtain a hermitian line bundle on $\mathcal{S}^*_{\mathrm{Kra}}$, denoted

$$\widehat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \| \cdot \|).$$

If f is actually weakly holomorphic, that is, if it belongs to $M^{!,\infty}_{2-n}(D, \chi)$, then $\Theta^{\mathrm{reg}}(f)$ is given by the logarithm of a Borcherds product. More precisely, we have the following theorem, which follows immediately from [Bor98, Theorem 13.3] and our construction of $\boldsymbol{\psi}(f)$ as the pullback of a Borcherds product, renormalized by (6.2.3), on an orthogonal Shimura variety.

Theorem 7.2.4. *Let $f \in M^{!,\infty}_{2-n}(D,\chi)$ be as in (5.2.2). The Borcherds product $\boldsymbol{\psi}(f)$ of Theorem 5.3.1 satisfies*

$$\Theta^{\mathrm{reg}}(f) = -\log \|\boldsymbol{\psi}(f)\|^2.$$

7.3. **Generating series of arithmetic special divisors.** We can now define arithmetic special divisors on $\mathcal{S}^*_{\mathrm{Kra}}$, and prove a modularity result for the corresponding generating series in the codimension one arithmetic Chow group. This result extends Theorem 7.1.5.

Recall our hypothesis that $n > 2$, and let m be a positive integer. As in [BF04, Proposition 3.11], or using Poincaré series, it can be shown that there exists a unique $f_m \in H^{\infty}_{2-n}(D,\chi)$ whose Fourier expansion at the cusp ∞ has the form

$$f_m = q^{-m} + O(1)$$

as $q \to 0$. According to Theorem 7.2.3, its regularized theta lift $\Theta^{\mathrm{reg}}(f_m)$ is a logarithmic Green function for $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$.

Denote by $\widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ the arithmetic Chow group [GS90] of rational equivalence classes of arithmetic divisors with $\mathbb{Q}$-coefficients. We allow the Green functions of our arithmetic divisors to have possible additional log-log error terms along the boundary of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, as in the theory of [BGKK07]. For $m > 0$ define an arithmetic special divisor

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = (\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m), \Theta^{\mathrm{reg}}(f_m)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$$

on $\mathcal{S}^*_{\mathrm{Kra}}$, and for $m = 0$ set

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

In the theory of arithmetic Chow groups one usually works on a regular scheme such as $\mathcal{S}^*_{\mathrm{Kra}}$. However, the codimension one arithmetic Chow group of $\mathcal{S}^*_{\mathrm{Pap}}$ makes perfect sense: one only needs to specify that it consists of rational equivalence classes of *Cartier* divisors on $\mathcal{S}^*_{\mathrm{Pap}}$ endowed with a Green function.

With this in mind one can use the equality

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)(\mathbb{C}) = 2 \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)(\mathbb{C})$$

in the complex fiber $\mathcal{S}^*_{\mathrm{Pap}}(\mathbb{C}) = \mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ to define arithmetic divisors

$$\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) = (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m), 2 \cdot \Theta^{\mathrm{reg}}(f_m)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})$$

for $m > 0$. For $m = 0$ we define

$$\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \widehat{\boldsymbol{\Omega}}^{-1} + (0, -2\log(D)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}),$$

where the metric on $\boldsymbol{\Omega}$ is induced from that on $\boldsymbol{\omega}$, again using $\boldsymbol{\Omega} \cong \boldsymbol{\omega}^2$ in the complex fiber.

Theorem 7.3.1. *The formal q-expansions*

$$\widehat{\phi}(\tau) = \sum_{m \geqslant 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]] \tag{7.3.1}$$

and

$$\sum_{m \geqslant 0} \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]]$$

are modular forms of level D, weight n, and character χ.

Proof. For any input form $f \in M^{!,\infty}_{2-n}(D, \chi)$ as in (5.2.2), the relation in the Chow group given by the Borcherds product $\boldsymbol{\psi}(f)$ is compatible with the Green functions, in the sense that

$$-\log \|\boldsymbol{\psi}(f)\|^2 = \sum_{m>0} c(-m) \cdot \Theta^{\mathrm{reg}}(f_m).$$

Indeed, this directly follows from $f = \sum_{m>0} c(-m) f_m$ and Theorem 7.2.4.

This observation allows us to simply repeat the argument of Theorems 7.1.4 and 7.1.5 on the level of arithmetic Chow groups. Viewing $\boldsymbol{\psi}(f)^2$ as a rational section of the metrized line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$, the arithmetic divisor

$$\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) \stackrel{\mathrm{def}}{=} \left(\mathrm{div}(\boldsymbol{\psi}(f)^2), -2\log \|\boldsymbol{\psi}(f)\|^2\right) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})$$

satisfies both

$$\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) = \widehat{\boldsymbol{\Omega}}^k_{\mathrm{Pap}} = -2k \cdot (0, \log(D)) - \sum_{r \mid D} \gamma_r c_r(0) \cdot \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0)$$

and

$$\begin{aligned}
&\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) \\
&\quad = \sum_{m>0} c(-m) \cdot \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - 2k \cdot (\mathrm{div}(\delta), 0) + 2 \sum_{r \mid D} \gamma_r c_r(0) \cdot \widehat{\mathcal{V}}_r \\
&\quad = \sum_{m>0} c(-m) \cdot \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - 2k \cdot (0, \log(D)) + 2 \sum_{r \mid D} \gamma_r c_r(0) \cdot \widehat{\mathcal{V}}_r,
\end{aligned}$$

where $\widehat{\mathcal{V}}_r$ is the the vertical divisor $\mathcal{V}_r = \sum_{p \mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}}$ endowed with the trivial Green function.

