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$\mathcal{D}$-elliptic sheaves and the Langlands correspondence

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Introduction

In a series of papers [Dr 1, Dr 2], Drinfeld introduced analogues of Shimura varieties for GL_d over a function field F of characteristic $p > 0$. Decomposing their ℓ-adic cohomology under the action of the Hecke operators he constructed very interesting Galois representations of F. In fact, for $d = 2$ he showed that the correspondence which to an automorphic representation associates the Galois representation on its eigenspace is, up to a Tate twist, a Langlands correspondence (equality of L-functions, ε-factors etc.). This is completely analogous to the classical case of modular curves over $\mathbb{Q}$ (the Shimura variety associated to GL_2). The essential difficulty in extending this result to general d lies in the non-compactness of Drinfeld's varieties. Our purpose in the present paper is to construct compact versions of Drinfeld varieties for central division algebras over F, to study their ℓ-adic cohomology and give applications to the global and local Langlands correspondence.

Drinfeld constructed his varieties as moduli spaces for two equivalent but different moduli problems, *elliptic modules* and *elliptic sheaves*. The equivalence of these two concepts was proved by Drinfeld [Dr 3] and Mumford. The idea of formulating variants of these moduli problems for a division algebra was proposed several years ago by one of us (U.St.). Here we will concentrate on the generalization of elliptic sheaves, to be called *$\mathcal{D}$-elliptic sheaves*. The concept of *$\mathcal{D}$-elliptic module* is closely related to Anderson's t-motives [An]; in spite of their elementary nature they will not play a role in this paper (comp., however, Sect. 3) since $\mathcal{D}$-elliptic sheaves are easier to study.

In the case of Drinfeld's varieties for GL_2 the Galois modules obtained by decomposing the cohomology can be characterized as having a certain simple ramification behaviour at a distinguished place ∞ of F (special representation)[1].

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[1] We do not discuss here Drinfeld's moduli variety of Shtuka's of rank 2 which, as Drinfeld has shown, yields all Galois modules without restriction

This distinguished place is part of the data needed to define Drinfeld's varieties (and our variants as well) and plays the role of an archimedian place. For general d this characterization conjecturally should continue to hold. This can indeed be proved [Lau 2] by a method due to Flicker and Kazhdan [Fl-Ka], by making use of certain unproved conjectures (Arthur's non-invariant trace formula for function fields, Deligne's conjecture on the Lefschetz fixed point formula). By contrast, the Galois modules obtained here from a division algebra have the ramification behaviour mentioned above at the distinguished place ∞, but are in addition ramified at the places where the division algebra ramifies. (We impose that the distinguished place ∞ is not a ramification place of the division algebra.) It may therefore be said that our variant of Drinfeld's construction yields fewer Galois modules but that due to the compactness of our varieties it can be pushed through for arbitrary d.

We now proceed to explain our main results. We fix a central simple algebra D of dimension d^2 over F and fix a place ∞ outside the ramification locus, Bad, of D. Let $\mathscr{D}$ be a maximal order of D. For an F-scheme S we introduce the concept of a $\mathscr{D}$-elliptic sheaf over S: this is essentially a vector bundle of rank d^2 on $X \times S$ equipped with an action of $\mathscr{D}$ and with a meromorphic $\mathscr{D}$-linear Frobenius, with pole at ∞ and satisfying some periodicity condition. (Here X denotes the smooth projective curve with function field F.) For a non-empty finite closed subscheme $I \subset X \setminus \{\infty\}$ we introduce the concept of a level-I-structure on a $\mathscr{D}$-elliptic sheaf over S. We also define an action of $\mathbb{Z}$ on the set $\mathscr{E}\ell\ell_{X,\mathscr{D},I}(S)$ of isomorphism classes of $\mathscr{D}$-elliptic sheaves with level-I-structure over S. We denote by $\mathscr{M}_I$ the set-valued functor on (Sch/F) which to an F-scheme S associates the factor set $\mathscr{M}_I = \mathscr{E}\ell\ell_{X,\mathscr{D},I}(S)/\mathbb{Z}$. Our *first main result* (4.1, 5.1, 6.2) is that *$\mathscr{M}_I$ is representable by a smooth quasi-projective algebraic variety of dimension $d-1$ over F and with good reduction at every place $o \notin \{\infty\} \cup$ Bad $\cup$ I. Furthermore, if D is a division algebra, then $\mathscr{M}_I$ is a projective variety*. The key tool in our proof of the above result is the canonical filtration of Harder, Narasimhan, Quillen and Tjurin which controls the instability of a vector bundle on an algebraic curve. For the compactness assertion we check the valuative criterion through a method used by Drinfeld in his analysis [Dr 8] of the degeneration behaviour of Shtuka's of rank 2. We note that if $(D, \mathscr{D}) = (\mathbb{M}_d(F), \mathbb{M}_d(\mathcal{O}_X))$ the concept of a $\mathscr{D}$-elliptic sheaf is essentially equivalent to Drinfeld's concept of an elliptic sheaf of rank d (Morita equivalence).

From now on we assume that D is a division algebra. The projective schemes $\mathscr{M}_I$ form for varying I a projective system on which the group $(D^\infty)^\times$ acts. (Here D^∞ denotes the adele ring of D outside ∞.) This allows us to define an action of $\mathrm{Gal}(\bar{F}/F) \times (D^\infty)^\times$ on the ℓ-adic cohomology groups

$$H^n = \varinjlim_{I} H^n(\mathscr{M}_I \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell) .$$

Denote by $(H^n)^{ss}$ the associated semi-simplification. For an infinite-dimensional irreducible admissible representation π^∞ of $(D^\infty)^\times$ denote by $V^n_{\pi^\infty}$ the π^∞-isotypical component of $(H^n)^{ss}$. Our *second main result* (14.9, 14.12, 16.5) states that *if there exists n with $V^n_{\pi^\infty} \neq (0)$ then the representation $\Pi = \mathrm{St}_\infty \otimes \pi^\infty$ of $D^\times_{\mathbb{A}}$ is automorphic*. (Here St_∞ denotes the Steinberg representation of $D^\times_\infty \simeq \mathrm{GL}_d(F_\infty)$.) *Conversely, for an automorphic representation Π of the form $\Pi = \mathrm{St}_\infty \otimes \Pi^\infty$ we have* $\dim(V^{d-1}_{\Pi^\infty}) = m(\Pi) \cdot d$ *and $V^n_{\Pi^\infty} = (0)$ for $n \neq d-1$*. (Here $m(\Pi)$ is the multiplicity of Π in the space of automorphic forms.) *Furthermore, $V^{d-1}_{\Pi^\infty}$ is the $m(\Pi)$-th power of an* *irreducible ℓ-adic representation W_{Π^∞} such that for any place $o \neq \infty$, $o \notin$ Bad for which Π_o is unramified, the Galois module W_{Π^∞} is unramified at o and we have for the trace of a power of the Frobenius*

$$\operatorname{tr}(\mathrm{Frob}_o^r; W_{\Pi^\infty}) = q_o^{r(d-1)/2}(z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r),$$

with $|z_i(\Pi_o)| = 1, i = 1, \ldots, d$ (Petersson conjecture). Here q_o denotes the number of elements in the residue field of o and $z_1(\Pi_o), \ldots, z_d(\Pi_o)$ are the Satake parameters of the unramified representation Π_o of $D_o^\times = \mathrm{GL}_d(F_o)$. The proof of this result is based on a description of the set of $\mathscr{D}$-elliptic sheaves in finite characteristic similar to the one by Honda-Tate for abelian varieties in finite characteristic. We follow closely Drinfeld [Dr 7] who obtained similar results in the case of elliptic sheaves and Shtuka's. We note that, due to the fact that GL_d has no L-indistinguishability, this is much simpler than the corresponding problem for general Shimura varieties. From this description we obtain an expression for the trace of the action of a Hecke operator times a power of the Frobenius on cohomology in terms of orbital integrals. This method, inaugurated by Ihara and greatly developed by Langlands and Kottwitz, is now completely standard. We use the Grothendieck-Lefschetz fixed point formula (which is simple since our varieties are compact), but the "fundamental lemma" which allows one to convert a twisted orbital integral into an orbital integral and is customarily invoked for Shimura varieties at this point is used only implicitly. Instead we follow Drinfeld using his classification of effective Dieudonné modules of height 1 and his calculation of the orbital integrals of the Hecke function corresponding to a power of the Frobenius. Another ingredient of our proof is the use of Kottwitz' Euler-Poincaré functions [Kot 2]. We then apply the Selberg trace formula (which is simple for a division algebra). At this point we obtain the assertions above as they pertain to the virtual Galois module $\sum(-1)^n V_{\Pi^\infty}^n$. However, to deduce the full result stated above we have to invoke an additional argument on "weights" which is based on Grothendieck's functional equation for L-functions and Deligne's purity theorems. In the original version of this paper this argument was also based on the strong Lefschetz theorem and used an ample invariant class. Since, as was pointed out to us by one of the referees, the existence of such a class is not obvious we indicate briefly how it follows from unpublished results of Drinfeld. Moreover, we rearranged our original proof so that we can base the argument alternatively on the strong Lefschetz theorem or on the classification of unitary representations of $\mathrm{GL}_d(F_o)$ due to Tadič. We note that for Π in the image of the global Jacquet-Langlands correspondence between automorphic representations of $D_{\mathbb{A}}^\times$ and $\mathrm{GL}_d(\mathbb{A})$ (i.e., for all Π if $d = 2$ or $d = 3$ [Ja-Pi-Sh 1] and conjecturally all Π for arbitrary d) these complicated arguments are not needed. Moreover, then $m(\Pi) = 1$ and Π and $V_{\Pi^\infty}^{d-1}$ are up to a Tate twist $(d-1)/2$ in global Langlands correspondence. This in particular applies to those Π used in our proof of the local Langlands correspondence. In our statement of our second main result we have restricted ourselves to the cohomology of the constant sheaf $\bar{\mathbb{Q}}_\ell$; we refer the reader to the main body of the text (13.8, 14.20) for a review of the problems posed by other local systems on our varieties.

Our *third main result* (15.7) is the local Langlands correspondence which establishes a *bijection between the set of isomorphism classes of irreducible ℓ-adic Galois representations of dimension d of a local field F of characteristic p* (with finite determinant character) *and the set of isomorphism classes of irreducible supercuspidal representations of* $\mathrm{GL}_d(F)$ (with finite central character). *This correspondence* *has all desired properties: preservation of L-functions, ε-factors (even of pairs)*, etc. It is in fact characterized by its properties [He 5]. To construct this correspondence we embed the local situation in a global one defined by a division algebra D and apply then our second main result. This method of obtaining a local correspondence from a global one has deep historical roots: it was in fact the method of the first proof of local class field theory, and has also been used by Deligne [De-Hu]. See also [Fl-Ka]. For the surjectivity of the correspondence we use Henniart's solution of the numerical Langlands conjecture.

We now describe briefly the contents of the various sections of this paper. After reviewing some well-known facts on central division algebras and their orders in Sect. 1 we introduce in Sect. 2 the concept of $\mathcal{D}$-elliptic sheaves and their level structures. Section 3 is not needed in the sequel. It contains Drinfeld's description of elliptic sheaves as vector bundles on the non-commutative projective line (or rather our variant for $\mathcal{D}$-elliptic sheaves). In Sects. 4, 5, 6 we construct the moduli space of $\mathcal{D}$-elliptic sheaves with level-I-structure and establish its geometric properties. Section 7 is devoted to the definition of Hecke correspondences. As in the Drinfeld case [Dr 2], the moduli variety of $\mathcal{D}$-elliptic sheaves admits coverings which replace the local systems on Shimura varieties given by a rational representation of the corresponding group. They are defined in Sect. 8. Sections 9 and 10 (together with Appendices A and B) give the description of the set of $\mathcal{D}$-elliptic sheaves in finite characteristic. In Sect. 11 the number of fixed points is given as an expression involving orbital integrals and in Sect. 12 the Lefschetz fixed point formula is invoked and the result of Sect. 11 is generalized from the constant local system to general local systems. In Sect. 13 this result is rewritten using the Selberg trace formula and the local zeta function of $\mathcal{M}_I$ is determined at a place of good reduction. Section 14 contains the proof of our global results and Sect. 15 their application to the local Langlands correspondence. Finally, in Sect. 16 we give some applications of our global results to the Tate conjectures for our varieties.

In the present paper we have left aside all questions related to places of bad reduction; in particular no mention is made of non-archimedian uniformization at the place ∞ (cf. however, (14.19)), or at a place where the Hasse invariant of D is equal to $1/d$. Also, we have restricted ourselves to the case of a maximal order although many of our results have analogues for general hereditary orders [(Cu-Re, (26.12)]. We hope to take up these questions in a sequel to this paper. We also hope to return to the subject by developing a theory of $\mathcal{D}$-Shtuka's (which bear the same relationship to Drinfeld's Shtuka's of rank d as $\mathcal{D}$-elliptic sheaves to his elliptic sheaves of rank d). We also do not touch here at all on rationality questions (of eigenvalues of Hecke operators etc.).

List of notations

F	global field of characteristic $p > 0$, with field of constants $\mathbb{F}_q$.
X	smooth projective irreducible curve over $\mathbb{F}_q$ corresponding to F.
$\lvert X\rvert$	set of closed points of X, identified with the set of places of F.
$\mathcal{O}_x, F_x$	completions of $\mathcal{O}_{X,x}$ resp. of F $(x\in\lvert X\rvert)$
$\kappa(x)$	residue field of $x\in\lvert X\rvert$
q_x	cardinality of $\kappa(x)$
ϖ_x	a uniformizer at $x\in\lvert X\rvert$
$x(t)$	valuation of $t\in F_x$ in $x\in\lvert X\rvert$

$\deg(x) \quad = \dim_{\mathbb{F}_q} \kappa(x)$

$\mathbb{A} \quad = \prod'_{x \in |X|} (F_x, \mathcal{O}_x)$ the adele ring of F

$\mathbb{A}^T \quad = \prod'_{x \notin T} (F_x, \mathcal{O}_x)$ the adele ring *outside* a set of places $T \subset |X|$

$\mathbb{A}_T \quad = \prod'_{x \in T} (F_x, \mathcal{O}'_x)$ the adele ring *inside* a set of places $T \subset |X|$

$F_T \quad = \mathbb{A}_T$ if T is finite.

For a finite closed subscheme $I \subset X$ we use

$\deg(I) \quad = \dim_{\mathbb{F}_q}(\mathcal{O}_I)$

$\mathcal{M}_I \quad = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_I$ for an $\mathcal{O}_X$-module $\mathcal{M}$.

The symbol D will stand for a central simple F-algebra of dimension d^2 over F. We use

$D_{\mathbb{A}} \quad = D \otimes_F \mathbb{A}$

$D^T \quad = D \otimes_F \mathbb{A}^T$

$D_T \quad = D \otimes_F \mathbb{A}_T$

$\mathrm{inv}_x(D)$ the Hasse invariant at $x \in |X|$

rn the reduced norm homomorphism rn: $D^\times \to F^\times$

$SL_1(D) \quad = \mathrm{Ker(rn)}$.

A similar notation will be used for simple central algebras over other local or global fields. For a ring R we denote by R^{op} the opposite ring and by $\mathbb{M}_d(R)$ the matrix ring of size d with entries in R.

All schemes, as well as their products and morphisms between them, are supposed to be over $\mathbb{F}_q$. If X and Y are schemes we write $X \times Y$ for their product over $\mathbb{F}_q$. A similar notation is employed for the tensor product over $\mathbb{F}_q$. For a scheme S, we denote by Frob_S its Frobenius endomorphism (over $\mathbb{F}_q$), which is the identity on the points and the q-th power map on functions. We also use

$\mathrm{frob}_q \quad \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ the arithmetic Frobenius.

$\mathrm{Frob}_x \quad = \mathrm{frob}_q^{-\deg(x)} \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\kappa(x))$ the geometric Frobenius ($x \in |X|$).

In the later sections we fix an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$, where $\bar{\mathbb{Q}}_\ell$ denotes an algebraic closure of $\mathbb{Q}_\ell$, $\ell \neq p$ (this is harmless for the purposes of this paper since we do not treat rationality questions; this apparent use of the axiom of choice can in fact be avoided, cf. [De 2, 1.2.11]).

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1 Central simple algebras over a function field

In this section we collect some well-known facts on orders in a finite-dimensional simple algebra D over the function field F with center equal to F. Let

$$\dim_F D = d^2 \ .$$

(1.1) Let $M \subset D$ be a finite subset containing an F-basis of D. For every $x \in |X|$ let

$$\mathcal{M}_x \subset D \otimes F_x$$

be the $\mathcal{O}_x$-submodule generated by M. Then for almost all x the algebra $D \otimes F_x$ is isomorphic to the matrix algebra $\mathbb{M}_d(F_x)$ and $\mathcal{M}_x$ is a maximal compact subring. (cf. [We, XI-1, Theorem 1]).

(1.2) Let $\infty \in X$ be a fixed place and let $A = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X)$ be the corresponding ring. An *order in D with respect to* ∞ is a finitely generated A-algebra $\mathcal{M} \subset D$ containing an F-basis of D. For $x \in |X|$, an *order in* $D \otimes F_x$ is a finitely generated $\mathcal{O}_x$-algebra $\mathcal{M}_x \subset D \otimes F_x$ containing an F_x-basis of $D \otimes F_x$. There is a one-to-one correspondence between the orders $\mathcal{M}$ in D with respect to ∞ and the set of orders for all $x \neq \infty$

$$\mathcal{M}_x \subset D \otimes F_x$$

such that there exists an F-basis M of D with

$$\mathcal{M}_x = \mathcal{O}_x \cdot M \quad \text{for almost all } x \neq \infty \ .$$

This correspondence associates to $\mathcal{M}$ the orders $\mathcal{M}_x = \mathcal{M} \otimes_A \mathcal{O}_x$ and conversely to $(\mathcal{M}_x)_{x \in |X| \setminus \{\infty\}}$ the order

$$\mathcal{M} = \bigcap_x (\mathcal{M}_x \cap D) \ .$$

(1.3) The set of orders $\mathcal{M}_x \subset D \otimes F_x$ for *all* $x \in |X|$ such that there exists an F-basis M of D with $\mathcal{M}_x = \mathcal{O}_x \cdot M$ for almost all x is in one-to-one correspondence with the set of locally free coherent $\mathcal{O}_X$-algebra sheaves $\mathcal{D}$ with stalk at the generic point equal to D. This correspondence associates to $\mathcal{D}$ the orders $\mathcal{D} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_x$ for all closed points x of X. Conversely, the set of orders $\mathcal{M}_x$, $x \in X$, defines the $\mathcal{O}_X$-algebra $\mathcal{D}$ with value on the open set U

$$\mathcal{D}(U) = \bigcap_{x \in U} (\mathcal{M}_x \cap D) \ .$$

(1.4) We now fix once and for all, a sheaf of algebras $\mathcal{D}$ as in (1.3). It follows from (1.3), (1.2) and (1.1) that there is a finite set of places

$$\mathrm{Bad} \subset |X|$$

such that for all $x \notin \mathrm{Bad}$, $D \otimes F_x$ is the matrix algebra $\mathbb{M}_d(F_x)$ and $\mathscr{D}_x$ a maximal compact subring isomorphic to $\mathbb{M}_d(\mathcal{O}_x)$. It then follows that locally around every $x \in X \setminus \mathrm{Bad}$ in the étale topology, the $\mathcal{O}_X$-algebra sheaf $\mathscr{D}$ is isomorphic to $\mathbb{M}_d(\mathcal{O}_X)$. In other words the points in $X \setminus \mathrm{Bad}$ are the unramified points for the pair $(D, \mathscr{D})$. A point x can be ramified if $D \otimes F_x$ is not the matrix algebra, or if $\mathscr{D}_x$ is not a maximal order.

(1.5) In the sequel we shall always assume that the distinguished place ∞ is unramified for $(D, \mathscr{D})$, i.e. $\infty \in X \setminus \mathrm{Bad}$. Furthermore we shall make the blanket assumption that $\mathscr{D}_x$ is a maximal order for all x. Much of what follows can be done under the assumption that $\mathscr{D}_x$ is a hereditary order for all x, and we shall point this out at the appropriate places.

2 The concept of a $\mathscr{D}$-elliptic sheaf. Level structures

(2.1) We fix a sheaf of $\mathcal{O}_X$-algebras $\mathscr{D}$ with generic fibre D as in the previous section and such that the distinguished place ∞ is unramified with respect to $(\mathscr{D}, D)$ (cf. (1.4)). Let S be a scheme ($= \mathbb{F}_q$-scheme).

(2.2) Definition. A *$\mathscr{D}$-elliptic sheaf over S is a sequence $(\mathscr{E}_i, j_i, t_i)$, $i \in \mathbb{Z}$, where* $\mathscr{E}_i$ are locally free $\mathcal{O}_{X \times S}$-modules of rank d^2 equipped with a right action of $\mathscr{D}$ compatible with the $\mathcal{O}_X$-action and where

$$j_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$$
$$t_i : {}^{\tau}\mathscr{E}_i \to \mathscr{E}_{i+1}$$

are injective $\mathcal{O}_{X \times S}$-linear homomorphisms compatible with $\mathscr{D}$-actions. Here ${}^{\tau}\mathscr{E}_i := (\mathrm{id}_X \times \mathrm{Frob}_S)^* \mathscr{E}_i$. The following conditions should hold:

(i) The diagrams

$$\begin{tikzcd}
\mathscr{E}_i \arrow[r, "j_i"] & \mathscr{E}_{i+1} \\
{}^{\tau}\mathscr{E}_{i-1} \arrow[u, "t_{i-1}"] \arrow[r, "{}^{\tau}j_{i-1}"] & {}^{\tau}\mathscr{E}_i \arrow[u, "t_i"']
\end{tikzcd}$$

are commutative.

(ii) $\mathscr{E}_{i+d \cdot \deg(\infty)} = \mathscr{E}_i(\infty) := \mathscr{E}_i \otimes_{\mathcal{O}_{X \times S}} (\mathcal{O}_X(\infty) \boxtimes \mathcal{O}_S)$ and the composite $\mathscr{E}_i \hookrightarrow \ldots \hookrightarrow \mathscr{E}_{i+d \cdot \deg(\infty)}$ is induced by the canonical injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\infty)$.

(iii) The direct image of $\mathscr{E}_i / \mathscr{E}_{i-1}$ by $\mathrm{pr}_S : X \times S \to S$ is a locally free $\mathcal{O}_S$-module of rank d.

(iv) Coker t_i is supported by the graph of a morphism $i_0 : S \to X$ and is the direct image of a locally free module on S by its graph $S \xrightarrow{(i_0, \mathrm{id}_S)} X \times S$; moreover i_0 satisfies $i_0(S) \subset X' := X \setminus \{\infty\} \setminus \mathrm{Bad}$. The morphism i_0 is called the *zero of* the *$\mathscr{D}$-elliptic sheaf*.

(2.3) Remarks. (a) We often write a $\mathscr{D}$-elliptic sheaf in the form

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathscr{E}_i \arrow[r, hook, "j_i"] & \mathscr{E}_{i+1} \arrow[r, hook] & \cdots \\
\cdots \arrow[r, hook] & {}^{\tau}\mathscr{E}_i \arrow[r, hook, "{}^{\tau}j_i"] \arrow[ur, "t_i"] & {}^{\tau}\mathscr{E}_{i+1} \arrow[r, hook] \arrow[ur, "t_{i+1}"] & \cdots
\end{tikzcd}$$

and refer to the first and second row of it.

(b) Since ∞ is an unramified place, $\mathscr{D} \otimes_{\mathcal{O}_X} \kappa(\infty) \cong \mathbb{M}_d(\kappa(\infty))$. If $\bar{s} \to S$ is a geometric point, $\mathscr{E}_i/\mathscr{E}_{i-1}$ is a module over

$$\mathbb{M}_d(\kappa(\infty) \otimes_{\mathbb{F}_q} \kappa(\bar{s}) \cong \bigoplus_{\mathrm{Hom}(\kappa(\infty),\kappa(\bar{s}))} \mathbb{M}_d(\kappa(\bar{s})) .$$

Therefore condition (iii) implies that the action factors through one factor and $\mathscr{E}_i/\mathscr{E}_{i-1}$ is the unique simple module of the corresponding matrix ring. Consider the support of $\mathscr{E}_i/\mathscr{E}_{i-1}$ (the closed subscheme of $X \times S$ defined by the annihilator ideal) and its pullback over $\bar{s}$. It is contained in $\infty \times \bar{s} = \mathrm{Spec}\,(\bigoplus_{\mathrm{Hom}(\kappa(\infty),\kappa(\bar{s}))} \kappa(\bar{s}))$ and corresponds actually to the factor singled out before. Therefore we see that supp $(\mathscr{E}_i/\mathscr{E}_{i-1})$ maps isomorphically to S and we may reformulate condition (iii) as follows:

(iii′) $\mathscr{E}_i/\mathscr{E}_{i-1}$ is a locally free sheaf of rank d on the graph of a morphism

$$i_{\infty,i}: S \to X$$

which factors through $\mathrm{Spec}(\kappa(\infty)) \hookrightarrow X$.

Note that the condition $i_{\infty,i}(S) = \{\infty\}$ is automatic since

$$\mathrm{supp}(\mathscr{E}_{i+1}/\mathscr{E}_i) \subset \mathrm{supp}(\mathscr{E}_i(\infty)/\mathscr{E}_i) = \infty \times S .$$

Also the condition on the rank can be weakened; it suffices to require that $(\mathscr{E}_i)_s \neq (\mathscr{E}_{i-1})_s$ for all $s \in S$, because

$$\sum_{i=1}^{d \cdot \deg(\infty)} \dim(\mathscr{E}_i/\mathscr{E}_{i-1})_s = d^2 \cdot \deg(\infty)$$

and because $\mathscr{E}_i/\mathscr{E}_{i-1}$, being a representation of the matrix ring, has dimension divisible by d.

(c) Similar remarks apply to condition (iv). Indeed, if $s \in S$ then $i_0(s) \in X'$ and hence $i_0^*(\mathscr{D})_s \simeq \mathbb{M}_d(\kappa(s))$. Therefore condition (iv) is equivalent to the following condition

(iv′) The direct image of Coker t_i is a locally free $\mathcal{O}_S$-module of rank d. The support of Coker t_i is disjoint from $(\{\infty\} \cup \mathrm{Bad}) \times S$.

(d) Since the inclusion $\mathscr{E}_{i-1} \hookrightarrow \mathscr{E}_i$ is an isomorphism over $(X \setminus \{\infty\}) \times S$ the data of all homomorphisms t_i are equivalent to giving a single one of them. This argument also shows that the morphism i_0 in condition (iv) is independent of the index i.

(e) Since the support of Coker t_i is disjoint from $\infty \times S$, we have isomorphisms

$${}^{\tau}(\mathscr{E}_i/\mathscr{E}_{i-1}) \to \mathscr{E}_{i+1}/\mathscr{E}_i .$$

In particular for the morphisms $i_{\infty,i}: S \to X$ we obtain

$$i_{\infty,i} \circ \mathrm{Frob}_S = i_{\infty,i+1} .$$

Note that, since $i_{\infty,i}(S) \subset \{\infty\}$, we have

$$i_{\infty,i} = i_{\infty,i+\deg(\infty)} .$$

(f) Suppose that $(D, \mathscr{D})$ is of the form $(\mathbb{M}_e(\tilde{D}), \mathbb{M}_e(\tilde{\mathscr{D}}))$ where $\dim_F(\tilde{D}) = (d/e)^2$. Then the concept of $\mathscr{D}$-elliptic sheaf can be interpreted, via Morita equivalence (cf. (9.5)) as "$\tilde{\mathscr{D}}$-elliptic sheaf of rank e (over $\tilde{\mathscr{D}}$)", the definition of which is left to the reader. When $\tilde{\mathscr{D}} = \mathcal{O}_X$ $(e = d)$, these objects were introduced by Drinfeld.

(2.4) Denote by $\mathcal{E}\ell\ell_{X,\mathscr{D}}(S) =: \mathcal{E}\ell\ell_{\mathscr{D}}(S)$ the category whose objects are the $\mathscr{D}$-elliptic sheaves over S and whose morphisms are the isomorphisms between $\mathscr{D}$-elliptic sheaves. If $S' \to S$ is an S-scheme, then a $\mathscr{D}$-elliptic sheaf $(\mathscr{E}_i, j_i, t_i)$ defines by pull-back a $\mathscr{D}$-elliptic sheaf over S'. This defines a fibered category

$$S \mapsto \mathcal{E}\ell\ell_{X,\mathscr{D}}(S)$$

over the category of $\mathbb{F}_q$-schemes, which obviously is a stack for the *fppf*-topology. The functor which associates to $(\mathscr{E}_i, j_i, t_i) \in \mathrm{ob}\,\mathcal{E}\ell\ell_{\mathscr{D}}(S)$ the morphism $i_0 : S \to X$ defines a morphism of stacks

$$\mathrm{zero} : \mathcal{E}\ell\ell_{X,\mathscr{D}} \to X$$

which factors through $X' \subset X$.

Remark. By imposing the condition that *zero* factors through X' we avoid problems connected with bad reduction; including the place ∞ would have meant dealing with "uniformisation à la Drinfeld"; including the ramified primes would have meant dealing with situations analogous to bad reduction in the number field case [Ra]. We hope to return to these questions in future work.

Similarly to i_0, the morphism $i_{\infty,0}$ above defines a morphism $\mathcal{E}\ell\ell_{\mathscr{D}} \to X$ which factors through $\mathrm{Spec}(\kappa(\infty))$. This is called the pole morphism,

$$\mathrm{pole}\colon \mathcal{E}\ell\ell_{X,\mathscr{D}} \to \mathrm{Spec}(\kappa(\infty)) .$$

The group $\mathbb{Z}$ acts on the stack $\mathcal{E}\ell\ell_{X,\mathscr{D}}$ by $[n](\mathscr{E}_i, j_i, t_i) = (\mathscr{E}'_i, j'_i, t'_i)$ with $\mathscr{E}'_i = \mathscr{E}_{i+n}$, $j'_i = j_{i+n}$, $t'_i = t_{i+n}$. On $\mathrm{Spec}(\kappa(\infty))$ the group $(\mathbb{Z}/\deg(\infty)\mathbb{Z})$ acts, 1 acts by Frob_∞ and we have the canonical group homomorphism $\mathbb{Z} \xrightarrow{c} \mathbb{Z}/\deg(\infty)\mathbb{Z}$.

Then the diagram

$$\begin{tikzcd}
\mathbb{Z} \times \mathcal{E}\ell\ell_{X,\mathscr{D}} \arrow[r] \arrow[d, "c \times \mathrm{pole}"] & \mathcal{E}\ell\ell_{X,\mathscr{D}} \arrow[d, "\mathrm{pole}"] \\
\mathbb{Z}/\deg(\infty)\mathbb{Z} \times \mathrm{Spec}\,\kappa(\infty) \arrow[r] & \mathrm{Spec}\,\kappa(\infty)
\end{tikzcd}$$

where the rows are given by the group actions is commutative.

(2.5) Let $I \subset X$ be a finite closed subscheme with $\infty \notin I$ and let $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathscr{D}$-elliptic sheaf with zero in $i_0(S)$ disjoint from I. Then $\mathscr{E}_{i|I\times S}$ and $t_{i|I\times S}$ are independent of $i \in \mathbb{Z}$. Let us denote them by $\mathscr{E}_{|I\times S}$ and $t_{|I\times S}\colon {}^{\tau}\mathscr{E}_{|I\times S} \to \mathscr{E}_{|I\times S}$. Moreover, $t_{|I\times S}$ is an isomorphism. We consider the functor

$$E_I : Sch/S \to H^0(I, \mathscr{D})\text{-right-modules}$$
$$T/S \mapsto \mathrm{Ker}(H^0(I \times T, t_{|I\times S} - \mathrm{id}_{\mathscr{E}_{|I\times S}}))$$

(where $t_{|I\times S}$ is considered as a q-linear map from $\mathscr{E}_{|I\times S}$ into itself).

(2.6) Lemma. *E_I is representable by a finite étale scheme over S in free $H^0(I, \mathscr{D})$-modules of rank* 1.

Proof. To any locally free sheaf of $\mathcal{O}_S$-modules $\mathscr{F}$ of constant finite rank n equipped with an isomorphism $\varphi : (\mathrm{Frob}_S)^* \mathscr{F} \xrightarrow{\sim} \mathscr{F}$ there is associated a functor $K(\mathscr{F}, \varphi)$ to the category of abelian groups

$$T/S \mapsto \mathrm{Ker}(H^0(I \times T, \varphi - \mathrm{id}_{\mathscr{F}})$$

(where we have again considered φ as a q-linear map from $\mathcal{F}$ into itself), comp. [Dr 6]. This functor is representable by a finite étale commutative group scheme over S of order q^n. Indeed, locally on S for the Zariski topology, $\mathcal{F} = \mathcal{O}_S^n$, φ is given by a matrix $\Phi \in \mathrm{GL}_n(H^0(S, \mathcal{O}_S))$ and $K(\mathcal{F}, \varphi)$ is representable by the group scheme

$$\left\{ X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{G}_{a,S}^n \,\middle|\, \Phi \cdot {}^{\tau}X = X \right\}$$

where

$${}^{\tau}X = \begin{pmatrix} x_1^q \\ \vdots \\ x_n^q \end{pmatrix}.$$

Then it follows that our E_I is representable by a finite étale scheme in $H^0(I, \mathcal{D})$-modules of order $|H^0(I, \mathcal{D})|$. It remains to show that this module is locally free of rank 1 over $H^0(I, \mathcal{D})$ and for this we can assume that the support of I is a single point x. It is easy to see that

$$E_x := \varprojlim_{I} E_I(\overline{\kappa(x)})$$

where I runs through all finite closed subschemes of X with support x, is a $\mathcal{D}_x$-module such that

$$E_I(\overline{\kappa(x)}) = E_x \otimes_{\mathcal{O}_x} \mathcal{O}_I \, .$$

($\mathcal{O}_I$ is a quotient of $\mathcal{O}_x$ as the support of I is x.) A standard argument [Dr 1, 2.2] shows therefore that the fact on the order of E_I already established implies that E_x is a free $\mathcal{O}_x$-module of rank d^2. Since $\mathcal{D}_x$ is a maximal order it follows that E_x is a free $\mathcal{D}_x$-module of rank one (comp. [Cu-Re, 26.24(iii)]). ☐

(2.7) Definition. Let $I \subset X$ be a finite closed subscheme. A *level-I structure on a $\mathcal{D}$-elliptic module $(\mathcal{E}_i, t_i, j_i)$ over S* is an isomorphism of $\mathcal{O}_{I\times S}$-modules

$$\iota : \mathcal{D}_I \boxtimes \mathcal{O}_S \xrightarrow{\sim} \mathcal{E}_{|I\times S}$$

compatible with the actions of $\mathcal{D}_I$ (by right translations on $\mathcal{D}_I \boxtimes \mathcal{O}_S$) and such that the following diagram is commutative:

$$\begin{tikzcd}
{}^{\tau}\mathcal{E}_{|I\times S} \arrow[rr, "t_{|I\times S}"] & & \mathcal{E}_{|I\times S} \\
& \mathcal{D}_I \boxtimes \mathcal{O}_S . \arrow[ul, "\tau_1"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

By definition, if the zero $i_0(S)$ of a $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_i, j_i, t_i)$ meets the support of I, then $(\mathcal{E}_i)$ does not possess any level-I-structure in our sense (but see [Dr 1]).

Denoting by $\mathcal{E}\ell\ell_{X,\mathcal{D},I}$ the corresponding stack of $\mathcal{D}$-elliptic sheaves with level-I-structure we obtain a commutative diagram of morphisms of stacks

$$\begin{tikzcd}
\mathcal{E}\ell\ell_{X,\mathcal{D},I} \arrow[r] \arrow[d, "r_I"] & X' \setminus I \arrow[d, hook] \\
\mathcal{E}\ell\ell_{X,\mathcal{D}} \arrow[r, "\text{zero}"] & X \setminus \{\infty, \mathbf{Bad}\} = X'
\end{tikzcd}$$

(where r_I is the omission of the level-I-structure.)

For any pairs $I \subset J \subset X \setminus \{\infty\}$ of finite closed subschemes, we have a morphism of stacks

$$r_{J,I} : \mathscr{E}\ell\ell_{X,\mathscr{D},J} \to \mathscr{E}\ell\ell_{X,\mathscr{D},I}$$

by restricting to I the level-J-structure and the obvious diagram commutes.

(2.8) By definition $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ enters in a 2-cartesian diagram of stacks

$$\begin{tikzcd}
\mathscr{E}\ell\ell_{X,\mathscr{D},I} \arrow[r] \arrow[d] & \mathscr{V}ec^{\bullet}_{X,\mathscr{D},I} \arrow[d, "{(\mathrm{id}, \mathrm{Frob})}"] \\
\mathrm{Hecke}_{X,\mathscr{D},I} \arrow[r] \arrow[d] & \mathscr{V}ec^{\bullet}_{X,\mathscr{D},I} \times \mathscr{V}ec^{\bullet}_{X,\mathscr{D},I} \\
X' \setminus I. &
\end{tikzcd}$$

Here $\mathscr{V}ec^{\bullet}_{X,\mathscr{D},I} = \mathscr{V}ec^{\bullet}_{\mathscr{D},I}$ is the stack classifying the sequences $(\mathscr{E}_i, j_i)$ as in the first row of a $\mathscr{D}$-elliptic sheaf satisfying the conditions (2.2)(ii) and (iii) together with a level-I-structure (definition obvious) and $\mathrm{Hecke}_{\mathscr{D},I}$ classifies the commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathscr{E}_i \arrow[r, hook, "j_i"] & \mathscr{E}_{i+1} \arrow[r, hook] & \cdots \\
\cdots \arrow[r, hook] & \mathscr{E}'_i \arrow[r, hook, "j'_i"] \arrow[ur, "t_i"] & \mathscr{E}'_{i+1} \arrow[r, hook] \arrow[ur, "t_{i+1}"] & \cdots
\end{tikzcd}$$

such that $(\mathscr{E}_i, j_i)$ satisfies the conditions (2.2)(ii) and (iii) and such that the t_i's satisfy the condition (2.2)(iv), together with a level-I-structure. Then it is clear that the sequence $(\mathscr{E}'_i, j'_i)$ together with the level-I-structure belongs to $\mathscr{V}ec^{\bullet}_{\mathscr{D},I}$.

(2.9) Let $(\mathscr{E}_i, j_i, t_i)$ be a $\mathscr{D}$-elliptic sheaf over S and let $\mathscr{L}$ be an invertible sheaf on X. Then it is clear that

$$(\mathscr{E}'_i, j'_i, t'_i) = (\mathscr{E}_i \otimes_{\mathcal{O}_S} \mathscr{L}, j_i \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathscr{L}}, t_i \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathscr{L}})$$

is again a $\mathscr{D}$-elliptic sheaf. So we have an action of the group $\mathrm{Pic}(X)$ on the stack $\mathscr{E}\ell\ell_{X,\mathscr{D}}$. Similarly, if $(\mathscr{E}_i, t_i)$ is equipped with a level-I-structure and if $\mathscr{L}$ is equipped with a level-I-structure (i.e. an isomorphism of $\mathcal{O}_X$-modules $\mathscr{L}_I \xrightarrow{\sim} \mathcal{O}_I$), then we obtain a level-I-structure on the new $\mathscr{D}$-elliptic sheaf $(\mathscr{E}'_i, j'_i, t'_i)$. So we have an action of the group $\mathrm{Pic}_I(X)$ on the stack $\mathscr{E}\ell\ell_{\mathscr{D},I}$.

Note that for all $s \in S$

$$\deg(\mathscr{E}'_i)_s = \deg(\mathscr{E}_i)_s + d^2 \cdot \deg(\mathscr{L})$$

and similarly for the Euler-Poincaré characteristics:

$$\chi(\mathscr{E}'_i)_s = \chi(\mathscr{E}_i)_s + d^2 \cdot \deg(\mathscr{L})\,.$$

3 $\mathscr{D}$-elliptic sheaves and vector bundles on the non-commutative projective line

In [Dr 3] Drinfeld shows that there is an equivalence of categories between the category of elliptic sheaves over S and the category of elliptic modules over S. In this section we consider the analogous question for $\mathscr{D}$-elliptic sheaves. Whereas in the case of elliptic sheaves one has to deal with the endomorphism ring of the additive group $\mathbb{G}_a$ we are here lead in a natural way into dealing with the endomorphism ring of $\mathbb{G}_a^d$. Therefore, there is an obvious relation to Anderson's t-motives [An]. Throughout this section we fix a base field L which we suppose perfect.

(3.1) We first recall briefly some basic facts about the skew polynomial ring $L[\tau]$ in the case that L is perfect, the commutation relation being as usual $\tau b = b^q \tau$.

The ring $L[\tau]$ admits a left and right Euclidean algorithm. Therefore, any left or right ideal is principal, and hence any finitely generated torsion-free module is free.

(3.2) Lemma. *The ring $L[\tau]$ satisfies the left and the right Ore condition. Hence it possesses a left and right skew field of fractions.*

Proof. The left Ore condition [Her] demands that for given f, $g \in L[\tau]$ we find $x, y \in L[\tau]$ such that

$$xf = yg .$$

Consider the homomorphism of left $L[\tau]$-modules

$$\begin{aligned} \varphi : L[\tau]^2 &\to L[\tau] \\ (x, y) &\mapsto xf - yg . \end{aligned}$$

For each non negative integer n, let

$$L[\tau]_n = \{a \in L[\tau];\, \deg_\tau(a) \leqq n\}$$

where $\deg_\tau(a)$ is the degree of the skew polynomial a. Then

$$\dim_L(L[\tau]_n) = n + 1$$

and

$$\varphi(L[\tau]_{\deg(g)+n} \oplus L[\tau]_{\deg(f)+n}) \subset L[\tau]_{\deg(f)+\deg(g)+n} .$$

Therefore

$$\operatorname{Ker}(\varphi) \cap (L[\tau]_{\deg(g)+n} \oplus L[\tau]_{\deg(f)+n}) \neq (0) .$$

Hence, $\operatorname{Ker}(\varphi)$ is non-trivial, as had to be shown. By a similar argument we get the right Ore condition for $[\tau]L = \{\sum_n \tau^n a_n\}$ which is equal to $L[\tau]$ as L is perfect. The fraction field is then formed as the ring of elements of the form $a^{-1} \cdot b$ or of the form $c \cdot d^{-1}$, $a, b, c, d \in L[\tau]$, $a \neq 0$, $d \neq 0$, comp. [Her, Theorem 7.1.1]. □

(3.3) Similar statements are true for the ring of skew power series $L[[\tau]]$. Its skew field of fractions may be identified with the field of skew Laurent series $L((\tau))$.

(3.4) Let $E = (\mathscr{E}_i, j_i, t_i)_{i \in \mathbf{Z}}$ be a $\mathscr{D}$-elliptic sheaf over $\operatorname{Spec}(L)$. We consider

$$P = H^0((X \setminus \{\infty\}) \otimes L, \mathscr{E}_i).$$

This is clearly independent of i since $\operatorname{supp}(\mathscr{E}_i/\mathscr{E}_{i-1}) \subset \{\infty\} \times \operatorname{Spec}(L)$. It is an $L[\tau]$-module, where the operation of τ is induced from $t_i : {}^\tau\mathscr{E}_i \to \mathscr{E}_{i+1}$.

(3.5) Lemma. *The $L[\tau]$-module P is free of rank d.*

Proof. We choose i large enough, such that the canonical homomorphism

$$\Gamma(X \otimes L, \mathscr{E}_i) \to \Gamma(X \otimes L, \mathscr{E}_i/\mathscr{E}_{i-1})$$

is surjective. This is possible because of the periodicity of $(\mathscr{E}_i, j_i, t_i)$ and the ampleness of the line bundle $\mathcal{O}_X(\infty)$. We choose an L-basis $e_1, \ldots, e_r$ of $\Gamma(X \otimes L, \mathscr{E}_i)$ such that $e_1, \ldots, e_d$ go to a basis of $\Gamma(X \otimes L, \mathscr{E}_i/\mathscr{E}_{i-1})$ under the above map.

Then it is immediate to check the following points:
(i) for all $j \geqq 0$ $\tau^j e_1, \ldots, \tau^j e_d$ are sections in $\Gamma(X \otimes L, \mathscr{E}_{i+j})$, such that their images in $\Gamma(X \otimes L, \mathscr{E}_{i+j}/\mathscr{E}_{i+j-1})$ form again a basis.
(ii) $\{e_1, \ldots, e_r\} \cup \{\tau^h e_i | 1 \leqq h \leqq j, 1 \leqq i \leqq d\}$ form a generating system of $\Gamma(X \otimes L, \mathscr{E}_{i+j})$ as an L-vector-space.
(iii) $\{e_1, \ldots, e_r\}$ is a generating system of $P = \varinjlim_{j \vartheta 1} \Gamma(X \otimes L, \mathscr{E}_{i+j})$ over $L[\tau]$.

Therefore in particular P is a finitely generated module over $L[\tau]$. Next we show that P is a torsion free $L[\tau]$-module. Suppose $f(\tau) \in L[\tau] \setminus \{0\}$ and $e \in P \setminus \{0\}$. We can find $i \in \mathbb{Z}$ such that $e \in \Gamma(X, \mathscr{E}_i)$ and $e \notin \Gamma(X, \mathscr{E}_{i-1})$. Then $\tau^j e \in \Gamma(X, \mathscr{E}_{i+j})$ and $\tau^j e \notin \Gamma(X_i, \mathscr{E}_{i+j-1})$ for $j \in \mathbb{N}$. Suppose the degree of $f(\tau)$ as a polynomial in τ is m. Then $f(\tau)e \in \Gamma(X, \mathscr{E}_{i+m})$ and $f(\tau)e \notin \Gamma(X, \mathscr{E}_{i+m-1})$. In particular $f(\tau)e \neq 0$. Therefore P is a torsion free $L[\tau]$-module of finite rank. Consequently, P is a free $L[\tau]$-module of finite rank. Now $P/\tau P$ is isomorphic to $\mathscr{E}_i/t_{i-1}({}^\tau\mathscr{E}_{i-1})$ (for any i). Therefore $P/\tau P$ is a d-dimensional L-vectorspace, and this shows that P is a free $L[\tau]$-module of rank d. □

(3.6) Let $O_D = H^0(X \setminus \{\infty\}, \mathscr{D})$. It is an order of D over $A = H^0(X \setminus \{\infty\}, \mathcal{O}_X)$. Then P has a natural structure of right O_D-module which commutes with the left action of $L[\tau]$. The underlying $A \otimes L$-module is locally free of rank d^2. Let us denote by

$$\varphi : O_D^{\mathrm{op}} \to \mathrm{End}_{L[\tau]}(P)$$

the corresponding homomorphism of $\mathbb{F}_q$-algebras. We obtain a commutative diagram of homomorphisms

$$\begin{tikzcd}
O_D^{\mathrm{op}} \arrow[r, "\varphi"] \arrow[d] & \mathrm{End}_{L[\tau]}(P) \arrow[d] \\
O_D^{\mathrm{op}} \otimes_A L \arrow[r] & \mathrm{End}_L(P/\tau P).
\end{tikzcd}$$

Here $A \to L$ is the *zero* of the $\mathscr{D}$-elliptic sheaf and the lower horizontal arrow is given by the operation of O_D on $\mathscr{E}_i/t_{i-1}({}^\tau\mathscr{E}_{i-1})$.

(3.7) Lemma. *For any $a \in O_D \cap D^\times$ the map $\varphi(a) : P \to P$ is injective and its cokernel $P/\varphi(a)P$ is a finite dimensional vector space over L of dimension* $\log_q |O_D/O_D \cdot a|$.

Proof. The map $\varphi(a)$ is $A \otimes L$-linear and $\varphi(a) \otimes_A \mathrm{id}_F : P \otimes_A F \to P \otimes_A F$ is bijective. (Note that a is invertible in $O_D \otimes_A F = D$.) So $\varphi(a)$ is injective and $\dim_L(P/\varphi(a)P) < \infty$. Now it is immediate to see that the map $\mathcal{O}_D \cap D^\times \ni a \mapsto \dim_L(P/\varphi(a)P) =: f(a)$ satisfies $f(ab) = f(a) + f(b)$. Therefore it extends to a homomorphism

$$D^\times \to \mathbb{Z}\,.$$

The same holds true for the map

$$O_D \cap D^\times \ni a \mapsto \log_q |O_D/O_D \cdot a|\,.$$

Both maps agree on $F^\times \subset D^\times$ because they obviously agree on $A\setminus\{0\}$ and both maps factor over $D^\times/[D^\times, D^\times]$. But by a well known result [Wa] $[D^\times, D^\times] = SL_1(D)$ and the reduced norm $D^\times \xrightarrow{\mathrm{rn}} F^\times$ by Eichler (cf. [Re, Theorem 34.8]) is in our situation surjective. Therefore we can factor our two maps through maps from $F^\times$ to $\mathbb{Z}$ which are identical on $(F^\times)^d \subset F^\times$. But then they agree on $F^\times$ itself. ☐

Remark. (i) With a little more effort it would be possible working first locally to avoid the somewhat difficult theorems above but we wanted a quick proof.
(ii) The lemma implies that $\varphi: O_D^{\mathrm{op}} \to \mathrm{End}_L(P)$ is injective.

(3.8) Corollary. *Let* $L(\tau)$ *be the skew-field of fractions of* $L[\tau]$ *and put* $V = L(\tau) \otimes_{L[\tau]} P$. *The homomorphism*

$$\varphi: O_D^{\mathrm{op}} \to \mathrm{End}_{L[\tau]}(P)$$

extends in a unique way to a homomorphism

$$\varphi: D^{\mathrm{op}} \to \mathrm{End}_{L(\tau)}(V)\,.$$

Proof. $L(\tau)$ is the union of free $L[\tau]$-modules and is therefore a flat module. Since for $a \in O_D \cap D^\times$ the homomorphism $\varphi(a)$ is injective, it follows that also

$$\mathrm{id} \otimes \varphi(a): L(\tau) \underset{L[\tau]}{\otimes} P \to L(\tau) \underset{L[\tau]}{\otimes} P$$

is injective. Since these are finite dimensional $L(\tau)$-vector-spaces it follows that $\mathrm{id} \otimes \varphi(a)$ is bijective and therefore we can extend φ. ☐

(3.9) Recall that $L[\tau]$ is the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_a$ over L (τ is the Frobenius endomorphism relative to $\mathbb{F}_q$). It follows that the functors

$$E \mapsto \mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$$

and

$$\mathrm{P} \mapsto \mathrm{Hom}_{L[\tau]}(P, \mathbb{G}_a)$$

are anti-equivalences of categories between unipotent groups over L, with $\mathbb{F}_q$-module structures, isomorphic to $\mathbb{G}_a^N$ for some N and free left $L[\tau]$-modules of finite rank. Therefore, fixing a basis of P as an $L[\tau]$-module, we can rewrite φ as an embedding

$$\varphi: O_D \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d)$$

of $\mathbb{F}_q$-algebras.

(3.10) Proposition. *For* $a \in \mathcal{O}_D \cap D^\times$ *the endomorphism* $\varphi(a)$ *of* $\mathbb{G}_{a,L}^d$ *is surjective with kernel a finite group scheme over* $\mathrm{Spec}(L)$ *of order* $|O_D/O_D \cdot a|$. *Moreover this kernel is étale over* $\mathrm{Spec}(L)$ *if and only if the zero of the* $\mathcal{D}$*-elliptic sheaf is disjoint from the divisor* $\mathrm{div}(\mathrm{rn}(a))$.

Proof. For any commutative L-algebra R, the R-valued points of $\mathrm{Ker}(\varphi(a))$ are given by

$$\begin{aligned}\mathrm{Ker}(\varphi(a))\ (\mathrm{Spec}\ R) &= \mathrm{Hom}_{L[\tau]}(P/\varphi(a)P, R) \\ &= \{u \in \mathrm{Hom}_L(P/\varphi(a)P, R);\, u(\tau x) = u(x)^q\}\,.\end{aligned}$$

Consider the finite-dimensional L-vector-space $P/\varphi(a)P$ as a locally free sheaf of rank $\log_q |O_D/O_D \cdot a|$ on $\operatorname{Spec}(L)$. The operation of τ endows this with the structure of a φ-sheaf on $\operatorname{Spec}(L)$ in the sense of Drinfeld [Dr 6] and $\operatorname{Ker} \varphi(a)$ is precisely the finite group scheme associated by Drinfeld to this φ-sheaf, as follows from the description above. To see then that all our assertions follow from Proposition 2.1 of loc. cit. it suffices to remark that

$$\tau : P/\varphi(a)P \to P/\varphi(a)P$$

is bijective if and only if $\varphi(a) : P/\tau P \to P/\tau P$ is bijective, i.e. if and only if the last condition in the assertion is satisfied. □

(3.11) Let $\mathscr{E}_{i,\infty} = H^0(\operatorname{Spec}(\mathscr{O}_\infty \hat{\otimes} L), \mathscr{E}_i)$. This is independent of i. Since $t_{i,\infty}$: ${}^\tau\mathscr{E}_{i,\infty} \to \mathscr{E}_{i+1,\infty}$ is an isomorphism, we obtain via the inclusions $j_{i,\infty} : \mathscr{E}_{i,\infty} \hookrightarrow \mathscr{E}_{i+1,\infty}$ a τ^{-1}-linear endomorphism of $\mathscr{E}_{i,\infty}$. We therefore obtain on $\mathscr{E}_{i,\infty}$ the structure of a $L[\tau^{-1}]$-module. This action is obviously continuous for the τ^{-1}-adic topology on $L[\tau^{-1}]$ and the ∞-adic topology on $\mathscr{E}_{i,\infty}$ and therefore $\mathscr{E}_{i,\infty}$ is even a $L[[\tau^{-1}]]$-module (for any $i \in \mathbb{Z}$). Nakayama's lemma and the fact that $\mathscr{E}_{i,\infty}/\tau^{-1}\mathscr{E}_{i,\infty}$ is a d-dimensional L-vectorspace imply that $\mathscr{E}_{i,\infty}$ is a free $L[[\tau^{-1}]]$-module of rank d. In addition $\mathscr{E}_{i,\infty}$ is a right-$\mathscr{D}_\infty$-module so that we get an $\mathbb{F}_q$-algebra homomorphism

$$\varphi_\infty : \mathscr{D}_\infty^{\mathrm{op}} \to \operatorname{End}_{L[[\tau^{-1}]]}(\mathscr{E}_{i,\infty})$$

which is injective as before.

(3.12) Proposition. *There are canonical isomorphisms*

$$F_\infty \underset{\mathscr{O}_\infty}{\otimes} \mathscr{E}_{i,\infty} \xleftarrow{\sim} L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} \mathscr{E}_{i,\infty} \xleftarrow{\sim} L((\tau^{-1})) \underset{L[\tau]}{\otimes} P .$$

Proof. To define the left arrow we shall first define the structure of a $L((\tau^{-1}))$-module on $F_\infty \otimes_{\mathscr{O}_\infty} \mathscr{E}_{i,\infty}$. Let ϖ_∞ denote a uniformising element at ∞. From the periodicity condition we have

$$\tau^{-d \cdot \deg(\infty)} \mathscr{E}_{i,\infty} = \varpi_\infty \cdot \mathscr{E}_{i,\infty} .$$

Therefore, if $e \in \mathscr{E}_{i,\infty}$, we define $e' \in \mathscr{E}_{i,\infty}$ by

$$\tau^{-d \cdot \deg(\infty)} e' = \varpi_\infty \cdot e$$

($\mathscr{E}_{i,\infty}$ is a free $L[[\tau^{-1}]]$-module) and put

$$\tau^{d \cdot \deg(\infty)}(1 \otimes e) = \varpi_\infty^{-1} \otimes e' .$$

Since $L((\tau^{-1})) = \bigcup\limits_{j=1}^{\infty} \tau^{j \cdot d \cdot \deg(\infty)} L[[\tau^{-1}]]$, this defines a $L((\tau^{-1}))$-linear map

$$L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} \mathscr{E}_{i,\infty} \to F_\infty \underset{\mathscr{O}_\infty}{\otimes} \mathscr{E}_{i,\infty}$$

extending the identity map of $\mathscr{E}_{i,\infty}$. It is obvious that the map is surjective. On the other hand, any $L((\tau^{-1}))$-submodule of $L((\tau^{-1})) \otimes_{L[[\tau^{-1}]]} \mathscr{E}_{i,\infty}$ has a non-trivial intersection with $\mathscr{E}_{i,\infty}$ and hence the kernel of this map is trivial. The first isomorphism is established. It also shows that the middle term is independent of i. The second arrow is induced from the inclusion

$$P \subset \bigcup_i \mathscr{E}_{i,\infty} = F_\infty \underset{\mathscr{O}_\infty}{\otimes} \mathscr{E}_{i,\infty} = F_\infty \underset{A}{\otimes} P .$$

The argument used in the proof of Corollary 3.8 shows that the operation of A on P extends to an operation of F on $L((\tau^{-1}))\otimes_{L[\tau]}P$. The topology on $F_\infty \otimes_{\mathcal{O}_\infty} \mathscr{E}_{i,\infty}$, defined by $\tau^{-n}\cdot\mathscr{E}_{i,\infty}$, $n\to\infty$, coincides with the topology defined by $\varpi_\infty^n \mathscr{E}_{i,\infty}$, $n\to\infty$. Since the image of the second arrow is a $L((\tau^{-1}))$-subvector space it is a closed subset. On the other hand it is clearly stable under the operation of F via its embedding in F_∞. Therefore, by continuity, the image is stable under F_∞ and therefore must be all of $F_\infty \otimes_A P \cong L((\tau^{-1}))\otimes_{L[[\tau^{-1}]]}\mathscr{E}_{i,\infty}$. But both sides of the second arrow are $L((\tau^{-1}))$-vector-spaces of dimension d, therefore the surjectivity of the map implies the injectivity. □

The following definition was suggested by Drinfeld in a letter to one of us [Dr 5].

(3.13) Definition. Let L be a perfect field. A *vector bundle of rank r over the non-commutative projective line over L, $\mathbb{P}_L^1(\tau)$,* is a free $L[\tau]$-module P of rank r together with a free $L[[\tau^{-1}]]$-submodule $W_\infty \subset L((\tau^{-1}))\otimes_{L[\tau]}P$ such that the induced map

$$L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} W_\infty \to L((\tau^{-1})) \underset{L[\tau]}{\otimes} P$$

is an isomorphism. A *homomorphism of vector bundles* $(P, W_\infty)\to(P', W'_\infty)$ is a pair of module homomorphisms

$$\varphi : P \to P',\ \varphi_\infty : W_\infty \to W'_\infty$$

making the obvious diagram commutative

$$\begin{tikzcd}
L((\tau^{-1})) \underset{L[\tau]}{\otimes} P \arrow[r, "\mathrm{id}\otimes\varphi"] & L((\tau^{-1})) \underset{L[\tau]}{\otimes} P' \\
L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} W_\infty \arrow[u] \arrow[r, "\mathrm{id}\otimes\varphi_\infty"] & L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} W'_\infty . \arrow[u]
\end{tikzcd}$$

Note that the vector bundles of variable rank over the non-commutative projective line over L form an exact category. We also need the following definition.

(3.14) Definition. A *coherent right $\mathscr{D}$-action on a vector bundle (P, W_∞) over the non-commutative projective line* is a commutative diagram of ring homomorphisms

$$\begin{tikzcd}
O_D^{\mathrm{op}} \arrow[r] \arrow[d, phantom, "\cap\,|"] & \mathrm{End}_{L[\tau]} P \arrow[d] \\
F_\infty \underset{A}{\otimes} O_D^{\mathrm{op}} = F_\infty \underset{\mathcal{O}_\infty}{\otimes} \mathscr{D}_\infty \arrow[r] & \mathrm{End}_{L((\tau^{-1}))}\big(L((\tau^{-1})) \underset{L[\tau]}{\otimes} P\big) \\
\mathscr{D}_\infty \arrow[r] \arrow[u, phantom, "\cup\,|"] & \mathrm{End}_{L[[\tau^{-1}]]}(W_\infty) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are the canonical ones and where, of course,

$$\mathrm{End}_{L((\tau^{-1}))}\big(L((\tau^{-1})) \underset{L[\tau]}{\otimes} P\big) = \mathrm{End}_{L((\tau^{-1}))}\big(L((\tau^{-1})) \underset{L[[\tau^{-1}]]}{\otimes} W_\infty\big).$$

To go further, we need some important properties of vector bundles over the non-commutative projective line.

(3.15) Definition. The *cohomology groups of a vector bundle* $E = (P, W_\infty)$ *over* $\mathbb{P}^1_L(\tau)$ are

(i) $H^0(\mathbb{P}^1(\tau), E) = (P \cap W_\infty)$ (intersection inside $L((\tau^{-1})) \otimes_{L[\tau]} P$ using the identification map).
(ii) $H^1(\mathbb{P}^1(\tau), E) = L((\tau^{-1})) \otimes_{L[\tau]} P/(P + W_\infty)$.

Remarks. (i) Given an exact sequence of vector bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

on $\mathbb{P}^1(\tau)$, it is an exercise to see that we have an exact cohomology sequence

$$0 \to H^0(\mathbb{P}^1(\tau), E_1) \to H^0(\mathbb{P}^1(\tau), E) \to H^0(\mathbb{P}^1(\tau), E_2) \to$$
$$\to H^1(\mathbb{P}^1(\tau), E_1) \to H^1(\mathbb{P}^1(\tau), E) \to H^1(\mathbb{P}^1(\tau), E_2) \to 0 .$$

Here the connecting homomorphism sends an element $s \in H^0(\mathbb{P}^1(\tau), E_2) = P(E_2) \cap W_\infty(E_2)$ into the residue class of $\tilde{s} - \tilde{s}_\infty \in L((\tau^{-1})) \otimes_{L[\tau]} P(E_1)/(P(E_1) + W_\infty(E_1))$, where $\tilde{s} \in P(E)$ is a lifting of $s \in P(E_2)$ and $\tilde{s}_\infty \in W_\infty(E)$ is a lifting of $s \in W_\infty(E_2)$.

(ii) We denote by $\mathcal{O}(n)$ $(n \in \mathbb{Z})$ the vector bundles $\mathcal{O}(n) = (L[\tau], \tau^n L[[\tau^{-1}]])$, with $\tau^n L[[\tau^{-1}]] \subset L((\tau^{-1}))$. One has

$$\dim_L H^0(\mathbb{P}^1(\tau), \mathcal{O}(n)) = \begin{cases} 0 & \text{for } n < 0 \\ n+1 & \text{for } n \geqq 0 \end{cases}$$

$$\dim_L H^1(\mathbb{P}^1(\tau), \mathcal{O}(n)) = \begin{cases} 0 & \text{for } n \geqq -1 \\ -n-1 & \text{for } n < -1 . \end{cases}$$

More generally, for any vector bundle $E = (P, W_\infty)$ on $\mathbb{P}^1(\tau)$ we denote by $E(n)$ $(n \in \mathbb{Z})$ the twisted vector bundle $(P, \tau^n W_\infty)$.

(3.16) Proposition. (i) *Any vector bundle of rank one is isomorphic to precisely one of the form* $\mathcal{O}(n)$.
(ii) *Any vector bundle of rank r is isomorphic to a direct sum of vector bundles of rank one.*
(iii) *For any vector bundle E over* $\mathbb{P}^1(\tau)$ *the cohomology groups are finite dimensional L-vector-spaces.*

Proof. (i) is trivial and (iii) is true for rank one bundles by the remark above. In general then (ii) implies (iii).

We prove (ii) by induction on the rank exactly as in the commutative case [Ok-Sch-Sp, Theorem 2.11]. Given a bundle E of rank r, we can find an exact sequence of vector bundles

$$0 \to \mathcal{O}(m) \to E \to E' \to 0 .$$

But $E' \cong \oplus_{i=1}^{r-1} \mathcal{O}(n_i)$ by induction hypothesis. This shows immediately that the possible m are bounded from above for if we had some other line subbundle

$$\mathcal{O}(m') \to E ,$$

it would either factorize through $\mathcal{O}(m)$ and then $m' \leqq m$ or it would give a non zero homomorphism (by composing) $\mathcal{O}(m') \to \mathcal{O}(n_i)$ for some i, $1 \leqq i \leqq r-1$, and then $m' \leqq n_i$.

Therefore by rescaling, we can assume that the m above is maximal and $m = 0$. Twisting by $\mathcal{O}(-1)$, we obtain an exact sequence of vector bundles

$$0 \to \mathcal{O}(-1) \to E(-1) \to E'(-1) = \bigoplus_{i=1}^{r-1} \mathcal{O}(n_i - 1) \to 0 .$$

By assumption $H^0(\mathbb{P}^1(\tau), E(-1)) = 0$ (otherwise we would have $\mathcal{O}(1) \subset E$). Furthermore by the remark (ii) above, $H^1(\mathbb{P}^1(\tau), \mathcal{O}(-1)) = 0$. But then the long exact cohomology sequence implies that

$$H^0\left(\mathbb{P}^1(\tau), \bigoplus_{i=1}^{r-1} \mathcal{O}(n_i - 1)\right) = \bigoplus_{i=1}^{r-1} H^0(\mathbb{P}^1(\tau), \mathcal{O}(n_i - 1))$$
$$= (0) .$$

Therefore $n_i \leqq 0$ for all $i = 1, \ldots, r-1$. But the extensions like

$$0 \to \mathcal{O} \to E \to E' \to 0$$

are classified as in the commutative case by the L-vector-space

$$\bigoplus_{i=1}^{r-1} H^1(\mathbb{P}^1(\tau), \mathcal{O}(-n_i)) = (0)$$

by remark (ii). This means that the sequence for E splits, therefore

$$E = \mathcal{O} \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}(n_i) . \qquad \square$$

(3.17) Theorem. *The functor*

$$(\mathscr{E}_i, j_i, t_i)_{i \in \mathbb{Z}} \mapsto (P := H^0(X \setminus \{\infty\}, \mathscr{E}_0), W_\infty := \mathscr{E}_{0,\infty})$$

defines an equivalence of categories between the category of $\mathscr{D}$-elliptic sheaves over L and the full subcategory of the category of vector bundles over $\mathbb{P}^1_L(\tau)$ with coherent right $\mathscr{D}$-action which satisfy the following conditions:

(i) *The induced homomorphism*

$$A \to O_D^{\mathrm{op}} \xrightarrow{\varphi} \mathrm{End}_{L[\tau]}(P) \to \mathrm{End}_L(P/\tau P)$$

equips $P/\tau P$ with an action of A which factors over the (central) action of L on $P/\tau P$ (and is therefore given by a ring homomorphism $A \to L$).
(ii) *P is finitely generated as an $A \otimes_{\mathbb{F}_q} L$-module.*
(iii) *W_∞ is finitely generated as an $\mathcal{O}_\infty \otimes_{\mathbb{F}_q} L$-module.*
(iv) *We have $\tau^{-d \cdot \deg(\infty)} W_\infty = \varpi_\infty W_\infty$.*

Proof. It is by now clear that the functor in question has its image in this subcategory. From the description of a locally free sheaf on a curve through lattices in its generic fibre, it follows easily that the functor is fully faithful. We have to prove the essential surjectivity.

Let (P, W_∞) be given. Let

$$P_\infty \subset L((\tau^{-1})) \bigotimes_{L[\tau]} P$$

be the F_∞-subspace generated by P (the structure of a F_∞-vector-space on $L((\tau^{-1}))\otimes_{L[\tau]}P$ comes from its $(F_\infty\otimes_A\mathcal{O}_D)$-right module structure). Since P is finitely generated as $(A\otimes_{\mathbb{F}_q}L)$-module, P_∞ is a finitely generated $(F_\infty\otimes_{\mathbb{F}_q}L)$-module. Since W_∞ is a finitely generated $(\mathcal{O}_\infty\otimes_{\mathbb{F}_q}L)$-module, so is $(W_\infty\cap P_\infty)$.

Claim. $W_\infty\cap P_\infty$ is an $(\mathcal{O}_\infty\otimes_{\mathbb{F}_q}L)$-lattice in P_∞, i.e. the canonical map

$$F_\infty\underset{\mathcal{O}_\infty}{\otimes}(W_\infty\cap P_\infty)\to P_\infty$$

is an isomorphism.

First of all, because $W_\infty\cap P_\infty$ is a finitely generated $(\mathcal{O}_\infty\otimes_{\mathbb{F}_q}L)$-module contained in the $F_\infty\otimes_{\mathbb{F}_q}L$-module P_∞, the map above is injective. If surjectivity did not hold, then $P_\infty/(W_\infty\cap P_\infty)$ would contain a $(F_\infty\otimes_{\mathbb{F}_q}L)$-module of positive dimension and therefore would have uncountable dimension as a L-vector-space. On the other hand

$$P_\infty/(W_\infty\cap P_\infty)\subset L((\tau^{-1}))\otimes_{L[\tau]}P/W_\infty\cong(L((\tau^{-1}))/L[[\tau^{-1}]])\otimes_{L[[\tau^{-1}]]}W_\infty$$

has countable L-dimension, a contradiction, whence the claim above.

Claim. P is an $A\otimes_{\mathbb{F}_q}L$-lattice in P_∞, i.e.

$$F_\infty\underset{A}{\otimes}P\to P_\infty$$

is an isomorphism.

Surjectivity is immediate from the definition of P_∞. Just as in the previous claim we have that for every n

$$\tau^{-n}W_\infty\cap P_\infty$$

is a $(\mathcal{O}_\infty\otimes_{\mathbb{F}_q}L)$-lattice in P_∞.

Since

$$\bigcap_{n\geqq 0}(\tau^{-n}W_\infty\cap P_\infty)=(0)$$

we see that $\tau^{-n}W_\infty\cap P_\infty$, $n\to\infty$, form a fundamental system of neighbourhoods of 0 in P_∞, considered as a topological $(F_\infty\otimes_{\mathbb{F}_q}L)$-module. If now injectivity did not hold, then $P\subset P_\infty$ would not be discrete and hence

$$P\cap(\tau^{-n}W_\infty\cap P_\infty)\neq(0)$$

for all n. However, Proposition 3.16 above implies that

$$P\cap\tau^{-n}W_\infty=(0)$$

for $n\geqq n_0$, n_0 sufficiently large, a contradiction. This proves the second claim.

We can now define vector bundles $\mathscr{E}_i$ over $X\otimes_{\mathbb{F}_q}L$ for all $i\in\mathbb{Z}$ by

$$H^0((X\setminus\{\infty\})\otimes_{\mathbb{F}_q}L,\mathscr{E}_i)=P$$

$$\mathscr{E}_{i,\infty}=\tau^iW_\infty\cap P_\infty\subset F_\infty\otimes_AP\,.$$

The inclusions $\mathscr{E}_i\subset\mathscr{E}_{i+1}$ are defined by the inclusions $\tau^iW_\infty\cap P_\infty\subset\tau^{i+1}W_\infty\cap P_\infty$ and the homomorphisms $t_i:{}^\tau\mathscr{E}_i\to\mathscr{E}_{i+1}$ from multiplication by τ,

$$\tau:\tau^iW_\infty\cap P_\infty\to\tau^{i+1}W_\infty\cap P_\infty\,.$$

The action of $\mathscr{D}$ on $\mathscr{E}_i$ comes from the operation of $\mathscr{D}$ on (P,W_∞).

It remains to show that $(\mathscr{E}_i, t_i)$ is a $\mathscr{D}$-elliptic sheaf and that (P, W_∞) is the image of $(\mathscr{E}_i, t_i)$ under the functor in question. However,

$$\mathscr{E}_i/\mathscr{E}_{i-1} = \tau^i W_\infty \cap P_\infty/\tau^{i-1} W_\infty \cap P_\infty$$

is an L-subvector-space of $\tau^i W_\infty/\tau^{i-1} W_\infty$. Since P_∞ is stable under the operation of $F_\infty \otimes_A O_D$, this subspace is stable under the operation of $\mathscr{D}_\infty$. Since $\tau^i W_\infty/\tau^{i-1} W_\infty$ is a simple $\mathscr{D}_\infty \otimes_{\mathbb{F}_q} L$-module, so is $\mathscr{E}_i/\mathscr{E}_{i-1}$. The remaining conditions on a $\mathscr{D}$-elliptic sheaf (periodicity, "tangent space" condition on Coker t_i) follow easily from the conditions imposed on (P, W_∞). The rest is clear. □

4 Moduli space: Smoothness

Our aim in this section will be to prove the following theorem.

(4.1) Theorem. *$\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ is an algebraic stack in the sense of Deligne-Mumford (cf. [De-Mu]) which is smooth of relative dimension $(d-1)$ over $X'\setminus I$.*

For the smoothness assertion in this theorem we shall apply the following lemma to the diagram in (2.8).

(4.2) Lemma. *Let S, U, V be smooth $\mathbb{F}_q$-schemes and let $\alpha: V \to U \times U$, $f: V \to S$ be $\mathbb{F}_q$-morphisms. We form the cartesian square*

$$\begin{tikzcd}
W \arrow[r, "\beta"] \arrow[d, hook, "j"'] \arrow[dd, bend right=60, "g"'] & U \arrow[d, hook, "i = (\mathrm{Frob}_U, \mathrm{id}_U)"] \\
V \arrow[r, "\alpha"] \arrow[d, "f"'] & U \times U \\
S. &
\end{tikzcd}$$

Let $w \in W$ and put $v = j(w)$. Assume that

$$(f, \mathrm{pr}_1 \circ \alpha): V \to S \times U$$

is smooth, of relative dimension n in v. Then g is smooth of relative dimension n in w.

Proof. Let $u = \beta(w)$ and $i(u) = (u', u)$. We have to check the transversality condition

$$\begin{aligned} T_{(u',u)}(U \times U) &= T_{u'}(U) \times T_u(U) \\ &= T_v(\alpha)(T_v(V/S)) + T_u(i)(T_u(U)) . \end{aligned}$$

However,

$$T_u(i)(T_u U) = \{0\} \times T_u(U)$$

and the composition

$$\begin{tikzcd}
T_v(V/S) \arrow[rr, "T_v(\alpha)"] \arrow[dr] & & T_{u'}(U) \times T_u(U) \arrow[dl, "\mathrm{pr}_1"] \\
& T_{u'}(U) &
\end{tikzcd}$$

is surjective by hypothesis, whence the assertion. □

(4.3) We shall first construct an open substack of $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ which will become "bigger" as $\deg(I)$ grows. Denote by $\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I}$ the open substack of $\mathscr{D}$-elliptic sheaves $(\mathscr{E}_i)$ over S such that $\mathscr{E}_0$ is stable as a vector bundle with level-I-structure i.e. such that for all geometric points s in S and for all locally free $\mathscr{O}_{X \times s}$-modules $\mathscr{F}$ properly contained in $(\mathscr{E}_0)_s$ we have

$$\frac{\deg(\mathscr{F}) - \deg(I)}{\mathrm{rk}(\mathscr{F})} < \frac{\deg((\mathscr{E}_0)_s) - \deg(I)}{\mathrm{rk}((\mathscr{E}_0)_s)}$$

(comp. [Se, 4.I. Définition 2]).

Denote by $\mathscr{V}ec_{X,I} = \mathscr{V}ec_I$ the stack of vector bundles of rank d^2 with level-I-structure. Then the subfunctor $\mathscr{V}ec_I^{\mathrm{st}}$ of such vector bundles which are stable in the above sense is representable by a disjoint union of quasi-projective schemes over $\mathbb{F}_q$ which at every point is smooth of dimension $d^4(g - 1 + \deg I)$ if $\deg(I) > 0$ (cf. [Se, 4.III]). The condition on the degree of I is needed to eliminate non-trivial automorphisms. Since this point is not explicit in loc.cit., we indicate the proof. Let f be an automorphism of a stable vector bundle $\mathscr{E}$ with level-I-structure such that f_I is the identity of $\mathscr{E}_I$. We consider the obvious diagram

$$\begin{tikzcd}
\mathscr{K} \arrow[r, hook] & \mathscr{E} \arrow[r, "f - \mathrm{id}_{\mathscr{E}}"] \arrow[d] & \mathscr{E} \\
& \mathscr{C} \arrow[r, hook] & \mathscr{I} \arrow[u, hook]
\end{tikzcd}$$

($\mathscr{K}$ = kernel, $\mathscr{C}$ = coimage, $\mathscr{I}$ = image for $f - \mathrm{id}_{\mathscr{E}}$). Then we have

$$\mathrm{length}(\mathscr{I}/\mathscr{C}) \geqq \mathrm{length}((\mathscr{I}/\mathscr{C})_I) = \deg(I)\,\mathrm{rk}(\mathscr{C})$$

and by the stability conditions for the submodules $\mathscr{K}$ and $\mathscr{I}$ of $\mathscr{E}$, we get that f is equal to the identity. Let $\mathscr{V}ec_{X,\mathscr{D},I} = \mathscr{V}ec_{\mathscr{D},I}$ be the stack classifying the vector bundles with an action of $\mathscr{D}$ (compatible with an action of $\mathscr{O}_X$) and with a $\mathscr{D}$-linear level-I-structure and denote by $\mathscr{V}ec^{\mathrm{st}}_{\mathscr{D},I}$ the inverse image of $\mathscr{V}ec_I^{\mathrm{st}}$ under the obvious morphism from $\mathscr{V}ec_{\mathscr{D},I}$ to $\mathscr{V}ec_I$ (after fixing a base of $\mathscr{D}_I$ as $\mathscr{O}_I$-module).

(4.4) Lemma. *The morphism*

$$\mathscr{V}ec_{X,\mathscr{D},I} \to \mathscr{V}ec_{X,I}$$

is relatively representable and affine.

Proof. To give a right $\mathscr{D}$-action on a vector bundle $\mathscr{E}$ on $X \times S$ is equivalent to giving a homomorphism of $\mathscr{O}_{X \times S}$-algebras

$$\mathscr{D}^{\mathrm{op}} \otimes \mathscr{O}_S \to \mathscr{E}nd_{\mathscr{O}_{X \times S}}(\mathscr{E})\,.$$

In particular, the morphism of stacks

$$\mathscr{V}ec_{\mathscr{D},I} \to \mathscr{V}ec_I$$

factors through the morphism of stacks

$$\mathscr{X} \to \mathscr{V}ec_I$$

where $\mathscr{X}$ is the stack classifying the vector bundles $\mathscr{E}$ over $X \times S$ with a level-I-structure and a homomorphism of $\mathscr{O}_{X \times S}$-modules

$$\mathscr{D}^{\mathrm{op}} \otimes \mathscr{O}_S \to \mathscr{E}nd_{\mathscr{O}_{X \times S}}(\mathscr{E})$$

and the resulting map

$$\mathcal{V}ec_{\mathcal{D},I} \to \mathcal{X}$$

is relatively representable and a closed immersion (the conditions that the above homomorphism be one of algebras and the $\mathcal{D}$-linearity on the level structure are clearly closed conditions). Now the morphism of stacks

$$\mathcal{X} \to \mathcal{V}ec_I$$

is relatively representable and affine. To see this, let $S \to \mathcal{V}ec_I$ be a morphism corresponding to a vector bundle $\mathcal{E}$ on $X \times S$. It is sufficient to show that the functor which to an S-scheme T associates the set

$$H^0(X \times T, \mathcal{E}nd_{\mathcal{O}_{X\times T}}(\mathcal{E}_T) \otimes_{\mathcal{O}_X} \mathcal{D}^\vee)$$

is representable. Here $\mathcal{E}_T = \mathcal{E}_{|X\times T}$. Let $\mathrm{pr}_T : X \times T \to T$ be the projection morphism and put $\mathcal{F} = \mathcal{E}nd_{\mathcal{O}_{X\times S}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{D}^\vee$. The above set is equal to $H^0(T, \mathrm{pr}_{T*}(\mathcal{F}_T))$. By Grothendieck duality (one-dimensional fibres of pr_T)

$$R\mathrm{pr}_{T*}(\mathcal{F}_T) = R\mathcal{H}om(R\mathrm{pr}_{T*}(\mathcal{F}_T^\vee \otimes \Omega_X^1[1]), \mathcal{O}_T) .$$

It follows that

$$\begin{aligned} H^0(T, \mathrm{pr}_{T*}(\mathcal{F}_T)) &= H^0(T, R\mathrm{pr}_{T*}(\mathcal{F}_T)) \\ &= H^0(T, R\mathcal{H}om(R\mathrm{pr}_{T*}(\mathcal{F}_T^\vee \otimes \Omega_X^1[1]), \mathcal{O}_T)) \\ &= \mathrm{Hom}(R\mathrm{pr}_{T*}(\mathcal{F}_T^\vee \otimes \Omega_X^1[1]), \mathcal{O}_T) \\ &= \mathrm{Hom}(R^1\mathrm{pr}_{T*}(\mathcal{F}_T^\vee \otimes \Omega_X^1), \mathcal{O}_T) . \end{aligned}$$

However, this is precisely the set of points of Grothendieck's functor $\mathbb{V}(\mathcal{G})$ with values in T (cf. [Gro]), for $\mathcal{G} = R^1\mathrm{pr}_{S*}(\mathcal{F}^\vee \otimes \Omega_X^1)$ because $R^1\mathrm{pr}_*$ commutes with base changes $T \to S$. The functor $\mathbb{V}(\mathcal{G})$ on (Sch/S) is always representable by an affine scheme over S (loc. cit.). □

We conclude that $\mathcal{V}ec^{\mathrm{st}}_{\mathcal{D},I}$ is representable by a disjoint union of quasi-projective schemes if $\deg(I) > 0$.

(4.5) Lemma. *The stack $\mathcal{V}ec_{\mathcal{D},I}$ is smooth over $\mathbb{F}_q$.*

Proof. We show this first for $\mathcal{V}ec_{\mathcal{D}}$ itself. Let S be the spectrum of a local artin ring R and let $\bar{S} \subset S$ be a closed subscheme defined by an ideal J with $J^2 = 0$. Let $(\bar{\mathcal{E}}, \bar{\iota})$ be a vector bundle on $X \times \bar{S}$ with a $\mathcal{D}$-action. The obstruction to extending $(\bar{\mathcal{E}}, \bar{\iota})$ to $(X \times S)$ lies in the cohomology group $\mathrm{Ext}^2_{\mathcal{D}}(\bar{\mathcal{E}}, \bar{\mathcal{E}} \otimes_{\mathcal{O}_S} J)$ [Il, Chap. IV, Proposition 3.1.5]. Consider the local-global spectral sequence for Ext. Since $\mathcal{D}_x$ is a maximal order, $\bar{\mathcal{E}}_x$ is a projective module of rank 1 over $\mathcal{D}_x \otimes \underline{R/J}$. It follows that all higher local Ext-groups $\mathrm{Ext}^i_{\mathcal{D}_x}(\bar{\mathcal{E}}_x, \bar{\mathcal{E}}_x \otimes J_x)$ vanish for $i > 0$. Since $\dim(X) = 1$ the obstruction group is trivial. This shows that $\mathcal{V}ec_{\mathcal{D}}$ is a smooth stack over $\mathbb{F}_q$. But $\mathcal{V}ec_{\mathcal{D},I} \to \mathcal{V}ec_{\mathcal{D}}$ is a torsor under the smooth group scheme given by the functor $T \mapsto (\mathcal{D}_I \otimes \mathcal{O}_T)^\times$. Therefore $\mathcal{V}ec_{\mathcal{D},I}$ is also smooth over $\mathbb{F}_q$. □

Remark. The argument above still works if we only assume that $\mathcal{D}_x$ is a hereditary order for all x (every $\mathcal{D}_x$-module which is free as an $\mathcal{O}_x$-module is projective [Cu-Re, 26.12 (ii)]. Therefore the results of the present section can be extended to this case. In particular, $\mathcal{V}ec^{\mathrm{st}}_{X,\mathcal{D},I}$ is representable by a smooth quasi-projective scheme if $\deg(I) > 0$.