Using the relation

$$0 = \widehat{\mathrm{div}}(\delta) = (\mathrm{div}(\delta), -\log|\delta^2|) = (\mathrm{div}(\delta), 0) - (0, \log(D))$$

in the arithmetic Chow group, we deduce that

$$0 = \sum_{m \geqslant 0} c(-m) \cdot \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) + \sum_{\substack{r \mid D \\ r > 1}} \gamma_r c_r(0) \left(\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2 \cdot \widehat{\mathcal{V}}_r\right).$$

With this relation in hand, both proofs go through verbatim. □

7.4. **Non-holomorphic generating series of special divisors.** In this subsection we discuss a non-holomorphic variant of the generating series (7.3.1), which is obtained by endowing the special divisors with other Green functions, namely with those constructed in [How12, How15] following the method of [Kud97b]. By combining Theorem 7.3.1 with a recent result of Ehlen and Sankaran [ES16], we show that the non-holomorphic generating series is also modular.

For every $m \in \mathbb{Z}$ and $v \in \mathbb{R}_{>0}$ define a divisor

$$\mathcal{B}_{\mathrm{Kra}}(m,v) = \frac{1}{4\pi v}\sum_{\Phi} \#\{x \in L_0 : \langle x,x\rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)$$

with real coefficients on $\mathcal{S}^*_{\mathrm{Kra}}$. Here the sum is over all K-equivalence classes of proper cusp label representatives Φ in the sense of §3.2, L_0 is the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2,0)$ defined by (3.1.4), and $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is the boundary divisor of Theorem 3.7.1. Note that $\mathcal{B}_{\mathrm{Kra}}(m,v)$ is trivial for all $m < 0$. We define classes in $\mathrm{Ch}^1_{\mathbb{R}}(\mathcal{S}^*_{\mathrm{Kra}})$, depending on the parameter v, by

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v) = \begin{cases} \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m,v) & \text{if } m \neq 0 \\ \boldsymbol{\omega}^{-1} + \mathrm{Exc} + \mathcal{B}_{\mathrm{Kra}}(0,v) & \text{if } m = 0. \end{cases}$$

Following [How12, How15, Kud97b], Green functions for these divisors can be constructed as follows. For $x \in V(\mathbb{R})$ and $z \in \mathcal{D}$ we put

$$R(x,z) = -2Q(x_z),$$

and we let

$$\beta(v) = \int_1^\infty e^{-vt}\frac{dt}{t}.$$

For $m \in \mathbb{Z}$ and $(v,z,g) \in \mathbb{R}_{>0} \times \mathcal{D} \times G(\mathbb{A}_f)$, we define a Green function

$$\Xi(m,v,z,g) = \sum_{\substack{x \in V \smallsetminus \{0\} \\ Q(x)=m}} \chi_{\widehat{L}}(g^{-1}x) \cdot \beta(2\pi v R(x,z)), \tag{7.4.1}$$

where $\chi_{\widehat{L}} \in S_L$ denotes the characteristic function of $\widehat{L}$. As a function of the variable (z,g), (7.4.1) is invariant under the left action of $G(\mathbb{Q})$ and under the right action of K, and so descends to a function on $\mathbb{R}_{>0} \times \mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. It was proved in [How15, Theorem 3.4.7] that $\Xi(m,v)$ is a logarithmic Green function for $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v)$ when $m \neq 0$. When $m = 0$ it is a logarithmic Green function for $\mathcal{B}_{\mathrm{Kra}}(0,v)$.

Consequently, we obtain arithmetic special divisors in $\widehat{\mathrm{Ch}}^1_{\mathbb{R}}(\mathcal{S}^*_{\mathrm{Kra}})$ depending on the parameter v by putting

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v) = \begin{cases} \big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v), \Xi(m,v)\big) & \text{if } m \neq 0 \\ \widehat{\boldsymbol{\omega}}^{-1} + (\mathcal{B}_{\mathrm{Kra}}(0,v), \Xi(0,v)) + (\mathrm{Exc}, -\log(Dv)) & \text{if } m = 0. \end{cases}$$

Note that for $m < 0$ these divisors are supported in the archimedian fiber.

Theorem 7.4.1. *The formal q-expansion*

$$\widehat{\phi}_{\text{non-hol}}(\tau) = \sum_{m\in\mathbb{Z}} \widehat{\mathcal{Z}}^{\text{tot}}_{\text{Kra}}(m,v)\cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{R}}(\mathcal{S}^*_{\text{Kra}})[[q]],$$

is a non-holomorphic modular form of level D, weight n, and character χ. Here $q = e^{2\pi i\tau}$ and $v = \mathrm{Im}(\tau)$.

Here the assertion about modularity is to be understood as in [ES16, Definition 4.11]. In our situation it reduces to the statement that there is a smooth function $s(\tau, z, g)$ on $\mathfrak{H}\times \mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$ with the following properties:

(1) in (z, g) the function $s(\tau, z, g)$ has at worst log-log-singularities at the boundary of $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$ (in particular it is a Green function for the trivial divisor);
(2) $s(\tau, z, g)$ transforms in τ as a non-holomorphic modular form of level D, weight n, and character χ;
(3) the difference $\widehat{\phi}_{\text{non-hol}}(\tau) - s(\tau, z, g)$ belongs to the space

$$M_n(D,\chi)\otimes_{\mathbb{C}} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\text{Kra}}) \oplus (R_{n-2}M_{n-2}(D,\chi))\otimes_{\mathbb{C}} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\text{Kra}}),$$

where R_{n-2} denotes the Maass raising operator as in Section 8.4.