(4.6) Lemma. *The natural morphism* (cf. (2.8))

$$(\mathscr{E}_i, j_i) \mapsto \mathscr{E}_0, \quad \mathscr{V}ec^{\bullet}_{X,\mathscr{D},I} \to \mathscr{V}ec_{X,\mathscr{D},I}$$

is relatively representable by a flag variety. In particular it is smooth.

Proof. Let $\mathscr{E}_0$ on $(X \times S)$ be given. Then the chain $(\mathscr{E}_i)_{\iota \in \mathbb{Z}}$ corresponds because of the periodicity condition (ii) to a flag of sub-sheaves

$$\{0\} \subset \bar{\mathscr{E}}_1 \subset \ldots \subset \bar{\mathscr{E}}_{d \cdot \deg(\infty)-1} \subset \mathscr{E}_0(\infty)/\mathscr{E}_0$$

such that the successive quotients are locally free as $\mathscr{O}_S$-sheaves and which are stable under the action of $\mathscr{D}$, i.e. under the action of $\mathscr{D} \otimes_{\mathscr{O}_X} \kappa(\infty) \simeq \mathbb{M}_d(\kappa(\infty))$. But under Morita equivalence this flag corresponds to a complete flag of locally free $\mathscr{O}_{\{\infty\} \times S}$-sub-modules of a locally free $\mathscr{O}_{\{\infty\} \times S}$-module of rank d such that the successive quotients are locally free. □

We conclude that the open substack $\mathscr{V}ec^{\bullet; \mathrm{st}}_{\mathscr{D}, I}$ of $\mathscr{V}ec^{\bullet}_{\mathscr{D}, I}$ is representable by a disjoint union of smooth quasiprojective schemes over $\mathbb{F}_q$ if $\deg(I) > 0$.

(4.7) Lemma. *The morphism*

$$\mathrm{Hecke}_{X,\mathscr{D},I} \to (X' \backslash I) \times \mathscr{V}ec^{\bullet}_{X,\mathscr{D},I}$$

(*given by the zero morphism and the first row*) *is relatively representable and smooth of relative dimension* $(d-1)$.

Proof. An object of $((X' \backslash I) \times \mathscr{V}ec^{\bullet}_{\mathscr{D},I})\,(S)$ corresponds to a morphism $i_0 : S \to X$ factoring through $(X' \backslash I)$ and a chain of vector bundles with $\mathscr{D}$-action on $(X \times S)$,

$$\ldots \subset \mathscr{E}_i \subset \mathscr{E}_{i+1} \subset \ldots$$

with a level-I-structure. To complete this by a second row $(\mathscr{E}'_i)$ with homomorphisms

$$t_i : \mathscr{E}'_i \to \mathscr{E}_{i+1}$$

satisfying the required conditions is equivalent to giving a locally free $\mathscr{O}_{X \times S}$-submodule $\mathscr{E}' \subset \mathscr{E}_0$ which is stable under the operation of $\mathscr{D}$ and such that $\mathscr{E}_0/\mathscr{E}'$ is a locally free sheaf of rank d on the graph of i_0. Here we used the fact that $\infty \notin i_0(S)$ (cf. (2.3), remark (d)). Let $\tilde{\imath}_0 : S \hookrightarrow (X \times S)$ be the closed embedding defined by the graph of i_0. Then the choice of $\mathscr{E}' \subset \mathscr{E}_0$ corresponds to the choice of an $\mathscr{O}_S$-submodule which is locally a direct summand of rank $d(d-1)$

$$\bar{\mathscr{E}}' \subset (\tilde{\imath}_0)^*(\mathscr{E})$$

and which is stable under $i_0{}^*(\mathscr{D})$. Since $i_0(S)$ misses all ramified places, $i_0{}^*(\mathscr{D})$ is an Azumaya algebra. It therefore follows that the scheme classifying the possible $\bar{\mathscr{E}}'$ is locally in the étale topology isomorphic to $\mathbb{P}^{d-1}$ (a Brauer-Severi scheme of dimension $(d-1)$ over S). In particular it is smooth over S. □

Summarizing the above lemmas, an application of Lemma 4.2 shows that the open substack $\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I}$ of $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ is representable by a disjoint union of quasi-projective schemes which is smooth of relative dimension $(d-1)$ over $X' \backslash I$ if $\deg(I) > 0$.

(4.8) We now finish the proof of Theorem 4.1. For any two closed finite subschemes $I \subset I' \subset X'$ with $\deg(I') > 0$ we have the morphism of stacks

$$r_{I',I} : \mathscr{E}\ell\ell_{\mathscr{D},I'} \to \mathscr{E}\ell\ell_{\mathscr{D},I}$$

which associates to a level-I'-structure its restriction to I. Thanks to (2.6), over $(X'\setminus I')$ this morphism is a torsor under the finite group

$$\mathrm{Ker}(\mathrm{GL}_1(H^0(I',\mathscr{D}))\to\mathrm{GL}_1(H^0(I,\mathscr{D})))\,.$$

(Note that the multiplicative group of the algebra acts on the set of level structures via

$$\iota\circ g:\mathscr{D}_{I\times S}\xrightarrow{g^*}\mathscr{D}_{I\times S}\xrightarrow{\iota}(\mathscr{E}_i)_{I\times S}\,.$$

Clearly

$$(r_{I',I})^{-1}(\mathscr{E}\ell\ell^{\mathrm{st}}_{\mathscr{D},I})\subset\mathscr{E}\ell\ell^{\mathrm{st}}_{\mathscr{D},I'}\,.$$

Therefore, the open substack $\mathscr{E}\ell\ell^{\mathrm{st}}_{\mathscr{D},I'}$ which is stable under $[\mathrm{Ker}(\mathrm{GL}_1(\mathscr{D}_{I'})\to\mathrm{GL}_1(\mathscr{D}_I))$, gives as a quotient in the sense of stacks [De-Mu] an open substack

$$\mathscr{E}\ell\ell^{\mathrm{st}}_{\mathscr{D},I'}/[\mathrm{Ker}(\mathrm{GL}_1(\mathscr{D}_{I'})\to\mathrm{GL}_1(\mathscr{D}_I))]\subset\mathscr{E}\ell\ell_{\mathscr{D},I}$$

which contains $\mathscr{E}\ell\ell^{\mathrm{st}}_{\mathscr{D},I'}\times_{(X\setminus I)}(X\setminus I')$. It clearly is an algebraic stack in the sense of Deligne-Mumford which is smooth of relative dimension $(d-1)$ over $X'\setminus I'$. Letting now I' vary over all finite closed subschemes of X' containing I, these open substacks cover $\mathscr{E}\ell\ell_{\mathscr{D},I}$. (Every vector bundle becomes stable for a sufficiently high level structure.) □

(4.9) Remark. One could also consider the scheme which is the quotient of $\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I'}$ by $\mathrm{Ker}(\mathrm{GL}_1(\mathscr{D}_{I'})\to\mathrm{GL}_1(\mathscr{D}_I))$. This is the coarse moduli scheme for the algebraic stack $\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I'}/[\mathrm{Ker}(\mathrm{GL}_1(\mathscr{D}_{I'})\to\mathrm{GL}_1(\mathscr{D}_I))]$. These coarse moduli schemes glue together and yield a coarse moduli scheme $Ell_{X,\mathscr{D},I}$ of $\mathscr{E}\ell\ell_{\mathscr{D},I}$. The morphism of algebraic stacks

$$\mathscr{E}\ell\ell_{X,\mathscr{D},I}\to Ell_{X,\mathscr{D},I}$$

is an isomorphism over $\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I}$ if $\deg(I)>0$ (see (4.3)).

(4.10) Remark. The representability and smoothness of $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ over $X'\setminus I$ may also be seen by checking Artin's conditions [De-Ra, III, Théorème 2.3], the main point being to check by deformation-theoretic methods that the morphism $\mathscr{E}\ell\ell_{\mathscr{D},I}\to X'\setminus I$ is formally smooth. We sketch the argument. Let S be the spectrum of a local artin ring and let $\bar{S}\subset S$ be the closed subscheme defined by an ideal J of square zero. Let $(\bar{\mathscr{E}}_i,\bar{j}_i,\bar{t}_i,\bar{\iota})\in\mathscr{E}\ell\ell_{\mathscr{D},I}(\bar{S})$ and let

$$\iota_0:S\to X'$$

be an extension of the "zero" $\bar{\iota}_0:\bar{S}\to X$ of $(\bar{\mathscr{E}}_i)$. We have to show that we may lift $(\bar{\mathscr{E}}_i)$ into $(\mathscr{E}_i,j_i,t_i,\iota)$ over S with zero section ι_0. For this we first note that the Frobenius $\mathrm{Frob}_S:S\to S$ factors through $\bar{S}\subset S$. Denoting by $\mathrm{Frob}_{S,\bar{S}}:S\to\bar{S}$ the resulting morphism we must have for a lifting $\mathscr{E}_i$ of $\bar{\mathscr{E}}_i$

$${}^{\tau}\mathscr{E}_i=(\mathrm{id}_X\times\mathrm{Frob}_{S,\bar{S}})^*(\bar{\mathscr{E}}_i)\,.$$

Therefore lifting $\bar{\mathscr{E}}_i$ to $\mathscr{E}_i$ is equivalent to finding an injective $\mathscr{D}$-linear homomorphism of locally free $\mathscr{O}_{X\times S}$-modules

$$t_{-1}:{}^{\tau}\mathscr{E}_{-1}\to\mathscr{E}_0$$

which lifts $\bar{t}_{-1}$ and such that $\operatorname{Coker}(t_{-1})$ is a locally free sheaf of rank d on the graph of i_0. Indeed, since $\infty \notin i_0(S)$, the rest of the data is then uniquely determined. Denoting by $\tilde{i}_0: S \to X \times S$ the section defined by the graph of i_0, giving t_{-1} is equivalent to giving

$$\tilde{i}_0{}^*({}^\tau\mathscr{E}_{-1}) \to \tilde{i}_0{}^*(\mathscr{E}_0)$$

which is $\tilde{i}_0^*(\mathscr{D})$-linear and such that the cokernel is a locally free $\mathscr{O}_S$-module of rank d which lifts the corresponding homomorphism of $\mathscr{O}_{\bar{S}}$-modules.

However $\tilde{i}_0^*(\mathscr{D})$ is an Azumaya algebra. Therefore by Morita-equivalence the problem may be considered as that of lifting a given direct summand of $\mathscr{O}_{\bar{S}}^d$ into a direct summand of $\mathscr{O}_S^d$. There is no obstruction to doing this. □

5 Moduli space: Boundedness

The aim of this section is to prove the following theorem.

(5.1) Theorem. *The algebraic stack $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ is the disjoint union of algebraic stacks of finite type over $X' \setminus I$. In fact, the substack of $\mathscr{D}$-elliptic sheaves $(\mathscr{E}_i, t_i)$ with fixed degree* $\deg(\mathscr{E}_0)$, *which is open and closed in $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ is of finite type over $X' \setminus I$, and actually is for $I \neq \emptyset$ a quasi-projective scheme.*

For the proof of this theorem we inspect the proof of the existence of $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ in Sect. 4. We had represented $\mathscr{E}\ell\ell_{X,\mathscr{D},I}$ as an increasing union of open substacks of the form

$$\mathscr{E}\ell\ell^{\mathrm{st}}_{X,\mathscr{D},I'}/[\operatorname{Ker}(\mathrm{GL}_1(\mathscr{D}_{I'}) \to \mathrm{GL}_1(\mathscr{D}_I))]$$

where we may take the finite subschemes I' to range over the finite subschemes (ordered by inclusion) with $\operatorname{supp}(I') = \operatorname{supp}(I)$. However, for $I \neq \emptyset$, there are no automorphisms, this action is free and the quotient is in fact a scheme quasi-projective over $X' \setminus I$. Therefore Theorem 5.1 is a consequence of the following result.

(5.2) Theorem. *There exists a constant c with the following property. Let $I \subset X'$ be a finite closed subscheme of degree $> c$. Let L be an algebraically closed field and let $(\mathscr{E}_i, j_i, t_i, \iota)$ be a $\mathscr{D}$-elliptic sheaf with level-I-structure over* $\operatorname{Spec}(L)$. *Then $\mathscr{E}_0$ is stable as a vector bundle with level-I-structure.*

(5.3) For the proof of Theorem 5.2 we need some general results from the theory of vector bundles on a smooth projective curve X over an algebraically closed field L which we proceed to recall. The term "vector bundle" will here be used as synonymous for locally free $\mathscr{O}_X$-module of finite rank. By a subbundle of a vector bundle $\mathscr{E}$ we understand an $\mathscr{O}_X$-submodule $\mathscr{F} \subset \mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is again a vector bundle. Recall that there is a one-to-one correspondence between the subbundles of $\mathscr{E}$ and the sub-$L(X)$-vector spaces of the generic fibre $\mathscr{E}_\eta$: to a subvector space $V \subset \mathscr{E}_\eta$ we associate the maximal submodule $\mathscr{F}$ of $\mathscr{E}$ with generic fibre equal to V. (The stalk of $\mathscr{F}$ at a point x of X is equal to $V \cap \mathscr{E}_x$–intersection inside $\mathscr{E}_\eta$). Recall [Gr, Sect. 3] that the *slope* of a non zero vector bundle $\mathscr{E}$ is defined as

$$\mu(\mathscr{E}) = \frac{\deg \mathscr{E}}{\operatorname{rk} \mathscr{E}}$$

and that a vector bundle $\mathscr{E}$ is called *semi-stable* if for all non zero subbundles $\mathscr{F}$ of $\mathscr{E}$

$$\mu(\mathscr{F}) \leqq \mu(\mathscr{E}) .$$

A remark which will be used repeatedly below is that if $\mathscr{E} \subset \mathscr{E}'$ is an inclusion of non zero vector bundles of the same rank then

$$\deg \mathscr{E}' = \deg \mathscr{E} + \dim(\mathscr{E}'/\mathscr{E})$$

and hence

$$\mu(\mathscr{E}') = \mu(\mathscr{E}) + \frac{\dim(\mathscr{E}'/\mathscr{E})}{\operatorname{rk} \mathscr{E}'} .$$

(5.4) Let $\mathscr{E}$ be a vector bundle on X. Then there exists a unique filtration of $\mathscr{E}$ by subbundles

$$(0) = \mathscr{E}^{(0)} \subset \mathscr{E}^{(1)} \subset \dots \quad \dots \subset \mathscr{E}^{(r)} = \mathscr{E}$$

with the following two properties

(i) $\mathscr{E}^{(j)}/\mathscr{E}^{(j-1)}$ is semi-stable for all $j = 1, \dots, r$
(ii) $\mu(\mathscr{E}^{(j)}/\mathscr{E}^{(j-1)}) > \mu(\mathscr{E}^{(j+1)}/\mathscr{E}^{(j)})$ for all $j = 1, \dots, r-1$.

This filtration is called the *Harder-Narasimhan filtration* or *canonical filtration* of $\mathscr{E}$ [Gr]. This filtration also has the following two properties [Gr, Proposition 3.3]. Put for an arbitrary vector bundle $\mathscr{E} \neq (0)$

$$\mu_{\max}(\mathscr{E}) = \max\{\mu(\mathscr{F}); (0) \neq \mathscr{F} \subset \mathscr{E}\}$$

$$\mu_{\min}(\mathscr{E}) = \min\{\mu(\mathscr{E}/\mathscr{F}); \mathscr{F} \subsetneqq \mathscr{E}\}$$

(iii) $\mathscr{E}^{(j)}/\mathscr{E}^{(j-1)}$ is the largest subbundle of $\mathscr{E}/\mathscr{E}^{(j-1)}$ with slope equal to $\mu_{\max}(\mathscr{E}/\mathscr{E}^{(j-1)})$.
(iv) $\mathscr{E}^{(j)}/\mathscr{E}^{(j-1)}$ is the largest quotient bundle of $\mathscr{E}^{(j)}$ with slope equal to $\mu_{\min}(\mathscr{E}^{(j)})$.

We also have to use the following result.

(v) Let $\mathscr{F} \subset \mathscr{E}$, $\mathscr{F} \neq (0)$, $\mathscr{E}$ be a subbundle with

$$\mu_{\max}(\mathscr{E}/\mathscr{F}) < \mu_{\min}(\mathscr{F}) .$$

Let

$$(0) = \mathscr{E}^{(0)} \subset \mathscr{E}^{(1)} \subset \dots \subset \mathscr{E}^{(s)} = \mathscr{F}$$

and

$$(0) = \mathscr{E}^{(s)}/\mathscr{F} \subset \mathscr{E}^{(s+1)}/\mathscr{F} \subset \dots \subset \mathscr{E}^{(r)}/\mathscr{F} = \mathscr{E}/\mathscr{F}$$

be the canonical filtrations of $\mathscr{F}$ and $\mathscr{E}/\mathscr{F}$ respectively. Then

$$(0) = \mathscr{E}^{(0)} \subset \mathscr{E}^{(1)} \subset \dots \subset \mathscr{E}^{(s-1)} \subset \mathscr{E}^{(s)} = \mathscr{F} \subset \mathscr{E}^{(s+1)} \subset \dots \subset \mathscr{E}^{(r)} = \mathscr{E}$$

is the canonical filtration of $\mathscr{E}$.

Let $\mathscr{E}$ be a vector bundle and $\mathscr{F} \subset \mathscr{E}$ a subbundle. We introduce

$$\operatorname{jump}_{\mathscr{E}}(\mathscr{F}) = \mu_{\min}(\mathscr{F}) - \mu_{\max}(\mathscr{E}/\mathscr{F}) .$$

A consequence of the previous results is

(vi) A subbundle $\mathscr{F}$ of $\mathscr{E}$ appears in the canonical filtration of $\mathscr{E}$ if and only if

$$\operatorname{jump}_{\mathscr{E}}(\mathscr{F}) > 0 .$$

(5.5) Let $\mathscr{L}$ be a line bundle on X, i.e. an invertible $\mathscr{O}_X$-module. Then for a non zero vector bundle $\mathscr{E}$

$$\mu(\mathscr{E} \otimes \mathscr{L}) = \mu(\mathscr{E}) + \deg \mathscr{L} ,$$

and for a subbundle $\mathscr{F}$ of $\mathscr{E}$

$$\operatorname{jump}_{\mathscr{E}}(\mathscr{F}) = \operatorname{jump}_{\mathscr{E} \otimes \mathscr{L}}(\mathscr{F} \otimes \mathscr{L}) .$$

In particular, if $(0) = \mathscr{E}^{(0)} \subset \mathscr{E}^{(1)} \subset \ldots \subset \mathscr{E}^{(r)} = \mathscr{E}$ is the canonical filtration of $\mathscr{E}$, then

$$(0) = \mathscr{E}^{(0)} \otimes \mathscr{L} \subset \mathscr{E}^{(1)} \otimes \mathscr{L} \subset \ldots \subset \mathscr{E}^{(r)} \otimes \mathscr{L} = \mathscr{E} \otimes \mathscr{L}$$

is the canonical filtration of $\mathscr{E} \otimes \mathscr{L}$.

(5.6) We now return to the proof of Theorem 5.2. We choose a finite set $d_1, \ldots d_R \in \Gamma(X \setminus \{\infty\}, \mathscr{D})$ which generates this $\Gamma(X \setminus \{\infty\}, \mathscr{O}_X)$-module. Since the orders of the poles are bounded we find a constant $t \geqq 1$ with

$$d_j \in \Gamma(X, \mathscr{D}(t \cdot \infty)) \quad (j = 1, \ldots, R) .$$

We now consider a $\mathscr{D}$-elliptic sheaf $(\mathscr{E}_i, t_i)$ over $\operatorname{Spec}(L)$. Since the field L plays no role in the arguments we drop it from the notation. Let us set $m := \deg(\infty)$. Since $\mathscr{E}_{i+d \cdot m} = \mathscr{E}_i(\infty)$ we find

$$d_j \cdot \mathscr{E}_i \subset \mathscr{E}_{i+tdm}$$

for all $j = 1, \ldots, R$ and $i \in \mathbb{Z}$.

(5.7) Proposition. *Let $(\mathscr{E}_i, t_i)$ be a $\mathscr{D}$-elliptic sheaf. Then for every $i \in \mathbb{Z}$ and every non zero proper subbundle $\mathscr{F} \subset \mathscr{E}_i$ one has*

$$\operatorname{jump}_{\mathscr{E}_i}(\mathscr{F}) \leqq (t + 2) \cdot d^2 \cdot m .$$

Proof. We argue by contradiction and assume that there is an i_o and a non-zero proper subbundle $\mathscr{F}$ of $\mathscr{E} = \mathscr{E}_{i_o}$ with

$$\operatorname{jump}_{\mathscr{E}}(\mathscr{F}) > (t + 2) \cdot d^2 \cdot m .$$

Let $\mathscr{F}_i \subset \mathscr{E}_i$ be the subbundle with generic fibre $\mathscr{F}_\eta$ (cf. (5.3)). By making use of the remark made at the end of (5.3) it follows easily that for i with $i_o \leqq i \leqq i_o + dm - 1$

$$|\operatorname{jump}_{\mathscr{E}_i}(\mathscr{F}_i) - \operatorname{jump}_{\mathscr{E}}(\mathscr{F})| \leqq 2 \cdot d^2 \cdot m .$$

Here we used that $\dim(\mathscr{E}_i / \mathscr{E}_{i-1}) = d$. Using the periodicity of the $\mathscr{E}_i$ and the remark on tensoring with a line bundle in (5.5) we conclude that this estimate holds for all $i \in \mathbb{Z}$. Using our assumption on $\mathscr{F} \subset \mathscr{E}$ we conclude that for all i

$$\operatorname{jump}_{\mathscr{E}_i}(\mathscr{F}_i) > t \cdot d^2 \cdot m .$$

In particular (cf. (5.4) (vi)), $\mathscr{F}_i$ occurs in the canonical filtration of $\mathscr{E}_i$.

(5.8) Lemma. *$\mathscr{F}_i$ is stable under the operation of $\mathscr{D}$ on $\mathscr{E}_i$.*

Proof. Consider the multiplication maps

$$(\cdot d_j) : \mathscr{E}_i \to \mathscr{E}_{i+tdm} .$$

Consider the canonical filtrations of $\mathscr{E}_i$ and $\mathscr{E}_{i+tdm}$

$$(0)\subset\mathscr{E}_i^{(1)}\subset\ldots\subset\mathscr{E}_i^{(r)}=\mathscr{E}_i$$

$$(0)\subset\mathscr{E}_{i+tdm}^{(1)}\subset\ldots\subset\mathscr{E}_{i+tdm}^{(s)}=\mathscr{E}_{i+tdm}\,.$$

We know that $\mathscr{F}_i$ and $\mathscr{F}_{i+tdm}$ appear in these filtrations,

$$\mathscr{F}_i=\mathscr{E}_i^{(\alpha)},\quad \mathscr{F}_{i+tdm}=\mathscr{E}_{i+tdm}^{(\beta)}\,.$$

Let $\tilde{\mathscr{F}}_i\subset\mathscr{E}_i$ be the subbundle with the same generic fibre as $\mathscr{E}_{i+tdm}^{(\beta+1)}$. Then $\mathscr{F}_i\subset\tilde{\mathscr{F}}_i$ and

$$|\mu(\tilde{\mathscr{F}}_i/\mathscr{F}_i)-\mu(\mathscr{E}_{i+tdm}^{(\beta+1)}/\mathscr{E}_{i+tdm}^{(\beta)})|\leqq td^2m$$

again using the fact that $\dim(\mathscr{E}_{i+1}/\mathscr{E}_i)=d$. On the other hand, by property (5.4) (iii) of the canonical filtration,

$$\begin{aligned}\mu(\tilde{\mathscr{F}}_i/\mathscr{F}_i)&\leqq\mu(\mathscr{E}_i^{(\alpha+1)}/\mathscr{E}_i^{(\alpha)})\\&<\mu(\mathscr{E}_i^{(\alpha)}/\mathscr{E}_i^{(\alpha-1)})-td^2m\end{aligned}$$

by our estimate on the *jump* at $\mathscr{F}_i$. Taking these two estimates together we obtain

$$\mu(\mathscr{E}_{i+tdm}^{(\beta+1)}/\mathscr{E}_{i+tdm}^{(\beta)})<\mu(\mathscr{E}_i^{(\alpha)}/\mathscr{E}_i^{(\alpha-1)})\,.$$

Using property (5.4) (ii) of the canonical filtration it follows for all terms preceding $\mathscr{F}_i=\mathscr{E}_i^{(\alpha)}$ and all terms following $\mathscr{F}_{i+tdm}^{(\beta+1)}$ that

$$\mu(\mathscr{E}_i^{(l)}/\mathscr{E}_i^{(l-1)})>\mu(\mathscr{E}_{i+tdm}^{(k)}/\mathscr{E}_{i+tdm}^{(k-1)})$$

($l=1,\ldots,\alpha$ and $k=\beta+1,\ldots,s$). Now we use the fact that for semi-stable bundles $\mathscr{E}$ and $\mathscr{F}$ with $\mu(\mathscr{E})>\mu(\mathscr{F})$ we have

$$\operatorname{Hom}(\mathscr{E},\mathscr{F})=0\,.$$

By an easy induction (ascending in l and descending in k) we conclude

$$d_j\cdot\mathscr{E}_i^{(\alpha)}\subset\mathscr{E}_{i+tdm}^{(\beta)}\,,$$

i.e.

$$d_j\cdot\mathscr{F}_i\subset\mathscr{F}_{i+tdm}\quad(j=1,\ldots,R)\,.$$

Since the elements d_j generate D over the function field it follows that $(\mathscr{F}_i)_\eta$ is a D-stable subspace of $(\mathscr{E}_i)_\eta$. Since $\mathscr{E}_i$ is $\mathscr{D}$-stable it therefore follows that $\mathscr{F}_i$ is $\mathscr{D}$-stable as well. The lemma is proved. □

(5.9) We now consider the behaviour of $\mathscr{F}_i$ under the pull back by Frobenius. Since the pull back of a semi-stable bundle under Frobenius is again semi-stable and with the same slope it follows from the characterization of the canonical filtration ((5.4) (i), (ii)) that the pull back of the canonical filtration is the canonical filtration of the pull back. It follows that ${}^\tau\mathscr{F}_i$ occurs in the canonical filtration of ${}^\tau\mathscr{E}_i$: ${}^\tau\mathscr{F}_i={}^\tau(\mathscr{E}_i^{(\alpha)})={}^\tau\mathscr{E}_i^{(\alpha)}$. Let γ be such that $\mathscr{F}_{i+1}=\mathscr{E}_{i+1}^{(\gamma)}$. Similar arguments to those used in the proof of Lemma 5.8 show now that

$$\mu({}^\tau\mathscr{E}_i^{(\alpha)}/{}^\tau\mathscr{E}_i^{(\alpha-1)})=\mu(\mathscr{E}_i^{(\alpha)}/\mathscr{E}_i^{(\alpha-1)})>\mu(\mathscr{E}_{i+1}^{(\gamma+1)}/\mathscr{E}_{i+1}^{(\gamma)})\,.$$

We again conclude as in the proof of Lemma 5.8 that

$$t_i({}^\tau\mathscr{F}_i)\subset\mathscr{F}_{i+1}\,.$$

(5.10) We now claim that the inclusions $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ are strict for all i. Indeed $\mathscr{F}_{i+1}/\mathscr{F}_i$ and $\mathscr{F}_{i+1}/t_i({}^\tau\mathscr{F}_i)$ are torsion sheaves with disjoint supports. Therefore

$$\mathscr{F}_{i+1} = \mathscr{F}_i + t_i({}^\tau\mathscr{F}_i) \quad \text{for all } i\,.$$

If now $\mathscr{F}_i = \mathscr{F}_{i+1}$ for some i, then it would follow that

$$\begin{aligned}\mathscr{F}_{i+2} &= \mathscr{F}_{i+1} + t_{i+1}({}^\tau\mathscr{F}_{i+1})\\ &= \mathscr{F}_i + t_i({}^\tau\mathscr{F}_i) = \mathscr{F}_{i+1} = \mathscr{F}_i\end{aligned}$$

and similarly for $i + 3, \ldots$. This contradicts the equality

$$\mathscr{F}_{i+dm} = \mathscr{F}_i(\infty)\,,$$

since $\mathscr{F}_i \neq (0)$. Since $\mathscr{E}_{i+1}/\mathscr{E}_i$ is a simple $\mathscr{D} \otimes \kappa(\infty)$-module we therefore conclude that $\mathscr{F}_{i+1}/\mathscr{F}_i = \mathscr{E}_{i+1}/\mathscr{E}_i$ and

$$\dim(\mathscr{F}_{i+1}/\mathscr{F}_i) = d$$

for all i. It follows that

$$\dim(\mathscr{F}_{i+dm}/\mathscr{F}_i) = d^2 m\,.$$

On the other hand,

$$\dim(\mathscr{F}_{i+dm}/\mathscr{F}_i) = \dim(\mathscr{F}_i(\infty)/\mathscr{F}_i) = \mathrm{rk}(\mathscr{F}_i)\cdot m\,.$$

It follows that $\mathrm{rk}(\mathscr{F}_i) = d^2 = \mathrm{rk}(\mathscr{E}_i)$, hence $\mathscr{F}_i = \mathscr{E}_i$ and this contradiction proves the Proposition 5.7.

Proof of Theorem 5.2 As our constant c we take

$$c = (d^2 - 1)^2 \cdot (t + 2) \cdot d^4 m\,.$$

Let I be a finite closed subscheme of degree $> c$. Let $\mathscr{F}$ be a non zero proper subbundle of $\mathscr{E}_0$. We have to show that

$$\frac{\deg \mathscr{F} - \deg I}{\mathrm{rk}\,\mathscr{F}} < \frac{\deg \mathscr{E}_0 - \deg I}{\mathrm{rk}\,\mathscr{E}_0}\,,$$

i.e. that

$$\mu(\mathscr{F}) - \mu(\mathscr{E}_0) < \deg I \cdot \left(\frac{1}{\mathrm{rk}\,\mathscr{F}} - \frac{1}{d^2}\right).$$

However $\mu(\mathscr{F}) \leqq \mu_{\max}(\mathscr{E}_0) = \mu(\mathscr{E}_0{}^{(1)})$, the slope of the first term in the canonical filtration of $\mathscr{E}_0$. Using the trivial estimate

$$\deg I \left(\frac{1}{\mathrm{rk}\,\mathscr{F}} - \frac{1}{d^2}\right) > c \cdot \frac{1}{d^2(d^2 - 1)} = (d^2 - 1)(t + 2) d^2 m$$

we therefore are reduced to proving

$$\mu(\mathscr{E}_0^{(1)}) - \mu(\mathscr{E}_0) \leqq (d^2 - 1)(t + 2) \cdot d^2 m\,.$$

However, if $(0) = \mathscr{E}_0^{(0)} \subset \mathscr{E}_0^{(1)} \subset \ldots \subset \mathscr{E}_0^{(r)} = \mathscr{E}_0$ is the canonical filtration of $\mathscr{E}_0$ we have

$$\mu(\mathscr{E}_0^{(1)}) \geqq \mu(\mathscr{E}_0) \geqq \mu(\mathscr{E}_0/\mathscr{E}_0^{(r-1)})$$

and

$$\begin{aligned}\mu(\mathscr{E}_0^{(1)}) - \mu(\mathscr{E}_0) &\leqq \mu(\mathscr{E}_0^{(1)}) - \mu(\mathscr{E}_0/\mathscr{E}_0^{(r-1)}) \\ &= \sum_{j=1}^{r-1} (\mu(\mathscr{E}_0^{(j)}/\mathscr{E}_0^{(j-1)}) - \mu(\mathscr{E}_0^{(j+1)}/\mathscr{E}_0^{(j)})) \\ &= \sum_{j=1}^{r-1} \operatorname{jump}_{\mathscr{E}_0}(\mathscr{E}_0^{(j)}) \\ &\leqq (d^2 - 1)\cdot(t+2)d^2 m .\end{aligned}$$

Here we used the fact that r is at most $d^2 = \operatorname{rk}(\mathscr{E}_0)$ and the estimate of Proposition 5.7. The theorem is proved. □

6 The valuative criterion of properness

Our aim in this section is to prove the following theorem.

(6.1) Theorem. *Assume that the algebra D is a division algebra. Then the morphism* (*cf.* (2.4))

$$\mathscr{E}\ell\ell_{X,\mathscr{D}}/\mathbb{Z} \to X' = X \setminus \{\infty\} \setminus \mathrm{Bad}$$

is proper.

Since the natural morphism $\mathscr{E}\ell\ell_{X,\mathscr{D},I} \to \mathscr{E}\ell\ell_{X,\mathscr{D}}$ when restricted over $X' \setminus I$ is finite, we may formulate the following corollary.

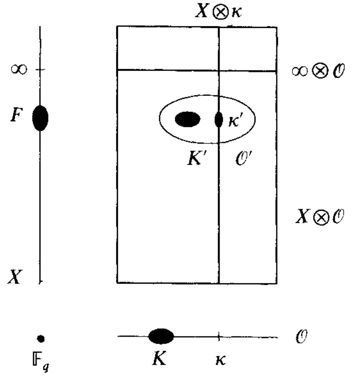
(6.2) Corollary. *Let I be a finite closed subscheme contained in* $X' = X \setminus \{\infty\}$. *The morphism*

$$\mathscr{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z} \to X' \setminus I$$

is proper.

We shall prove Theorem 6.1 by checking the valuative criterion of properness. Consider the open substack of $\mathscr{D}$-elliptic sheaves $(\mathscr{E}_i, j_i, t_i)$ with $0 \leqq \chi(X_s, (\mathscr{E}_0)_s) \leqq d^2 \deg(\infty) - 1$. The tensor operation by $\mathscr{O}_X(\infty)$ (cf. (2.9)) allows us to present $\mathscr{E}\ell\ell_{X,\mathscr{D}}$ as a disjoint sum of copies enumerated by the integers of this open substack. Therefore, by Sect. 5, it follows that $\mathscr{E}\ell\ell_{X,\mathscr{D}}/\mathbb{Z}$ is of finite type over X'. From its construction (Sect. 4) it follows that $\mathscr{E}\ell\ell_{X,\mathscr{D}}$ is an increasing union of separated open substacks, hence is separated and therefore the morphism $\mathscr{E}\ell\ell_{X,\mathscr{D}} \to X'$ is separated.

(6.3) We introduce the following notations. Let $\mathscr{O} \supseteq \mathbb{F}_q$ be a complete discrete valuation ring with perfect residue field κ. Let K be its field of fractions and denote by ϖ a uniformizing element. Let $\mathscr{O}'$ be the local ring of the generic point of the curve $X \otimes \kappa$, considered as a point of the scheme $X \otimes \mathscr{O}$. Then $\mathscr{O}'$ is a discrete valuation ring (not complete), with ϖ as uniformizing element. The residue field $\kappa' = \operatorname{Frac}(F \otimes \kappa)$ of $\mathscr{O}'$ is the function field of $X \otimes \kappa$ and the field of fractions $K' = \operatorname{Frac}(F \otimes K)$ is the function field of $X \otimes K$.



(6.4) Since the direct image and inverse image functors for the inclusion induce an equivalence of the categories of locally free sheaves on the punctured and unpunctured spectrum of a two-dimensional regular local ring we have an equivalence of categories between the category of locally free sheaves $\tilde{\mathscr{F}}$ on $X \otimes \mathcal{O}$ and the category of pairs $(\mathscr{F}, N)$ where $\mathscr{F}$ is a locally free sheaf on $X \otimes K$ and where N is an $\mathcal{O}'$-lattice in $\mathscr{F}_{K'}$ (cf. [Dr 8, 3.1]). (To $\tilde{\mathscr{F}}$ corresponds $\mathscr{F} = \tilde{\mathscr{F}} | X \otimes K$ and $N = H^0(\mathrm{Spec}(\mathcal{O}'), \tilde{\mathscr{F}})$.) An $\mathcal{O}'$-lattice in a finite dimensional K'-vector space is a $\mathcal{O}'$-submodule of finite type containing a K'-basis.

(6.5) Let $(\mathscr{E}_i, j_i, t_i)$ be a $\mathscr{D}$-elliptic sheaf over K. Then $(\mathscr{E}_i)_{K'}$ is independent of i and will be denoted by V. The morphisms t_i induce an $(\mathrm{id}_F \otimes \mathrm{Frob}_K)$-linear endomorphism φ of V with $K' \cdot \varphi(V) = V$. Then V is a free $D \otimes_F K'$-module of rank 1 and φ is D-linear. Our aim will be to construct $\mathcal{O}'$-lattices adapted to φ. Let V be any finite-dimensional K'-vector space with a $(\mathrm{id}_F \otimes \mathrm{Frob}_K)$-linear endomorphism φ with $K' \cdot \varphi(V) = V$.

We recall [Dr 8,3.1] that Drinfeld calls an $\mathcal{O}'$-lattice $M \subset V$ *admissible* if $\varphi(M) \subset M$ and if the induced endomorphism

$$\bar{\varphi} : M/\varpi M \to M/\varpi M$$

is not nilpotent, i.e. if $\varphi^{\dim(V)} M \not\subset \varpi M$. Note that a φ-invariant lattice is *not* admissible if and only if $\varphi^n(M) \to 0$ as $n \to \infty$ (i.e. for any lattice N there exists an n_0 such that $\varphi^n(M) \subset N$ for $n \geq n_0$).

(6.6) Proposition (Drinfeld [Dr8, 3.2]). (i) *There exists a φ-invariant lattice $M_0 \subset V$ containing all other φ-invariant lattices. If M_0 is not admissible then there are no admissible lattices in V.*
(ii) *After replacing K by a finite extension L (and V by $V \otimes_K L$ and φ by $\varphi \otimes_{\mathrm{Frob}_K} \mathrm{Frob}_L$) there exist admissible lattices in V.*

Remark. In fact the proof of Drinfeld is formulated only in the case when $\dim(V) = 2$ (the case of interest to him) but it is perfectly valid in general.

(6.7) Let now $(\mathscr{E}_i, j_i, t_i)$ be a $\mathscr{D}$-elliptic sheaf over K and consider (V, φ) as before. After replacing K by a finite extension, we may assume that the maximal φ-invariant lattice $M_0 \subset V$ is admissible. Since φ is D-linear it follows from the

maximality of M_0 that M_0 is D-stable. Using now the equivalence of categories mentioned in (5.4) we obtain a commutative diagram of homomorphisms of locally free sheaves on $X \otimes_{\mathbb{F}_q} \mathcal{O}$ (note that ${}^{\tau}\tilde{\mathscr{E}}_i = ({}^{\tau}\mathscr{E}_i)^{\sim}$.)

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{\mathscr{E}}_i \arrow[r, "\tilde{j}_i"'] & \tilde{\mathscr{E}}_{i+1} \arrow[r] & \tilde{\mathscr{E}}_{i+2} \arrow[r] & \cdots \\
\cdots \arrow[r] & {}^{\tau}\tilde{\mathscr{E}}_i \arrow[r] \arrow[ur, "\tilde{t}_i"'] & {}^{\tau}\tilde{\mathscr{E}}_{i+1} \arrow[r] \arrow[ur, "\tilde{t}_{i+1}"'] & {}^{\tau}\tilde{\mathscr{E}}_{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

Since $(\mathscr{D} \otimes K)^{\sim} = \mathscr{D} \otimes \mathcal{O}$, these sheaves are equipped with a $\mathscr{D}$-action and all homomorphisms are $\mathscr{D}$-linear. Furthermore, $\tilde{\mathscr{E}}_{i+d\deg(\infty)} = \tilde{\mathscr{E}}_i(\infty)$. Hence the homomorphisms $\tilde{j}_i$ are injective and the argument in (2.3.b), counting the dimensions of the representations $(\tilde{\mathscr{E}}_i/\tilde{\mathscr{E}}_{i-1}) \otimes \kappa$ of $\mathbb{M}_d(\kappa(\infty))$, implies that these dimensions are all equal to d and therefore $\tilde{\mathscr{E}}_i/\tilde{\mathscr{E}}_{i-1}$ is a free $\mathcal{O}$-module of rank d.

(6.8) We now assume in addition that the zero morphism $i_0 : \operatorname{Spec} K \to X'$ extends to a morphism $\tilde{i}_0 : \operatorname{Spec} \mathcal{O} \to X'$. We distinguish two cases

First case $\qquad \mathcal{O}' \cdot \varphi(M_0) = M_0$

Second case $\qquad \mathcal{O}' \cdot \varphi(M_0) \subsetneqq M_0$.

(6.9) Lemma. *In the first case $(\tilde{\mathscr{E}}_i, \tilde{j}_i, \tilde{t}_i)$ is a $\mathscr{D}$-elliptic sheaf over* Spec $\mathcal{O}$. *If D is a division algebra the second case does not occur.*

Proof. Assume that we are in the first case. It only has to be checked that Coker $\tilde{t}_i$ is a locally free sheaf on the graph of $\tilde{i}_0$. However, the stalk at the generic point $\operatorname{Spec}(\kappa')$ of $X \otimes \kappa$ of Coker $\tilde{t}_i$ is equal to the cokernel of the induced endomorphism of $\bar{M}_0 = M_0/\varpi \cdot M_0$,

$$\bar{\varphi} : \bar{M}_0 \to \bar{M}_0 .$$

Since we are in the first case and using the fact that κ is perfect, $\bar{\varphi}$ is surjective. Hence $\operatorname{Spec}(\kappa') \notin \operatorname{Supp} \operatorname{Coker} \tilde{t}_i$, which implies that Supp Coker $\tilde{t}_i$ is the graph of $\tilde{i}_0$. Comparing Euler-Poincaré characteristics

$$\begin{aligned} \dim_\kappa \operatorname{Coker} \tilde{t}_i \otimes \kappa = \chi(\tilde{\mathscr{E}}_{i+1} \otimes \kappa) - \chi({}^{\tau}\tilde{\mathscr{E}}_i \otimes \kappa) &= \chi(\mathscr{E}_{i+1}) - \chi({}^{\tau}\mathscr{E}_i) \\ &= \dim_K \operatorname{Coker} t_i , \end{aligned}$$

we conclude that Coker $\tilde{t}_i$ is a flat $\mathcal{O}$-module of finite rank, which proves our claim in this case.