Proof. Theorem 4.13 of [ES16] states that the difference

$$\widehat{\phi}_{\text{non-hol}}(\tau) - \widehat{\phi}(\tau) \tag{7.4.2}$$

is a non-holomorphic modular form of level D, weight n, and character χ, valued in $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\text{Kra}})$. Hence the assertion follows from Theorem 7.3.1. □

Remark 7.4.2. According to Theorem 4.18 of [ES16], the difference (7.4.2) has trivial holomorphic projection. Therefore the generating series $\widehat{\phi}_{\text{non-hol}}(\tau)$ and $\widehat{\phi}(\tau)$ define the same arithmetic theta lift

$$S_n(D,\chi) \to \widehat{\mathrm{Ch}}^1_{\mathbb{R}}(\mathcal{S}^*_{\text{Kra}}).$$

8. Appendix: some technical calculations

We collect some technical arguments and calculations. Strictly speaking, none of these are essential to the proofs in the body of the text. We explain the connection between the fourth roots of unity γ_p defined by (5.3.1) and the local Weil indices appearing in the theory of the Weil representation, provide alternative proofs of Propositions 6.1.2 and 6.3.3, and explain in greater detail how Proposition 6.3.1 is deduced from the formulas of [Kud16].

8.1. **Local Weil indices.**

In this subsection, we explain how the quantity γ_p defined in (5.3.1) is related to the local Weil representation.

Let $L \subset V$ be as in §6.1, and recall that $S_L = \mathbb{C}[L'/L]$ is identified with a subspace of $S(V(\mathbb{A}_f))$ by sending $\mu \in L'/L$ to the characteristic function ϕ_μ of $\mu + \widehat{L} \subset V(\mathbb{A}_f)$.

As $\dim_{\mathbb{Q}} V = 2n$ and D is odd, the representation ω_L of $\mathrm{SL}_2(\mathbb{Z})$ on S_L is the pullback via

$$\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \prod_{p|D} \mathrm{SL}_2(\mathbb{Z}_p)$$

of the representation

$$\omega_L = \bigotimes_{p|D} \omega_p,$$

where $\omega_p = \omega_{L_p}$ is the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $S_{L_p} \subset S(V_p)$. These Weil representations are defined using the standard global additive character $\psi = \otimes_p \psi_p$ which is trivial on $\widehat{\mathbb{Z}}$ and on $\mathbb{Q}$ and whose restriction to $\mathbb{R} \subset \mathbb{A}$ is given by $\psi(x) = \exp(2\pi i x)$. Recall that, for $a \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$,

$$\begin{aligned}
\omega_p(n(b))\phi(x) &= \psi_p(bQ(x)) \cdot \phi(x) \\
\omega_p(m(a))\phi(x) &= \chi_{\boldsymbol{k},p}^n(a) \cdot |a|_p^n \cdot \phi(ax) \\
\omega_p(w)\phi(x) &= \gamma_p \int_{V_p} \psi_p(-[x,y]) \cdot \phi(y)\, dy, \quad w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},
\end{aligned}$$

where $\gamma_p = \gamma_p(L)$ is the Weil index of the quadratic space V_p with respect to ψ_p and $\chi_{\boldsymbol{k},p}$ is the quadratic character of $\mathbb{Q}_p^\times$ corresponding to $\boldsymbol{k}_p$. Note that dy is the self-dual measure with respect to the pairing $\psi_p([x,y])$.

Lemma 8.1.1.

(1) *When restricted to the subspace $S_{L_p} \subset S(V_p)$, the action of $\gamma \in \mathrm{SL}_2(\mathbb{Z}_p)$ depends only on the image of γ in $\mathrm{SL}_2(\mathbb{F}_p)$.*

(2) *The Weil index is given by*

$$\gamma_p = \epsilon_p^{-n} \cdot (D,p)_p^n \cdot \mathrm{inv}_p(V_p)$$

where $(a,b)_p$ is the Hilbert symbol for $\mathbb{Q}_p$ and $\mathrm{inv}_p(V_p)$ is the invariant of V_p in the sense of (1.7.3).

Proof. (i) It suffices to check this on the generators. We omit this.
(ii) We can choose an $O_{\boldsymbol{k},p}$-basis for L_p such that the matrix for the hermitian form is $\mathrm{diag}(a_1, \ldots, a_n)$, with $a_j \in \mathbb{Z}_p^\times$. The matrix for the bilinear form $[x,y] = \mathrm{Tr}_{K_p/\mathbb{Q}_p}(\langle x,y\rangle)$ is then $\mathrm{diag}(2a_1, \ldots, 2a_n, 2Da_1, \ldots, 2Da_n)$. Then, according to the formula for β_V in [Kud94, p. 379], we have

$$\gamma_p^{-1} = \gamma_{\mathbb{Q}_p}(\frac{1}{2} \cdot \psi_p \circ V) = \prod_{j=1}^n \gamma_{\mathbb{Q}_p}(a_j \psi_p) \cdot \gamma_{\mathbb{Q}_p}(Da_j \psi_p),$$

where we note that, in the notation there, $x(w) = 1$, and $j = j(w) = 1$. Next by Proposition A.11 of the Appendix to [RR93], for any $\alpha \in \mathbb{Z}_p^\times$, we have $\gamma_{\mathbb{Q}_p}(\alpha\psi_p) = 1$ and

$$\gamma_{\mathbb{Q}_p}(\alpha p \psi_p) = \left(\frac{-\alpha}{p}\right) \cdot \epsilon_p = (-\alpha, p)_p \cdot \epsilon_p.$$

Here note that if $\eta = \alpha p\psi_p$, then the resulting character $\bar{\eta}$ of $\mathbb{F}_p$ is given by

$$\bar{\eta}(\bar{a}) = \psi_p(p^{-1}a) = e(-p^{-1}a).$$

and $\gamma_{\mathbb{F}_p}(\bar{\eta}) = \left(\frac{-1}{p}\right) \cdot \epsilon_p$. Thus

$$\gamma_p = \epsilon_p^{-n} \cdot (-D/p, p)_p^n \cdot (\det(V), p)_p,$$

as claimed. □

8.2. **A direct proof of Proposition 6.1.2.** The proof of Proposition 6.1.2, which expresses the Fourier coefficients of the vector valued form $\tilde{f}$ in terms of those of the scalar valued form $f \in M^!_{2-n}(D,\chi)$, appealed to the more general results of [Sch09]. In some respects, it is easier to prove Proposition 6.1.2 from scratch than it is to extract it from [*loc. cit.*]. This is what we do here.

Recall that $\tilde{f}$ is defined from f by the induction procedure of (6.1.1), and that the coefficients $\tilde{c}(m,\mu)$ in its Fourier expansion (6.1.2) are indexed by $m \in \mathbb{Q}$ and $\mu \in L'/L$. Recall that, for $r \mid D$, $rs = D$,

$$W_r = \begin{pmatrix} r\alpha & \beta \\ D\gamma & r\delta \end{pmatrix} = R_r \begin{pmatrix} r & \\ & 1 \end{pmatrix}, \quad R_r = \begin{pmatrix} \alpha & \beta \\ s\gamma & r\delta \end{pmatrix} \in \Gamma_0(s).$$

Note that

$$\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma(D) \simeq \prod_{p\mid D} B_p\backslash \mathrm{SL}_2(\mathbb{F}_p), \tag{8.2.1}$$

so this set has order $\prod_{p\mid D}(p+1)$. A set of coset representatives is given by

$$\bigsqcup_{\substack{r\mid D \\ c \,(\mathrm{mod}\, r)}} R_r \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}.$$

Now, using (3.3.1), we have

$$\begin{aligned} \left(f\big|_{2-n} R_r \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}\right)(\tau) &= \left(f\big|_{2-n} W_r \begin{pmatrix} r^{-1} & r^{-1}c \\ & 1 \end{pmatrix}\right)(\tau) \\ &= \chi_r(\beta)\chi_s(\alpha) \sum_{m \gg -\infty} r^{\frac{n}{2}-1} c_r(m) \cdot e^{\frac{2\pi i m(\tau+c)}{r}} \end{aligned} \tag{8.2.2}$$

On the other hand, the image of the inverse of our coset representative on the right side of (8.2.1) has components

$$\begin{cases} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & -\beta \\ -s\gamma & \alpha \end{pmatrix} & \text{if } p \mid r \\ \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} r\delta & -\beta \\ 0 & \alpha \end{pmatrix} & \text{if } p \mid s. \end{cases}$$

Note that $r\alpha\delta - s\beta\gamma = 1$. Then, as elements of $\mathrm{SL}_2(\mathbb{F}_p)$, we have

$$\begin{cases} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \alpha\beta \\ & 1 \end{pmatrix} & \text{if } p \mid r \\ \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & -\alpha\beta \\ & 1 \end{pmatrix} & \text{if } p \mid s. \end{cases}$$

The element on the second line just multiplies $\phi_{0,p}$ by $\chi_p(\alpha)$. For the element on the first line, the factor on the right fixes ϕ_0 and

$$\omega_p \left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right) \phi_0 = \gamma_p \, p^{-\frac{n}{2}} \sum_{\mu \in L_p'/L_p} \phi_\mu.$$

Thus, the element on the first line carries $\phi_{0,p}$ to

$$\chi_p(\beta)\gamma_p \, p^{-\frac{n}{2}} \sum_{\mu \in L_p'/L_p} \psi_p(-c\, Q(\mu))\, \phi_\mu.$$

Recall from (6.1.3) that for $\mu \in L'/L$, r_μ is the product of the primes $p \mid D$ such that $\mu_p \neq 0$. Thus

$$\omega_L \left(R_r \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \right)^{-1} \phi_0 = \chi_s(\alpha)\chi_r(\beta)\, \gamma_r \, r^{-\frac{n}{2}} \sum_{\substack{\mu \in L'/L \\ r_\mu \mid r}} e^{2\pi i c Q(\mu)} \phi_\mu. \tag{8.2.3}$$

Taking the product of (8.2.2) and (8.2.3) and summing on c and on r, we obtain

$$\begin{aligned} &\sum_{r \mid D} \gamma_r \cdot r^{-1} \sum_{c \,(\mathrm{mod}\, r)} \sum_{\substack{\mu \in L'/L \\ r_\mu \mid r}} e^{2\pi i c Q(\mu)} \phi_\mu \sum_{m \gg -\infty} c_r(m) e^{\frac{2\pi i m(\tau + c)}{r}} \\ &= \sum_{r \mid D} \gamma_r \sum_{\substack{\mu \in L'/L \\ r_\mu \mid r}} \phi_\mu \sum_{\substack{m \gg -\infty \\ \frac{m}{r} + Q(\mu) \in \mathbb{Z}}} c_r(m) \, q^{\frac{m}{r}} \\ &= \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} \sum_{\substack{\mu \in L'/L \\ m + Q(\mu) \in \mathbb{Z}}} \sum_{\substack{r \\ r_\mu \mid r \mid D}} \gamma_r c_r(mr) \phi_\mu \, q^m \end{aligned}$$

This gives the claimed general expression for $\tilde{c}(m, \mu)$ and completes the proof of Proposition 6.1.2.