We now assume that we are in the second case. Then the endomorphism $\bar{\varphi}$ of the κ'-vector space $\bar{M}_0$ (recall that $\bar{\varphi}$ is $\mathrm{id}_F \otimes \mathrm{Frob}_\kappa$-linear) is neither surjective nor nilpotent.

We consider the flag of κ'-subvector spaces

$$\bar{M}_0 \supsetneqq \operatorname{Im}(\bar{\varphi}) \supsetneqq \operatorname{Im}(\bar{\varphi}^2) \supsetneqq \ldots \supsetneqq \operatorname{Im}(\bar{\varphi}^n) = \operatorname{Im}(\bar{\varphi}^{n+1}) = \ldots \supsetneqq (0) .$$

The sequence becomes stationary with non-trivial end term.

On the other hand, $\operatorname{Im}(\bar{\varphi}^i)/\operatorname{Im}(\bar{\varphi}^{i+1})$ is a $D \otimes_F \kappa'$-module, hence its dimension as a vector space over $\kappa' = F \otimes \kappa$ is divisible by d. Hence it follows that $n \leqq d - 1$, and that putting $\bar{N} = \operatorname{Im}(\bar{\varphi}^n)$,

$$\dim \bar{N} = rd, \quad 0 < r < d .$$

Let $\bar{\mathscr{E}}_i$ be the restriction of $\bar{\tilde{\mathscr{E}}}_i$ to the special fibre $X \otimes \kappa$. Its stalk at the generic point is $\bar{M}_0$. Let

$$\bar{\mathscr{F}}_i \subset \bar{\mathscr{E}}_i$$

be the locally free $\mathcal{O}_{X \otimes \kappa}$-submodule generated by $\bar{N}$, i.e. the maximal locally free $\mathcal{O}_{X \otimes \kappa}$-submodule of $\bar{\mathscr{E}}_i$ with generic stalk equal to $\bar{N} \subset \bar{M}_0$. (The stalk of $\bar{\mathscr{F}}_i$ at a point x' of $X \otimes \kappa$ is equal to $(\bar{\mathscr{E}}_i)_{x'} \cap \bar{N}$–intersection inside $\bar{M}_0$.) By the maximality property of $\bar{\mathscr{F}}_i$ this is a $\mathscr{D} \otimes \kappa$-submodule of $\bar{\mathscr{E}}_i$ and we have a cartesian square

$$\begin{tikzcd}
\bar{\mathscr{E}}_i \arrow[r, hook] & \bar{\mathscr{E}}_{i+1} \\
\bar{\mathscr{F}}_i \arrow[r, hook] \arrow[u, hook] & \bar{\mathscr{F}}_{i+1} \arrow[u, hook]
\end{tikzcd}$$

and $\bar{\mathscr{F}}_{i-d\cdot\deg(\infty)} = \bar{\mathscr{F}}_i(-\infty)$. The successive quotients $\bar{\mathscr{F}}_i/\bar{\mathscr{F}}_{i-1}$ are $(\mathscr{D} \otimes \kappa(\infty)) \otimes \kappa$-modules. Counting dimensions and taking into account that

$$r \cdot d \cdot \deg(\infty) = \dim(\bar{\mathscr{F}}_i/\bar{\mathscr{F}}_i(-\infty)) < \dim(\bar{\mathscr{E}}_i/\bar{\mathscr{E}}_i(-\infty)) = d^2 \cdot \deg(\infty)$$

we conclude that there exists an index i with $\bar{\mathscr{F}}_i = \bar{\mathscr{F}}_{i+1}$. Consider the following commutative diagram

$$\begin{tikzcd}
& \bar{\mathcal{E}}_i \arrow[rr] & & \bar{\mathcal{E}}_{i+1} \\
{}^{\tau}\bar{\mathcal{E}}_{i-1} \arrow[ur, "\bar{t}_{i-1}"] \arrow[rr] & & {}^{\tau}\bar{\mathcal{E}}_i \arrow[ur, "\bar{t}_i"] & \\
& \bar{\mathcal{F}}_i \arrow[uu] \arrow[rr, equal] & & \bar{\mathcal{F}}_{i+1} \arrow[uu] \\
{}^{\tau}\bar{\mathcal{F}}_{i-1} \arrow[uu] \arrow[ur, dashed] \arrow[rr] & & {}^{\tau}\bar{\mathcal{F}}_i \arrow[uu] \arrow[ur, dashed] &
\end{tikzcd}$$

Here the broken oblique arrows arise from the maximality properties of $\bar{\mathscr{F}}_i$ and $\bar{\mathscr{F}}_{i+1}$. We therefore obtain a homomorphism

$$t : {}^{\tau}\bar{\mathscr{F}}_i \rightarrow \bar{\mathscr{F}}_i$$

whose stalk at the generic point is equal to

$$\bar{\varphi} | \bar{N} : \bar{N} \rightarrow \bar{N} .$$

This last homomorphism is bijective, therefore t is injective. Since $\deg(\bar{\mathscr{F}}_i) = \deg({}^{\tau}\bar{\mathscr{F}}_i)$, we conclude that t is an isomorphism. Applying now Drinfeld's Galois descent lemma [Dr 7, Proposition 1.1]) we conclude that $\bar{\mathscr{F}}_i$ is of the form

$$\bar{\mathscr{F}}_i = \mathscr{F}' \otimes \kappa ,$$

where $\mathcal{F}'$ is a locally free sheaf on X. Furthermore $\mathcal{F}'$ is a $\mathcal{D}$-module and its rank over $\mathcal{O}_X$ is equal to $r \cdot d = \dim \bar{N}$. Then the generic stalk $\mathcal{F}'_F$ is a D-module of dimension $rd < d^2$ over F. If D is a division algebra such a module cannot exist and this contradiction proves the lemma. □

Therefore, $\mathcal{E}\ell\ell_{X,\mathcal{D}}/\mathbb{Z} \to X'$ satisfies the valuative criterion of properness and the Theorem 6.1 is proved.

7 Hecke correspondences

(7.1) Let T be a finite set of places containing $\{\infty\}$. We form

$$\mathcal{E}\ell\ell^T_{X,\mathcal{D}} = \varprojlim_{I \cap T = \emptyset} \mathcal{E}\ell\ell_{X,\mathcal{D},I} .$$

This is again an algebraic stack and even a scheme. The canonical morphism $\mathcal{E}\ell\ell^T_{X,\mathcal{D}} \to X \setminus \{\infty\}$ factors through the morphism $X_{(T)} \setminus \{\infty\} \to X \setminus \{\infty\}$, where $X_{(T)} \to X$ is the localization of X along T. Let

$$\mathcal{O}^T = \prod_{x \notin T} \mathcal{O}_x \subset \mathbb{A}^T = \prod_{x \notin T}{}' (F_x, \mathcal{O}_x) .$$

We embed $F^\times$ diagonally into $(\mathbb{A}^T)^\times$.

A section of $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$ over S is equivalent to the data of a $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_i, j_i, t_i)$ over S and *a* $\mathcal{D}$-linear isomorphism

$$\iota^T : (\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}^T) \boxtimes \mathcal{O}_S \xrightarrow{\sim} \mathcal{E}_{\bar{i}\,\mathcal{O}_{X \times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S)$$

with an obvious compatibility with its pullback under τ. There is an obvious right action of $(\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}^T)^\times$ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$ (composition with ι^T of the action of $(\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}^T)^\times$ on $\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}^T$ by left multiplication).

(7.2) Let $\mathrm{Pic}_{X,I}(\mathbb{F}_q)$ be the set of invertible sheaves on X with a level-I-structure. We had defined in (2.7) an operation of $\mathrm{Pic}_{X,I}(\mathbb{F}_q)$ on $\mathcal{E}\ell\ell_{\mathcal{D},I}$. Let

$$\mathrm{Pic}^T_X(\mathbb{F}_q) = \varprojlim_{I \cap T = \emptyset} \mathrm{Pic}_{X,I}(\mathbb{F}_q) .$$

We obtain in the same way an action of $\mathrm{Pic}^T_X(\mathbb{F}_q)$ on $\mathcal{E}\ell\ell^T_{\mathcal{D}}$. We let $(F^\times)^T$ be the set of elements of $F^\times$, which are units at all places contained in T. Denoting $\mathcal{O}^\times_T = \prod_{x \in T} \mathcal{O}^\times_x$ we have the identification

$$\begin{aligned} \mathrm{Pic}^T_X(\mathbb{F}_q) &\cong F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times_T \\ &\cong (F^\times)^T \backslash (\mathbb{A}^T)^\times \end{aligned}$$

and obtain an action of $(F^\times)^T \backslash (\mathbb{A}^T)^\times$ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$. (Under this identification, the idèle with component in $x \in |X| \setminus T$ equal to a uniformizing element ϖ_x and all other components equal to 1, corresponds to the invertible sheaf $\mathcal{O}_X(-x)$ with the obvious level structures outside x and the level structure induced by ϖ_x at x.)

(7.3) We shall define an action of the semigroup $\Gamma = (D^T)^\times \cap (\mathcal{D} \otimes \mathcal{O}^T)$ which extends the action of $(\mathcal{D} \otimes \mathcal{O}^T)^\times$. We leave to the reader to check that the actions of

Γ and $(F^\times)^T\backslash(\mathbb{A}^T)^\times$ agree on $(\mathbb{A}^T)^\times \cap \Gamma = (\mathbb{A}^T)^\times \cap \mathcal{O}^T$. So, the actions of $(\mathcal{D}\otimes\mathcal{O}^T)^\times$ and $(F^\times)^T\backslash(\mathbb{A}^T)^\times$ come from an action of $(D^T)^\times$ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$.

(7.4) Let $g = g^T \in \Gamma$ and $(\mathcal{E}_i, j_i, t_i, \iota^T) \in \mathcal{E}\ell\ell^T_{\mathcal{D}}(S)$. We wish to define $g(\mathcal{E}_i, j_i, t_i, \iota^T) =: (\mathcal{E}'_i, j'_i, t'_i, (\iota^T)')$. We have the diagrams (for all $i\in\mathbb{Z}$)

$$\begin{tikzcd}
\mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S) \arrow[d, dashed, ":=[g]"'] & (\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}^T)\boxtimes\mathcal{O}_S \arrow[l, "\sim", "\iota^T"'] \arrow[d, "g\cdot"] \\
\mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \otimes \mathcal{O}_S) & (\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}^T)\boxtimes\mathcal{O}_S \arrow[l, "\sim", "\iota^T"']
\end{tikzcd}$$

where we can fill in the broken arrow $[g]$ in a unique way. We consider the cartesian diagrams

$$\begin{tikzcd}
\mathcal{E}'_i \arrow[r, dashed, "\alpha_i"] \arrow[d, dashed, "\beta_i"'] & \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S) \arrow[d, "{[g]}"] \\
\mathcal{E}_i \arrow[r, "\text{can}"'] & \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S)\,.
\end{tikzcd}$$

They define $\mathcal{E}'_i$, $i\in\mathbb{Z}$. The definition of j'_i and t'_i is obvious. The level structure $(\iota^T)'$ is defined as the composition

$$(\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}^T)\boxtimes\mathcal{O}_S \xrightarrow[\iota^T]{\sim} \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S) \xleftarrow[\alpha_i\otimes\mathrm{id}]{\sim} \mathcal{E}'_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S)\,.$$

We therefore obtain a commutative diagram

$$\begin{tikzcd}
(\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}^T)\boxtimes\mathcal{O}_S \arrow[r, "(\iota^T)'"] \arrow[d, "g\cdot"'] & \mathcal{E}'_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S) \arrow[d, "\beta_i\otimes\mathrm{id}"] \\
(\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}^T)\boxtimes\mathcal{O}_S \arrow[r, "\iota^T"] & \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}^T \boxtimes \mathcal{O}_S)\,.
\end{tikzcd}$$

This defines the action of Γ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$. If $g\in\Gamma$ is even an element of $(\mathcal{D}\otimes\mathcal{O}^T)^\times$, the $g\cdot$ above and therefore also $[g]$ is an isomorphism, which implies that the β_i are also isomorphisms and therefore the action of $g\in(\mathcal{D}\otimes\mathcal{O}^T)^\times$ coincides with that defined in (7.1).

(7.5) Let $K^T\subset(D^T)^\times$ be an open compact subgroup and let $g^T\in(D^T)^\times$. On a finite level the above construction defines a correspondence over Spec(F)

$$\begin{tikzcd}
 & \mathcal{E}\ell\ell^T_{\mathcal{D}}/K^T\cap(g^T)^{-1}K^Tg^T \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\
\mathcal{E}\ell\ell^T_{\mathcal{D}}/K^T & & \mathcal{E}\ell\ell^T_{\mathcal{D}}/K^T \arrow[ll]
\end{tikzcd}$$

where the morphism c_1 (resp. c_2) is induced from the inclusion $K^T\cap(g^T)^{-1}K^Tg^T\subset K^T$ (resp. $K^T\cap(g^T)^{-1}K^Tg^T \xhookrightarrow{\mathrm{Ad}(g^T)} K^T$).

These morphisms are extended over Spec $\mathcal{O}_{X,x}$, $x\in X\setminus T$, as soon as the x-component of g is trivial, and are then finite and étale.

8 Level structures at infinity

In [Dr 4] Drinfeld gives a construction of a pro-Galois covering of the moduli stack of elliptic sheaves and relates it to the corresponding covering of the moduli stack of elliptic modules [Dr 2]. In this section, we show that the same construction works in our context. As a matter of fact, since the $\mathcal{O}_\infty$-algebra $\mathcal{D}_\infty$ splits, up to Morita equivalence, we are in Drinfeld's framework.

Throughout this whole section we fix a uniformizer ϖ_∞ of $\mathcal{O}_\infty$ and an isomorphism of $\mathcal{O}_\infty$-algebras $\mathcal{D}_\infty \simeq \mathbb{M}_d(\mathcal{O}_\infty)$.

(8.1) Let S be a scheme (over $\mathbb{F}_q$) and let $E = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathcal{D}$-elliptic sheaf over S. For each $i\in\mathbb{Z}$, let

$$\mathcal{E}_i^\vee = \mathcal{H}om_{\mathcal{O}_{X\times S}}(\mathcal{E}_i, \mathcal{O}_{X\times S})$$

be the dual of $\mathcal{E}_i$ and let

$$\check{M}_i = \mathcal{E}^\vee_{-i}|(X\times S)^\wedge_\infty .$$

Here $(X\times S)^\wedge_\infty$ is the completion of $X\times S$ along $\{\infty\}\times S$ and $\check{M}_i$ is a locally free $\mathcal{O}_\infty \hat{\otimes} \mathcal{O}_S$- module of constant rank d^2. The j_i's define an inductive system

$$\ldots \hookrightarrow \check{M}_i \hookrightarrow \check{M}_{i+1} \hookrightarrow \ldots$$

and we shall denote by $\check{N}$ its limit. Then $\check{N}$ is a locally free $F_\infty \hat{\otimes} \mathcal{O}_S$-module of constant rank d^2. We identify the $\check{M}_i$'s with their images in $\check{N}$. The t_i's induce compatible isomorphisms

$$\check{M}_i \xrightarrow{\sim} {}^\tau\check{M}_{i+1} = (\mathcal{O}_\infty \hat{\otimes}_{\mathbb{F}_q} \mathrm{Frob}_S)^*(\check{M}_{i+1})$$

($i\in\mathbb{Z}$) and in the limit an isomorphism

$$\check{N} \xrightarrow{\sim} {}^\tau\check{N} = (F_\infty \hat{\otimes}_{\mathbb{F}_q} \mathrm{Frob}_S)^*(\check{N})$$

which maps $\check{M}_i$ onto ${}^\tau\check{M}_{i+1}$. We define

$$\check{\psi} : {}^\tau\check{N} \to \check{N}$$

as the inverse of the above isomorphism. We have

$$\check{\psi}({}^\tau\check{M}_i) = \check{M}_{i-1} \subset \check{M}_i$$

for each $i\in\mathbb{Z}$.

The right actions of $\mathcal{D}_\mathcal{D}$ on the $\mathcal{E}_i$'s induce a left action of $\mathcal{D}_\infty$ on $\check{N}$. This action commutes with $\check{\psi}$ and, for each $i\in\mathbb{Z}$, stabilizes $\check{M}_i\subset\check{N}$. Thanks to our identification of $\mathcal{D}_\infty$ with $\mathbb{M}_d(\mathcal{O}_\infty)$, we get canonical splittings

$$(\check{N}, \check{\psi}) = (\check{N}', \check{\psi}')^d$$

and

$$\check{M}_i = (\check{M}'_i)^d .$$

Now, $\check{N}'$ is a locally free $F_\infty \hat{\otimes} \mathcal{O}_S$-module of constant rank d and $\check{M}'_i\subset\check{N}'$ is a locally free $\mathcal{O}_\infty \hat{\otimes} \mathcal{O}_S$-submodule of constant rank d, for each $i\in\mathbb{Z}$. It follows from the properties of $\mathcal{D}$-elliptic sheaves that we have

$$\varpi_\infty \check{M}'_i = \check{M}'_{i-\deg(\infty)d} \subset \ldots \subset \check{M}'_{i-1} \subset \check{M}'_i$$

$$\check{\psi}'({}^{\tau}M'_i) = {}^{\tau}M'_{i-1}\ ,$$

so that

$$\varpi_{\infty}\check{M}'_i \subset \check{\psi}'({}^{\tau}\check{M}'_i) \subset \check{M}'_i\ ,$$

and that the quotient $\kappa(\infty) \otimes \mathcal{O}_S$-module

$$\check{M}'_i/\varpi_{\infty}\check{M}'_i \twoheadrightarrow \check{M}'_i/\check{\psi}'({}^{\tau}\check{M}'_i)$$

(viewed as an $\mathcal{O}_{\{\infty\}\times S}$-module) is supported on the graph of a $\mathbb{F}_q$-morphism of schemes

$$i_{\infty, i}: S \to \{\infty\}$$

and is locally free of constant rank one on its support, for each $i \in \mathbb{Z}$. Moreover, for each $i \in \mathbb{Z}$,

$$i_{\infty, i+1} = i_{\infty, i} \circ \mathrm{Frob}_S$$

and $i_{\infty, 0}$ is the pole of E (see (2.2) and (2.3)).

(8.2) Let S be a scheme with an $\mathbb{F}_q$-morphism of schemes

$$i_{\infty, 0}: S \to \{\infty\}\ .$$

Then any other $\mathbb{F}_q$-morphism of schemes from S to $\{\infty\}$ is equal to

$$i_{\infty, i} = i_{\infty, 0} \circ \mathrm{Frob}_S^i$$

for a unique $i \in \mathbb{Z}/\deg(\infty)\mathbb{Z}$.

We will associate to $(S, i_{\infty, 0})$ as before a triple

$$(N, \psi, M) = (N_{d,1}(i_{\infty, 0}), \psi_{d,1}(i_{\infty, 0}), M_{d,1}(i_{\infty, 0}))$$

where N is a free $F_{\infty} \hat{\otimes} \mathcal{O}_S$-module of constant rank d, where

$$\psi: {}^{\tau}N \xrightarrow{\sim} N$$

is an isomorphism of $F_{\infty} \hat{\otimes} \mathcal{O}_S$-modules and where M is a free $\mathcal{O}_{\infty} \hat{\otimes} \mathcal{O}_S$-submodule of constant rank d such that

$$\varpi_{\infty} M \subset \psi({}^{\tau}M) \subset M$$

$$\psi^{d \cdot \deg(\infty)}({}^{\tau^{d \deg(\infty)}}M) = \varpi_{\infty} M$$

and such that the quotient $\kappa(\infty) \otimes \mathcal{O}_S$-module

$$M/\varpi_{\infty} M \twoheadrightarrow M/\psi({}^{\tau}M)$$

is supported on the graph of

$$i_{\infty, 0}: S \to \{\infty\}$$

and is free of constant rank one on its support.

For N we simply take

$$N = (F_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S)^d = \bigoplus_{i=0}^{\deg(\infty)-1} (F_\infty \hat{\otimes}_{\kappa(\infty), i_{\infty,i}^{\cdot}} \mathcal{O}_S)^d$$

with canonical basis $(e_{ij})_{0 \leqq i \leqq \deg(\infty)-1, 1 \leqq j \leqq d}$.

Then we define ψ by its effect on the canonical basis:

$$\psi(e_{ij}) = e_{i+1,j} \quad (\forall i = 0, \ldots, \deg(\infty) - 2, \quad \forall j = 1, \ldots, d)$$

and

$$\psi(e_{\deg(\infty)-1,j}) = \begin{cases} \varpi_\infty e_{0,d} & \text{if } j = 1 \\ e_{0,j-1} & \text{if } j = 2, \ldots, d\,. \end{cases}$$

Finally, we set

$$M = (\mathcal{O}_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S)^d = \bigoplus_{i=0}^{\deg(\infty)-1} (\mathcal{O}_\infty \hat{\otimes}_{\kappa(\infty), i_{\infty,i}^{\cdot}} \mathcal{O}_S)^d\,.$$

(8.3) Let $\kappa(\infty)_d$ be an extension of degree d of $\kappa(\infty)$. Let us set

$$F_{\infty,d} = F_\infty \hat{\otimes}_{\kappa(\infty)} \kappa(\infty)_d$$

and

$$\sigma_{\infty,d} = F_\infty \hat{\otimes}_{\kappa(\infty)} \mathrm{frob}_q^{\deg(\infty)}\,.$$

Let $F_{\infty,d}[\tau_\infty]$ be the polynomial algebra over $F_{\infty,d}$ in the non commutative variable τ_∞, with commmutation rule

$$\tau_\infty \cdot a = \sigma_{\infty,d}^{-1}(a) \cdot \tau_\infty$$

for each $a \in F_{\infty,d}$. The element $\tau_\infty^d - \varpi_\infty$ of $F_{\infty,d}[\tau_\infty]$ is central. It is well known that

$$\bar{D}_\infty := F_{\infty,d}[\tau_\infty]/(\tau_\infty^d - \varpi_\infty)$$

is "the" central division algebra over F_∞ with invariant $-1/d$ and admits

$$\bar{\mathcal{D}}_\infty := \mathcal{O}_{\infty,d}[\tau_\infty]/(\tau_\infty^d - \varpi_\infty)$$

as maximal order. Here $\mathcal{O}_{\infty,d} \subset F_{\infty,d}$ is the ring of integers.

If $\lambda : S \to \mathrm{Spec}(\kappa(\infty)_d)$ is a $\mathbb{F}_q$-morphism of schemes, we can construct an embedding of F_∞-algebras

$$\lambda^* : \bar{D}_\infty \hookrightarrow \mathrm{End}(N_{d,1}(i_{\infty,0}), \psi_{d,1}(i_{\infty,0}))$$

in the following way. Here $i_{\infty,0}$ is the composed map

$$S \xrightarrow{\lambda} \mathrm{Spec}(\kappa(\infty)_d) \xrightarrow{can} \{\infty\}$$

and $\mathrm{End}(N, \psi)$ is the F_∞-algebra of $F_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S$-linear endomorphisms of N commuting with ψ. For any $\alpha \in \kappa(\infty)_d$, the image of $1 \hat{\otimes} \alpha \in F_{\infty,d} \subset \bar{D}_\infty$ by this embedding is given by

$$\lambda^*(1 \hat{\otimes} \alpha)(e_{ij}) = (1 \hat{\otimes} \lambda^* (\mathrm{frob}_q^{i-j\deg(\infty)}(\alpha)))e_{ij}$$

($\forall i = 0, \ldots, \deg(\infty) - 1$; $\forall j = 1, \ldots, d$) and the image of $\tau_\infty \in \bar{D}_\infty$ by λ^* is given by

$$\lambda^*(\tau_\infty)(e_{ij}) = \begin{cases} \varpi_\infty e_{i,d} & \text{if } j = 1 \\ e_{i,j-1} & \text{if } j = 2, \ldots, d \end{cases}$$

($\forall i = 0, \ldots, \deg(\infty) - 1$)

If λ and λ' are two $\mathbb{F}_q$-morphisms of schemes from S to $\mathrm{Spec}\,(\kappa(\infty)_d)$ which lift $i_{\infty,0}$, we have

$$\lambda' = \lambda \circ \mathrm{Frob}_S^{n\deg(\infty)}$$

for some $n \in \mathbb{Z}/d\mathbb{Z}$ and it is easy to see that

$$\lambda'^* = \lambda^* \circ \mathrm{Ad}(\tau_\infty^{-n})\,.$$

For any λ as before, it is also clear that $\lambda^*(\bar{D}_\infty)$ maps $M_{d,1}(i_{\infty,0}) \subset N_{d,1}(i_{\infty,0})$ into itself.

(8.4) Definition. Let $E = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$ be a $\mathcal{D}$-elliptic sheaf over S. Let $\check{M}'_0$, $\check{\psi}'$ and $i_{\infty,0}$ be defined as in (8.1). Then *a level structure at infinity on E* is a pair (λ, α) where

$$\lambda : S \to \mathrm{Spec}(\kappa(\infty)_d)$$

is a $\mathbb{F}_q$-morphism of schemes which lifts the pole $i_{\infty,0}$ of E and where

$$\alpha : M_{d,1}(i_{\infty,0}) \xrightarrow{\sim} \check{M}'_0$$

is an isomorphism of $\mathcal{O}_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S$-modules such that the following diagram commutes

$$\begin{tikzcd} {}^{\tau}M_{d,1}(i_{\infty,0}) \arrow[r, "{}^{\tau}\alpha", "\sim"'] \arrow[d, "\psi_{d,1}(i_{\infty,0})"'] & {}^{\tau}\check{M}'_0 \arrow[d, "\check{\psi}'"] \\ M_{d,1}(i_{\infty,0}) \arrow[r, "\alpha", "\sim"'] & \check{M}'_0\,. \end{tikzcd}$$

We have an obvious notion of isomorphisms between $\mathcal{D}$-elliptic sheaves over S with level structure at infinity. Let $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}(S)$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}(S)$) be the category of $\mathcal{D}$-elliptic sheaves over S with a level structure at infinity (resp. with a level-I-structure and a level structure at ∞; here I is a finite subscheme of $X\backslash\{\infty\}$). Then, the obvious notion of pullback gives us a fibered category

$$S \mapsto \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}(S)$$

(resp. $$S \mapsto \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}(S))$$

over the category of $\mathbb{F}_q$-schemes S, which is clearly a stack for the *fpqc* topology. We will denote this stack by $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$). We have a forgetful morphism of stacks

$$\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}} \xrightarrow{r_\infty} \mathcal{E}\ell\ell_{X,\mathcal{D}}$$

(resp. $$\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I} \xrightarrow{r_{\infty,I}} \mathcal{E}\ell\ell_{X,\mathcal{D},I})$$

which maps $(E,(\lambda,\alpha))$ into E and a morphism of stacks

$$\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}} \xrightarrow{\lambda} \mathrm{Spec}(\kappa(\infty)_d)$$

(resp. $$\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I} \xrightarrow{\lambda_I} \mathrm{Spec}(\kappa(\infty)_d))$$

which maps $(E,(\lambda,\alpha))$ into λ and which lifts $i_{\infty,0}\circ r_\infty$ where $i_{\infty,0}$ is the pole.

It is clear that $r_{\infty,I}$ is the base change of r_∞ by the canonical morphism of stacks

$$r_I : \mathcal{E}\ell\ell_{X,\mathcal{D},I} \to \mathcal{E}\ell\ell_{X,\mathcal{D}}\cdot .$$

Therefore, if $I \subset J \subset X\backslash\{\infty\}$ are two finite closed subschemes, we have a 2-commutative diagram of stacks

$$\begin{tikzcd}
\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},J} \arrow[r, "\tilde{r}_{J,I}"] \arrow[d, "r_{\infty,J}"'] & \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I} \arrow[r, "\tilde{r}_I"] \arrow[d, "r_{\infty,I}"'] & \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}} \arrow[d, "r_\infty"'] \\
\mathcal{E}\ell\ell_{X,\mathcal{D},J} \arrow[r, "r_{J,I}"] & \mathcal{E}\ell\ell_{X,\mathcal{D},I} \arrow[r, "r_I"] & \mathcal{E}\ell\ell_{X,\mathcal{D}}
\end{tikzcd}$$

with 2-cartesian squares. Moreover

$$\lambda_I = \lambda \circ \tilde{r}_I \, .$$

(8.5) The stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (and therefore the stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$) has a natural structure of pro-stack. Indeed, to give the isomorphism

$$\alpha : M_{d,1}(i_{\infty,0}) \xrightarrow{\sim} \check{M}'_0$$

of $\mathcal{O}_\infty \hat{\otimes} \mathcal{O}_S$-modules is the same as to give the projective system

$$(\alpha_n = \alpha \text{ modulo } \varpi_\infty^{n+1})_{n \geqq 0}$$

of isomorphisms of $\mathcal{O}_\infty/(\varpi_\infty^{n+1}) \hat{\otimes} \mathcal{O}_S$-modules and α commutes with the ψ's if and only if each α_n commutes with the ψ's modulo ϖ_∞^{n+1}. In other words, $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ is the projective limit of the stacks of $\mathcal{D}$-elliptic sheaves with a level structure at infinity modulo ϖ_∞^{n+1}. From the definitions, it is clear that the projections $r_\infty, r_{\infty,I}, \tilde{r}_I$ and $\tilde{r}_{J,I}$ are continuous morphisms of pro-stacks.

(8.6) On the pro-stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (and therefore on the pro-stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$), we have continuous right actions of the pro-finite group $\bar{\mathcal{D}}_\infty^\times$ and the finite group $\mathbb{Z}/d\mathbb{Z}$: $\delta \in \bar{\mathcal{D}}_\infty^\times$ maps $(E,(\lambda,\alpha))$ into $(E,(\lambda,\alpha\circ\lambda^*(\delta)))$ and $n \in \mathbb{Z}/d\mathbb{Z}$ maps $(E,(\lambda,\alpha))$ into $(E,(\lambda\circ\mathrm{Frob}_S^{n\deg(\infty)},\alpha))$. As we have

$$(\lambda\circ\mathrm{Frob}_S^{n\deg(\infty)})^* = \lambda^*\circ\mathrm{Ad}(\tau_\infty^{-n})$$

these two actions induce a continuous right action of the profinite group

$$\bar{\mathcal{D}}_\infty^\times \rtimes \mathbb{Z}/d\mathbb{Z} \, .$$

Here $n \in \mathbb{Z}/d\mathbb{Z}$ acts on $\bar{\mathcal{D}}_\infty^\times$ by $\mathrm{Ad}(\tau_\infty^{-n})$. But we can identify this semi-direct product with the pro-finite group

$$\bar{\mathcal{D}}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$$

$(\varpi_\infty \in F_\infty^\times \subset \bar{D}_\infty^\times)$: we identify $(\delta, n) \in \bar{\mathcal{D}}_\infty^\times \rtimes \mathbb{Z}/d\mathbb{Z}$ with $\delta\tau_\infty^{-n} \in D_\infty^\times$ modulo $\varpi_\infty^{\mathbb{Z}}$. So we get a continuous right action of the pro-finite group $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on the pro-stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (and therefore on the pro-stack $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$).

From its definition, it follows that this right action of $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ on $\widetilde{\mathcal{E}\ell\ell}_{X,\mathscr{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathscr{D},I}$) commutes with the projection r_∞ (resp. $r_{\infty,I}$) and that the projection λ (resp. λ_I) is equivariant if we let $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ act on $\mathrm{Spec}(\kappa(\infty)_d)$ through its quotient

$$-\infty\circ\mathrm{rn}:\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty\twoheadrightarrow\mathbb{Z}/d\mathbb{Z}$$

$(\mathrm{Gal}(\kappa(\infty)_d/\kappa(\infty))=\mathbb{Z}/d\mathbb{Z})$. Here rn denotes the reduced norm.

For each $I\subset J\subset X\backslash\{\infty\}$ as before, it is also clear that $\tilde{r}_{J,I}$ is $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$-equivariant.

(8.7) Let $\infty\in T\subset|X|$ be a finite set of places. Let us denote by

$$r^{\mathrm{T}}_\infty:\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}\to\mathcal{E}\ell\ell^T_{X,\mathscr{D}}$$

the projective limit of the $r_{\infty,I}$ for $I\cap T=\emptyset$ (see (7.1)), with its continuous right action of $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$. Using that r^T_∞ is the base change of r_∞ by the canonical projection

$$r^T:\mathcal{E}\ell\ell^T_{X,\mathscr{D}}\to\mathcal{E}\ell\ell_{X,\mathscr{D}}\,,$$

we get a continuous right action of $(\mathscr{D}^T)^\times$ on $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$ which lifts the action of $(\mathscr{D}^T)^\times$ on $\mathcal{E}\ell\ell^T_{X,\mathscr{D}}$ and which commutes with the right action of $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$.

Using isogenies prime to T as in (7.4) we can extend this action of $(\mathscr{D}^T)^\times$ on $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$ to a continuous right action of the semi-group $(D^T)^\times\cap\mathscr{D}^T$ on $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$ which lifts the action of the same semi-group on $\mathcal{E}\ell\ell^T_{X,\mathscr{D}}$ defined in loc. cit. Indeed, if $E_1\xrightarrow{u}E_2$ is an isogeny prime to T between two $\mathscr{D}$-elliptic sheaves over S, u induces an isomorphism of $F_\infty\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_S$-modules

$$\check{N}'_2\xrightarrow{\sim}\check{N}'_1$$

which commutes with the $\check{\psi}$'s and which maps $\check{M}'_{2,0}$ onto $\check{M}'_{1,0}$ (see (8.1)). Obviously, the actions of $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ and of $(D^T)^\times\bigcap\mathscr{D}^T$ on $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$ commute.

On the other hand, we can also lift the action (7.2) of

$$\mathrm{Pic}^T_X(\mathbb{F}_q)\ =(F^\times)^T\backslash(\mathbb{A}^T)^\times$$

on $\mathcal{E}\ell\ell^T_{X,\mathscr{D}}$ to $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$. Indeed, if we tensor a $\mathscr{D}$-elliptic sheaf E over S by a line bundle $\mathscr{L}$ over X with a canonical identification of $\mathscr{L}_\infty$ with $\mathcal{O}_\infty$ this has no effect at all on the triple $(\check{N}',\check{\psi}',\check{M}'_0)$ (see (8.1)).

Now arguing as in (7.4), we get easily:

(8.8) Proposition. *There is a continuous right action of* $(D^T)^\times$ *on* $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$ *which lifts the action of* $(D^T)^\times$ *on* $\mathcal{E}\ell\ell^T_{X,\mathscr{D}}$ *defined in* (7.4) *and which commutes with the continuous right action of* $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ *on* $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$.

Moreover, if we consider the product of these two actions, the subgroup

$$(F^\times)^T\subset(\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty)\times(D^T)^\times$$

(diagonally embedded; $(F^\times)^T\subset(\mathbb{A}^T)^\times\subset(D^T)^\times$ *and* $(F^\times)^T\subset F^\times_\infty\subset\bar{D}^\times_\infty$*) acts trivially on* $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathscr{D}}$.

As in (7.5), for any open compact subgroup $K^T \subset (D^T)^\times$ and any $g^T \in (D^T)^\times$, we have a Hecke correspondance

$$\begin{tikzcd} & \widetilde{\mathcal{E}\ell\ell}^T_{X,\mathcal{D}}/(K^T \cap (g^T)^{-1} K^T g^T) \arrow[dl, "\tilde{c}_1"'] \arrow[dr, "\tilde{c}_2"] & \\ \widetilde{\mathcal{E}\ell\ell}^T_{X,\mathcal{D}}/K^T & & \widetilde{\mathcal{E}\ell\ell}^T_{X,\mathcal{D}}/K^T . \arrow[ll, dashed] \end{tikzcd}$$

(8.9) On $\mathcal{E}\ell\ell_{X,\mathcal{D}}$ (resp. $\mathcal{E}\ell\ell_{X,\mathcal{D},I}$) we also have the action of $\mathbb{Z}$ by translation (see 2.4)). It can be lifted to $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$) in the following way. Let $(E,(\lambda,\alpha))$ be a $\mathcal{D}$-elliptic sheaf with a level structure at infinity over S. Then $1 \in \mathbb{Z}$ maps $E = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ onto $E^\dagger = (\mathcal{E}_{i+1}, j_{i+1}, t_{i+1})_{i\in\mathbb{Z}}$. Let us set

$$\lambda^\dagger = \lambda \circ \mathrm{Frob}_S .$$

From α we get an isomorphism of $\mathcal{O}_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S$-modules

$$\alpha_1 : \psi_{d,1}(i_{\infty,0})({}^\tau M_{d,1}(i_{\infty,0})) \xrightarrow{\sim} \check{\psi}'({}^\tau \check{M}'_0) = \check{M}'_{-1}$$

which commutes again with the ψ's. But we have an isomorphism of $\mathcal{O}_\infty \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_S$-modules

$$\mathrm{can} : M_{d,1}(i_{\infty,1}) \xrightarrow{\sim} \psi_{d,1}(i_{\infty,0})({}^\tau M_{d,1}(i_{\infty,0}))$$

which commutes with the ψ's. It is defined by

$$\mathrm{can}(e_{i,j}(1)) = e_{i+1,j}(0)\,(\forall i = 0, \ldots, \deg(\infty) - 2, \quad \forall j = 1, \ldots, d)$$

and

$$\mathrm{can}(e_{\deg(\infty)-1,j}(1)) = \begin{cases} \varpi_\infty e_{0,d}(0) & \text{if } j = 1 \\ e_{0,j-1}(0) & \text{if } j = 2, \ldots, d \end{cases}$$

where $(e_{i,j}(0))$ and $(e_{i,j}(1))$ are the canonical bases of $N_{d,1}(i_{\infty,0})$ and $N_{d,1}(i_{\infty,1})$ respectively (see (8.2)). So we can set

$$\alpha^\dagger = \alpha_1 \circ \mathrm{can} .$$

Then we let $1 \in \mathbb{Z}$ act on $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$) by

$$(E,(\lambda,\alpha)) \mapsto (E^\dagger,(\lambda^\dagger,\alpha^\dagger)) .$$

We let the reader check that this action of $\mathbb{Z}$ on $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$) commutes with the action of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$. As this action of $\mathbb{Z}$ on the stacks $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$'s commutes with the transition maps $\tilde{r}_I$ and $\tilde{r}_{J,I}$, we get for any finite set of places $\infty \in T \subset |X|$ an action of $\mathbb{Z}$ on $\widetilde{\mathcal{E}\ell\ell}^T_{X,\mathcal{D}}$ which lifts the action of $\mathbb{Z}$ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D}}$ and which commutes with the action of $(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}) \times (D^T)^\times$.

(8.10) Theorem. *The morphism of pro-stacks*

$$r_\infty : \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}} \to \mathcal{E}\ell\ell_{X,\mathcal{D}}$$

with the continuous right action of the pro-finite group $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ is representable and is a pro-finite, pro-etale and pro-Galois covering with pro-finite Galois group $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$.

(8.11) Corollary. *The same statement is satisfied by*

$$r_{\infty,I} : \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I} \to \mathcal{E}\ell\ell_{X,\mathcal{D},I} .$$

Moreover, $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D}}$ (resp. $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I}$) is a scheme.

Proof of the theorem. We can factor r_∞ through

$$\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D}} \xrightarrow{(r_\infty,\lambda)} \mathscr{E}\ell\ell_{X,\mathscr{D}} \otimes_{\kappa(\infty)} \kappa(\infty)_d \to \mathscr{E}\ell\ell_{X,\mathscr{D}}$$

where $\mathscr{E}\ell\ell_{X,\mathscr{D}}$ is viewed as a $\kappa(\infty)$-stack by the pole map. Therefore, it is enough to prove that the morphism of pro-stacks

$$(r_\infty, \lambda): \widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D}} \to \mathscr{E}\ell\ell_{X,\mathscr{D}} \otimes_{\kappa(\infty)} \kappa(\infty)_d$$

is representable and is a pro-finite, pro-etale and pro-Galois covering with profinite Galois group $\bar{\mathscr{D}}_\infty^\times$.

Let us begin with the representability. Let S be a scheme (over $\mathbb{F}_q$), let $E = (\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathscr{D}$-elliptic sheaf over S and let $\lambda: S \to \operatorname{Spec}(\kappa(\infty)_d)$ be a $\mathbb{F}_q$-morphism of schemes which lifts the pole $i_{\infty,0}: S \to \{\infty\}$ of E. For each nonnegative interger n, let J_n be the *fppf* sheaf

$$\mathscr{I}\!som_{\mathcal{O}_\infty/(\varpi_\infty^{n+1})\otimes_{\mathbb{F}_q}\mathcal{O}_S}(M_{d,1}(i_{\infty,0})/\varpi_\infty^{n+1} M_{d,1}(i_{\infty,0}), \check{M}'_0/\varpi_\infty^{n+1}\check{M}'_0)$$

over S. We can organize the J_n's into a projective system

$$\cdots \to J_{n+1} \to J_n \to \cdots \to J_1 \to J_0 \to J_{-1} = S$$

where the transition map $J_{n+1} \to J_n$ is the reduction modulo ϖ_∞^{n+1}. The J_n's are clearly representable by S-schemes and the transition maps are all affine and locally finitely presented. The sheaf of level structure at infinity modulo ϖ_∞^{n+1} on (E, λ) is the subsheaf

$$G_n \subset J_n$$

defined by the commutation with the ψ's. Therefore, G_n is representable by a closed subscheme of J_n and the ideal of $\mathcal{O}_{J_n}$ defining G_n is locally of finite type. If we organize the G_n's into a projective system

$$\cdots \to G_{n+1} \to G_n \to \cdots \to G_1 \to G_0 \to G_{-1} = S$$

where the transition map $G_{n+1} \to G_n$ is induced by the transition map $J_{n+1} \to J_n$, we see that

(i) each G_n is representable by a S-scheme and the transition maps are all affine and locally finitely presented;
(ii) the projective limit of this system, which is the sheaf of level structures at infinity on (E, λ) is representable by a S-scheme, affine over S.

Now, in order to finish the proof of the theorem, it suffices to check the following assertions for any S, E and λ as before:

(a) locally for the *fpqc* topology on S, $(\check{M}'_0, \psi')$ is isomorphic to $(M_{d,1}(\iota_0), \psi_{d,1}(\iota_0))$;
(b) the sheaf of F_∞-algebras

$$\mathscr{E}\!nd(M_{d,1}(\iota_0), \psi_{d,1}(\iota_0))$$

for the *fpqc* topology on the category of S-schemes is constant with value $\bar{\mathscr{D}}_\infty$.