8.3. **A more detailed proof of Proposition 6.3.1.** In this section, we explain in more detail how to obtain the product formula of Proposition 6.3.1 from the general formula given in [Kud16].

For our weakly holomorphic S_L-valued modular form $\tilde{f}$ of weight $2 - n$, with Fourier expansion given by (6.1.2), the corresponding meromorphic Borcherds form $\Psi(\tilde{f})$ on $\tilde{\mathcal{D}}^+$ has a product formula [Kud16, Corollary 2.3] in a neighborhood of the 1-dimensional boundary component associated to L_{-1}. It is given as a product of 4 factors, labeled (a), (b), (c) and (d).

We note that, in our present case, there is a basic simplification in factor (b) due to the restriction on the support of the Fourier coefficients of $\tilde{f}$. More precisely, for $m > 0$, $\tilde{c}(-m, \mu) = 0$ for $\mu \notin L$, and $\tilde{c}(-m, 0) = c(-m)$. In particular, if $x \in L'$ with $[x, \mathrm{e}_{-1}] = [x, \mathrm{f}_{-1}] = 0$, then $Q(x) = Q(x_0)$, where x_0 is the $(L_0)_{\mathbb{Q}}$ component of x. If $x_0 \neq 0$, then $Q(x) > 0$, and $\tilde{c}(-Q(x), \mu) = 0$ for $\mu \notin L$. The factors for $\Psi(\tilde{f})$ are then given by:
(a)

$$\prod_{\substack{x \in L' \\ [x, \mathrm{f}_{-1}] = 0 \\ [x, \mathrm{e}_{-1}] > 0 \\ \mathrm{mod}\ L \cap \mathbb{Q}\mathrm{f}_{-1}}} \left(1 - e^{-2\pi i [x,w]}\right)^{\tilde{c}(-Q(x), x)}.$$

(b)

$$P_1(w_0, \tau_1) \stackrel{\mathrm{def}}{=} \prod_{\substack{x \in L_0 \\ [x, W_0] > 0}} \left(\frac{\vartheta_1(-[x,w], \tau_1)}{\eta(\tau_1)} \right)^{c(-Q(x))},$$

where W_0 is a Weyl chamber in $V_0(\mathbb{R})$, as in [Kud16, §2].
(c)

$$P_0(\tau_1) \stackrel{\mathrm{def}}{=} \prod_{\substack{x \in \mathfrak{d}^{-1} L_{-1}/L_{-1} \\ x \neq 0}} \left(\frac{\vartheta_1(-[x,w], \tau_1)}{\eta(\tau_1)} \, e^{\pi i [x,w] \cdot [x, \mathrm{e}_1]} \right)^{\tilde{c}(0,x)/2}$$

(d) and

$$\kappa \, \eta(\tau_1)^{\tilde{c}(0,0)} \, q_2^{I_0},$$

where κ is a scalar of absolute value 1, and

$$I_0 = -\sum_m \sum_{\substack{x \in L' \cap (L_{-1})^{\perp} \\ \mathrm{mod}\ L_{-1}}} \tilde{c}(-m, x) \, \sigma_1(m - Q(x)).$$

The factors given in Proposition 6.3.1 are for the form

$$\tilde{\boldsymbol{\psi}}_g(f) \stackrel{\mathrm{def}}{=} (2\pi i)^{\tilde{c}(0,0)} \Psi(2\tilde{f})$$

The quantity q_2 in [Kud16] is our $e(\xi)$, and τ_1 there is our τ.

Recall from (3.9.5) that $\mathfrak{d}^{-1} L_{-1} = \mathbb{Z}\mathrm{e}_{-1} + D^{-1}\mathbb{Z}\mathrm{f}_{-1}$, so that, in factor (c), the product runs over vectors $D^{-1} b \, \mathrm{f}_{-1}$, with $b \pmod{D}$ nonzero. For these vectors $[x, \mathrm{e}_1] = 0$. In the formula for I, x runs over vectors of the form

$$x = -\frac{b}{D} \mathrm{f}_{-1} + x_0,$$

with $x_0 \in \mathfrak{d}^{-1} L_0$. But, again, if $x_0 \neq 0$, $Q(x) = Q(x_0) > 0$ and $\tilde{c}(-Q(x), x) = 0$ unless $b = 0$, and so the sum in that term runs over $x_0 \in L_0$ $x_0 \neq 0$ and over $-\frac{b}{D}\mathrm{f}_{-1}$'s.

Thus the factors for $\tilde{\boldsymbol{\psi}}_g(f)$ are given by:

(a)

$$\prod_{\substack{x\in L' \\ [x,\mathrm{f}_{-1}]=0 \\ [x,\mathrm{e}_{-1}]>0 \\ \mathrm{mod}\ L\cap\mathbb{Q}\mathrm{f}_{-1}}} \left(1-e^{-2\pi i[x,w]}\right)^{2\,\tilde{c}(-Q(x),x)},$$

(b)

$$P_1(w_0,\tau_1) \stackrel{\mathrm{def}}{=} \prod_{\substack{x_0\in L_0 \\ x_0\neq 0}} \left(\frac{\vartheta_1(-[x_0,w],\tau_1)}{\eta(\tau_1)} \right)^{c(-Q(x_0))},$$

(c)

$$P_0(\tau_1) \stackrel{\mathrm{def}}{=} \prod_{\substack{b\in\mathbb{Z}/D\mathbb{Z} \\ b\neq 0}} \left(\frac{\vartheta_1(-[x,w],\tau_1)}{\eta(\tau_1)} \right)^{\tilde{c}(0,\frac{b}{D}f_{-1})},$$

(d) and, setting $k=\tilde{c}(0,0)$,

$$\kappa^2\,(\,2\pi i\,\eta^2(\tau))^k\,q_2^{2I_0},$$

where κ is a scalar of absolute value 1, and

$$I_0=-2\sum_{m>0}\sum_{x_0\in L_0} c(-m)\,\sigma_1(m-Q(x_0))+\frac{1}{12}\sum_{b\in\mathbb{Z}/D\mathbb{Z}}\tilde{c}(0,\frac{b}{D}\mathrm{f}_{-1}).$$

Here note that for $\tilde{\boldsymbol{\psi}}_g(f)=(2\pi i)^{\tilde{c}(0,0)}\Psi(2\tilde{f})$ we have multiplied the previous expression by 2.

Finally recall

$$w=-\xi\mathrm{e}_{-1}+(\tau\xi-Q(w_0))\mathrm{f}_{-1}+w_0+\tau\mathrm{e}_1+f_1.$$

If $[x,f_{-1}]=0$, then x has the form

$$x=-a\mathrm{e}_{-1}-\frac{b}{D}\mathrm{f}_{-1}+x_0+c\mathrm{e}_1,$$

so that

$$[x,w]=-c\,\xi+[x_0,w_0]-a\tau-\frac{b}{D},$$

and

$$Q(x)=-ac+Q(x_0).$$

Using these values, the formulas given in Proposition 6.3.1 follow immediately.

8.4. **A direct proof of Proposition 6.3.3.** Here we give a direct proof of Proposition 6.3.3, which does not rely on Corollary 6.3.2. We begin by recalling some general facts about derivatives of modular forms.

We let $q\frac{d}{dq}$ be the Ramanujan theta operator on q-series. Recall that the image under $q\frac{d}{dq}$ of a holomorphic modular form g of weight k is in general not a modular form. However, the function

$$D(g) = q\frac{d}{dq}(g) - \frac{k}{12}gE_2 \tag{8.4.1}$$

is a holomorphic modular form of weight $k+2$ (see e.g. [BHY], Section 4.2). Here

$$E_2(\tau) = -24\sum_{m\geqslant 0}\sigma_1(m)q^m$$

denotes the non-modular Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. In particular $\sigma_1(0) = -\frac{1}{24}$. We extend σ_1 to rational arguments by putting $\sigma_1(r) = 0$ if $r \notin \mathbb{Z}_{\geqslant 0}$. If $R_k = 2i\frac{\partial}{\partial\tau} + \frac{k}{v}$ denotes the Maass raising operator, and

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi v}$$

is the non-holomorphic (but modular) Eisenstein series of weight 2, we also have

$$D(g) = -\frac{1}{4\pi}R_k(g) - \frac{k}{12}gE_2^*.$$

Proposition 8.4.1. *Let $f \in M^{!,\infty}_{2-n}(D,\chi)$ as in (5.2.2). The integer*

$$I = \frac{1}{12}\sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}}\tilde{c}(0,\alpha) - 2\sum_{m>0}c(-m)\sum_{x\in L_0}\sigma_1(m-Q(x)).$$

defined in Proposition 6.3.1 is equal to the integer

$$\mathrm{mult}_\Phi(f) = \frac{1}{n-2}\sum_{x\in L_0}c(-Q(x))Q(x)$$

defined by (5.2.4).

Proof. Consider the $S_{L_0}^\vee$-valued theta function

$$\Theta_0(\tau) = \sum_{x\in L_0'}q^{Q(x)}\chi^\vee_{x+L_0} \in M_{n-2}(\omega^\vee_{L_0}).$$

Applying the above construction (8.4.1) to Θ_0 we obtain an $S^\vee_{L_0}$-valued modular form

$$D(\Theta_0) = \sum_{x\in L_0'}Q(x)q^{Q(x)}\chi^\vee_{x+L_0} - \frac{n-2}{12}\Theta_0E_2^* \in M_n(\omega^\vee_{L_0})$$

of weight n. For its Fourier coefficients we have

$$D(\Theta_0) = \sum_{\nu \in L_0'/L_0} \sum_{m \geqslant 0} b(m,\nu) q^m \chi_\nu^\vee$$

$$b(m,\nu) = \sum_{\substack{x \in \nu + L_0 \\ Q(x)=m}} Q(x) + 2(n-2) \sum_{x \in \nu + L_0} \sigma_1(m - Q(x)).$$

As in [BHY15, (4.8)], an S_L-valued modular form F induces an S_{L_0}-valued form F_{L_0}. If we denote by F_μ the components of F with respect to the standard basis (χ_μ) of S_L, we have

$$F_{L_0,\nu} = \sum_{\alpha \in \mathfrak{d}^{-1} L_{-1}/L_{-1}} F_{\nu+\alpha} \tag{8.4.2}$$

for $\nu \in L_0'/L_0$.