These two assertions are well known (if S is the spectrum of a field, see (B.3) and (B.10) for a reformulation and references). □

9 $\mathscr{D}$-elliptic sheaves of finite characteristic: Description up to isogeny

Let o be a place of F which is distinct from ∞. In [Dr 2] and [Dr 7], Drinfeld gives a description of the set of isomorphism classes of elliptic modules of characteristic o and shtukas with pole ∞ and zero o over an algebraically closed field. In this section, our purpose is to give a similar description of the set of isomorphism classes of $\mathscr{D}$-elliptic sheaves of characteristic o over an algebraically closed field, at least when $o \notin \mathrm{Bad}$.

In the whole section the place o of F, $o \neq \infty$ and $o \notin \mathrm{Bad}$, is fixed. We denote by k an algebraic closure of $\kappa(o)$. We identify $(D_x, \mathscr{D}_x)$ with $(\mathbb{M}_d(F_x), \mathbb{M}_d(\mathcal{O}_x))$ for $x = \infty, o$.

(9.1) From now on, we shall use the term $\mathscr{D}$-elliptic sheaf of characteristic o over k to mean $\mathscr{D}$-elliptic sheaf over $\mathrm{Spec}(k)$ such that the zero is the canonical morphism

$$\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(o)) \hookrightarrow X .$$

Let us recall that such an object is given by a sequence $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where $\mathscr{E}_i$ is a locally free $\mathcal{O}_{X\otimes_{\mathbb{F}_q}k}$-module of rank d^2, equipped with a right action of $\mathscr{D}$ which extends the $\mathcal{O}_X$-action, and where

$$j_i : \mathscr{E}_i \hookrightarrow \mathscr{E}_{i+1}$$
$$t_i : {}^{\tau}\mathscr{E}_i \hookrightarrow \mathscr{E}_{i+1}$$

are injective $\mathscr{D}$-linear homomorphisms (${}^{\tau}\mathscr{E}_i = (X \otimes_{\mathbb{F}_q} \mathrm{frob}_q)^* \mathscr{E}_i$ where $\mathrm{frob}_q \in \mathrm{Gal}(k/\mathbb{F}_q)$ is the arithmetic Frobenius element with respect to $\mathbb{F}_q$). Moreover, for each $i \in \mathbb{Z}$, the following conditions hold:

(i) The diagram

$$\begin{tikzcd}
\mathscr{E}_i \arrow[r, hook, "j_i"] & \mathscr{E}_{i+1} \\
{}^{\tau}\mathscr{E}_{i-1} \arrow[u, hook, "t_{i-1}"'] \arrow[r, hook, "{}^{\tau}j_{i-1}"] & {}^{\tau}\mathscr{E}_i \arrow[u, hook, "t_i"']
\end{tikzcd}$$

commutes;

(ii) $\mathscr{E}_{i+d\cdot\deg(\infty)} = \mathscr{E}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)$ and the inclusion

$$\mathscr{E}_i \xhookrightarrow{j_i} \mathscr{E}_{i+1} \xhookrightarrow{j_{i+1}} \cdots \hookrightarrow \mathscr{E}_{i+d\cdot\deg(\infty)} = \mathscr{E}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)$$

is induced by the canonical one $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\infty)$;

(iii) $\dim_k H^0(X \otimes k, \mathrm{Coker}\, j_i) = d$;

(iv) the support of $\mathrm{Coker}\, t_i$ is the image of the graph of the canonical morphism

$$\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(o)) \hookrightarrow X .$$

To each $\mathscr{D}$-elliptic sheaf $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ of characteristic o over k, we can attach a φ-space (V, φ) and an F-algebra homomorphism

$$\imath : D^{\mathrm{op}} \to \mathrm{End}(V, \varphi)$$

in the following way. (The terminology is that of Appendix A.) Let V be the generic fibre of $\mathscr{E}_0$, i.e. the fibre of $\mathscr{E}_0$ at the generic point $\eta \otimes k$ of $X \otimes k$. It is a free $D \otimes k$-module of rank 1. Thanks to the j_i's we can identify V with the generic fibre of $\mathscr{E}_i$ for any $i \in \mathbb{Z}$. Then, the t_i's induce a bijective $F \otimes \mathrm{frob}_q$-semilinear map $\varphi: V \to V$ and (V, φ) is a φ-space over k. The action of D on V commutes with φ and gives the required homomorphism ι.

(9.2) Definition. The triple (V, φ, ι) is called *the generic fibre of the $\mathscr{D}$-elliptic sheaf* $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$. Two $\mathscr{D}$-elliptic sheaves (of characteristic o) over k are said to be *isogeneous* (or in the same isogeny class) if their generic fibres are isomorphic.

If x is a place of F, to any $\mathscr{D}$-elliptic sheaf (of characteristic o) over k, $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, we can also attach a Dieudonné F_x-module (V_x, φ_x) with an F_x-algebra homomorphism

$$\iota_x: D_x^{\mathrm{op}} \to \mathrm{End}(V_x, \varphi_x)$$

and a $\mathscr{D}_x$-lattice (i.e. a lattice $M_x \subset V_x$ which is stable under $\iota_x(\mathscr{D}_x^{\mathrm{op}})$). If (V, φ, ι) is the generic fibre of the $\mathscr{D}$-elliptic sheaf, by definition, we have $(V_x, \varphi_x) = (F_x \hat{\otimes}_F V, F_x \hat{\otimes}_F \varphi)$, ι_x is induced by ι and we have $M_x = H^0(\mathrm{Spec}(\mathcal{O}_x \hat{\otimes} k), \mathscr{E}_0)$.

(9.3) Lemma. *The Dieudonné F_x-modules (V_x, φ_x) and the lattices M_x have the following properties*:

(i) *if $x = \infty$, we have*

$$\varphi_\infty(M_\infty) \supset M_\infty$$
$$\varphi_\infty^{d\cdot\deg(\infty)}(M_\infty) = \varpi_\infty^{-1} M_\infty$$
$$\dim_k(\varphi_\infty(M_\infty)/M_\infty) = d$$

for any uniformizer ϖ_∞ of $\mathcal{O}_\infty$;
(ii) *if $x = o$, we have*

$$\varpi_o M_o \subset \varphi_o(M_o) \subset M_o$$

for any uniformizer ϖ_o of $\mathcal{O}_o$ and the $\kappa(o) \otimes k$-module $M_o/\varphi_o(M_o)$ is of length d and is supported on the connected component of $\mathrm{Spec}(\kappa(o) \otimes k)$ *which corresponds to the given embedding $\kappa(o) \hookrightarrow k$*;
(iii) *if $x \neq \infty, o$, we have*

$$\varphi_x(M_x) = M_x;$$

(iv) *some* (*and thus each*) *basis of the $F \otimes k$-vector space V belongs to and generates the $\mathcal{O}_x \hat{\otimes} k$-submodules M_x of V_x for all except finitely many places $x \neq \infty, o$ of F.*
Conversely, we let the reader check that:

(9.4) Proposition. *The above constructions define a bijection between the set of isomorphism classes of $\mathscr{D}$-elliptic sheaves $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ (of characteristic o) over k and the set of isomorphism classes of pairs $((V, \varphi, \iota), (M_x)_{x\in|X|})$ where (V, φ) is a φ-space of rank d^2 over $F \otimes k$, $\iota: D^{\mathrm{op}} \to \mathrm{End}(V, \varphi)$ is an F-algebra homomorphism and $(M_x)_{x\in|X|}$ is a collection of $\mathscr{D}_x$-lattices in the Dieudonné F_x-modules $(V_x, \varphi_x) = (F_x \hat{\otimes}_F V, F_x \hat{\otimes}_F \varphi)$ which satisfy the properties* (9.3) (i) *to* (9.3) (iv).

Now we will classify all the isogeny classes of $\mathscr{D}$-elliptic sheaves (of characteristic o) over k. Let $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathscr{D}$-elliptic sheaf (of characteristic o) over k and let (V, φ, ι) be its generic fibre.

(9.5) Let R be a unitary ring and let e be a positive integer. Let us denote by $E^{ij}\in\mathbb{M}_e(R)$ the elementary matrix with the (i, j)-th entry equal to 1 and all other entries equal to 0. If $\tilde{M}$ is a right R-module we can view $(\tilde{M})^e$ as a right $\mathbb{M}_e(R)$-module ($\mathbb{M}_e(R)$ acts by right multiplication on row vectors with entries in $\tilde{M}$). Then, the functor $\tilde{M}\mapsto\tilde{M}^e$ from the category of right R-modules to the category of right $\mathbb{M}_e(R)$-modules is an equivalence of categories with quasi-inverse

$$M\mapsto ME^{11}$$

(the map

$$M\to(ME^{11})^e$$

$$m\mapsto(mE^{11},\dots,mE^{e1})$$

is an isomorphism of right $\mathbb{M}_e(R)$-modules.) (Morita equivalence.)

(9.6) Lemma. *The φ-space (V,φ) is isotypical, i.e. isomorphic to $(W,\psi)^n$ for some irreducible φ-space (W,ψ) and some positive integer n.*

Proof. We can assume that D is of the form $\mathbb{M}_e(\tilde{D})$ where $\tilde{D}$ is a central division algebra over F with $\dim_F(\tilde{D})=(d/e)^r$ for some positive integer e. For simplicity we shall also assume that $\mathscr{D}$ is of the form $\mathbb{M}_e(\tilde{\mathscr{D}})$, where $\tilde{\mathscr{D}}$ is a sheaf of $\mathcal{O}_X$-algebras with generic fibre $\tilde{D}$ such that $\tilde{\mathscr{D}}_x$ is a maximal order for all x. Then, via Morita equivalence,

$$(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}=((\tilde{\mathscr{E}}_i,\tilde{j}_i,\tilde{t}_i)_{i\in\mathbb{Z}})^e$$

where $(\tilde{\mathscr{E}}_i,\tilde{j}_i,\tilde{t}_i)_{i\in\mathbb{Z}}$ is a "$\tilde{\mathscr{D}}$-elliptic sheaf of rank e (of characteristic o) over k" and

$$(V,\varphi,\iota)=(\tilde{V},\tilde{\varphi},\tilde{\iota})^e$$

where $(\tilde{V},\tilde{\varphi},\tilde{\iota})$ is the generic fibre of $(\tilde{\mathscr{E}}_i,\tilde{j}_i,\tilde{t}_i)_{i\in\mathbb{Z}}$.

Now, the argument is very similar to the one used to prove Theorem 6.1. Let $(\tilde{W},\tilde{\psi})$ be a non trivial φ-subspace of $(\tilde{V},\tilde{\varphi})$. For each integer i, let $\tilde{\mathscr{F}}_i\subset\tilde{\mathscr{E}}_i$ be the unique coherent $\mathcal{O}_{X\otimes k}$-submodule such that $(\tilde{\mathscr{F}}_i)_\eta\subset(\tilde{\mathscr{E}}_i)_\eta$ is equal to $\tilde{W}\subset\tilde{V}$ and such that $\tilde{\mathscr{E}}_i/\tilde{\mathscr{F}}_i$ is torsion free. Obviously, we have $\tilde{j}_i(\tilde{\mathscr{F}}_i)\subset\tilde{\mathscr{F}}_{i+1}$, $\tilde{\mathscr{F}}_{i+\mathbf{deg}(\infty)(d/e)}=\tilde{\mathscr{F}}_i\otimes_{\mathcal{O}_X}\mathcal{O}_X(\infty)$ and $\tilde{t}_i({}^\tau\tilde{\mathscr{F}}_i)\subset\tilde{\mathscr{F}}_{i+1}$ for each $i\in\mathbb{Z}$.

Now, let us assume that $\tilde{W}\subset\tilde{V}$ is stable under the action of $\tilde{D}$. Then $\tilde{\mathscr{F}}_i\subset\tilde{\mathscr{E}}_i$ is stable under the action of $\tilde{\mathscr{D}}$ for each $i\in\mathbb{Z}$. But

$$H^0(\mathrm{Spec}(\mathcal{O}_\infty\hat{\otimes}k),\tilde{\mathscr{E}}_i/\tilde{\mathscr{E}}_{i-1})$$

is a simple $\tilde{\mathscr{D}}_\infty$-module and

$$H^0(\mathrm{Spec}(\mathcal{O}_\infty\hat{\otimes}k),\tilde{\mathscr{F}}_i/\tilde{\mathscr{F}}_{i-1})\subset H^0(\mathrm{Spec}(\mathcal{O}_\infty\hat{\otimes}k),\tilde{\mathscr{E}}_i/\tilde{\mathscr{E}}_{i-1})$$

is a $\tilde{\mathscr{D}}_\infty$-submodule. Therefore either $\tilde{\mathscr{F}}_{i-1}=\tilde{\mathscr{F}}_i$ or the length of $\tilde{\mathscr{F}}_i/\tilde{\mathscr{F}}_{i-1}$ is d/e. As the length of $\tilde{\mathscr{F}}_{\mathbf{deg}(\infty)(d/e)}/\tilde{\mathscr{F}}_0$ is equal to

$$\dim_{F\otimes k}(\tilde{W})\cdot\deg(\infty)<(d/e)\cdot\deg(\infty)\,,$$

there exists at least one $i\in\{1,\dots,(d/e)\cdot\deg(\infty)\}$ such that $\tilde{\mathscr{F}}_{i-1}=\tilde{\mathscr{F}}_i$.

Now, if $\tilde{\mathscr{F}}_{i-1}=\tilde{\mathscr{F}}_i$ for some $i\in\mathbb{Z}$, we have

$$\tilde{t}_{i-1}({}^\tau\tilde{\mathscr{F}}_{i-1})\subset\tilde{\mathscr{F}}_i=\tilde{\mathscr{F}}_{i-1}$$

and, by comparing the degrees, we get

$$\tilde{t}_{i-1}({}^\tau\tilde{\mathscr{F}}_{i-1})=\tilde{\mathscr{F}}_{i-1}\,.$$

In other words, $\tilde{\mathscr{F}}_{i-1}$ is endowed with descent data from k to $\mathbb{F}_q$ in the sense of [Dr 6 (1.1)]. Moreover by hypothesis the descent data are compatible with the $\tilde{\mathscr{D}}$-action. Therefore, by loc.cit., there exists a unique coherent $\mathcal{O}_X$-module $\bar{\mathscr{F}}_{i-1}$ with a $\tilde{\mathscr{D}}$-action such that $\tilde{\mathscr{F}}_{i-1}$ with its $\tilde{\mathscr{D}}$-action and descent data is the base change of $\bar{\mathscr{F}}_{i-1}$ from $\mathbb{F}_q$ to k. In particular, there exists a unique $\tilde{D}$-module of finite type $\bar{W}(=(\bar{\mathscr{F}}_{i-1})_\eta)$ such that

$$(\tilde{W}, \tilde{\varphi}) \cong (\bar{W} \otimes_{\mathbb{F}_q} k, \bar{W} \otimes_{\mathbb{F}_q} \mathrm{Frob}_q)\,.$$

But $\tilde{D}$ is a central division algebra of dimension $(d/e)^2$ over F, so $\dim_F(\bar{W})$ is a multiple of $(d/e)^2$. This contradicts our hypothesis that $\tilde{W}$ is a non trivial subspace of $\tilde{V}$. Therefore, the triple $(\tilde{V}, \tilde{\varphi}, \tilde{\imath})$ is irreducible and the pair $(\tilde{V}, \tilde{\varphi})$ is isotypical. The lemma follows. □

(9.7) Let x be a good place of F for the pair $(D, \mathscr{D})$ and let us fix an identification of the pair $(D_x, \mathscr{D}_x)$ with $(\mathbb{M}_d(F_x), \mathbb{M}_d(\mathcal{O}_x))$. Then the Dieudonné F_x-module (V_x, φ_x) and its lattice M_x (see (9.3)) admit a canonical splitting

$$(V_x, \varphi_x) = (V'_x, \varphi'_x)^d$$
$$M_x = M'^d_x$$

where (V'_x, φ'_x) is a Dieudonné F_x-module and $M'_x \subset V'_x$ is a lattice in such way that the action ι_x of D_x (resp. $\mathscr{D}_x$) on (V_x, φ_x) (resp. M_x) becomes the natural right action of $\mathbb{M}_d(F_x)$ (resp. $\mathbb{M}_d(\mathcal{O}_x)$) on $(V'_x, \varphi'_x)^d$ (resp. M'^d_x). In particular, for $x = \infty, o$ we get, using the terminology of Appendix B:

(9.8) Lemma. (i) *The Dieudonné F_∞-module $(V'_\infty, \varphi'_\infty)$ is isomorphic to $(N_{d,-1}, \psi_{d,-1})$ and the lattice $M'_\infty \subset V'_\infty$ satisfies the following properties:*

$$\begin{cases} M'_\infty \subset \varphi'_\infty(M'_\infty) \\ (\varphi'_\infty)^{d\cdot\deg(\infty)}(M'_\infty) = \varpi_\infty^{-1} M'_\infty \\ \dim_k(\varphi'_\infty(M'_\infty)/M'_\infty) = 1 \end{cases}$$

for any uniformizer ϖ_∞ of $\mathcal{O}_\infty$.
(ii) *The Dieudonné F_o-module (V'_o, φ'_o) is isomorphic to $(N_{1,0}, \psi_{1,0})^{d-h} \oplus (N_{h,1}, \psi_{h,1})$ for some integer h with $0 < h \leqq d$ and the lattice $M'_o \subset V'_o$ satisfies the following properties:*

$$\begin{cases} \varpi_o M'_o \subset \varphi'_o(M'_o) \subset M'_o \\ \dim_k(M'_o/\varphi'_o(M'_o)) = 1 \end{cases}$$

for any uniformizer ϖ_o of $\mathcal{O}_o$ and the support of $M'_o/\varphi'_o(M'_o)$ is the connected component of

$$\mathrm{Spec}(\kappa(o) \otimes k) \subset \mathrm{Spec}(\mathcal{O}_o \otimes k)$$

which corresponds to the given embedding $\kappa(o) \hookrightarrow k$.

Proof. The properties of the lattices M'_∞ and M'_o follow immediately from the corresponding properties of the lattices M_∞ and M_o (see (9.3)). Then the lemma follows from (B.7) and (B.8). □

(9.9) Proposition. *Let $(\tilde{F}, \tilde{\Pi})$ be the φ-pair which is associated to the φ-space (V, φ)* (see (A.4)). *Then $(\tilde{F}, \tilde{\Pi})$ has the following properties:*

(i) *$\tilde{F}$ is a field and $[\tilde{F}:F]$ divides d;*
(ii) *$F_\infty \otimes_F \tilde{F}$ is a field and, if $\tilde{\infty}$ is the unique place of $\tilde{F}$ which divides ∞, we have $\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi}) = -[\tilde{F}:F]/d$;*
(iii) *there exists a unique place $\tilde{o} \neq \tilde{\infty}$ of $\tilde{F}$ such that $\tilde{o}(\tilde{\Pi}) \neq 0$; moreover, $\tilde{o}$ divides o;*
(iv) *we have $h = d[\tilde{F}_{\tilde{o}}:F_o]/[\tilde{F}:F]$ where h is the positive integer of* (9.8) (ii).

Moreover, if (W, ψ) is the irreducible φ-space which corresponds to $(\tilde{F}, \tilde{\Pi})$ (see (A.6)) *then the φ-space (V, φ) is (non-canonically) isomorphic to $(W, \psi)^d$.*

Proof. Let n and (W, ψ) be as in (9.6) and let $(\tilde{F}, \tilde{\Pi})$ be the φ-pair which is associated to (W, ψ) (see (A.4)). Then, thanks to (A.6),

$$d^2/n = \dim_{F \otimes k}(W) = d(\tilde{\Pi})[\tilde{F}:F] ,$$

$\tilde{F}$ is a field and $\mathrm{End}(W, \psi)$ is a central division algebra over $\tilde{F}$ of dimension $d(\tilde{\Pi})^2$ and, for each place $\tilde{x}$ of $\tilde{F}$,

$$\mathrm{inv}_{\tilde{x}}(\mathrm{End}(W, \psi)) \equiv -\deg(\tilde{x})\tilde{x}(\tilde{\Pi}) \text{ (modulo } \mathbb{Z}) .$$

From the diagonal embedding $\tilde{F} \subset \mathrm{End}(W, \psi) \subset \mathbb{M}_n(\mathrm{End}(W, \psi)) = \mathrm{End}(V, \varphi)$ we get an embedding $F_\infty \otimes_F \tilde{F} \subset \mathrm{End}(V_\infty, \varphi_\infty)$. But, with the notations of (9.8), we have $\mathrm{End}(V_\infty, \varphi_\infty) = \mathbb{M}_d(\mathrm{End}(V'_\infty, \varphi'_\infty))$ and $F_\infty \otimes_F \tilde{F} \subset \mathbb{M}_d(\mathrm{End}(V'_\infty, \varphi'_\infty))$ commutes with $D_\infty^{\mathrm{op}} = \mathbb{M}_d(F_\infty) \subset \mathbb{M}_d((\mathrm{End}(V'_\infty, \varphi'_\infty))$. Therefore, we have

$$F_\infty \otimes_F \tilde{F} \subset \mathrm{End}(V'_\infty, \varphi'_\infty) \subset \mathbb{M}_d(\mathrm{End}(V'_\infty, \varphi'_\infty))$$

and $F_\infty \otimes_F \tilde{F}$ is a field over F_∞ of a degree which divides d. ($\mathrm{End}(V'_\infty, \varphi'_\infty)$ is a central division algebra over F_∞ with invariant $1/d$.) Moreover, (W_∞, ψ_∞) is non canonically isomorphic to $(V'_\infty, \varphi'_\infty)^e$ for some positive integer e and $d = ne$ (the category of Dieudonné F_∞-modules is semi-simple and $(V'_\infty, \varphi'_\infty)$ is irreducible, see (B.3) and (9.8) (i)). So, thanks to (B.4), we have

$$-1/d = \deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi})/[\tilde{F}:F] ,$$

where $\tilde{\infty}$ is the unique place of $\tilde{F}$ which divides ∞ ($[\tilde{F}_{\tilde{\infty}}:F_\infty] = [\tilde{F}:F]$). The parts (i) and (ii) of the proposition are now proved.

Similarly, we have

$$F_o \otimes_F \tilde{F} \subset \mathrm{End}(V_o, \varphi_o) = \mathbb{M}_d(\mathrm{End}(V'_o, \varphi'_o))$$
$$\mathrm{End}(V'_o, \varphi'_o) \simeq \mathrm{End}(N_{h,1}, \psi_{h,1}) \times \mathbb{M}_{d-h}(F_o)$$

(see (9.8(ii)) and

$$F_o \otimes_F \tilde{F} \subset \mathbb{M}_d(\mathrm{End}(N_{h,1}, \psi_{h,1})) \times \mathbb{M}_d(\mathbb{M}_{d-h}(F_o))$$

commutes with

$$D_o^{\mathrm{op}} = \mathbb{M}_d(F_o) \subset \mathbb{M}_d(\mathrm{End}(N_{h,1}, \psi_{h,1})) \times \mathbb{M}_d(\mathbb{M}_{d-h}(F_o)) .$$

Therefore we have

$$F_o \otimes_F \tilde{F} \subset \mathrm{End}(N_{h,1}, \psi_{h,1}) \times \mathbb{M}_{d-h}(F_o)$$

and there exists one and only one place $\tilde{o}$ of F dividing o such that

$$\tilde{F}_{\tilde{o}} \subset \mathrm{End}(N_{h,1}, \psi_{h,1})$$

and

$$(V'_{\tilde{o}}, \varphi'_{\tilde{o}}) \simeq (N_{h,1}, \psi_{h,1}) .$$

Moreover $(W_{\tilde{o}}, \psi_{\tilde{o}})$ is non canonically isomorphic to $(V'_{\tilde{o}}, \varphi'_{\tilde{o}})^e$ where $e = d/n$ (n divides d). So, thanks to (B.4), we have

$$1/h = \deg(\tilde{o})\tilde{o}(\tilde{\Pi})/[\tilde{F}_{\tilde{o}} : \mathrm{F}_o]$$

and

$$he = d(\tilde{\Pi})[\tilde{F}_{\tilde{o}} : F_o] .$$

If $\tilde{x}$ is a place of $\tilde{F}$ which is distinct of $\tilde{\infty}$ and $\tilde{o}$, then $(W_{\tilde{x}}, \psi_{\tilde{x}})$ is some positive power of $(N_{1,0}, \psi_{1,0})$ (if $\tilde{x}$ divides o, $(W_{\tilde{x}}, \psi_{\tilde{x}})$ is isomorphic to $(V'_{\tilde{x}}, \varphi'_{\tilde{x}})^e$; if $\tilde{x}$ does not divide o, the lattice $M_x \subset V_x$ (see (9.3)) satisfies $\varphi_x(M_x) = M_x$ and we can apply (B.6) and (B.3)). So, thanks to (B.4), we have $\tilde{x}(\tilde{\Pi}) = 0$.

The part (iii) and (iv) of the proposition are also proved $(\deg(\tilde{o})\tilde{o}(\tilde{\Pi}) = -\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi}) = [\tilde{F} : F]/d)$.

The last assertion of the proposition is now easy to prove: $d(\tilde{\Pi})$ is the l.c.m. of the denominators of $\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi})$ and $\deg(\tilde{o})\tilde{o}(\tilde{\Pi})$, but

$$\deg(\tilde{o})\tilde{o}(\tilde{\Pi}) = -\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi}) = 1/(d/[\tilde{F} : F]) ,$$

so $d(\tilde{\Pi}) = d/[\tilde{F} : F]$, $e = 1$ $(de = d(\tilde{\Pi})[\tilde{F} : F])$ and $n = d$. □

(9.10) Corollary. *Let* $(\tilde{F}, \tilde{\Pi})$ *be the* φ*-pair which is associated to the* φ*-space* (V, φ) *(see* (A.4)*). Then* $\mathrm{End}(V, \varphi, \iota)$ *is a central division algebra over* $\tilde{F}$ *of dimension* $(d/[\tilde{F} : F])^2$ *and with invariant*

$$\mathrm{inv}_{\tilde{x}}(\mathrm{End}(V, \varphi, \iota)) = \begin{cases} [\tilde{F} : F]/d & \textit{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F} : F]/d & \textit{if } \tilde{x} = \tilde{o} \\ [\tilde{F}_{\tilde{x}} : F_x]\mathrm{inv}_x(D) & \textit{otherwise} \end{cases}$$

for each place x *of* F *and each place* $\tilde{x}$ *of* $\tilde{F}$ *dividing* x.

In particular, for each place x *of* F *and each place* $\tilde{x}$ *of* $\tilde{F}$ *dividing* x, *we have*

$$(d[\tilde{F}_{\tilde{x}} : F_x]/[\tilde{F} : F])\mathrm{inv}_x(D) \in \mathbb{Z} .$$

Proof. By definition, $\mathrm{End}(V, \varphi, \iota)$ is the centralizer of $\iota(D^{\mathrm{op}})$ in $\mathrm{End}(V, \varphi)$. But

$$\mathrm{End}(V, \varphi) = \mathbb{M}_d(\mathrm{End}(W, \psi))$$

and $\mathrm{End}(W, \psi)$ is a central division algebra over $\tilde{F}$ of dimension $(d/[\tilde{F} : F])^2$ and with invariants

$$\mathrm{inv}_{\tilde{x}}(\mathrm{End}(W, \psi)) = \begin{cases} [\tilde{F} : F]/d & \text{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F} : F]/d & \text{if } \tilde{x} = \tilde{o} \\ 0 & \text{otherwise} \end{cases}$$

for each place $\tilde{x}$ of $\tilde{F}$ (see (9.9) and (A.6)). Therefore, $\mathrm{End}(V, \varphi, \iota)$ is a central simple algebra over $\tilde{F}$ of dimension $(d/[\tilde{F} : F])^2$ and with invariants as required (see [Re]). Such a central simple algebra over $\tilde{F}$ is obviously a division algebra $(\tilde{F}_{\tilde{\infty}} \otimes_{\tilde{F}} \mathrm{End}(V, \varphi, \iota)$ and $\tilde{F}_{\tilde{o}} \otimes_{\tilde{F}} \mathrm{End}(V, \varphi, \iota)$ are division algebras) and the corollary follows. □

(9.11) Definition. A (D, ∞, o)-*type* is a φ-pair $(\tilde{F}, \tilde{\Pi})$ which satisfies the following properties:

(i) $\tilde{F}$ is a field and $[\tilde{F}:F]$ divides d;

(ii) $F_\infty \otimes_F \tilde{F}$ is a field and, if $\tilde{\infty}$ is the unique place of $\tilde{F}$ which divides ∞, we have

$$\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi}) = -[\tilde{F}:F]/d;$$

(iii) there exists a unique place $\tilde{o} \neq \tilde{\infty}$ of $\tilde{F}$ such that $\tilde{o}(\tilde{\Pi}) \neq 0$; moreover $\tilde{o}$ divides o;

(iv) for each place x of F and each place $\tilde{x}$ of $\tilde{F}$ dividing x, we have

$$(d[\tilde{F}_{\tilde{x}}:F_x]/[\tilde{F}:F])\mathrm{inv}_x(D) \in \mathbb{Z}\,.$$

We have seen how to associate a (D, ∞, o)-type to the generic fibre of a $\mathscr{D}$-elliptic sheaf (of characteristic o) over k. Conversely, let $(\tilde{F}, \tilde{\Pi})$ be a (D, ∞, o)-type.

(9.12) Construction. We will associate to $(\tilde{F}, \tilde{\Pi})$ a triple $(V, \varphi, \imath)$ (well defined up to isomorphism) where (V, φ) is a φ-space (over k) and $\imath: D^{\mathrm{op}} \to \mathrm{End}(V, \varphi)$ is a F-algebra homomorphism.

Let (W, ψ) be "the" irreducible φ-space which corresponds to the φ-pair $(\tilde{F}, \tilde{\Pi})$ (see (A.6)) and let Δ be "the" central division algebra over $\tilde{F}$ with invariants

$$\mathrm{inv}_{\tilde{x}}\Delta = \begin{cases} [\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{o} \\ [\tilde{F}_{\tilde{x}}:F_x]\mathrm{inv}_x(D) & \text{otherwise} \end{cases}$$

for each place x of F and each place $\tilde{x}$ of $\tilde{F}$ dividing x. Thanks to (9.11) (iv), we have

$$\dim_{\tilde{F}}(\Delta) = (d/[\tilde{F}:F])^2\,.$$

Then, $D^{\mathrm{op}} \otimes_F \Delta$ and $\mathbb{M}_d(\mathrm{End}(W, \psi))$ are central simple algebras over $\tilde{F}$ of the same dimension $d^4/[\tilde{F}:F]^2$ and with the same invariants

$$\begin{cases} [\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{o} \\ 0 & \text{otherwise} \end{cases}$$

for each place $\tilde{x}$ of $\tilde{F}$ (see (A.6)). Therefore, these two $\tilde{F}$-algebras are isomorphic and, thanks to the Skolem-Noether theorem, an isomorphism between them is unique up to an inner automorphism of $\mathbb{M}_d(\mathrm{End}(W, \psi))$. Let us choose such an isomorphism

$$\alpha: D^{\mathrm{op}} \otimes_F \Delta \xrightarrow{\sim} \mathbb{M}_d(\mathrm{End}(W, \psi))$$

and let us set $(V, \varphi) = (W, \psi)^d$. Then

$$\imath: D^{\mathrm{op}} \xrightarrow{\delta \mapsto \delta \otimes 1} D^{\mathrm{op}} \otimes_F \Delta \xrightarrow[\sim]{\alpha} \mathbb{M}_d(\mathrm{End}(W, \psi)) = \mathrm{End}(V, \varphi)$$

is a F-algebra homomorphism and the commutant of $\imath(D^{\mathrm{op}})$ in $\mathrm{End}(V, \varphi)$ is the image of Δ by the $\tilde{F}$-algebra homomorphism

$$\Delta \xrightarrow{\delta \mapsto 1 \otimes \delta} D^{\mathrm{op}} \otimes_F \Delta \xrightarrow[\sim]{\alpha} \mathbb{M}_d(\mathrm{End}(W, \psi)) = \mathrm{End}(V, \varphi)\,.$$

The isomorphism class of the triple (V, φ, ι) is clearly independent of the choices of (W, ψ), Δ and α. ☐

(9.13) Theorem. *The composed map*

$$(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \mapsto (V, \varphi, \iota) \mapsto (\tilde{F}, \tilde{\Pi}) ,$$

where $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ is a $\mathscr{D}$-elliptic sheaf (of characteristic o) over k, (V, φ, ι) is its generic fibre and $(\tilde{F}, \tilde{\Pi})$ is the corresponding (D, ∞, o)-type, induces a bijection from the set of isogeny classes of $\mathscr{D}$-elliptic sheaves (of characteristic o) over k onto the set of isomorphism classes of (D, ∞, o)-types.

Moreover, the inverse bijection is induced by the construction (9.12).

Proof. Most of the theorem is already proved. The only non trivial part which is left is the surjectivity of the map. More precisely, let $(\tilde{F}, \tilde{\Pi})$ be a (D, ∞, o)-type and let (V, φ, ι) be "the" triple which corresponds to $(\tilde{F}, \tilde{\Pi})$ by the construction (9.12). We want to prove that there exists at least one $\mathscr{D}$-elliptic sheaf $(\mathscr{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ (of characteristic o) over k with generic fibre isomorphic to (V, φ, ι).

Thanks to (9.4), we are reduced to finding a collection $(M_x)_{x\in|X|}$ of $\mathscr{D}_x$-lattices which satisfy the properties (9.3) (i) to (iv). Let us fix a basis of the $F\otimes k$-vector space V. Then there exists a finite set of places Σ of F containing ∞, o and Bad such that, if $x\in|X|\backslash\Sigma$, the $\mathcal{O}_x$-submodule M_x of V_x generated by this basis is a $\mathscr{D}_x$-lattice with $\varphi_x(M_x) = M_x$. If $x\in\Sigma\backslash\{\infty, o\}$, (V_x, φ_x) is isomorphic to $(V_x^{\varphi_x}\hat{\otimes} k, V_x^{\varphi_x}\hat{\otimes}\,\mathrm{frob}_q)$ (see (9.11) (iii), (B.4) and (B.6)) and any finitely generated $\mathscr{D}_x$-submodule of the free D_x-module of rank one, $V_x^{\varphi_x}$, induces a $\mathscr{D}_x$-lattice $M_x\subset V_x$ such that $\varphi_x(M_x) = M_x$. Finally, if $x = \infty$ (resp. $x = o$), we have seen that

$$(V_\infty, \varphi_\infty) = (V'_\infty, \varphi'_\infty)^d$$

(resp. $\qquad (V_o, \varphi_o) = (V'_o, \varphi'_o)^d)$

as a module over $D_\infty = \mathbb{M}_d(F_\infty)$ (resp. $D_o = \mathbb{M}_d(F_o)$). But thanks to (B.4), we have

$$(V'_\infty, \varphi'_\infty) \simeq (N_{d,-1}, \psi_{d,-1})$$

and

$$(V'_o, \varphi'_o) \simeq (N_{1,0}, \psi_{1,0})^{d-h} \oplus (N_{h,1}, \psi_{h,1})$$

where

$$h = d[\tilde{F}_{\tilde{o}}:F_o]/[\tilde{F}:F]$$

and thanks to (B.8) there exists a lattice $M'_\infty\subset V'_\infty$ (resp. $M'_o\subset V'_o$) such that

$$M'_\infty \subset \varphi'_\infty(M'_\infty)$$

$$(\varphi'_\infty)^{d\cdot\deg(\infty)}(M'_\infty) = \varpi_\infty^{-1} M'_\infty$$

$$\dim_k(\varphi'_\infty(M'_\infty)/M'_\infty) = 1$$

(resp. $\qquad \varpi_o M'_o \subset \varphi'_o(M'_o) \subset M'_o$

$$\dim_k (M'_o/\varphi'_o(M'_o)) = 1$$

and the support of $M'_o/\varphi'_o(M'_o)$ is the connected component of

$$\mathrm{Spec}(\kappa(o)\otimes k) \subset \mathrm{Spec}(\mathcal{O}_o\otimes k)$$

which corresponds to the given embedding $\kappa(o) \hookrightarrow k$)

for some uniformizer ϖ_∞ (resp. ϖ_o) of $\mathcal{O}_\infty$ (resp. $\mathcal{O}_o$). We can take $M_\infty = M'^d_\infty$ (resp. $M_o = M'^d_o$) to complete our collection of $\mathcal{D}_x$-lattices. □

10 $\mathcal{D}$-elliptic sheaves in finite characteristic: Description of an isogeny class

Let us fix a (D, ∞, o)-type $(\tilde{F}, \tilde{\Pi})$. We will describe the set

$$\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \subset \mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)$$

of isomorphism classes of $\mathcal{D}$-elliptic sheaves (of characteristic o) over k which are in the isogeny class corresponding to $(\tilde{F}, \tilde{\Pi})$.

(10.1) Let us fix a φ-space with D-action (V, φ, ι) with (D, ∞, o)-type $(\tilde{F}, \tilde{\Pi})$ (see (9.12)) and let $\Delta = \mathrm{End}(V, \varphi, \iota)$. For each place x of F we have the corresponding Dieudonné F_x-module with D_x-action

$$(V_x, \varphi_x, \iota_x) = (F_x \hat{\otimes}_F V, F_x \hat{\otimes}_F \varphi, F_x \hat{\otimes}_F \iota) .$$

Let $\mathcal{Y}_x$ be the set of $\mathcal{D}_x$-lattices M_x in V_x which satisfy the property (9.3)(i) if $x = \infty$ (resp. (9.3)(ii) if $x = o$, resp. (9.3)(iii) if $x \neq \infty, o$) and let

$$\mathcal{Y}_\emptyset^{\infty,o} \subset \prod_{x \neq \infty, o} \mathcal{Y}_x$$

be the set of families of lattices which satisfy the extra condition (9.3)(iv). Then we have a natural left action of $\Delta^\times$ on the set

$$\mathcal{Y}_{\mathbb{A},\emptyset} := \mathcal{Y}_\infty \times \mathcal{Y}_\emptyset^{\infty,o} \times \mathcal{Y}_o$$

and it follows from (9.4) that we have a natural bijection

$$\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^\times \backslash \mathcal{Y}_{\mathbb{A},\emptyset} .$$

(10.2) Now, we will give a more concrete description of $\mathcal{Y}_{\mathbb{A},\emptyset}$. Let us begin with its part $\mathcal{Y}_\emptyset^{\infty,o}$. Let $D^{\infty,o}$ be the restricted product of the D_x's with respect to the $\mathcal{D}_x$'s for all places $x \neq \infty, o$. Let $(M_x \subset V_x)_{x \neq \infty,o}$ be a base point of $\mathcal{Y}_\emptyset^{\infty,o}$. Then, we can form the restricted product $(V^{\infty,o}, \varphi^{\infty,o})$ of the (V_x, φ_x)'s with respect to the M_x's for all places $x \neq \infty, o$. Thanks to the condition (9.3)(iv), this restricted product $(V^{\infty,o}, \varphi^{\infty,o})$ is independent of the choice of the base point $(M_x \subset V_x)_{x \neq \infty,o}$ and, thanks to the condition (9.3)(iii), the canonical map

$$(V^{\infty,o})^{\varphi^{\infty,o}} \hat{\otimes}_{\mathbb{F}_q} k \to V^{\infty,o}$$

is bijective. Obviously, $(V^{\infty,o})^{\varphi^{\infty,o}}$ is a free right $D^{\infty,o}$-module of rank one. Let us fix a basis of this $D^{\infty,o}$-module. Then we get a left action of $(D^{\infty,o})^\times$ on $\mathcal{Y}_\emptyset^{\infty,o}$ ($D^{\infty,o}$ acts by left multiplication on the right $D^{\infty,o}$-module $V^{\infty,o} = D^{\infty,o} \hat{\otimes} k$). As

$$\mathcal{D}^{\infty,o} = \prod_{x \neq \infty, o} \mathcal{D}_x$$

is a maximal order of $D^{\infty,o}$, this action is transitive and our choice of a base point of $\mathcal{Y}_\emptyset^{\infty,o}$ gives an identification

$$\mathcal{Y}_\emptyset^{\infty,o} \xrightarrow{\sim} (D^{\infty,o})^\times / (\mathcal{D}^{\infty,o})^\times .$$

Moreover, the action of Δ on the right $D^{\infty,o}$-module $(V^{\infty,o})^{\varphi^{\infty,o}} = D^{\infty,o}$ gives an embedding $\Delta \hookrightarrow D^{\infty,o}$ through which the left action of $\Delta^{\times}$ on $\mathcal{Y}_{\emptyset}^{\infty,o}$ factors in an obvious way if we use the above description of $\mathcal{Y}_{\emptyset}^{\infty,o}$.