Let $\tilde{f} \in M^!_{2-n}(\omega_L)$ be the S_L-valued form corresponding to f, as in (6.1.1). Using (8.4.2) we obtain

$$\tilde{f}_{L_0} \in M^!_{2-n}(\omega_{L_0})$$

with Fourier expansion

$$\tilde{f}_{L_0} = \sum_{\nu,m} \sum_{\alpha \in \delta^{-1} I/I} \tilde{c}(m, \nu+\alpha) q^m \chi_{\nu+L_0}.$$

We consider the natural pairing between the S_{L_0}-valued modular form $\tilde{f}_{L_0}$ of weight $2-n$ and the $S_{L_0}^\vee$-valued modular form $D(\Theta_0)$ of weight n,

$$(\tilde{f}_{L_0}, D(\Theta_0)) \in M_2^!(\mathrm{SL}_2(\mathbb{Z})).$$

By the residue theorem, the constant term of the q-expansion vanishes, and so

$$\sum_{m \geqslant 0} \sum_{\substack{\nu \in L_0'/L_0 \\ \alpha \in \delta^{-1} I/I}} \tilde{c}(-m, \nu+\alpha) b(m,\nu) = 0. \tag{8.4.3}$$

We split this up in the sum over $m > 0$ and the contribution from $m = 0$. Employing Proposition 6.1.2, we obtain that the sum over $m > 0$ is equal to

$$\sum_{m>0} c(-m) b(m,0).$$

For the contribution of $m = 0$ we notice

$$b(0,\nu) = \begin{cases} -\frac{n-2}{12}, & \nu = 0 \in L_0'/L_0, \\ 0, & \nu \neq 0. \end{cases}$$

Hence this part is equal to

$$-\frac{n-2}{12} \sum_{\alpha \in \mathfrak{d}^{-1} L_{-1}/L_{-1}} \tilde{c}(0,\alpha).$$

Inserting the two contributions into (8.4.3), we obtain

$$\begin{aligned}
0 &= \sum_{m>0} c(-m)b(m,0) - \frac{n-2}{12} \sum_{\alpha \in \mathfrak{d}^{-1}L_{-1}/L_{-1}} \tilde{c}(0,\alpha) \\
&= \sum_{m>0} c(-m)\Big(\sum_{\substack{x\in L_0 \\ Q(x)=m}} Q(x) + 2(n-2) \sum_{x\in L_0} \sigma_1(m - Q(x))\Big) \\
&\quad - \frac{n-2}{12} \sum_{\alpha \in \mathfrak{d}^{-1}L_{-1}/L_{-1}} \tilde{c}(0,\alpha) \\
&= \sum_{x\in L_0} c(-Q(x))Q(x) + 2(n-2) \sum_{m>0} c(-m) \sum_{x\in L_0} \sigma_1(m-Q(x)) \\
&\quad - \frac{n-2}{12} \sum_{\alpha \in \mathfrak{d}^{-1}L_{-1}/L_{-1}} \tilde{c}(0,\alpha) \\
&= (n-2)\mathrm{mult}_\Phi(f) - (n-2)I.
\end{aligned}$$

This concludes the proof of the proposition. □

Now we verify directly the other claim of Proposition 6.3.3: the function

$$P_1(\tau, w_0) = \prod_{m>0} \prod_{\substack{x\in L_0 \\ Q(x)=m}} \Theta\big(\tau, \langle w_0, x\rangle\big)^{c(-m)}$$

satisfies the transformation law (3.9.14) with respect to the translation action of $\mathfrak{b}L_0$ on the variable w_0.

First recall that, for $a,\, b \in \mathbb{Z}$,

$$\Theta(\tau, z + a\tau + b) = \exp\big(-\pi i a^2\tau - 2\pi i a z + \pi i(b-a)\big) \cdot \Theta(\tau, z).$$

If we write $\alpha = a\tau + b$ and $\tau = u + iv$, then

$$a = \frac{\mathrm{Im}(\alpha)}{v} = \frac{\alpha - \bar{\alpha}}{2iv}, \qquad b = \mathrm{Re}(\alpha) - \frac{u}{v}\,\mathrm{Im}(\alpha).$$

Thus

$$\frac{1}{2}a^2\tau + az + \frac{1}{2}(a-b) = \frac{1}{4iv}(\alpha-\bar{\alpha})\alpha + \frac{1}{2iv}(\alpha - \bar{\alpha})z + \frac{1}{2}(a - b - ab).$$

For z and w in $\mathbb{C}$, write

$$R(z,w) = R_\tau(z,w) = B_\tau(z,w) - H_\tau(z,w) = \frac{1}{v}z(w - \bar{w}).$$

Then

$$\frac{1}{4v}(\alpha-\bar{\alpha})\alpha + \frac{1}{2v}(\alpha-\bar{\alpha})z = \frac{1}{2}R(z,\alpha) + \frac{1}{4}R(\alpha,\alpha),$$

and we can write

$$\Theta(\tau, z+\alpha) = \exp(-\pi R(z,\alpha) - \frac{\pi}{2}R(\alpha,\alpha)) \cdot \exp(\pi i(a-b-ab))^{-1}\,\Theta(\tau,z).$$

We will consider the contribution of the $\frac{1}{2}(a-b-ab)$ term separately.

For $\beta \in V_0$, we have $\langle w_0+\beta, x\rangle = \langle w_0, x\rangle + \langle \beta, x\rangle$. Suppose that for all $x \in L_0$, we have $\langle \beta, x\rangle = a\tau + b$ for a and b in $\mathbb{Z}$. Writing $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\tau$, this is precisely the condition that $\beta \in \mathfrak{b}\, L_0$. Then we obtain a factor

$$\exp\left(-\pi \sum_{m>0} \sum_{\substack{x\in L_0\\ Q(x)=m}} c(-m)\left[R\big(\langle w_0,x\rangle,\langle\beta,x\rangle\big) + \frac{R\big(\langle\beta,x\rangle,\langle\beta,x\rangle\big)}{2}\right]\right).$$

Expanding the sum and using the hermitian version of Borcherds' quadratic identity from the proof of Proposition 5.2.2, we have

$$\begin{aligned}
&\sum_{x\in L_0} \frac{c(-Q(x))}{v}\left[\langle w_0,x\rangle\langle\beta,x\rangle - \langle w_0,x\rangle\langle x,\beta\rangle + \frac{\langle\beta,x\rangle\langle\beta,x\rangle}{2} - \frac{\langle\beta,x\rangle\langle x,\beta\rangle}{2}\right]\\
&= -\frac{1}{v}\left(\langle w_0,\beta\rangle + \frac{1}{2}\langle\beta,\beta\rangle\right)\cdot\frac{1}{2n-4}\cdot\sum_{x\in L_0} c(-Q(x))\,[x,x]\\
&= -\frac{1}{v}\left(\langle w_0,\beta\rangle + \frac{1}{2}\langle\beta,\beta\rangle\right)\cdot \mathrm{mult}_\Phi(f).
\end{aligned}$$

Thus, using $I = \mathrm{mult}_\Phi(f)$, we have a contribution of

$$\exp\left(\frac{\pi\langle w_0,\beta\rangle}{v} + \frac{\pi\langle\beta,\beta\rangle}{2v}\right)^I$$

to the transformation law.

Next we consider the quantity

$$\begin{aligned}
a&-b-ab\\
&= \frac{\mathrm{Im}(\alpha)}{v} - \mathrm{Re}(\alpha) - \frac{u\,\mathrm{Im}(\alpha)}{v} - \frac{\mathrm{Im}(\alpha)}{v}\left(\mathrm{Re}(\alpha) - \frac{u\,\mathrm{Im}(\alpha)}{v}\right)\\
&= \frac{\alpha-\bar\alpha}{2iv} - \frac{(\alpha+\bar\alpha)}{2} - \frac{u(\alpha-\bar\alpha)}{2iv} - \frac{\alpha-\bar\alpha}{2iv}\left(\frac{(\alpha+\bar\alpha)}{2} - \frac{u(\alpha-\bar\alpha)}{2iv}\right).
\end{aligned}$$

This will contribute $\exp(-\pi i A)$, where A is defined as the sum

$$\sum_{x\neq 0} c(-Q(x))\left[\frac{\alpha-\bar\alpha}{2iv} - \frac{\alpha+\bar\alpha}{2} - \frac{u(\alpha-\bar\alpha)}{2iv} - \frac{\alpha-\bar\alpha}{2iv}\left(\frac{(\alpha+\bar\alpha)}{2} - \frac{u(\alpha-\bar\alpha)}{2iv}\right)\right]$$

where $\alpha = \langle\beta,x\rangle$. Since x and $-x$ both occur in the sum, the linear terms vanish and

$$A = \sum_{x\neq 0} c(-Q(x))\left[-\frac{\alpha-\bar\alpha}{2iv}\left(\frac{(\alpha+\bar\alpha)}{2} - \frac{u(\alpha-\bar\alpha)}{2iv}\right)\right].$$

Using the hermitian version of Borcherds quadratic identity, as in the proof of Proposition 5.2.2, we obtain

$$A = \frac{uI}{2v^2}\cdot\langle\beta,\beta\rangle.$$

Thus we have

$$\begin{aligned} & P_1(\tau, w_0+\beta) \\ = \quad & P_1(\tau, w_0) \cdot \exp\left(\frac{\pi}{v}\langle w_0, \beta\rangle + \frac{\pi}{2v}\langle \beta, \beta\rangle\right)^I \cdot \exp\left(\frac{-2\pi i u\langle \beta, \beta\rangle}{4v^2}\right)^I. \end{aligned}$$

Finally, we recall the conjugate linear isomorphism $L_{-1} \cong \mathfrak{b}$ of (3.9.11) defined by $e_{-1} \mapsto \tau$ and $f_{-1} \mapsto 1$. As

$$\mathfrak{d}^{-1} L_{-1} = \mathbb{Z} e_{-1} + D^{-1}\mathbb{Z} f_{-1},$$

we have $-\delta^{-1}\tau = a\tau + D^{-1}b$ for some $a, b \in \mathbb{Z}$, and hence

$$\tau = -D^{-1}b(a+\delta^{-1})^{-1}.$$

This gives $u/v = a\,D^{\frac{1}{2}}$. Also, using

$$\delta e_{-1} = -Dae_{-1} - b\,f_{-1},$$

we have

$$\frac{1}{2}(1+\delta)\,e_{-1} = \frac{1}{2}(1-Da)\,e_{-1} - \frac{1}{2}b\,f_{-1} \in \mathbb{Z}e_{-1} + \mathbb{Z}f_{-1} = L_{-1}.$$

Thus a is odd and b is even. Recall that $\mathrm{N}(\mathfrak{b}) = 2v/\sqrt{D}$. Thus

$$\frac{u}{4v^2} = \frac{aD^{\frac{1}{2}}}{2\mathrm{N}(\mathfrak{b})D^{\frac{1}{2}}},$$

and, since $\langle \beta, \beta\rangle \in \mathrm{N}(\mathfrak{b})$, we have

$$\exp\Big(-\frac{2\pi i u\langle \beta, \beta\rangle}{4v^2}\Big) = \exp\Big(-\frac{\pi i\langle \beta, \beta\rangle}{\mathrm{N}(\mathfrak{b})}\Big) = \pm 1.$$

The transformation law is then

$$P_1(\tau, w_0+\beta) = \exp\left(\frac{\pi}{v}\langle w_0, \beta\rangle + \frac{\pi}{2v}\langle \beta, \beta\rangle - i\pi\,\frac{\langle \beta, \beta\rangle}{\mathrm{N}(\mathfrak{b})}\right)^I \cdot P_1(\tau, w_0),$$

as claimed in Proposition 6.3.3.

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