Next, let us consider $\mathcal{Y}_{\infty}$. We have a canonical splitting

$$(V_{\infty}, \varphi_{\infty}) = (V'_{\infty}, \varphi'_{\infty})^{d}$$

thanks to the action of $D_{\infty} = \mathbb{M}_{d}(F_{\infty})$ and $\mathcal{Y}_{\infty}$ can be identified with the set of lattices M'_{∞} in V'_{∞} such that

$$\begin{cases} M'_{\infty} \subset \varphi'_{\infty}(M'_{\infty}) \\ (\varphi'_{\infty})^{d \cdot \deg(\infty)}(M'_{\infty}) = \varpi_{\infty}^{-1} M'_{\infty} \\ \dim_{k}(\varphi'_{\infty}(M'_{\infty})/M'_{\infty}) = 1 \, . \end{cases}$$

But, thanks to (B.10), this set of lattices is a principal homogeneous set under $\mathbb{Z}$ ($m \in \mathbb{Z}$ maps M'_{∞} into $(\varphi'_{\infty})^{m}(M'_{\infty})$). Let us identify (non canonically) $\mathcal{Y}_{\infty}$ with $\mathbb{Z}$. Moreover, the action of Δ on $(V_{\infty}, \varphi_{\infty})$ commutes with the action of $D_{\infty} = \mathbb{M}_{d}(F_{\infty})$ and is therefore induced by an action of Δ on $(V'_{\infty}, \varphi'_{\infty})$. So we get an embedding

$$\Delta \hookrightarrow \mathrm{End}(V'_{\infty}, \varphi'_{\infty}) \, .$$

Thanks to (B.11), $\mathrm{End}(V'_{\infty}, \varphi'_{\infty})$ acts on $\mathcal{Y}_{\infty}$ through the homomorphism

$$\mathrm{End}(V'_{\infty}, \varphi'_{\infty})^{\times} \xrightarrow{\mathrm{rn}} F_{\infty}^{\times} \xrightarrow{-\deg(\infty)\infty(-)} \mathbb{Z} \, .$$

Therefore, $\Delta^{\times}$ acts on $\mathcal{Y}_{\infty}$ through the homomorphism

$$\Delta^{\times} \hookrightarrow \mathrm{End}(V'_{\infty}, \varphi'_{\infty})^{\times} \xrightarrow{\mathrm{rn}} F_{\infty}^{\times} \xrightarrow{-\deg(\infty)\infty(-)} \mathbb{Z} \, ,$$

which is nothing else than the homomorphism

$$\Delta^{\times} \xrightarrow{\mathrm{rn}} \tilde{F}^{\times} \xrightarrow{-\deg(\tilde{\infty})\tilde{\infty}(-)} \mathbb{Z} \, .$$

The description of $\mathcal{Y}_{o}$ with its left action of $\Delta^{\times}$ is similar and we let the reader check the details. Using the action of $D_{o} = \mathbb{M}_{d}(F_{o})$ we can split (V_{o}, φ_{o}) into

$$(V_{o}, \varphi_{o}) = (V'_{o}, \varphi'_{o})^{d} \, .$$

Using the action of $\tilde{F}$ we can split (V'_{o}, φ'_{o}) into

$$(V'_{o}, \varphi'_{o}) = (V_{o}'^{\tilde{o}}, \varphi_{o}'^{\tilde{o}}) \oplus (V'_{\tilde{o}}, \varphi'_{\tilde{o}}) \, .$$

Now, we can identify $\mathcal{Y}_{o}$ with the product $\mathcal{Y}_{o}^{\tilde{o}} \times \mathcal{Y}_{\tilde{o}}$, where $\mathcal{Y}_{o}^{\tilde{o}}$ is the set of lattices $M_{o}'^{\tilde{o}}$ in $V_{o}'^{\tilde{o}}$ such that

$$\varphi_{o}'^{\tilde{o}}(M_{o}'^{\tilde{o}}) = M_{o}'^{\tilde{o}}$$

and where $\mathcal{Y}_{\tilde{o}}$ is the set of lattices $M'_{\tilde{o}}$ in $V'_{\tilde{o}}$ such that

$$\begin{cases} \varpi_{o} M'_{\tilde{o}} \subset \varphi'_{\tilde{o}}(M'_{\tilde{o}}) \subset M'_{\tilde{o}} \\ \dim_{k}(M'_{\tilde{o}}/\varphi'_{\tilde{o}}(M'_{\tilde{o}})) = 1 \end{cases}$$

and the support of $M'_{\tilde{o}}/\varphi'_{\tilde{o}}(M'_{\tilde{o}})$ is the connected component of

$$\mathrm{Spec}(\kappa(o) \otimes k) \subset \mathrm{Spec}(\mathcal{O}_{o} \otimes k)$$

which corresponds to the given embedding $\kappa(o) \hookrightarrow k$ (see (B.8)). Then, as for $\mathcal{Y}_{\emptyset}^{\infty,o}$, we get an identification

$$\mathcal{Y}_{o}^{\tilde{o}} \xrightarrow{\sim} \mathrm{GL}_{d-h}(F_{o})/\mathrm{GL}_{d-h}(\mathcal{O}_{o})$$

and an embedding of algebras

$$\Delta \hookrightarrow \mathbb{M}_{d-h}(F_o)$$

through which the action of $\Delta^\times$ on $\mathcal{Y}_o^{\tilde{o}}$ factors in an obvious way if we use the above identification. Here

$$h = [\tilde{F}_{\tilde{o}} : F_o] d / [\tilde{F} : F]$$

(see (9.9)). As $\mathcal{Y}_\infty$, $\mathcal{Y}_{\tilde{o}}$ is a principal homogeneous set under $\mathbb{Z}$ ($m \in \mathbb{Z}$ maps $M'_{\tilde{o}}$ into $(\varphi'_{\tilde{o}})^{m \cdot \deg(o)}(M'_{\tilde{o}})$; the extra component $\deg(o)$ comes from the condition on the support and we can identify (non canonically) $\mathcal{Y}_{\tilde{o}}$ with $\mathbb{Z}$. The action of $\Delta^\times$ on $\mathcal{Y}_{\tilde{o}} = \mathbb{Z}$ factors through the homomorphism

$$\Delta^\times \hookrightarrow \mathrm{End}(V'_{\tilde{o}}, \varphi'_{\tilde{o}})^\times \xrightarrow{\mathrm{rn}} F_o^\times \xrightarrow{o(-)} \mathbb{Z} ,$$

which is nothing else than the homomorphism

$$\Delta^\times \xrightarrow{\mathrm{rn}} \tilde{F}^\times \xrightarrow{\deg(\tilde{o})/\deg(o))\tilde{o}(-)} \mathbb{Z} .$$

(10.3) We can summarize the above results in the following way. Let us start with our (D, ∞, o)-type $(\tilde{F}, \tilde{\Pi})$ and our algebraic closure k of $\kappa(o)$. Then we have the two places $\tilde{\infty}$, $\tilde{o}$ of $\tilde{F}$, the integer $h = [\tilde{F}_{\tilde{o}} : F_o] d / [\tilde{F} : F]$, "the" central division algebra Δ over $\tilde{F}$ with invariants

$$\mathrm{inv}_{\tilde{x}}(\Delta) = \begin{cases} [\tilde{F} : F]/d & \text{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F} : F]/d & \text{if } \tilde{x} = \tilde{o} \\ [\tilde{F}_{\tilde{x}} : F_x] \mathrm{inv}_x(D) & \text{otherwise} \end{cases}$$

for each place x of F and each place $\tilde{x}$ of $\tilde{F}$ dividing x.

Then, let us arbitrarily fix an embedding of $\mathbb{A}^{\infty, o}$-algebras

$$\mathbb{A}^{\infty, o} \otimes_F \tilde{F} \hookrightarrow D^{\infty, o} ,$$

an embedding of F_∞-algebras

$$\tilde{F}_{\tilde{\infty}} \hookrightarrow \mathrm{End}(N_{d,-1}, \psi_{d,-1}) ,$$

where $\mathrm{End}(N_{d,-1}, \psi_{d,-1})$ is "the" central division algebra over F_∞ with invariant $1/d$, an embedding of F_o-algebras

$$\tilde{F}_o^{\tilde{o}} \hookrightarrow \mathbb{M}_{d-h}(F_o)$$

and an embedding of F_o-algebras

$$\tilde{F}_{\tilde{o}} \hookrightarrow \mathrm{End}(N_{h,1}, \psi_{h,1}) ,$$

where $\mathrm{End}(N_{h,1}, \psi_{h,1})$ is "the" central division algebra over F_o with invariant $-1/h$ (this is always possible and different choices of these embeddings are conjugate). Now, we can and we will identify $\mathbb{A}^{\infty, o} \otimes_F \Delta$ (resp. $\Delta_{\tilde{\infty}}$, resp. $\Delta_o^{\tilde{o}}$, resp. $\Delta_{\tilde{o}}$) with the centralizer of the image of $\mathbb{A}^{\infty, o} \otimes_F \tilde{F}$ (resp. $\tilde{F}_{\tilde{\infty}}$, resp. $\tilde{F}_o^{\tilde{o}}$, resp. $\tilde{F}_{\tilde{o}}$) by the above embedding. In particular, we get group homomorphisms

$$\begin{cases} \Delta^\times \hookrightarrow (D^{\infty, o})^\times \\ \Delta^\times \hookrightarrow \mathrm{End}(N_{d,-1}, \psi_{d,-1})^\times \xrightarrow{\mathrm{rn}} F_\infty^\times \xrightarrow{-\deg(\infty)\infty(-)} \mathbb{Z} \\ \Delta^\times \hookrightarrow \mathrm{GL}_{d-h}(F_o) \\ \Delta^\times \hookrightarrow \mathrm{End}(N_{h,1}, \psi_{h,1})^\times \xrightarrow{\mathrm{rn}} F_o^\times \xrightarrow{o(-)} \mathbb{Z} \end{cases}$$

(the second and the last ones coincide with

$$\Delta^{\times} \xrightarrow{\mathrm{rn}} \tilde{F}^{\times} \xrightarrow{-\deg(\tilde{\infty})\tilde{\infty}(-)} \mathbb{Z}$$

and

$$\Delta^{\times} \xrightarrow{\mathrm{rn}} \tilde{F}^{\times} \xrightarrow{\deg(\tilde{o})/\deg(o))o(-)} \mathbb{Z}$$

respectively) through which $\Delta^{\times}$ acts on

$$\begin{cases} Y_{\emptyset}^{\infty,o} = (D^{\infty,o})^{\times}/(\mathscr{D}^{\infty,o})^{\times} \\ Y_{\infty} = \mathbb{Z} \\ Y_{o}^{\tilde{o}} = \mathrm{GL}_{d-h}(F_{o})/\mathrm{GL}_{d-h}(\mathscr{O}_{o}) \\ Y_{\tilde{o}} = \mathbb{Z} \end{cases}$$

and therefore on

$$Y_{\mathbb{A},\emptyset} = Y_{\infty} \times Y_{\emptyset}^{\infty,o} \times Y_{o}^{\tilde{o}} \times Y_{\tilde{o}} \,.$$

The above constructions give a (non canonical) bijection

$$\mathscr{E}\ell\ell_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^{\times}\backslash Y_{\mathbb{A},\emptyset} \,.$$

In itself this statement cannot be used. But we will now make it more precise and more useful by looking at the structures that we have on $\mathscr{E}\ell\ell_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$.

(10.4) Let us begin with the action of the Hecke operators. Let $I \subset X$ be a finite closed subscheme such that $I \cap \{\infty, o\} = \emptyset$ and let $\mathscr{I} \subset \mathscr{O}_X$ be the corresponding ideal. Then we have a Galois etale finite covering

$$r_{I,\emptyset,o}\colon \mathscr{E}\ell\ell_{X,\mathscr{D},I,o} \longrightarrow \mathscr{E}\ell\ell_{X,\mathscr{D},o}$$

with Galois group

$$K_{I}^{\infty,o} = \mathrm{Ker}\,((\mathscr{D}^{\infty,o})^{\times} \to (\mathscr{I}^{\infty,o}\mathscr{D}^{\infty,o}\backslash\mathscr{D}^{\infty,o})^{\times}) \,.$$

In terms of the description (9.4) of $\mathscr{E}\ell\ell_{X,\mathscr{D},o}(k)$, the map

$$r_{I,\emptyset,o}(k)\colon \mathscr{E}\ell\ell_{X,\mathscr{D},I,o}(k) \to \mathscr{E}\ell\ell_{X,\mathscr{D},o}(k)$$

can be described in the following way. The set $\mathscr{E}\ell\ell_{X,\mathscr{D},I,o}(k)$ is the set of isomorphism classes of triples

$$((V, \varphi, \imath), (M_x)_{x\in|X|}, (\alpha_x)_{x\in I})$$

where $(V, \varphi, \imath)$ and $(M_x)_{x\in|X|}$ are as in (9.4) and where

$$\alpha_x\colon \mathscr{I}_x\mathscr{D}_x\backslash\mathscr{D}_x \xrightarrow{\sim} \mathscr{I}_x M_x^{\varphi_x}\backslash M_x^{\varphi_x}$$

is an isomorphism of right $\mathscr{D}_x$-modules for each $x\in I$ (recall that, for each $x \neq \infty, o$, the canonical map

$$(M_x^{\varphi_x} \hat{\otimes}_{\mathbb{F}_q} k,\ M_x^{\varphi_x} \hat{\otimes}_{\mathbb{F}_q} \mathrm{frob}_q) \to (M_x, \varphi_x)$$

is an isomorphism); $r_{I,\emptyset,o}(k)$ maps

$$((V, \varphi, \imath), (M_x)_{x\in|X|}, (\alpha_x)_{x\in I})$$

into

$$((V, \varphi, \imath), (M_x)_{x\in|X|}) \,.$$

Therefore, in terms of the above description of the isogeny class $\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$, the restriction

$$r_{I,\emptyset,o}(k)_{(\tilde{F},\tilde{\Pi})}:\mathcal{E}\ell\ell_{X,\mathcal{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})}\to\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$$

of $r_{I,\emptyset,o}(k)$ to this isogeny class can be described in the following way. There is a (noncanonical) bijection between $\mathcal{E}\ell\ell_{X,\mathcal{D},I,\mathrm{o}}(k)_{(\tilde{F},\tilde{\Pi})}$ and the quotient set,

$$\Delta^{\times}\backslash Y_{\mathbb{A},I}$$

where

$$Y_{\mathbb{A},I}=Y_{\infty}\times Y_{I}^{\infty,o}\times Y_{o}^{\tilde{o}}\times Y_{\tilde{o}}$$

and

$$Y_{I}^{\infty,o}=(D^{\infty,o})^{\times}/K_{I}^{\infty,o}$$

($\Delta^{\times}$ acts through the embedding $\Delta^{\times}\hookrightarrow(D^{\infty,o})^{\times}$ on $Y_{I}^{\infty,o}$ and as before on Y_{∞}, $Y_{o}^{\tilde{o}}$, $Y_{\tilde{o}}$); $r_{I,\emptyset,o}(k)_{(\tilde{F},\tilde{\Pi})}$ maps

$$\Delta^{\times}[m_{\infty},h^{\infty,o}K_{I}^{\infty,o},h_{o}^{\tilde{o}}\mathrm{GL}_{d-h}(\mathcal{O}_{o}),m_{\tilde{o}}]$$

into

$$\Delta^{\times}[m_{\infty},h^{\infty,o}K_{\emptyset}^{\infty,o},h_{o}^{\tilde{o}}\mathrm{GL}_{d-h}(\mathcal{O}_{o}),m_{\tilde{o}}]$$

$(K_{\emptyset}^{\infty,o}=(\mathcal{D}^{\infty,o})^{\times})$.

Morever, if $J\subset X$ is another finite closed subscheme such that $J\cap\{\infty,o\}=\emptyset$ and if $I\subset J$, we have a Galois etale finite covering

$$r_{J,I,o}:\mathcal{E}\ell\ell_{X,\mathcal{D},J,o}\to\mathcal{E}\ell\ell_{X,\mathcal{D},I,o}$$

with Galois group

$$K_{I}^{\infty,o}/K_{J}^{\infty,o}$$

such that

$$r_{J,\emptyset,o}=r_{I,\emptyset,o}\circ r_{J,I,o}\ .$$

Then, $r_{J,I,o}(k)$ maps $\mathcal{E}\ell\ell_{X,\mathcal{D},J,o}(k)_{(\tilde{F},\tilde{\Pi})}$ into $\mathcal{E}\ell\ell_{X,\mathcal{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ and it is clear that we can choose the bijections

$$\mathcal{E}\ell\ell_{X,\mathcal{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})}\xrightarrow{\sim}\Delta^{\times}\backslash Y_{\mathbb{A},I}$$

and

$$\mathcal{E}\ell\ell_{X,\mathcal{D},J,o}(k)_{(\tilde{F},\tilde{\Pi})}\xrightarrow{\sim}\Delta^{\times}\backslash Y_{\mathbb{A},J}$$

as above in such way that the restriction $r_{J,I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ of $r_{J,I,o}(k)$ is induced by the canonical map $Y_{J}^{\infty,o}\to Y_{I}^{\infty,o}$ and the identity maps of Y_{∞}, $Y_{o}^{\tilde{o}}$ and $Y_{\tilde{o}}$.

Finally, let T be a finite set of places of F containing $\{\infty,o\}$. Let us set

$$\mathcal{E}\ell\ell_{X,\mathcal{D},o}^{T}(k)_{(\tilde{F},\tilde{\Pi})}=\varprojlim_{I\cap T=\emptyset}(\mathcal{E}\ell\ell_{X,\mathcal{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})},r_{J,I,o}(k)_{(\tilde{F},\tilde{\Pi})})\ .$$

Then we leave it to the reader to check that we can find a bijection of $\mathcal{E}\ell\ell_{X,\mathcal{D},o}^{T}(k)_{(\tilde{F},\tilde{\Pi})}$ on the quotient set

$$\Delta^{\times}\backslash Y_{\mathbb{A}}^{T}$$

where

$$Y_{\mathbb{A}}^{T}=Y_{\infty}\times Y^{\infty,o,T}\times Y_{o}^{\tilde{o}}\times Y_{\tilde{o}}$$

and

$$Y^{\infty,o,T}=(D^{\infty,o})^{\times}/\mathcal{D}_{T\backslash\{\infty,o\}}^{\times}=(D^{T})^{\times}\times D_{T\backslash\{\infty,o\}}^{\times}/\mathcal{D}_{T\backslash\{\infty,o\}}^{\times})$$

such that the right action of $(D^T)^\times$ on $\mathcal{E}\ell\ell^T_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ which is described in (7.4) is identified with the obvious right action of $(D^T)^\times$ on $\Delta^\times\backslash Y^T_{\mathbb{A}}$ (here $(D^T)^\times$ acts by right translation on itself and by the identity on the other factors of $Y^T_{\mathbb{A}}$).

Now let $K^T \subset (D^T)^\times$ be an open compact subgroup. For any $g^T \in (D^T)^\times$, the Hecke correspondence

$$\begin{tikzcd} & \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}/(K^T \cap (g^T)^{-1}K^T g^T) \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}/K^T & & \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}/K^T \arrow[ll, dashed] \end{tikzcd},$$

which is described in (7.5), induces a correspondence

$$\begin{tikzcd} & \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}/(K^T \cap (g^T)^{-1}K^T g^T) \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}/K^T & & \mathcal{E}\ell\ell^T_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}/K^T \arrow[ll, dashed] \end{tikzcd}.$$

Then, this last correspondence is isomorphic to the correspondence

$$\begin{tikzcd} & \Delta^\times\backslash Y^T_{\mathbb{A}}/(K^T \cap (g^T)^{-1}K^T g^T) \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \Delta^\times\backslash Y^T_{\mathbb{A}}/K^T & & \Delta^\times\backslash Y^T_{\mathbb{A}}/K^T \arrow[ll, dashed] \end{tikzcd}$$

where c_1 is induced by the identity of $Y^T_{\mathbb{A}}$ and c_2 is induced by the right action of $(g^T)^{-1}$ on $Y^T_{\mathbb{A}}$.

In practice, we will take $T = \{\infty, o\}$.

(10.5) In terms of the description $\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^\times\backslash Y_{\mathbb{A}}$ or $\mathcal{E}\ell\ell^T_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^\times\backslash Y^T_{\mathbb{A}}$, the pole map is simply induced by the canonical map

$$Y_\infty = \mathbb{Z} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z} = \mathrm{Spec}(\kappa(\infty))(k)$$

(as we have fixed an origin in Y_∞, we have a corresponding origin in $\mathrm{Hom}_{\mathbb{F}_q\text{-alg}}(\kappa(\infty), k)$).

Similarly, the action of $\mathbb{Z}$ by translations of the indices on the moduli space of $\mathcal{D}$-elliptic sheaves is induced by the action by translations of $\mathbb{Z}$ on $Y_\infty = \mathbb{Z}$. Indeed, in terms of the description (9.4) of the moduli space, $n \in \mathbb{Z}$ maps

$$((V, \varphi, \iota), (M_x)_{x\in|X|})$$

into

$$((V, \varphi, \iota), \varphi^n_\infty(M_\infty), (M_x)_{x\in|X|\setminus\{\infty\}}) .$$

Let $\mathrm{Frob}_o \in \mathrm{Gal}(k/\kappa(o))$ be the geometric Frobenius ($\mathrm{Frob}_o = \mathrm{frob}_q^{-\deg(o)}$). It acts on the $\mathcal{D}$-elliptic sheaves of characteristic o over k by

$$(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \mapsto {}^{\tau^{\deg(o)}}(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} .$$

So, in terms of the description (9.4), it maps

$$((V, \varphi, \iota), (M_x)_{x\in|X|})$$

into

$$((V, \varphi, \iota), \varphi^{\deg(o)}_\infty(M_\infty), (M_x)_{x\in|X|\setminus\{\infty, o\}}, \varphi^{\deg(o)}_o(M_o)) .$$

Therefore, in terms of the description $\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^{\times}\backslash Y_{\mathbb{A}}$ or $\mathcal{E}\ell\ell^{T}_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^{\times}/Y^{T}_{\mathbb{A}}$, the action of Frob_o is induced by the translation of $(\deg(o), 1)$ on $Y_{\infty} \times Y_{\tilde{o}} = \mathbb{Z} \times \mathbb{Z}$.

(10.6) Let us now consider the level structures at infinity on $\mathcal{D}$-elliptic sheaves of characteristic o over k. We have a pro-finite, pro-etale and pro-Galois covering

$$r_{\infty,o} : \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},o} \to \mathcal{E}\ell\ell_{X,\mathcal{D},o}$$

with pro-finite Galois group $\bar{D}^{\times}_{\infty}/\varpi^{\mathbb{Z}}_{\infty}$. In terms of the description (9.4), the map

$$r_{\infty,o}(k) : \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},o}(k) \to \mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)$$

can be described in the following way. For each

$$((V, \varphi, \imath), (M_x)_{x \in |X|}) \in \mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)$$

let

$$i_{\infty,0} : \mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(\infty))$$

be the pole of this $\mathcal{D}$-elliptic sheaf, i.e. the support of the $\kappa(\infty) \otimes k$-module $\varphi_{\infty}(M_{\infty})/M_{\infty}$, and let $\check{M}'_{\infty}$ be the dual of the free $\mathcal{O}_{\infty} \hat{\otimes} k$-module of constant rank d, M'_{∞}, and $\check{\psi}'_{\infty} : \check{M}'_{\infty} \to \check{M}'_{\infty}$ be the restriction to $\check{M}'_{\infty}$ of the $F_{\infty} \hat{\otimes} \mathrm{frob}_q$-semilinear map $(\check{\varphi}'_{\infty})^{-1} : \check{V}'_{\infty} \to \check{V}'_{\infty}$.

Here we have split $(V_{\infty}, \varphi_{\infty})$ and M_{∞} into $(V'_{\infty}, \varphi'_{\infty})^{d}$ and $(M'_{\infty})^{d}$ using the identification $\mathcal{D}_{\infty} = \mathbb{M}_{d}(\mathcal{O}_{\infty})$ and $\check{\varphi}'_{\infty} : \check{V}'_{\infty} \to \check{V}'_{\infty}$ is the dual map of φ'_{∞}. Then, the set $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},o}(k)$ is the set of isomorphism classes of triples

$$((V, \varphi, \imath), (M_x)_{x \in |X|}, (\lambda, \alpha))$$

where $((V, \varphi, \imath), (M_x)_{x \in |X|})$ belongs to $\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)$ and

$$\lambda : \mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(\infty)_d)$$

is a lifting of $i_{\infty,0}$ and

$$\alpha : M_{d,1}(i_{\infty,0}) \xrightarrow{\sim} \check{M}'_{\infty}$$

is an isomorphism of $\mathcal{O}_{\infty} \hat{\otimes} k$-modules which commutes with the ψ's; $r_{\infty,o}(k)$ maps

$$((V, \varphi, \imath), (M_x)_{x \in |X|}, (\lambda, \alpha))$$

into

$$((V, \varphi, \imath), (M_x)_{x \in |X|}) ,$$

and

$$\bar{D}^{\times}_{\infty} = \mathcal{D}^{\times}_{\infty} \rtimes \mathbb{Z}/d\mathbb{Z}$$

acts on the pair (λ, α) as described in (8.6).

Therefore, in terms of the description $\Delta^{\times}\backslash Y_{\mathbb{A},\emptyset}$ of the isogeny class $\mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$, the restriction

$$r_{\infty,o}(k)_{(\tilde{F},\tilde{\Pi})} : \widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \to \mathcal{E}\ell\ell_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$$

of $r_{\infty,o}(k)$ to this isogeny class can be described in the following way. We set

$$\tilde{Y}_{\infty} = Y_{\infty} \times (\bar{D}^{\times}_{\infty}/\varpi^{\mathbb{Z}}_{\infty}) = \mathbb{Z} \times (\bar{D}^{\times}_{\infty}/\varpi^{\mathbb{Z}}_{\infty})$$

with the action by right translations of $\bar{D}^{\times}_{\infty}/\varpi^{\mathbb{Z}}_{\infty}$ on its second factor. We let $\Delta^{\times}$ act on $\tilde{Y}_{\infty}$ (left action) in the following way. We have fixed an embedding of algebras

$$\Delta \hookrightarrow \mathrm{End}(N_{d,-1}, \psi_{d,-1}) .$$

Let us identify $\mathrm{End}(N_{d,-1}, \psi_{d,-1})$ with $(\bar{D}_\infty)^{\mathrm{op}}$ (both are central division algebras over F_∞ with invariant $1/d$), so that we get an embedding of algebras $\Delta \hookrightarrow (\bar{D}_\infty)^{\mathrm{op}}$ and an embedding of groups $\Delta^\times \hookrightarrow ((\bar{D}_\infty)^{\mathrm{op}})^\times \xrightarrow{\sim} \bar{D}_\infty^\times$.

Here the last isomorphism is given by $g \mapsto g^{-1}$. Then $\Delta^\times$ acts on the first factor $Y_\infty = \mathbb{Z}$ by translations through the group homomorphism

$$\Delta^\times \hookrightarrow \bar{D}_\infty^\times \xrightarrow{\mathrm{rn}} F_\infty^\times \xrightarrow{\deg(\infty)\infty(-)} \mathbb{Z}$$

and on the second factor $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ by left translations through the group homomorphism $\Delta^\times \hookrightarrow \bar{D}_\infty^\times$. Note that the group homomorphism

$$\Delta^\times \hookrightarrow \mathrm{End}(N_{d,-1}, \psi_{d,-1})^\times \xrightarrow{\mathrm{rn}} F_\infty^\times \xrightarrow{-\deg(\infty)\infty(-)} \mathbb{Z}$$

coincides with the one given above.

Again, we leave it to the reader to construct a bijection of $\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ onto the quotient set $\Delta^\times\backslash\tilde{Y}_{\mathbb{A},\emptyset}$, where $\tilde{Y}_{\mathbb{A},\emptyset} = \tilde{Y}_\infty \times Y_\emptyset^{\infty,o} \times Y_o^{\tilde{o}} \times Y_{\tilde{o}}$, such that the right action of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on $\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the right action of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on $\tilde{Y}_\infty$ that we have described above and such that the map $r_{\infty,o}(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the projection of $\tilde{Y}_\infty$ onto its first factor Y_∞.

Similarly, we have $(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})$-equivalent (non canonical) bijections

$$\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^\times\backslash\tilde{Y}_{\mathbb{A},I}$$

and

$$\widetilde{\mathscr{E}\ell\ell}^T_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^\times\backslash\tilde{Y}^T_{\mathbb{A}}$$

for any finite closed subscheme $I \subset X\backslash\{\infty, o\}$ and any finite set T of places of F containing $\{\infty, o\}$. Here, we have set

$$\tilde{Y}_{\mathbb{A},I} = \tilde{Y}_\infty \times Y_I^{\infty,o} \times Y_o^{\tilde{o}} \times Y_{\tilde{o}}$$

and

$$\tilde{Y}^T_{\mathbb{A}} = \tilde{Y}_\infty \times Y^{\infty,o,T} \times Y_o^{\tilde{o}} \times Y_{\tilde{o}}\,.$$

The maps $r_{\infty,I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ and $r^T_{\infty,o}(k)_{(\tilde{F},\tilde{\Pi})}$ are induced by the projection of $\tilde{Y}_\infty$ onto its first factor Y_∞. The maps $\tilde{r}_{I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ and more generally $\tilde{r}_{J,I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ are induced by the canonical maps $Y_I^{\infty,o} \twoheadrightarrow Y_\emptyset^{\infty,o}$ and $Y_J^{\infty,o} \twoheadrightarrow Y_I^{\infty,o}$. The action of $(D^T)^\times$ on $\widetilde{\mathscr{E}\ell\ell}^T_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the action of $(D^T)^\times$ on $Y^{\infty,o,T}$.

As in (10.4), we can give an explicit description of the Hecke correspondences for $\widetilde{\mathscr{E}\ell\ell}^T_{X,\mathscr{D},o}$ in each isogeny class.

The map

$$\lambda_o(k)_{(\tilde{F},\tilde{\Pi})}: \widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})} \to \mathrm{Spec}(\kappa(\infty)_d)$$

and the similar maps $\lambda_{I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ and $\lambda_o^T(k)_{(\tilde{F},\tilde{\Pi})}$ are induced by the map

$$\tilde{Y}_\infty = \mathbb{Z} \times \bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/d\cdot\deg(\infty)\mathbb{Z}$$

which sends (n, δ) into

$$n - \deg(\infty)\infty(\mathrm{rn}(\delta)) \quad (\text{modulo } d\cdot\deg(\infty)\mathbb{Z})$$

(as we have fixed an origin in $\tilde{Y}_\infty$, we have a corresponding origin in $\mathrm{Hom}_{\mathbb{F}_q\text{-alg}}(\kappa(\infty)_d, k)$).

The action (8.9) of $\mathbb{Z}$ on $\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ or on $\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ or on $\widetilde{\mathscr{E}\ell\ell}^T_{X,\mathscr{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the action by translations of $\mathbb{Z}$ on the first factor $Y_\infty = \mathbb{Z}$ of $\tilde{Y}_\infty$.

The action of the geometric Frobenius Frob_o on $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ or on $\widetilde{\mathcal{E}\ell\ell}_{X,\mathcal{D},I,o}(k)_{(\tilde{F},\tilde{\Pi})}$ or on $\widetilde{\mathcal{E}\ell\ell}^{T}_{X,\mathcal{D},o}(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the translation of $(\deg(o), 1)$ on $Y_\infty \times Y_{\tilde{o}} = \mathbb{Z} \times \mathbb{Z} \subset \tilde{Y}_\infty \times Y_{\tilde{o}}$.

11 Counting fixed points

As in Sects. 9 and 10, we fix a place $o \neq \infty$, $o \notin \mathrm{Bad}$, of F and an algebraic closure k of $\kappa(o)$. We fix an open compact subgroup $K^{\infty,o} \subset (D^{\infty,o})^\times$ and an open *normal* subgroup $\bar{K}_\infty \subset \bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$. We assume that $K^{\infty,o}$ is *small enough*: for example, we assume that $K^{\infty,o} \subset K^{\infty,o}_I$ for some non empty finite closed subscheme $I \subset X \backslash \{\infty, o\}$.

We consider the proper and smooth scheme

$$M = \widetilde{\mathcal{E}\ell\ell}^{\{\infty,o\}}_{X,\mathcal{D},o}/(\mathbb{Z} \times \bar{K}_\infty \times K^{\infty,o})$$

over $\kappa(o)$ (see (8.9)). On M we have a right action of the finite group $(\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty)/\bar{K}_\infty$ and a right action of the Hecke correspondences

$$\begin{tikzcd} & M(g^{\infty,o}) \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ M & & M \arrow[ll, dashed] \end{tikzcd}$$

where

$$M(g^{\infty,o}) = \widetilde{\mathcal{E}\ell\ell}^{\{\infty,o\}}_{X,\mathcal{D},o}/(\mathbb{Z} \times \bar{K}_\infty \times (K^{\infty,o} \cap (g^{\infty,o})^{-1} K^{\infty,o} g^{\infty,o}))$$

and $g^{\infty,o}$ runs through $(D^{\infty,o})^\times$ (see (8.8)). These two actions commute.

On the set $M(k)$ of k-rational points of M we also have an action of the geometric Frobenius $\mathrm{Frob}_o \in \mathrm{Gal}(k/\kappa(o))$ which commutes with the actions of $(\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty)/\bar{K}_\infty$ and the Hecke correspondences.

Let us fix $\bar{g}_\infty \in \bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ and $g^{\infty,o} \in (D^{\infty,o})^\times$. For any non negative integer r, let us denote by

$$\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})$$

the fixed point set of the product of the actions of $\bar{g}_\infty \bar{K}_\infty$, the Hecke correspondence associated to $g^{\infty,o}$ and Frob^r_o. In other words, $\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})$ is the set of $m \in M(g^{\infty,o})(k)$ such that

$$\mathrm{Frob}^r_o(c_1(m)) \cdot \bar{g}_\infty \bar{K}_\infty = c_2(m)\ .$$

(11.1) Lemma. *If* $r > 0$, *the fixed point set* $\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})$ *is finite. Moreover, each fixed point in* $\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})$ *has multiplicity one.*

Proof. If $r > 0$, the graph of Frob^r_o is transversal to any correspondence

$$\begin{tikzcd} & M' \arrow[dl, "c'_1"'] \arrow[dr, "c'_2"] & \\ M & & M \arrow[ll, dashed] \end{tikzcd}$$

with c'_1 and c'_2 etale. □

Our goal in this section is to compute the number of elements

$$\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o})$$

of $\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})$ when $r > 0$. *From now on we will assume* $r > 0$.

(11.2) Let us recall that $M(k)$ can be decomposed into isogeny classes. For each (D, ∞, o)-type $(\tilde{F}, \tilde{\Pi})$ we have a corresponding isogeny class

$$M(k)_{(\tilde{F},\tilde{\Pi})} \subset M(k)$$

and a non canonical bijection

$$M(k)_{(\tilde{F},\tilde{\Pi})} \xrightarrow{\sim} \Delta^{\times} \backslash \tilde{Y}_{\mathbb{A}}^{\{\infty,o\}} / (\mathbb{Z} \times \bar{K}_{\infty} \times K^{\infty,o}) ,$$

where

$$\tilde{Y}_{\mathbb{A}}^{\{\infty,o\}} = \mathbb{Z} \times (\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}) \times (D^{\infty,o})^{\times} \times (\mathrm{GL}_{d-h}(F_o)/\mathrm{GL}_{d-h}(\mathcal{O}_o)) \times \mathbb{Z} .$$

Here, the action of $\mathbb{Z}$ (resp. $\bar{K}_{\infty} \subset \bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$, resp. $K^{\infty,o} \subset (D^{\infty,o})^{\times}$) on $\tilde{Y}_{\mathbb{A}}^{\{\infty,o\}}$ is the action of right translations on the factor $Y_{\infty} = \mathbb{Z}$ (resp. $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$, resp. $(D^{\infty,o})^{\times}$). Moreover, the action of Frob_o^r on $M(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the translation $(r\deg(o), r)$ on the factor $Y_{\infty} \times Y_{\tilde{o}} = \mathbb{Z} \times \mathbb{Z}$ of $\tilde{Y}_{\mathbb{A}}^{\{\infty,o\}}$, the correspondence

$$\begin{tikzcd} & M(g^{\infty,o})(k)_{(\tilde{F},\tilde{\Pi})} \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ M(k)_{(\tilde{F},\tilde{\Pi})} & & M(k)_{(\tilde{F},\tilde{\Pi})} \arrow[ll, dashed] \end{tikzcd}$$

is induced by the correspondence

$$\begin{tikzcd} & (D^{\infty,o})^{\times}/(K^{\infty,o} \cap (g^{\infty,o})^{-1} K^{\infty,o} g^{\infty,o}) \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ (D^{\infty,o})^{\times}/K^{\infty,o} & & (D^{\infty,o})^{\times}/K^{\infty,o} \arrow[ll, dashed] \end{tikzcd}$$

where

$$c_1(h^{\infty,o}(K^{\infty,o} \cap (g^{\infty,o})^{-1} K^{\infty,o} g^{\infty,o})) = h^{\infty,o} K^{\infty,o}$$

$$c_2(h^{\infty,o}(K^{\infty,o} \cap (g^{\infty,o})^{-1} K^{\infty,o} g^{\infty,o})) = h^{\infty,o} (g^{\infty,o})^{-1} K^{\infty,o}$$

on the factor $(D^{\infty,o})^{\times}$ of $\tilde{Y}_{\mathbb{A}}^{\{\infty,o\}}$ and the action of $(\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}})/\bar{K}_{\infty}$ on $M(k)_{(\tilde{F},\tilde{\Pi})}$ is induced by the action by right translations of $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$ on itself.

In particular, we can also decompose $\mathrm{Fix}_r(\bar{g}_{\infty}, g^{\infty,o})$ into isogeny classes

$$\mathrm{Fix}_r(\bar{g}_{\infty}, g^{\infty,o})_{(\tilde{F},\tilde{\Pi})} \subset M(g^{\infty,o})(k)_{(\tilde{F},\tilde{\Pi})}$$

and

$$\Delta^{\times}[\bar{h}_{\infty}\bar{K}_{\infty}, h^{\infty,o}(K^{\infty,o} \cap (g^{\infty,o})^{-1} K^{\infty,o} g^{\infty,o}), h_o^{\tilde{o}}\mathrm{GL}_{d-h}(\mathcal{O}_o), m_{\tilde{o}}]$$

is in $\mathrm{Fix}_r(\bar{g}_{\infty}, g^{\infty,o})_{(\tilde{F},\tilde{\Pi})}$ if and only if there exists $\delta \in \Delta^{\times}$, $\bar{k}_{\infty} \in \bar{K}_{\infty}$, $k^{\infty,o} \in K^{\infty,o}$ and $k_o^{\tilde{o}} \in \mathrm{GL}_{d-h}(\mathcal{O}_o)$ such that

$$\begin{cases} \bar{h}_{\infty}\bar{g}_{\infty} = \delta \bar{h}_{\infty}\bar{k}_{\infty} \\ h^{\infty,o} = \delta h^{\infty,o}(g^{\infty,o})^{-1} k^{\infty,o} \\ h_o^{\tilde{o}} = \delta h_o^{\tilde{o}} k_o^{\tilde{o}} \\ m_{\tilde{o}} + r = (\deg(\tilde{o})/\deg(o))\tilde{o}(\mathrm{rn}(\delta)) + m_{\tilde{o}} \end{cases}$$

i.e. if and only if there exists $\delta\in\Delta^\times$ such that

$$\begin{cases}(\bar h_\infty)^{-1}\delta\bar h_\infty\in\bar g_\infty\bar K_\infty\\ (h^{\infty,o})^{-1}\delta h^{\infty,o}\in K^{\infty,o}g^{\infty,o}\\ (h_o^{\tilde o})^{-1}\delta h_o^{\tilde o}\in\mathrm{GL}_{d-h}(\mathcal{O}_o)\\ \deg(\tilde o)\tilde o(\mathrm{rn}(\delta))=r\deg(o)\,.\end{cases}$$

(11.3) Lemma. *For any $h^{\infty,o}\in(D^{\infty,o})^\times$, $h_o^{\tilde o}\in\mathrm{GL}_{d-h}(F_{\tilde o})$, the only $\delta\in\Delta^\times$ such that*

$$\begin{cases}(h^{\infty,o})^{-1}\delta h^{\infty,o}\in K^{\infty,o}\\ (h_o^{\tilde o})^{-1}\delta h_o^{\tilde o}\in\mathrm{GL}_{d-h}(\mathcal{O}_o)\\ \tilde o(\mathrm{rn}(\delta))=0\end{cases}$$

is the identity element (recall that $K^{\infty,o}$ is small enough).

Proof. See [Lau 2, (3.2.6)]. □

(11.4) Let $\Delta^\times_\natural$ be a system of representatives of the conjugacy classes in $\Delta^\times$. Let us say that $\delta\in\Delta^\times$ is *r-admissible* (*at the place o*) if

$$\deg(\tilde o)\tilde o(\mathrm{rn}(\delta))=r\deg(o)$$

and δ is conjugate to an element of $\mathrm{GL}_{d-h}(\mathcal{O}_o)$ in $\mathrm{GL}_{d-h}(F_o)$. Clearly, this is a property of the conjugacy class of δ. For each $\delta\in\Delta^\times$, let

$$\Delta^\times_\delta=\{\delta'\in\Delta^\times\,|\,\delta'\delta=\delta\delta'\}$$

be its centralizer in $\Delta^\times$. The map

$$h^{\infty,o}(K^{\infty,o}\cap(g^{\infty,o})^{-1}K^{\infty,o}g^{\infty,o})\mapsto h^{\infty,o}K^{\infty,o}$$

from the set of classes satisfying

$$(h^{\infty,o})^{-1}\delta h^{\infty,o}\in K^{\infty,o}g^{\infty,o}$$

to the set of classes satisfying

$$(h^{\infty,o})^{-1}\delta h^{\infty,o}\in K^{\infty,o}g^{\infty,o}K^{\infty,o}$$

is clearly bijective. Therefore we have proved:

(11.5) Proposition. *For each (D,∞,o)-type $(\tilde F,\tilde\Pi)$, $\mathrm{Fix}_r(\bar g_\infty,g^{\infty,o})_{(\tilde F,\tilde\Pi)}$ is the disjoint union over the δ's in $\Delta^\times_\natural$ which are r-admissible of the sets of double classes*

$$\Delta^\times_\delta\,[\bar h_\infty\bar K_\infty,h^{\infty,o}K^{\infty,o},h_o^{\tilde o}\mathrm{GL}_{d-h}(\mathcal{O}_o),m_{\tilde o}]$$

which satisfy

$$\begin{cases}(\bar h_\infty)^{-1}\delta\bar h_\infty\in\bar g_\infty\bar K_\infty\\ (h^{\infty,o})^{-1}\delta h^{\infty,o}\in K^{\infty,o}g^{\infty,o}K^{\infty,o}\\ (h_o^{\tilde o})^{-1}\delta h_o^{\tilde o}\in\mathrm{GL}_{d-h}(\mathcal{O}_o)\,.\end{cases}$$

(11.6) For each $\delta\in\Delta^\times$, let $(\bar D_\infty)^\times_\delta$ (resp. $(D^{\infty,o})^\times_\delta$, resp. $\mathrm{GL}_{d-h}(F_o)_\delta$) be the centralizer of δ in $\bar D^\times_\infty$ (resp. $(D^{\infty,o})^\times$, resp. $\mathrm{GL}_{d-h}(F_o)$). Let $d\bar h_\infty$ (resp. $dh^{\infty,o}$, resp. $dh_o^{\tilde o}$) be

the Haar measure on $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ (resp. $(D^{\infty,o})^\times$, resp. $\mathrm{GL}_{d-h}(F_o)$) which is normalized by

$$\mathrm{vol}(\bar{K}_\infty, d\bar{h}_\infty) = 1$$

(resp. $$\mathrm{vol}(K^{\infty,o}, dh^{\infty,o}) = 1$$

resp. $$\mathrm{vol}(\mathrm{GL}_{d-h}(\mathcal{O}_o), dh_o^{\tilde{o}}) = 1) .$$

Let $dm_{\tilde{o}}$ be the counting measure on $Y_{\tilde{o}} = \mathbb{Z}$. Let $d\bar{h}_{\infty,\delta}$ (resp. $dh_\delta^{\infty,o}$, resp. $dh_{o,\delta}^{\tilde{o}}$) be an arbitrary Haar measure on $(\bar{D}_\infty)_\delta^\times/\varpi_\infty^{\mathbb{Z}}$ (resp. $(D^{\infty,o})_\delta^\times$, resp. $\mathrm{GL}_{d-h}(F_o)_\delta$). Let $d\delta'$ be the counting measure on $\Delta_\delta^\times$. Note that all the above groups are unimodular.

Let $\bar{f}_\infty$ (resp. $f^{\infty,o}$, resp. $f_o^{\tilde{o}}$) be the characteristic function of $\bar{g}_\infty \bar{K}_\infty$ in $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ (resp. $K^{\infty,o}g^{\infty,o}K^{\infty,o}$ in $(D^{\infty,o})^\times$, resp. $\mathrm{GL}_{d-h}(\mathcal{O}_o)$ in $\mathrm{GL}_{d-h}(F_o)$).

We can introduce the orbital integrals

$$O_\delta(\bar{f}_\infty, d\bar{h}_{\infty,\delta}) = \int_{(\bar{D}_\infty)_\delta^\times \backslash \bar{D}_\infty^\times} \bar{f}_\infty((\bar{h}_\infty)^{-1}\delta\bar{h}_\infty)\frac{d\bar{h}_\infty}{d\bar{h}_{\infty,\delta}} ,$$

$$O_\delta(f^{\infty,o}, dh_\delta^{\infty,o}) = \int_{(D^{\infty,o})_\delta^\times \backslash (D^{\infty,o})^\times} f^{\infty,o}((h^{\infty,o})^{-1}\delta h^{\infty,o})\frac{dh^{\infty,o}}{dh_\delta^{\infty,o}}$$

and

$$O_\delta(f_o^{\tilde{o}}, dh_{o,\delta}^{\tilde{o}}) = \int_{\mathrm{GL}_{d-h}(F_o)_\delta \backslash \mathrm{GL}_{d-h}(F_o)} f_o^{\tilde{o}}((h_o^{\tilde{o}})^{-1}\delta h_o^{\tilde{o}})\frac{dh_o^{\tilde{o}}}{dh_{o,\delta}^{\tilde{o}}} .$$

They are absolutely convergent.

We can also introduce the volume

$$\mathrm{vol}\left(\Delta_\delta^\times \backslash [((\bar{D}_\infty)_\delta^\times/\varpi_\infty^{\mathbb{Z}}) \times (D^{\infty,o})_\delta^\times \times \mathrm{GL}_{d-h}(F_o)_\delta \times \mathbb{Z}], \frac{d\bar{h}_{\infty,\delta} \times dh_\delta^{\infty,o} \times dh_{o,\delta}^{\tilde{o}} \times dm_{\tilde{o}}}{d\delta'}\right) .$$

(11.7) Lemma. *For each r-admissible* $\delta \in \Delta^\times$, *the embeddings of F-algebras*

$$\Delta \hookrightarrow \bar{D}_\infty^{\mathrm{op}}$$

$$\Delta \hookrightarrow D^{\infty,o}$$

and

$$\Delta \hookrightarrow \mathbb{M}_{d-h}(F_o)$$

induce group isomorphisms between the centralizers of δ

$$(\Delta_\infty)_\delta^\times \xrightarrow{\sim} (\bar{D}_\infty)_\delta^\times$$

$$(\Delta^{\infty,o})_\delta^\times \xrightarrow{\sim} (D^{\infty,o})_\delta^\times$$

and

$$(\Delta_o^{\tilde{o}})_\delta^\times \xrightarrow{\sim} \mathrm{GL}_{d-h}(F_o)_\delta .$$

In particular, the Haar measure $d\bar{h}_{\infty,\delta} \times dh_\delta^{\infty,o} \times dh_{o,\delta}^{\tilde{o}}$ *induces a Haar measure* $d\delta'^{\tilde{o}}$ *on* $(\Delta^{\tilde{o}})_\delta^\times/\varpi_\infty^{\mathbb{Z}} = ((\Delta_\infty)_\delta^\times/\varpi_\infty^{\mathbb{Z}}) \times (\Delta^{\infty,o})_\delta^\times \times (\Delta_o^{\tilde{o}})_\delta^\times$ *and the above volume is equal to*

$$\mathrm{vol}\left(\Delta_\delta^\times \backslash [((\Delta^{\tilde{o}})_\delta^\times/\varpi_\infty^{\mathbb{Z}}) \times \mathbb{Z}], \frac{d\delta'^{\tilde{o}} \times dm_{\tilde{o}}}{d\delta'}\right) .$$

Proof. See [Lau 2, (3.3.4)]. □

Then it is clear that the Proposition 11.5 implies:

(11.8) Proposition. *For each (D, ∞, o)-type $(\tilde{F}, \tilde{\Pi})$, the number*

$$\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o})_{(\tilde{F},\tilde{\Pi})}$$

of elements of $\mathrm{Fix}_r(\bar{g}_\infty, g^{\infty,o})_{(\tilde{F},\tilde{\Pi})}$ *is equal to*

$$\sum_\delta \mathrm{vol}\left(\Delta_\delta^\times \backslash [((\Delta^{\tilde{o}})_\delta^\times / \varpi_\infty^{\mathbb{Z}}) \times \mathbb{Z}], \frac{d\delta'^{\tilde{o}} \times dm_{\tilde{o}}}{d\delta'}\right) O_\delta(\bar{f}_\infty, d\bar{h}_{\infty,\delta}) O_\delta(f^{\infty,o}, dh_\delta^{\infty,o}) O_\delta(f_o^{\tilde{o}}, dh_{o,\delta}^{\tilde{o}})$$

where δ runs through the set of all r-admissible elements in $\Delta_\natural^\times$.

(11.9) Let $D_\natural^\times$ be a system of representatives of the conjugacy classes in $D^\times$. Let $\gamma \in D^\times$ and let $F' = F[\gamma] \subset D$. Let us say that γ is *elliptic at the place ∞* of F if $F_\infty \otimes_F F'$ is a field, i.e. if there exists only one place ∞' of F' dividing ∞. Let us say that γ is *r-admissible at the place o* of F if $o(\det \gamma) = r$ and there exists a place o' of F' dividing o such that

$$o'(\gamma) \neq 0 \quad \text{and} \quad x'(\gamma) = 0$$

for all other places of F' dividing o. As in [Lau 2, (3.4)], we have a natural bijection

$$\{\gamma \in D_\natural^\times \mid \gamma \text{ is elliptic at } \infty \text{ and } r\text{-admissible at } o\}$$

$$\xrightarrow{\sim} \coprod_{(\tilde{F},\tilde{\Pi})} (\delta \in \Delta_\natural^\times \mid \delta \text{ is } r\text{-admissible at } o\}$$

where $(\tilde{F}, \tilde{\Pi})$ runs through a system of representatives of the isomorphism classes of (D, ∞, o)-types such that $[\tilde{F}:F]$ divides d and where Δ is attached to $(\tilde{F}, \tilde{\Pi})$ as before.

It is defined as follows. Let $\gamma \in D_\natural^\times$ be elliptic at ∞ and r-admissible at o. Let $F' = F[\gamma]$ with its two places ∞' and o' as before. Let $\Pi' \in F'$ be such that

$$\infty'(\Pi') \neq 0, \quad o'(\Pi') \neq 0 \quad \text{and} \quad x'(\Pi') = 0$$

for all other places x' of F'. We set

$$\tilde{F} = \bigcap_{\substack{n \in \mathbb{Z} \\ n \neq 0}} F[\Pi'^n] \subset F'$$

and we denote by $\tilde{\infty}$ and $\tilde{o}$ the places of $\tilde{F}$ which are induced by ∞' and o' respectively. There exists $n \in \mathbb{Z}$, $n \neq 0$, such that $\Pi'^n \in \tilde{F}$ and we set

$$\tilde{\Pi} = \Pi'^n \otimes \alpha \in \tilde{F}^\times \otimes \mathbb{Q},$$

where $\alpha \in \mathbb{Q}$ is determined by the condition

$$\deg(\tilde{\infty})\tilde{\infty}(\tilde{\Pi}) = -[\tilde{F}:F]/\alpha.$$

Then $(\tilde{F}, \tilde{\Pi})$ is a (D, ∞, o)-type and $[\tilde{F}:F]$ divides d. Let Δ be the corresponding central division algebra over $\tilde{F}$. The invariants of Δ are given by

$$\mathrm{inv}_{\tilde{x}}(\Delta) = \begin{cases} [\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{\infty} \\ -[\tilde{F}:F]/d & \text{if } \tilde{x} = \tilde{o} \\ [\tilde{F}_{\tilde{x}}:F_x]\mathrm{inv}_x(D) & \text{otherwise} \end{cases}$$

and we have

$$\dim_{\tilde{F}}(\Delta) = (d/[\tilde{F}:F])^2.$$

Therefore, as

$$\tilde{F}_{\tilde{\infty}} \otimes_{\tilde{F}} F' = F'_{\infty'} \quad \text{and} \quad \tilde{F}_{\tilde{o}} \otimes_{\tilde{F}} F' = F'_{o'}$$

are fields and as $F' \subset D$, we can find an embedding of $\tilde{F}$-algebras $F' \hookrightarrow \Delta$ and all these embeddings are conjugate in Δ. In particular, we get $\delta \in \Delta_{\natural}^{\times}$ which is conjugate to the image of $\gamma \in F'$ in Δ by the above embedding. The desired bijection maps γ into $((\tilde{F}, \tilde{\Pi}), \delta)$ (see loc. cit. for more details).

(11.10) Let $\gamma \in D_{\natural}^{\times}$ be elliptic at ∞ and r-admissible at o and let $((\tilde{F}, \tilde{\Pi}), \delta)$ be the corresponding triple (see (11.9)).

At the place ∞ of F, we can view $\bar{D}_{\infty}^{\times}$ as an inner twist of $D_{\infty}^{\times} = \mathrm{GL}_d(F_{\infty})$ and if $\bar{\gamma} \in \bar{D}_{\infty}^{\times}$ is the transfer of $\gamma \in D_{\infty}^{\times}$ by the inner twisting ($\bar{\gamma}$ is well defined up to conjugacy), $\bar{\gamma}$ and the image of $\delta \in \Delta^{\times}$ in $\bar{D}_{\infty}^{\times}$ are conjugate in $\bar{D}_{\infty}^{\times}$. Moreover, we can identify the centralizer $(\bar{D}_{\infty})_{\bar{\gamma}}^{\times}$ of $\bar{\gamma}$ in $\bar{D}_{\infty}^{\times}$ with $(\Delta_{\infty})_{\delta}^{\times}$.

As $\Delta^{\infty,o}$ is the centralizer of $\tilde{F}$ in $D^{\infty,o}$, the centralizer $(D^{\infty,o})_{\gamma}^{\times}$ of γ in $(D^{\infty,o})^{\times}$ coincides with $(\Delta^{\infty,o})_{\delta}^{\times}$ (we have $\tilde{F} \subset F' = F[\gamma] = \tilde{F}[\delta] \subset \Delta$).

At the place o, the situation is more complicated. As o' is the unique place of F' which divides $\tilde{o}$, we have

$$d[F'_{o'} : F_o]/[F' : F] = d[\tilde{F}_{\tilde{o}} : F_o]/[\tilde{F} : F] = h\,.$$

As γ is r-admissible at o, up to conjugacy we can assume that

$$\gamma = (\gamma_o^{o'}, \gamma_{o'}) \in \mathrm{GL}_{d-h}(F_o) \times \mathrm{GL}_h(F_o) \subset \mathrm{GL}_d(F_o)$$

where

$$F_o[\gamma_o^{o'}] = (F')_o^{o'} \subset \mathbb{M}_{d-h}(F_o)$$

and

$$F_o[\gamma_{o'}] = F'_{o'} \subset \mathbb{M}_h(F_o)\,.$$

Here, $\mathrm{GL}_{d-h}(F_o) \times \mathrm{GL}_h(F_o)$ is viewed as a standard Levi subgroup of $\mathrm{GL}_d(F_o)$. Now, as $\Delta_o^{\tilde{o}}$ is the centralizer of $\tilde{F}_o^{\tilde{o}}$ in $\mathrm{GL}_{d-h}(F_o)$, the centralizer $\mathrm{GL}_{d-h}(F_o)_{\gamma_o^{o'}}$ of $\gamma_o^{o'}$ in $\mathrm{GL}_{d-h}(F_o)$ coincides with $(\Delta_o^{\tilde{o}})_{\delta}^{\times}$.

(11.11) We can rewrite the formula (11.8) for $\mathrm{Lef}_r(\bar{g}_{\infty}, g^{\infty,o})$ as follows. Let us fix arbitrary Haar measures $d\bar{h}_{\infty,\bar{\gamma}}$, $dh_{\gamma}^{\infty,o}$ and $dh_{o,\gamma}^{o'}$ on $(\bar{D}_{\infty})_{\bar{\gamma}}^{\times}$, $(D^{\infty,o})_{\gamma}^{\times}$ and $\mathrm{GL}_{d-h}(F_o)_{\gamma_o^{o'}}$ respectively. Thanks to (11.10), they induce a Haar measure

$$d\delta'^{\tilde{o}} = d\bar{h}_{\infty,\delta} \times dh_{\delta}^{\infty,o} \times dh_{o,\delta}^{\tilde{o}}$$

on

$$(\Delta^{\tilde{o}})_{\delta}^{\times}/\varpi_{\infty}^{\mathbb{Z}} = ((\Delta_{\infty})_{\delta}^{\times}/\varpi_{\infty}^{\mathbb{Z}}) \times (\Delta^{\infty,o})_{\delta}^{\times} \times (\Delta_o^{\tilde{o}})_{\delta}^{\times}$$

and we can consider the volume

$$\mathrm{vol}\left(\Delta_{\delta}^{\times} \backslash [((\Delta^{\tilde{o}})_{\delta}^{\times}/\varpi_{\infty}^{\mathbb{Z}}) \times \mathbb{Z}], \frac{d\delta'^{\tilde{o}} \times dm_{\tilde{o}}}{d\delta'}\right).$$

We also have the orbital integrals

$$O_{\bar{\gamma}}(\bar{f}_{\infty}, d\bar{h}_{\infty,\bar{\gamma}}) = \int_{(\bar{D}_{\infty})_{\bar{\gamma}}^{\times} \backslash \bar{D}_{\infty}^{\times}} \bar{f}_{\infty}((\bar{h}_{\infty})^{-1} \bar{\gamma} \bar{h}_{\infty}) \frac{d\bar{h}_{\infty}}{d\bar{h}_{\infty,\bar{\gamma}}}\,,$$

$$O_{\gamma}(f^{\infty,o}, dh_{\gamma}^{\infty,o}) = \int_{(D^{\infty,o})_{\gamma}^{\times} \backslash (D^{\infty,o})^{\times}} f^{\infty,o}((h^{\infty,o})^{-1} \gamma h^{\infty,o}) \frac{dh^{\infty,o}}{dh_{\gamma}^{\infty,o}}$$

and

$$O_{\gamma_o^{o'}}(f_o^{o'}, dh_{o,\gamma}^{o'}) = \int\limits_{\mathrm{GL}_{d-h}(F_o)_{\gamma_o^o}\backslash \mathrm{GL}_{d-h}(F_o)} f_o^{o'}((h_o^{o'})^{-1}\gamma_o^{o'}h_o^{o'})\,\frac{dh_o^{o'}}{dh_{o,\gamma}^{o'}},$$

where $\bar{f}_\infty$, $f^{\infty,o}$, $d\bar{h}_\infty$, $dh^{\infty,o}$ are defined as before and $f_o^{o'}$, $dh_o^{o'}$ are new notations for $f_o^{\tilde{o}}$, $dh_o^{\tilde{o}}$ respectively. Then:

(11.12) Proposition. *$Lef_r(\bar{g}_\infty, g^{\infty,o})$ is equal to*

$$\sum_\gamma \mathrm{vol}\left(\varDelta_\delta^\times\backslash[((\varDelta^{\tilde{o}})_\delta^\times/\varpi_\infty^{\mathbb{Z}})\times\mathbb{Z}], \frac{d\delta'^{\tilde{o}}\times dm_{\tilde{o}}}{d\delta'}\right)$$

$$\cdot O_{\bar{\gamma}}(\bar{f}_\infty, d\bar{h}_{\infty,\bar{\gamma}})O_\gamma(f^{\infty,o}, dh_\gamma^{\infty,o})O_{\gamma_o^{o'}}(f_o^{o'}, dh_{o,\gamma}^{o'})$$

where γ runs through the set of elements in $D_\natural^\times$ which are elliptic at ∞ and r-admissible at o.

(11.13) Let $\gamma\in D_\natural^\times$ be elliptic at ∞ and r-admissible at o and let $F' = F[\gamma]\subset D$ with its two places ∞' and o' as before. Then we have

$$F_o\subset F'_{o'} = F_o[\gamma_{o'}]\subset \mathbb{M}_h(F_o)$$

and if we choose an isomorphism of F_o-vector-spaces

$$F_o^h\simeq(F'_{o'})^{h/[F'_{o'}:F_o]}$$

we get an identification of the centralizer of $\gamma_{o'}$ in $\mathbb{M}_h(F_o)$ with

$$\mathbb{M}_{h/[F'_{o'}:F_o]}(F'_{o'})\subset\mathbb{M}_h(F_o)\,.$$

In particular, the centralizer $\mathrm{GL}_h(F_o)_{\gamma_{o'}}$ of $\gamma_{o'}$ in $\mathrm{GL}_h(F_o)$ has a natural structure of $F'_{o'}$-group scheme and is non canonically isomorphic to $\mathrm{GL}_{h/[F'_{o'}:F_o]}(F'_{o'})$ as a $F'_{o'}$-group scheme.

If $((\tilde{F}, \tilde{\Pi}), \delta)$ is the image of γ by the bijection (11.9), the centralizer $(\varDelta_{\tilde{o}})_\delta^\times$ of δ in $\varDelta_{\tilde{o}}^\times$ also has a structure of $F'_{o'}$-group scheme ($F'_{o'} = \tilde{F}_{\tilde{o}}[\delta]$) and is an inner twist of $\mathrm{GL}_h(F_o)_{\gamma_{o'}}$ as a $F'_{o'}$-group scheme. Let $dh_{o',\gamma}$ be the Haar measure on

$$\mathrm{GL}_h(F_o)_{\gamma_{o'}}\simeq\mathrm{GL}_{h/[F'_o:F_o]}(F'_{o'})$$

which is normalized by

$$\mathrm{vol}(\mathrm{GL}_{h/[F'_{o'}:F_o]}(\mathcal{O}'_{o'}), dh_{o',\gamma}) = 1$$

(here $\mathcal{O}'_{o'}\subset F'_{o'}$ is the ring of integers). Let $d\delta'_{\tilde{o}}$ be the transfer of the Haar measure $dh_{o',\gamma}$ from $\mathrm{GL}_h(F_o)_\gamma$ to its inner twist $(\varDelta_{\tilde{o}}^\times)_\delta$ (over $F'_{o'}$) (see [Kot 2]). Thanks to [Ro] (see also [Lau 2, (4.6.4)]), $d\delta'_{\tilde{o}}$ is the Haar measure on $(\varDelta_{\tilde{o}})_\delta^\times$ which gives the volume

$$\mu = \frac{1}{(q^{\deg(o')} - 1)\cdots(q^{\deg(o')((h/[F'_o:F_o]) - 1)} - 1)}$$

to the group of units of the maximal order of $(\varDelta_{\tilde{o}})_\delta = \{\delta'\in\varDelta_{\tilde{o}}|\delta'\delta = \delta\delta'\}$.

(11.14) Lemma. *Let $d\delta'_{\mathbb{A}}$ be the Haar measure $d\delta'^{\tilde{o}}\times d\delta'_{\tilde{o}}$ on $(\varDelta_{\mathbb{A}})_\delta^\times/\varpi_\infty^{\mathbb{Z}} = ((\varDelta^{\tilde{o}})_\delta^\times/\varpi_\infty^{\mathbb{Z}})\times(\varDelta_{\tilde{o}})_\delta^\times$. Then*

$$\mathrm{vol}\left(\varDelta_\delta^\times\backslash[((\varDelta^{\tilde{o}})_\delta^\times/\varpi_\infty^{\mathbb{Z}})\times\mathbb{Z}], \frac{d\delta'^{\tilde{o}}\times dm_{\tilde{o}}}{d\delta'}\right)$$

is equal to

$$\frac{\deg(o')}{\deg(o)\mu}\operatorname{vol}\left(\Delta_{\delta}^{\times}\backslash(\Delta_{\mathbb{A}})_{\delta}^{\times}/\varpi_{\infty}^{\mathbb{Z}}, \frac{d\delta'_{\mathbb{A}}}{d\delta'}\right).$$

Proof. We have a $\Delta_{\delta}^{\times}$-equivariant group homomorphism

$$(\Delta_{\tilde{o}})_{\delta}^{\times} \xrightarrow{\mathrm{rn}} (F'_{o'})^{\times} \xrightarrow{(\deg(o')/\deg(o))o'(-)} \mathbb{Z}$$

with kernel the group of units of the maximal order of $(\Delta_{\tilde{o}})_{\delta}$ and with cokernel $\deg(o)\mathbb{Z}/\deg(o')\mathbb{Z}$. ($(\Delta_{\tilde{o}})_{\delta}$ is a central division algebra over $F'_{o'}$ and for $\delta'\in(\Delta_{\tilde{o}})_{\delta}^{\times}\subset\Delta_{\tilde{o}}^{\times}$, we have

$$\deg(o')o'(\mathrm{rn}\,\delta') = \deg(\tilde{o})\tilde{o}(\mathrm{rn}\,\delta')$$

where rn is the reduced norm for the central division algebra $(\Delta_{\tilde{o}})_{\delta}^{\times}$ over $F'_{o'}$ on the left side of the formula and rn is the reduced norm for the central division algebra $\Delta_{\tilde{o}}^{\times}$ over $\tilde{F}_{\tilde{o}}$ on the right side of the formula). □

(11.15) Following Drinfeld (see [Ka] or [Lau 2, (4.2.5)]), let us consider the function

$$f_o : \mathrm{GL}_d(F_o) \to \mathbb{Z}$$

such that

$$f_o(g_o) = 0$$

unless $g_o \in \mathbb{M}_d(\mathcal{O}_o) \cap \mathrm{GL}_d(F_o)$ and $o(\det g_o) = r$ and such that

$$f_o(g_o) = (1 - q^{\deg(o)}) \cdots (1 - q^{\deg(o)(\rho - 1)})$$

if $g_o \in \mathbb{M}_d(\mathcal{O}_o) \cap \mathrm{GL}_d(F_o)$, $o(\det g_o) = r$ and ρ is the nullity of the reduction $\bar{g}_o \in \mathbb{M}_d(\kappa(o))$ of g_o modulo the maximal ideal of $\mathcal{O}_o$. Let us recall that f_o is the *Hecke function* on $\mathrm{GL}_d(F_o)$, i.e. the $\mathrm{GL}_d(\mathcal{O}_o)$-bi-invariant function with compact support on $\mathrm{GL}_d(F_o)$, such that its Satake transform is equal to

$$f_o^{\vee}(z) = q^{\deg(o)r(d-1)/2}(z_1^r + \cdots + z_d^r)$$

(see [Lau 2, (4.2.6)]).

Drinfeld has computed the orbital integrals of f_o. Let us review his results. Let dh_o be the Haar measure on $\mathrm{GL}_d(F_o)$ which is normalized by

$$\operatorname{vol}(\mathrm{GL}_d(\mathcal{O}_o), dh_o) = 1\ .$$

For any $\gamma \in \mathrm{GL}_d(F_o)$, let $dh_{o,\gamma}$ be an arbitrary Haar measure on the centralizer $\mathrm{GL}_d(F_o)_\gamma$ of γ in $\mathrm{GL}_d(F_o)$ (this centralizer is always unimodular, see [Lau 2, (4.8.6)]). Then we can consider the orbital integral

$$O_\gamma(f_o, dh_{o,\gamma}) = \int_{\mathrm{GL}_d(F_o)_\gamma\backslash \mathrm{GL}_d(F_o)} f_o(h_o^{-1}\gamma h_o)\,\frac{dh_o}{dh_{o,\gamma}}$$

(it is always absolutely convergent, see [De–Ka–Vi] or [Lau 2, (4.8.9)]).

(11.16) Theorem (Drinfeld). (i) *The orbital integral $O_\gamma(f_o, dh_{o,\gamma})$ vanishes unless there exists a positive integer $h \leq d$, an elliptic element $\gamma' \in \mathrm{GL}_h(F_o)$ with $o(\det\gamma') = r$ and an element $\gamma'' \in \mathrm{GL}_{d-h}(\mathcal{O}_o)$ such that γ is conjugate to*

$$\begin{pmatrix}\gamma' & 0\\ 0 & \gamma''\end{pmatrix} \in \mathrm{GL}_d(F_o)$$

in $\mathrm{GL}_d(F_o)$.

(ii) *Let h be a positive integer with $h \leqq d$. Let $\gamma' \in \mathrm{GL}_h(F_o)$ be elliptic with $o(\det\gamma') = r$ and let $\gamma'' \in \mathrm{GL}_{d-h}(\mathcal{O}_o)$. Let*

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix} \in \mathrm{GL}_d(F_o)\,.$$

We can identify $\mathrm{GL}_d(F_o)_\gamma$ with the product of the centralizers

$$\mathrm{GL}_h(F_o)_{\gamma'} \times \mathrm{GL}_{d-h}(F_o)_{\gamma''}\,,$$

and we can identify $\mathrm{GL}_h(F_o)_{\gamma'}$ with

$$\mathrm{GL}_{h/[F'_{o'}:F_o]}(F'_{o'})\,,$$

where $F'_{o'} = F_o[\gamma'] \subset \mathbb{M}_h(F_o)$. Let us normalize $dh_{o,\gamma}$ in the following way. We take

$$dh_{o,\gamma} = dh'_{o,\gamma} \times dh''_{o,\gamma}$$

where $dh'_{o,\gamma}$ is the Haar measure on $\mathrm{GL}_{h/[F'_{o'}:F_o]}(F'_{o'})$ which gives the volume one to the maximal compact subgroup

$$\mathrm{GL}_{h/[F'_o:F_o]}(\mathcal{O}'_{o'}) \subset \mathrm{GL}_{h/[F'_o:F_o]}(F'_{o'})$$

and where $dh''_{o,\gamma}$ is an arbitrary Haar measure on $\mathrm{GL}_{d-h}(F_o)_{\gamma''}$. Then

$$O_\gamma(f_o, dh_{o,\gamma}) = (1 - q^{\deg(o')}) \cdots (1 - q^{\deg(o')((h/[F'_{o'}:F_o]) - 1)}) \frac{\deg(o')}{\deg(o)} O_{\gamma''}(f''_o, dh''_{o,\gamma})$$

where f''_o is the characteristic function of $\mathrm{GL}_{d-h}(\mathcal{O}_o)$ in $\mathrm{GL}_{d-h}(F_o)$.

Proof. See [Lau 2, (4.6.1) and (4.8.13)]. □

Note that we need this theorem only for closed γ in $\mathrm{GL}_d(F_o)$ (i.e., γ semi-simple but not necessarily geometrically semi-simple). Indeed, if $\gamma \in D^\times \subset D^\times_o = \mathrm{GL}_d(F_o)$, $F_o[\gamma] \subset \mathbb{M}_d(F_o)$ is a product of fields (F_o is separable over F). Therefore, $\mathrm{GL}_d(F_o)_\gamma$ is obviously unimodular and the orbital integral $O_\gamma(f_o, dh_{o,\gamma})$ is obviously convergent (the orbit of γ in $\mathrm{GL}_d(F_o)$ is closed).

(11.17) It follows from (11.14) and (11.16) that we can rewrite the formula (11.12) for $\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o})$ in the following way

$$\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o}) = \sum_\gamma \mathrm{vol}\left(\Delta^\times_\delta \backslash (\Delta_{\mathbb{A}})^\times_\delta / \varpi^{\mathbb{Z}}_\infty, \frac{d\delta'_{\mathbb{A}}}{d\delta'}\right) \varepsilon_\infty(\gamma) O_{\bar{\gamma}}(\bar{f}_\infty, d\bar{h}_{\infty,\bar{\gamma}}) O_\gamma(f^\infty, dh^\infty_\gamma)$$

where γ runs through the set of elements in $D^\times_\natural$ which are elliptic at ∞. Here we have set $f^\infty = f^{\infty,o} f_o$ with f_o as in (11.15) and

$$dh^\infty_\gamma = dh^{\infty,o}_\gamma \times dh^{o'}_{o,\gamma} \times dh_{o',\gamma}$$

and the sign

$$\varepsilon_\infty(\gamma) = (-1)^{(d/[F_\infty[\gamma]:F_\infty]) - 1}$$

is the Kottwitz sign at ∞ of γ (see [Kot 1]). Indeed, for any $\gamma \in D^\times_\natural$ which is elliptic at ∞, we have

$$O_\gamma(f_o, dh_{o,\gamma}) = 0$$

unless γ is r-admissible at o. Moreover, if γ is r-admissible at o, we have

$$O_\gamma(f_o, dh_{o,\gamma}) = \varepsilon_o(\gamma) \frac{\deg(o')}{\deg(o)\mu} O_{\gamma^{o'}_o}(f^{o'}_o, dh^{o'}_{o,\gamma})\,,$$

where

$$\varepsilon_o(\gamma) = (-1)^{(h/[F'_{o'}:F_o])-1}$$

is the Kottwitz sign at o of γ. But

$$h = d[F'_{o'}:F_o]/[F':F]$$

(see (11.10)) and

$$[F':F] = [F'_{\infty'}:F_\infty]$$

as ∞' is the unique place of $F' = F[\gamma]$ which divides ∞. So

$$\varepsilon_o(\gamma) = \varepsilon_\infty(\gamma)$$

(product formula for ε!, see [Kot 1]).

(11.18) If $\gamma \in D_\natural^\times$ is elliptic at ∞ and r-admissible at o and if $((\tilde{F}, \tilde{\Pi}), \Delta)$ is its image by the bijection (11.9), the centralizer $D_\gamma^\times$ of γ in $D^\times$ and $\Delta_\delta^\times$ have both a natural structure of group scheme over $F' = F[\gamma] = \tilde{F}[\delta]$ and $D_\gamma^\times$ is an inner twist of $\Delta_\delta^\times$ over F'. The same is true locally. The Haar measure dh_γ^∞ on $(D^\infty)_\gamma^\times$ is obviously the transfer of the Haar measure $d\delta'^\infty$ on $(\Delta^\infty)_\delta^\times$ by this inner twisting. Let $dh_{\infty,\gamma}$ be the transfer of the Haar measure $d\bar{h}_{\infty,\gamma}$ from $(\Delta_\infty)_\delta^\times/\varpi_\infty^{\mathbb{Z}}$ to its inner twist $(D_\infty)_\gamma^\times/\varpi_\infty^{\mathbb{Z}}$. Then it follows from Weil's computations of Tamagawa numbers of $\Delta_\delta^\times$ and $D_\gamma^\times$ that

$$\mathrm{vol}\left(\Delta_\delta^\times\backslash(\Delta_{\mathbb{A}})_\delta^\times/\varpi_\infty^{\mathbb{Z}}, \frac{d\delta'_{\mathbb{A}}}{d\delta'}\right) = \mathrm{vol}\left(D_\gamma^\times\backslash(D_{\mathbb{A}})_\gamma^\times/\varpi_\infty^{\mathbb{Z}}, \frac{dh_{\mathbb{A},\gamma}}{dh_\gamma}\right)$$

where dh_γ is the counting measure on $D_\gamma^\times$ (see [Lau 2, (3.5)] for example). Therefore, we get:

(11.19) Theorem. *Let $\bar{f}_\infty$ be the characteristic function of $\bar{g}_\infty \bar{K}_\infty \subset \bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$, let $f^{\infty,o}$ be the characteristic function of $K^{\infty,o}g^{\infty,o}K^{\infty,o}$ in $(D^{\infty,o})^\times$ and let f_o be the Hecke function on $\mathrm{GL}_d(F_o)$ with Satake transform*

$$f_o^\vee(z) = q^{\deg(o)r(d-1)/2}(z_1^r + \cdots + z_d^r)\,.$$

We set

$$f^\infty = f^{\infty,o}f_o\,.$$

Let $d\bar{h}_\infty$ be the Haar measure on $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ which is normalized by $\mathrm{vol}(\bar{K}_\infty, d\bar{h}_\infty) = 1$ and let dh^∞ be the Haar measure on $(D^\infty)^\times$ which is normalized by $\mathrm{vol}(K^{\infty,o} \times \mathscr{D}_o^\times, dh^\infty) = 1$.
For each $\gamma \in D_\natural^\times$ which is elliptic at ∞, let

$$dh_{\mathbb{A},\gamma} = dh_{\infty,\gamma} \times dh_\gamma^\infty$$

be an arbitrary Haar measure on its centralizer $(D_{\mathbb{A}})_\gamma^\times/\varpi_\infty^{\mathbb{Z}}$ in $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ and let dh_γ be the counting measure on $D_\gamma^\times$. We can transfer the conjugacy class of γ in $D_\infty^\times$ to a conjugacy class in the inner twist $\bar{D}_\infty^\times$ of $D_\infty^\times$ (as a F_∞-group scheme); let $\bar{\gamma} \in \bar{D}_\infty^\times$ be a representative of this conjugacy class. The centralizer $(\bar{D}_\infty)_{\bar{\gamma}}^\times$ of $\bar{\gamma}$ in $\bar{D}_\infty^\times$ is an inner twist of $(D_\infty)_\gamma^\times$ (as a $F_\infty[\gamma]$-group scheme); let $d\bar{h}_{\infty,\bar{\gamma}}$ be the transfer of the Haar measure $dh_{\infty,\gamma}$ from $(D_\infty)_\gamma^\times/\varpi_\infty^{\mathbb{Z}}$ to its inner twist $(\bar{D}_\infty)_{\bar{\gamma}}^\times/\varpi_\infty^{\mathbb{Z}}$. Let

$$\varepsilon_\infty(\gamma) = (-1)^{(d/[F_\infty[\gamma]:F_\infty])-1}$$

be the Kottwitz sign at ∞ of γ.

Then we have

$$\mathrm{Lef}_{r}(\bar{g}_{\infty}, g^{\infty,o}) = \sum_{\gamma} \mathrm{vol}\left(D_{\gamma}^{\times}\backslash(D_{\mathbb{A}})_{\gamma}^{\times}/\varpi_{\infty}^{\mathbb{Z}}, \frac{dh_{\mathbb{A},\gamma}}{dh_{\gamma}}\right)\varepsilon_{\infty}(\gamma)O_{\bar{\gamma}}(\bar{f}_{\infty}, d\bar{h}_{\infty,\bar{\gamma}})O_{\gamma}(f^{\infty}, dh_{\gamma}^{\infty})$$

where γ runs through the set of elements in $D_{\natural}^{\times}$ which are elliptic at ∞.

12 The Lefschetz fixed point formula

From now on we assume that D is a division algebra. We fix a prime number ℓ distinct from the characteristic p of $\mathbb{F}_q$ and we fix an algebraic closure $\bar{\mathbb{Q}}_{\ell}$ of the field $\mathbb{Q}_{\ell}$ of ℓ-adic numbers. We fix an irreducible representation

$$\rho_{\infty}: \bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}} \to \mathrm{GL}(L)$$

on a finite dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space L which is defined over a finite extension E_{λ} of $\mathbb{Q}_{\ell}$ in $\bar{\mathbb{Q}}_{\ell}$ and which is continuous for the pro-finite topology on $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$ and the ℓ-adic topology on $\mathrm{GL}(L)$. Then, it is well known that ρ_{∞} factors through a finite quotient $(\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}})/\bar{K}_{\infty}$ of $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$ ($\bar{K}_{\infty}$ is a normal open subgroup of $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$).

We fix a non empty finite closed subscheme I of $X\backslash\{\infty\}$. Then we have the proper and smooth scheme of pure relative dimension $d-1$

$$\mathcal{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z} \to X\backslash(\{\infty\}\cup \mathrm{Bad}\cup I)$$

and its pro-finite, pro-etale, and pro-Galois covering

$$r_{\infty,I}: \widetilde{\mathcal{E}\ell\ell}_{X,\mathscr{D},I}/\mathbb{Z} \to \mathcal{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z}$$

with pro-finite Galois group $\bar{D}_{\infty}^{\times}/\varpi_{\infty}^{\mathbb{Z}}$. Here $\mathbb{Z}$ acts by translation of the indices on the $\mathscr{D}$-elliptic sheaves (see (2.4)). The pair $(r_{\infty,I}, \rho_{\infty})$ defines a locally constant $\bar{\mathbb{Q}}_{\ell}$-sheaf $\mathscr{L}_{\rho_{\infty,I}}$ on $\mathcal{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z}$.

If $\eta = \mathrm{Spec}(F)$ is the generic point of X and if $\bar{F}$ is an algebraic closure of F, we can consider the ℓ-adic cohomology groups

$$H_{\eta,I}^{n} = H^{n}((\mathcal{E}\ell\ell_{X,\mathscr{D},I,\eta}/\mathbb{Z})\otimes_{F}\bar{F}, \mathscr{L}_{\rho_{\infty,I}})$$

($n\in\mathbb{Z}$). Each $H_{\eta,I}^{n}$ is a finite-dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space, with a rational structure over E_{λ} induced by the rational structure of ρ_{∞} over E_{λ}. In fact, $H_{\eta,I}^{n} = 0$ unless $0 \leqq n \leqq 2d-2$. On each $H_{\eta,I}^{n}$, we have an action of $\mathrm{Gal}(\bar{F}/F)$ which is defined over E_{λ} and continuous for the Krull topology on $\mathrm{Gal}(\bar{F}/F)$ and the ℓ-adic topology on $H_{\eta,I}^{n}$.

Let $\mathscr{H}_{I}^{\infty}$ be the $\mathbb{Q}$-algebra of locally constant functions with compact supports

$$f^{\infty}: (D^{\infty})^{\times} \to \mathbb{Q}$$

which are K_{I}^{∞}-bi-invariant (i.e. invariant by left and right translations under K_{I}^{∞}). Here the product is the convolution product with respect to the Haar measure dh^{∞} on $(D^{\infty})^{\times}$ which gives the volume 1 to the open compact subgroup $K_{I}^{\infty}\subset(D^{\infty})^{\times}$. A basis of $\mathscr{H}_{I}^{\infty}$ as a $\mathbb{Q}$-vector space is given by the characteristic functions

$$1_{K_{I}^{\infty}g^{\infty}K_{I}^{\infty}}$$

of the double classes $K_{I}^{\infty}g^{\infty}K_{I}^{\infty}\subset(D^{\infty})^{\times}$ when g^{∞} runs through a system of representatives of these double classes. For each $g^{\infty}\in(D^{\infty})^{\times}$, we have a Hecke correspondence (see (8.8))

$$\begin{tikzcd}
& \widetilde{\mathscr{E}\ell\ell}^{\{\infty\}}_{X,\mathscr{D},\eta}/(\mathbb{Z}\times(K_I^{\infty}\cap(g^{\infty})^{-1}K_I^{\infty}g^{\infty})) \arrow[dl, "\tilde{c}_1"'] \arrow[dr, "\tilde{c}_2"] & \\
\widetilde{\mathscr{E}\ell\ell}^{\{\infty\}}_{X,\mathscr{D},\eta}/(\mathbb{Z}\times K_I^{\infty}) & & \widetilde{\mathscr{E}\ell\ell}^{\{\infty\}}_{X,\mathscr{D},\eta}/(\mathbb{Z}\times K_I^{\infty}) . \arrow[ll, dashed]
\end{tikzcd}$$

This correspondence acts on each $H^n_{\eta,I}$ and this action depends only on the double class $K_I^{\infty}g^{\infty}K_I^{\infty}$. In fact we get an action of the $\mathbb{Q}$-algebra $\mathscr{H}_I^{\infty}$ on $H^n_{\eta,I}$ if we let $1_{K_I^{\infty}g^{\infty}K_I^{\infty}}$ act by this correspondence. This action of $\mathscr{H}_I^{\infty}$ on $H^n_{\eta,I}$ is also defined over E_{λ}. The actions of $\mathrm{Gal}(\bar{F}/F)$ and $\mathscr{H}_I^{\infty}$ on the $H^n_{\eta,I}$'s commute.

Our goal is to determine the virtual representation

$$H^{*}_{\eta,I}=\sum_{n\in\mathbb{Z}}(-1)^n H^n_{\eta,I}$$

of $\mathrm{Gal}(\bar{F}/F)\times\mathscr{H}_I^{\infty}$. For this it suffices to compute its trace. This is the purpose of this section.

(12.1) Thanks to the proper base change theorem and the local acyclicity of smooth morphisms, the action of $\mathrm{Gal}(\bar{F}/F)$ on the $H^n_{\eta,I}$'s ($n\in\mathbb{Z}$) is unramified at each place $o\neq\infty$, $o\notin\mathrm{Bad}$, $o\notin I$ of F.

More precisely, let $o\neq\infty$, $o\notin\mathrm{Bad}$, $o\notin I$ be a place of F and let us choose a diagram

$$\begin{array}{ccccccc}
\bar{F} & \subset & \bar{F}_o & \supset & \bar{\mathcal{O}}_o & \twoheadrightarrow & k \\
\cup & & \cup & & \cup & & \cup \\
F & \subset & F_o & \supset & \mathcal{O}_o & \twoheadrightarrow & \kappa(o)
\end{array}$$

where $\bar{F}_o$ is an algebraic closure of F_o, $\bar{\mathcal{O}}_o$ is the normalization of $\mathcal{O}_o$ in $\bar{F}_o$ and k is the residue field of the local ring $\bar{\mathcal{O}}_o$ (k is an algebraic closure of $\kappa(o)$). We can consider the ℓ-adic cohomology groups

$$H^n_{o,I}=H^n((\mathscr{E}\ell\ell_{X,\mathscr{D},I,o}/\mathbb{Z})\otimes_{\kappa(o)}k,\ \mathscr{L}_{\rho_{\infty,I}})$$

($n\in\mathbb{Z}$). Each $H^n_{o,I}$ is a finite dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space, with a rational structure over E_{λ} induced by the rational structure of ρ_{∞} over E_{λ}. In fact, $H^n_{o,I}=0$ unless $0\leqq n\leqq 2d-2$. On each $H^n_{o,I}$ we have an action of $\mathrm{Gal}(k/\kappa(o))$ which is defined over E_{λ} and continuous for the Krull topology on $\mathrm{Gal}(k/\kappa(o))$ and the ℓ-adic topology on $H^n_{o,I}$.

Then, we have a canonical isomorphism of $\bar{\mathbb{Q}}_{\ell}$-vector spaces

$$H^n_{\eta,I}\xrightarrow{\sim}H^n_{o,I}\,,$$

for each $n\in\mathbb{Z}$, which is compatible with the rational structures over E_{λ} and the actions of $\mathrm{Gal}(\bar{F}_o/F_o)$ (we have a canonical embedding $\mathrm{Gal}(\bar{F}_o/F_o)\subset\mathrm{Gal}(\bar{F}/F)$ and a canonical epimorphism $\mathrm{Gal}(\bar{F}_o/F_o)\twoheadrightarrow\mathrm{Gal}(k/\kappa(o))$; see [SGA 4, XVI (2.2)]).

On each $H^n_{o,I}$ we also have an action of the Hecke operators. In fact, as we have not studied the bad reduction of the modular varieties of $\mathscr{D}$-elliptic sheaves, we do not have an action of the full $\mathbb{Q}$-algebra $\mathscr{H}_I^{\infty}$ on $H^n_{o,I}$ but only an action of its $\mathbb{Q}$-subalgebra $\mathscr{H}_I^{\infty,o}\subset\mathscr{H}_I^{\infty}$ of locally constant functions with compact support

$$f^{\infty,o}:(D^{\infty,o})^{\times}\to\mathbb{Q}$$

which are $K_I^{\infty,o}$-bi-invariant. Here the product is the convolution product with respect to the Haar measure $dh_I^{\infty,o}$ on $(D^{\infty,o})^\times$ which gives the volume 1 to the open compact subgroup $K_I^{\infty,o} \subset (D^{\infty,o})^\times$ and the embedding $\mathscr{H}_I^{\infty,o} \subset \mathscr{H}_I^\infty$ maps $f^{\infty,o}$ onto $f^{\infty,o} 1_{\mathrm{GL}_d(\mathscr{O}_o)}$. The characteristic function

$$1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}}$$

acts on $H_{o,I}^n$ as the Hecke correspondence

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathscr{E}\ell\ell}^{\{\infty,o\}}_{X,\mathscr{D},\mathrm{o}}/(\mathbb{Z} \times (K_I^{\infty,o} \cap (g^{\infty,o})^{-1} K_I^{\infty,o} g^{\infty,o})) \arrow[dl, "\bar{c}_1"'] \arrow[dr, "\bar{c}_2"] & \\ \widetilde{\mathscr{E}\ell\ell}^{\{\infty,o\}}_{X,\mathscr{D},\mathrm{o}}/(\mathbb{Z} \times K_I^{\infty,o}) & & \widetilde{\mathscr{E}\ell\ell}^{\{\infty,o\}}_{X,\mathscr{D},\mathrm{o}}/(\mathbb{Z} \times K_I^{\infty,o}) \arrow[ll, dashed] \end{tikzcd}$$

for each $g^{\infty,o} \in (D^{\infty,o})^\times$ and each $n \in \mathbb{Z}$. It is clear that the action of $\mathscr{H}_I^{\infty,o}$ on the $H_{o,I}^n$'s ($n \in \mathbb{Z}$) is defined over E_λ and commutes with the action of $\mathrm{Gal}(k/\kappa(o))$. It is also clear that the above isomorphisms $H_{\eta,I}^n \xrightarrow{\sim} H_{o,I}^n$ ($n \in \mathbb{Z}$) are $\mathscr{H}_I^{\infty,o}$-equivariant ($\mathscr{H}_I^{\infty,o} \subset \mathscr{H}_I^\infty$). In other words, for each place $o \neq \infty$, $o \notin \mathrm{Bad}$, $o \notin I$, of F, the restriction of the virtual representation $H_{\eta,I}^*$ of $\mathrm{Gal}(\bar{F}/F) \times \mathscr{H}_I^\infty$ to

$$\mathrm{Gal}(\bar{F}_o/F_o) \times \mathscr{H}_I^{\infty,o} \subset \mathrm{Gal}(\bar{F}/F) \times \mathscr{H}_I^\infty$$

is uniquely determined by the alternating traces

$$\mathrm{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}}; H_{o,I}^*)$$

($r \in \mathbb{Z}$, $g^{\infty,o} \in (D^{\infty,o})^\times$), as $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$ and as the characteristic functions $1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}}$ generate $\mathscr{H}_I^{\infty,o}$ as a $\mathbb{Q}$-vector space. It is even enough to know these traces for all positive integers r and all $g^{\infty,o} \in (D^{\infty,o})^\times$. Here we have set

$$H_{o,I}^* = \sum_{n \in \mathbb{Z}} (-1)^n H_{o,I}^n .$$

(12.2) Let us fix a place $o \neq \infty$, $o \notin \mathrm{Bad}$, $o \notin I$, of F and a positive integer r. Let us consider the ℓ-adic cohomology groups

$$\tilde{H}_{o,I}^n = H_o^n((\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},I,o}/(\mathbb{Z} \times \bar{K}_\infty)) \otimes_{\kappa(o)} k, \mathbb{Q}_\ell)$$

($n \in \mathbb{Z}$) where $\bar{K}_\infty$ is an open normal subgroup of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on which ρ_∞ is trivial. Each $\tilde{H}_{o,I}^n$ is a finite dimensional $\mathbb{Q}_\ell$-vector space and $\tilde{H}_{o,I}^n = 0$ unless $0 \leqq n \leqq 2d - 2$. On each $\tilde{H}_{o,I}^n$ we have commuting actions of the finite group $(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty$, of $\mathrm{Gal}(k/\kappa(o))$ and of $\mathscr{H}_I^{\infty,o}$. The action of $(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty$ is induced by the action of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on $\widetilde{\mathscr{E}\ell\ell}^{\{\infty,o\}}_{X,\mathscr{D},I,o}$, the action of $\mathrm{Gal}(k/\kappa(o))$ is continuous for the Krull topology on $\mathrm{Gal}(k/\kappa(o))$ and the ℓ-adic topology on $\tilde{H}_{o,I}^n$ and the action of $\mathscr{H}_I^{\infty,o}$ is induced by the Hecke correspondences as before.

By definition of $\mathscr{L}_{\rho_{\infty,I}}$, we have canonical isomorphisms of $\bar{\mathbb{Q}}_\ell$-vector spaces

$$H_{o,I}^n \xrightarrow{\sim} (\tilde{H}_{o,I}^n \otimes_{\mathbb{Q}_\ell} L)^{(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty}$$

where $(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty$ acts through ρ_∞ on L. In particular, we have

$$\mathrm{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}}; H_{o,I}^*)$$
$$= \frac{1}{|(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty|} \sum_{\bar{g}_\infty \bar{K}_\infty \in (\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty} \mathrm{tr}(\bar{g}_\infty \times \mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}}; \tilde{H}_{o,I}^*) \mathrm{tr}(\rho_\infty(\bar{g}_\infty)) \cdot$$

(12.3) Now, thanks to (11.1), we can apply the Lefschetz trace formula [SGA5, III(4.11.3)] to compute the traces

$$\operatorname{tr}(\bar{g}_\infty \times \mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}};\ \tilde{H}_{o,I}^*)$$

and we get that this trace is equal to

$$\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o})$$

with the notation of (11.1). Therefore, we have proved:

(12.4) Proposition. *With the above notations, we have*

$$\operatorname{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}};\ H_{o,I}^*) = \frac{1}{|(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty|} \cdot \sum_{\bar{g}_\infty \bar{K}_\infty \in (\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}})/\bar{K}_\infty} \mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o}) \operatorname{tr}(\rho_\infty(\bar{g}_\infty))$$

for each positive integer r and each $g^{\infty,o} \in (D^{\infty,o})^\times$.

Replacing $\mathrm{Lef}_r(\bar{g}_\infty, g^{\infty,o})$ by its formula (11.19), we get finally

(12.5) Theorem. *We have*

$$\operatorname{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o} g^{\infty,o} K_I^{\infty,o}};\ H_{o,I}^*) = \sum_\gamma \operatorname{vol}\left(D_\gamma^\times \backslash (D_{\mathbb{A}})_\gamma^\times / \varpi_\infty^{\mathbb{Z}}, \frac{dh_{\mathbb{A},\gamma}}{dh_\gamma}\right) \cdot \varepsilon_\infty(\gamma) O_{\bar{\gamma}}(\bar{f}_\infty, d\bar{h}_{\infty,\gamma}) O_\gamma(f^\infty, dh_\gamma^\infty)$$

where γ runs through the set of elements in $D_{\natural}^\times$ which are elliptic at ∞. Here, the Haar measures dh^∞, $dh_{\mathbb{A},\gamma}$, dh_γ^∞, $d\bar{h}_{\infty,\gamma}$, dh_γ and the functions $f^{\infty,o}$, f_o, f^∞ are chosen as in (11.19) *for $K^{\infty,o} = K_I^{\infty,o}$. Now the Haar measure $d\bar{h}_\infty$ is arbitrary and we have set*

$$\bar{f}_\infty = \frac{\chi_{\rho_\infty}}{\operatorname{vol}(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}, d\bar{h}_\infty)},$$

where χ_{ρ_∞} is the character of the representation ρ_∞ (a locally constant function on $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$).

Note that the product $\bar{f}_\infty d\bar{h}_\infty$ is independent of the choice of $d\bar{h}_\infty$.

13 The Selberg trace formula

In this section we shall replace the factor $\varepsilon_\infty(\gamma) \cdot O_{\bar{\gamma}}(\bar{f}_\infty, dh_{\infty,\gamma})$ in the expression (12.5) by an orbital integral of a function f_∞ on $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$. We shall suppose that ρ_∞ is the trivial representation. (For the general case, compare the remarks (13.8) at the end of this section). We shall show in this case that one may take for f_∞ the weakly cuspidal Euler-Poincaré function of [Lau 2, Sect. 5]. (It follows a posteriori that any of Kottwitz's Euler-Poincaré functions [Kot 2, Sect. 2], which depend on the choice of a set of representatives of the orbits of $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ on the set of facets in the building, will have the required properties and this is indeed proved in Kottwitz's paper [Kot 2, Theorems 2 and 2′], in the case of a p-adic field). We briefly recall the results of [Lau 2, Sect. 5].

(13.1) We choose an identification $D_\infty \simeq \mathbb{M}_d(F_\infty)$. Let T denote the group of diagonal matrices and B the group of upper triangular matrices in GL_d. Let Δ be the set of simple roots of (T, B). To any subset $I \subset \Delta$ there is associated a standard parabolic subgroup P_I, as well as a standard parahoric subgroup $\mathscr{P}_I^\circ$ contained in $\mathrm{GL}_d(\mathcal{O}_\infty)$. (We have $\mathscr{P}_\emptyset^\circ = \mathscr{B}^\circ$ the standard Iwahori subgroup and $\mathscr{P}_\Delta^\circ = \mathrm{GL}_d(\mathcal{O}_\infty)$ the standard maximal compact subgroup.) Let $\mathscr{P}_I$ denote the normalizer of $\mathscr{P}_I^\circ$ in $\mathrm{GL}_d(F_\infty)$. The group $\mathrm{GL}_d(F_\infty)$ acts on the building of $\mathrm{SL}_d(F_\infty)$ and $\mathscr{P}_I^\circ$ (resp. $\mathscr{P}_I$) is the pointwise stabilizer (resp. the stabilizer) of a facet σ_I. Let

$$\chi_I : \mathscr{P}_I \to \{\pm 1\}$$

be the sign character of the permutation representation afforded by the vertices of σ_I. We extend χ_I to all of $\mathrm{GL}_d(F_\infty)$ by setting it zero outside of $\mathscr{P}_I$. The weakly cuspidal Euler-Poincaré function [Lau 2, 5.1.2], is defined by the following expression

$$f_\infty = \sum_{I \subset \Delta} (-1)^{|\Delta - I|} \cdot \frac{\chi_I}{(|\Delta - I| + 1) \cdot \mathrm{vol}(\mathscr{P}_I^\circ, dh_\infty)} .$$

It is a function on $\mathrm{GL}_d(F_\infty)$ whose value in g only depends on the image of g in $\mathrm{PGL}_d(F_\infty)$. If the Haar measure dh_∞ on $\mathrm{GL}_d(F_\infty)$ is multiplied by a scalar the function f_∞ is divided by that scalar.

(13.2) Theorem. *For some Haar measure $d\bar{h}_\infty$ on $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$, let*

$$\bar{f}_\infty \equiv \frac{1}{\mathrm{vol}(\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}, d\bar{h}_\infty)} ,$$

a constant function on $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$, and define f_∞ as above (since $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ is an inner twisting of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ as an F_∞-group scheme we could take the Haar measure dh_∞ to be the transfer of $d\bar{h}_\infty$, but this does not matter).

(i) *The orbital integrals of f_∞ for non-elliptic elements are* 0. *Let $\gamma \in D_\infty^\times$ be elliptic and let $\bar{\gamma} \in \bar{D}_\infty^\times$ be its transfer by the inner twisting ($\bar{\gamma}$ is well-defined up to conjugacy). Then the centralizers $(D_\infty)_\gamma^\times$ and $(\bar{D}_\infty)_{\bar{\gamma}}^\times$ are inner twistings of one another (as $F_\infty[\gamma] = F_\infty[\bar{\gamma}]$-group schemes) and we choose Haar measures $dh_{\infty,\gamma}$ and $d\bar{h}_{\infty,\bar{\gamma}}$ on them which are transfers of one another. Then*

$$O_\gamma(f_\infty, dh_{\infty,\gamma}) = \varepsilon_\infty(\bar{\gamma}) \cdot O_{\bar{\gamma}}(\bar{f}_\infty, d\bar{h}_{\infty,\bar{\gamma}}) = \frac{\varepsilon_\infty(\bar{\gamma})}{\mathrm{vol}((\bar{D}_\infty^\times)_{\bar{\gamma}}/\varpi_\infty^{\mathbb{Z}}, d\bar{h}_{\infty,\bar{\gamma}})} .$$

Here as before $\varepsilon_\infty(\gamma) = \varepsilon_\infty(\bar{\gamma})$ is the Kottwitz sign

$$\varepsilon_\infty(\gamma) = (-1)^{d/[F_\infty[\gamma]:F_\infty] - 1} .$$

(Note that both sides of the above identity are independent of the choice of Haar measures $d\bar{h}_\infty$ and dh_∞.)

(ii) *Let π_∞ be a unitary irreducible representation of $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$. Then* tr $\pi_\infty(f_\infty) = 0$ *except in the following two cases: the trace of f_∞ on the trivial representation is* 1 *and the trace of f_∞ on the Steinberg representation* St_∞ *is $(-1)^{d-1}$.*

Proof. For (i) we refer to [Lau 2, 5.1.3(i), (iii)]. For (ii) we refer to [Kot 2, Theorem 2′], noting that the blanket assumption made in that paper that the characteristic of the ground field be zero is not used in its proof. In [Kot2], Kottwitz considers $D_\infty^\times/F_\infty^\times$, not $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$. However, this latter case reduces immediately to the former. Indeed, f_∞ is invariant by translation under $F_\infty^\times/\varpi_\infty^{\mathbb{Z}}$, hence if tr $\pi_\infty(f_\infty) \neq 0$ it follows that π_∞ factors through $D_\infty^\times/F_\infty^\times$. □

(13.3) Recall from the list of notations that we have fixed an isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$. We now insert the above expression for the factor $\varepsilon_\infty(\gamma)\cdot O_{\bar{\gamma}}(\bar{f}_\infty, d\bar{h}_\infty)$ in the formula of (12.5). Putting $f = f_\infty \cdot f^\infty$ we obtain therefore the following expression

$$(13.4)\quad \mathrm{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o}\cdot g^{\infty,o}\cdot K_I^{\infty,o}};\, H^*_{o,I}) = \sum_{\gamma\in D_\natural^\times} \mathrm{vol}\left(D_\gamma^\times\backslash(D_{\mathbb{A}})_\gamma^\times/\varpi_\infty^{\mathbb{Z}}, \frac{dh_{\mathbb{A},\gamma}}{dh_\gamma}\right) \cdot O_\gamma(f, dh_{\mathbb{A},\gamma}) \,.$$

This is indeed the formula of (12.5) since, due to the vanishing of the orbital integrals of f_∞ on non-elliptic elements of $D_\infty^\times$, this sum effectively only ranges over those $\gamma\in D_\natural^\times$ which are elliptic at ∞.

The sum appearing above is nothing but one side of the Selberg trace formula. More precisely, let $\mathscr{A}(D^\times\backslash D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}})$ be the space of locally constant functions, equipped with the right regular representation of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$. Since D is a division algebra the coset space $D^\times\backslash D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ is compact. (Note that the coset space $F^\times\backslash \mathbb{A}^\times/\varpi_\infty^{\mathbb{Z}}$ is finite). Therefore this space is admissible and decomposes as a direct sum of irreducible admissible representations with finite multiplicities,

$$\mathscr{A}(D^\times\backslash D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}) = \bigoplus_{\Pi} m(\Pi)\cdot\Pi \,.$$

Here Π ranges over the irreducible admissible representations of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ and $m(\Pi)$ denotes the multiplicity. If $m(\Pi) > 0$ then Π is called automorphic. The compactness of the coset space also implies that the operator induced by a locally constant function with compact support on $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ has a trace. The usual manipulation (integration of the kernel function over the diagonal) yields the Selberg trace formula. For any $f\in C_c^\infty(D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}})$ of the form $f = 1_{K^S}\cdot f_S$ where S is a finite set of places with $\{\infty\}\cup\mathrm{Bad}\subset S$ so that $(D^S)^\times = \mathrm{GL}_d(\mathbb{A}^S)$, where $K^S = \prod_{x\notin S}\mathrm{GL}_d(\mathcal{O}_x)$ is the canonical maximal compact subgroup and where $f_S\in C_c^\infty(D_S^\times/\varpi_\infty^{\mathbb{Z}})$, we have

(13.5)

$$\begin{aligned}\mathrm{tr}\left(f;\, \mathscr{A}\left(D^\times\backslash D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}};\, \frac{dh_{\mathbb{A}}}{dh}\right)\right) &= \sum_{\Pi} m(\Pi)\,\mathrm{tr}\,\Pi(f)\\ &= \sum_{\gamma\in D_\natural} \mathrm{vol}\left(D_\gamma^\times\backslash(D_{\mathbb{A}})_\gamma^\times/\varpi_\infty^{\mathbb{Z}}, \frac{dh_{\mathbb{A},\gamma}}{dh_\gamma}\right)\cdot O_\gamma(f, dh_{\mathbb{A},\gamma})\,,\end{aligned}$$

with the choice of f as in (13.4) and the same choices of Haar measures (dh_γ is the counting measure on $D_\gamma^\times$). Putting together (13.4) and (13.5), we get:

(13.6) Proposition. *We keep the notations of* (12.5) *and introduce the function* $f = f_\infty\cdot f^\infty$ *on* $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ *with* f_∞ *as in* (13.2). *Then*

$$\mathrm{tr}(\mathrm{Frob}_o^r \times 1_{K_I^{\infty,o}\cdot g^{\infty,o}K_I^{\infty,o}};\, H^*_{o,I}) = \sum_{\Pi} m(\Pi)\cdot\mathrm{tr}\,\Pi(f)\,.$$

Using (13.2) (ii) we therefore obtain the following conclusion.

(13.7) Corollary. *Let I be a non-empty finite closed subscheme of $X\backslash\{\infty\}$ and let $o \neq \infty$, $o\notin$* Bad, *$o\notin I$ be a place of F. For an automorphic representation Π*

occurring with multiplicity $m(\Pi)$ *in* $\mathscr{A}(D^\times\backslash D^\times_{\mathbb{A}}/\varpi^{\mathbb{Z}}_\infty)$ *we introduce*

$$\chi_\infty(\Pi)=\chi(\Pi_\infty)=\begin{cases}1 & \textit{if } \Pi_\infty \textit{ is the trivial representation}\\ (-1)^{d-1} & \textit{if } \Pi_\infty \textit{ is the Steinberg representation}\\ 0 & \textit{in all other cases}\,.\end{cases}$$

and

$$\chi^{\infty,o}_I(\Pi)=\chi(\Pi^{\infty,o},K^{\infty,o}_I)=\dim((\Pi^{\infty,o})^{K^{\infty,o}_I})\,.$$

Then for the local factor at o *of the Hasse-Weil zeta function of* $\mathscr{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z}$ *there is the following expression*

$$Z_o(\mathscr{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z},s)=(\det(1-q^{-\deg(o)s}\cdot\mathrm{Frob}_o;R\Gamma((\mathscr{E}\ell\ell_{X,\mathscr{D},I,o}/\mathbb{Z})\otimes_{\kappa(o)}k,\bar{\mathbb{Q}}_\ell)))^{-1}$$

$$=\prod_{\Pi,\,\dim(\Pi^{K_o}_o)=1}L(\Pi_o,s-(d-1)/2)^{m(\Pi)\cdot\chi_\infty(\Pi)\cdot\chi^{\infty,o}_I(\Pi)}\,.$$

The factors on the right hand side are the standard L-functions of unramified representations of $\mathrm{GL}_d(F_o)$ *and* $K_o=\mathrm{GL}_d(\mathcal{O}_o)$ *is the canonical maximal compact subgroup.*

Proof. The (completely standard) proof proceeds by regarding both sides as formal power series in $T=q^{-\deg(o)s}$ and taking $T\cdot\frac{d}{dT}\log$ of both sides, and finally comparing coefficients in front of T^r, $r\geqq 1$. This reduces the assertion to proving for every $r\geqq 1$

$$\mathrm{tr}(\mathrm{Frob}^r_o;H^*_{o,I})=\sum_{\Pi,\,\dim(\Pi^{K_o}_o)=1}m(\Pi)\cdot\chi_\infty(\Pi)\cdot\chi^{\infty,o}_I(\Pi)$$
$$\cdot(q^{\deg(o)r(d-1)/2}\cdot(z_1(\Pi_o)^r+\cdots+z_d(\Pi_o)^r))\,.$$

Here $(z_1(\Pi_o),\cdots,z_d(\Pi_o))\in(\mathbb{C}^\times)^d/\mathfrak{S}_d$ is associated as usual to the unramified representation, cf. [Ca, 4.4], (comp. (14.5) below) and we have used the definition of the standard L-function in the unramified case. However, $\chi^{\infty,o}_I(\Pi)=\mathrm{tr}\,\Pi^{\infty,o}(1_{K^{\infty,o}_I})$, as we have normalized the Haar measure on $(D^{\infty,o})^\times$ such that $K^{\infty,o}_I$ gets volume 1, and by (13.2) $\chi_\infty(\Pi)=\mathrm{tr}\,\Pi_\infty(f_\infty)$. Finally, as we have normalized the Haar measure on $D^\times_o$ such that the maximal compact subgroups get volume 1, we have for f_o as in (11.15),

$$\mathrm{tr}\,\Pi_o(f_o)=q^{\deg(o)r(d-1)/2}\cdot(z_1(\Pi_o)^r+\cdots+z_d(\Pi_o)^r)\,,$$

cf. [Ca, 4.4], (comp. (14.5) below). Therefore the right hand side of the identity above is equal to $\sum m(\Pi)\cdot\mathrm{tr}\,\Pi(f)$ and the assertion follows from (13.6). □

(13.8) We conclude this chapter with some remarks on the case of a general representation ρ_∞. *We do not claim to have a proof for all of the statements below* but rather hope that the specialists in this area can provide us with guidance to the literature. For each representation ρ_∞ of $\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ there should exist a unique irreducible square-integrable representation π_∞ of $D^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ uniquely characterized by the following relation between characters on elliptic regular elements. If $\gamma\in D^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ is elliptic regular (i.e. $F_\infty[\gamma]$ is a separable field extension of degree d of F_∞) and if γ corresponds to $\bar{\gamma}\in\bar{D}^\times_\infty/\varpi^{\mathbb{Z}}_\infty$ under the inner twisting (as F_∞-group scheme), then

$$\chi_{\pi_\infty}(\gamma)=(-1)^{d-1}\cdot\chi_{\rho_\infty}(\bar{\gamma})\,.$$

(generalized Jacquet-Langlands correspondence, compare [De–Ka–Vi; Ro; He 1, Appendix]). Following [Be–Ze 2], we can represent this representation π_∞ as a generalized Steinberg representation. More precisely, given π_∞, there exists a positive integer d' and $t \in \frac{1}{2}\mathbb{Z}$ with $(2t+1)d' = d$ and a supercuspidal representation π'_∞ of $\mathrm{GL}_{d'}(F_\infty)$ such that π_∞ is the unique irreducible submodule $\mathrm{St}_{2t+1}(\pi'_\infty)$ of the induced representation

$$\pi'_\infty(t) \times \cdots \times \pi'_\infty(-t)$$

(comp. [Mo–Wa, I 3]). The numbers in parentheses refer to the powers of the twist by $|\det|$. Furthermore, this induced representation has a unique irreducible quotient module which we shall denote by $\mathrm{Speh}_{2t+1}(\pi'_\infty)$. It is unitary. (In loc. cit., I.5., this is denoted by $I(\pi'_\infty, -t, t)$. It is a special case of the p-adic analogue of a Speh module). There are two extreme cases to this construction. If π_∞ is supercuspidal, then $t = 0$ and $\pi_\infty = \mathrm{Speh}_1(\pi_\infty) = \mathrm{St}_1(\pi_\infty)$. If π_∞ is the Steinberg representation, or equivalently, ρ_∞ is the trivial representation, then $d' = 1$, $\pi'_\infty = 1'_\infty$ is the trivial representation, $\pi_\infty = \mathrm{St}_d(1'_\infty) = \mathrm{St}_\infty$, and $\mathrm{Speh}_d(1'_\infty) = 1_\infty$ is the trivial representation.

The analogue of the function f_∞ of Theorem 13.2 would be a function f_{π_∞} on $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$, locally constant with compact support, and with the following properties.

(i) The non-elliptic orbital integrals of f_{π_∞} are zero. For elliptic $\gamma \in D_\infty^\times$ with corresponding $\bar{\gamma} \in \bar{D}_\infty^\times$

$$O_\gamma(f_{\pi_\infty}, dh_{\infty,\gamma}) = \varepsilon_\infty(\bar{\gamma}) \cdot O_{\bar{\gamma}}(\bar{f}_\infty, dh_{\infty,\bar{\gamma}}) .$$

Here $\bar{f}_\infty$ is defined as in (12.5) and the Haar measures are chosen as in (13.2).
(ii) For a unitary irreducible representation $\tilde{\pi}_\infty$ of $D_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ we have

$$\mathrm{tr}\,\tilde{\pi}_\infty(f_{\pi_\infty}) = \begin{cases} (-1)^{d-1} & \text{if } \tilde{\pi}_\infty \simeq \pi_\infty = \mathrm{St}_{2t+1}(\pi'_\infty) \\ (-1)^{d-1-2t} & \text{if } \tilde{\pi}_\infty \simeq \mathrm{Speh}_{2t+1}(\pi'_\infty) \\ 0 & \text{in all other cases .} \end{cases}$$

Furthermore, if π_∞ is supercuspidal it should be possible to take for f_{π_∞} a matrix coefficient of π_∞. (In the case of characteristic 0 this is indeed the case, cf. [Ro].)

Assuming all this, the conclusion of (13.6) holds without further modification, with the understanding that $H^*_{o,I}$ denotes cohomology with coefficients in the local system $\mathscr{L}_{\rho_\infty,I}$ corresponding to ρ_∞ (cf. the beginning of Sect. 12.). Similarly, corollary (13.7.) holds where on the left side there appears the L-function of $\mathscr{L}_{\rho_\infty,I}$ (in various degrees of cohomology) and where on the right hand side the definition of $\chi_\infty(\Pi)$ has to be modified in the obvious way (cf. (ii) above).

14 On the construction of global Galois representations associated to automorphic representations of the division algebra

We recall (cf. beginning of Sect. 12) that, starting from an irreducible representation ρ_∞ of $\bar{D}_\infty^\times/\varpi_\infty^{\mathbb{Z}}$ in a finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector space we had constructed a locally constant $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_{\rho_\infty,I}$ on $\mathscr{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z}$, for any non-empty finite closed subscheme I of $X \backslash \{\infty\}$. We denote the corresponding ℓ-adic cohomology groups by $H^n_{\eta,I}$ (cf. loc. cit.). Each $H^n_{\eta,I}$ is a representation of $\mathrm{Gal}(\bar{F}/F) \times \mathscr{H}_I^\infty$ where $\mathscr{H}_I^\infty$ denotes the Hecke algebra over $\mathbb{Q}$ of K_I^∞-biinvariant functions with compact supports on $(D^\infty)^\times$. We denote by $(H^n_{\eta,I})^{\mathrm{ss}}$ the associated semi-simplification as representation of $\mathrm{Gal}(\bar{F}/F)\times\mathscr{H}^\infty_I$ (the direct sum of all irreducible subquotients) and consider the virtual representation

$$H^*_{\eta,I}=\sum_n(-1)^n H^n_{\eta,I}=\sum_n(-1)^n(H^n_{\eta,I})^{\mathrm{ss}}.$$

A formula for the trace of this virtual representation on certain elements was given in Sect. 12.

(*14.1*) Recall that each $g^\infty\in(D^\infty)^\times$ defines a Hecke correspondence on the tower formed by

$$\widetilde{\mathscr{E}\ell\ell}_{X,\mathscr{D},I,\eta}/\mathbb{Z}\quad(\emptyset\neq I\subset X\backslash\{\infty\}).$$

We therefore obtain an action of $\mathrm{Gal}(\bar{F}/F)\times(D^\infty)^\times$ on the direct limit

$$H^n_\eta:=\varinjlim_I H^n_{\eta,I}.$$

Here the index system is formed by the non empty finite closed subschemes $I\subset X\backslash\{\infty\}$ and the transition homomorphisms are injective (existence of the trace morphism in étale cohomology).

Fix I. We consider $H^n_{\eta,I}$ as a subvectorspace of H^n_η. The action of $\mathrm{Gal}(\bar{F}/F)$ on $H^n_{\eta,I}$ is the induced action. We have

$$H^n_{\eta,I}=(H^n_\eta)^{K^\infty_I}$$

(subspace of invariants). The action of $\mathscr{H}^\infty_I$ on $H^n_{\eta,I}$ coincides with the induced action of $\mathscr{H}^\infty_I$ on the K^∞_I-invariant vectors. The finite-dimensionality of the cohomology groups $H^n_{\eta,I}$ implies therefore that H^n_η is an admissible representation of $(D^\infty)^\times$.

(*14.2*) We also introduce the semi-simplification $(H^n_\eta)^{\mathrm{ss}}$ of the representation H^n_η of $\mathrm{Gal}(\bar{F}/F)\times(D^\infty)^\times$. Since passing to the invariants under a compact group in a representation over a field of characteristic 0 is an exact functor, we have

$$(H^n_\eta)^{\mathrm{ss}}=\varinjlim_I(H^n_{\eta,I})^{\mathrm{ss}}.$$

We decompose $(H^n_\eta)^{\mathrm{ss}}$ into isotypic components under the action of $(D^\infty)^\times$:

$$(H^n_\eta)^{\mathrm{ss}}=\bigoplus_{\pi^\infty}V^n_{\pi^\infty}\otimes\pi^\infty\ .$$

Here π^∞ ranges over the irreducible admissible representations of $(D^\infty)^\times$ and

$$V^n_{\pi^\infty}=\mathrm{Hom}_{(D^\infty)^\times}(\pi^\infty,(H^n_\eta)^{\mathrm{ss}})$$

is a semi-simple $\mathrm{Gal}(\bar{F}/F)$-module. We call $V^n_{\pi^\infty}$ the (*global*) *Galois representation associated to the irreducible admissible representation* π^∞ *of* $(D^\infty)^\times$. We also introduce the virtual Galois representation

$$V^*_{\pi^\infty}:=\sum_n(-1)^n V^n_{\pi^\infty}.$$

Then

$$H^*_\eta := \sum_n (-1)^n H^n_\eta = \sum_n (-1)^n (H^n_\eta)^{ss} = \sum_{\pi^\infty} V^*_{\pi^\infty} \otimes \pi^\infty \ ,$$

as virtual representations of $\mathrm{Gal}(\bar{F}/F) \times (D^\infty)^\times$.

(*14.3*) Let $o \neq \infty$, $o \notin \mathrm{Bad}$, be a place of F. Then we have the equality of subspaces

$$\varinjlim_{o \notin I} H^n_{\eta, I} = (H^n_\eta)^{\mathrm{GL}_d(\mathcal{O}_o)}.$$

Here we have chosen an identification $D_o \simeq \mathbb{M}_d(F_o)$; any other choice would replace the subspace on the right by a conjugate under $D_o^\times$. It follows from (12.1) that this Galois module is unramified at o. Therefore, appealing to the isotypic decomposition in (14.2) above we obtain the following statement:

(14.4) Lemma. *Let π^∞ be an irreducible admissible representation of $(D^\infty)^\times$ and let $o \neq \infty$, $o \notin$ Bad, be a place of F such that the local component π_o^∞ of π^∞ at o is unramified (existence of a vector invariant under a maximal compact subgroup of $D_o^\times$). Then the Galois representation $V^n_{\pi^\infty}$ is unramified in o, for every n.*

(*14.5*) Fix a place o, and choose an identification $D_o \simeq \mathbb{M}_d(F_o)$. Let π_o be an irreducible admissible representation of $D_o^\times$ which is unramified, i.e. possesses a non-zero vector invariant under $\mathrm{GL}_d(\mathcal{O}_o)$. Then $\dim(\pi_o^{\mathrm{GL}_d(\mathcal{O}_o)}) = 1$ and π_o is the unique unramified component of an induced representation of the form

$$\mathrm{Ind}(\mathrm{GL}_d(F_o), B(F_o); \mu_1, \dots, \mu_d),$$

where $B \subset \mathrm{GL}_d$ is the standard Borel subgroup of upper triangular matrices and where $\mu_1, \dots, \mu_d$ are unramified quasi-characters of $F_o^\times$. The d-tuple $(\mu_1, \dots, \mu_d)$ is uniquely determined up to permutation, and hence so is the d-tuple of elements of $\mathbb{C}^\times$,

$$(z_1(\pi_o), \dots, z_d(\pi_o)) := (\mu_1(\varpi_o), \dots, \mu_d(\varpi_o)).$$

Moreover, this d-tuple up to permutation is independent of the identification of D_o with $\mathbb{M}_d(F_o)$. Furthermore, if f_o is a Hecke function, i.e. a $\mathrm{GL}_d(\mathcal{O}_o)$-biinvariant function with compact support on $\mathrm{GL}_d(F_o)$, with Satake transform $f_o^\vee$, then

$$\mathrm{tr}\, \pi_o(f_o) = f_o^\vee (z_1(\pi_o), \dots, z_d(\pi_o))$$

(cf. [Ca]). Here the Haar measure dg_o on $\mathrm{GL}_d(F_o)$ is normalized so that $\mathrm{GL}_d(\mathcal{O}_o)$ gets volume 1.

(*14.6*) *Starting from now and till the end of* (14.19) *we assume that ρ_∞ is the trivial representation of $\bar{D}_\infty^\times$.*

Fix a place $o \neq \infty$, $o \notin$ Bad, and let $K_o \subset D_o^\times$ be a maximal compact subgroup. We consider the virtual representation of $\mathrm{Gal}(\bar{F}_o/F_o) \times (D^{\infty, o})^\times$ on

$$(H^*_\eta)^{K_o} = \sum_n (-1)^n ((H^n_\eta)^{K_o})^{ss}.$$

The isotypic decomposition (14.2) yields

$$(H_\eta^*)^{K_o} = \sum_{\substack{\pi^\infty \\ \pi_o^{K_o} \neq (o)}} V_{\pi^\infty}^* \otimes \pi^{\infty,o}$$

(recall that $\dim(\pi_o^{K_o}) = 1$ if $\pi_o^{K_o} \neq (0)$).

Fix a finite set T of places of F. For each finite set S of places of F such that $S \cap T = \emptyset$, we consider the convolution algebra $\mathscr{C}_c^\infty(D_S^\times)$ of locally constant functions with compact support on $D_S^\times$. We have fixed a Haar measure dg^T on $(D^T)^\times$ and a splitting of dg^T as a product of local Haar measures so that we have a splitting $dg^T = dg_S\, dg^{T \cup S}$. If $S \subset S', S' \cap T = \emptyset$, we have a homomorphism of algebras

$$\mathscr{C}_c^\infty(D_S^\times) \to \mathscr{C}_c^\infty(D_{S'}^\times), f_S \mapsto f_S\, 1_{K_{S'-S}}\,.$$

We have fixed a compact subgroup $K = \prod_x K_x$ of $D_{\mathbb{A}}^\times$ such that K_x is a maximal compact subgroup for almost every x and we assume that $\mathrm{vol}(K_x, dg_x) = 1$ for almost every x. The Hecke algebra $\mathscr{H}((D^T)^\times)$ is the direct limit of the algebras $\mathscr{C}_c^\infty(D_S^\times)$ for these transition homomorphisms. Then we can view each admissible representation of $(D^T)^\times$ as a non degenerate $\mathscr{H}((D^T)^\times)$-module.

(14.7) Proposition. *Let $f^{\infty,o} \in \mathscr{H}((D^{\infty,0})^\times)$. For any $r \in \mathbb{Z}$, there is an equality of traces*

$$\sum_{\substack{\pi^\infty \\ \pi_o^{K_o} \neq (o)}} \mathrm{tr}(\pi^{\infty,o}(f^{\infty,o}))\,\mathrm{tr}(\mathrm{Frob}_o^r; V_{\pi^\infty}^*)$$

$$= \sum_{\substack{\Pi \\ \Pi_\infty \simeq 1_\infty \mathrm{or}\, \mathrm{St}_\infty \\ \Pi_o^{K_o} \neq (o)}} \chi(\Pi_\infty) m(\Pi)\, \mathrm{tr}(\Pi^{\infty,o}(f^{\infty,o}))\, q^{\deg(o) r(d-1)/2} (z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r)$$

where Π runs through the automorphic irreducible representations of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$. Here $\chi(\Pi_\infty) = 1$ if Π_∞ is isomorphic to the trivial representation 1_∞ of $D_\infty^\times$ and $\chi(\Pi_\infty) = (-1)^{d-1}$ if Π_∞ is isomorphic to the Steinberg representation St_∞ of $D_\infty^\times$.

Proof. Fix a finite closed subscheme $I \subset X \setminus \{\infty, o\}$. The lemma immediately follows from its variant where $r > 0$, where $f^{\infty,o}$ is $K_I^{\infty,o}$-biinvariant and where on both sides we impose in addition the existence of $K_I^{\infty,o}$-invariant vectors. But then we may take $f^{\infty,o}$ to be equal to $1_{K_I^{\infty,o}.g^{\infty,o}.K_I^{\infty,o}}$ and we may normalize the Haar measure so that it gives $K_{I,x}^\infty$ the volume 1 for each place $x \neq \infty$ of F. The left hand side in this variant equals the left hand side of the identity in (13.6). We use the determination of the traces of f_∞ in (13.2) (ii). The lemma therefore follows from (13.6), bearing in mind (cf. (14.5)) that, for our choice of function f_o (cf. (11.19)), we have

$$\mathrm{tr}\, \Pi_o(f_o) = q^{\deg(o) r(d-1)/2} (z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r). \qquad \square$$

(14.8) Lemma. *Let Π be an automorphic irreducible representation of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ and let T be a set of places of F such that $T \not\subset$ Bad. Then the following conditions are equivalent:*

(i) *Π is 1-dimensional and there exists a character χ: $F^\times \backslash \mathbb{A}^\times/\varpi_\infty^{\mathbb{Z}} \to \mathbb{C}^\times$ such that $\Pi \simeq \chi \circ \det$;*

(ii) *Π_T is finite dimensional.*

Moreover, if they are satisfied, we have $m(\Pi) = 1$.

Proof. This lemma is well known. Let us recall its proof. Obviously (i) $\Rightarrow$ (ii). Conversely, if Π_T is finite dimensional, there exists a place $x \notin \mathrm{Bad}$ of F such that Π_x is finite dimensional. As $D_x^\times \simeq \mathrm{GL}_d(F_x)$, this implies that Π_x is 1-dimensional, so that Π_x is trivial on $\mathrm{SL}_d(F_x) \subset \mathrm{GL}_d(F_x) \simeq D_x^\times$. Now, if $f \in \Pi \subset \mathscr{A}(D^\times \backslash D_{\mathbb{A}}^\times / \varpi_\infty^{\mathbb{Z}})$, we have $f(\gamma g_x g) = f(g g^{-1} g_x g) = (\Pi(g^{-1} g_x g)(f))(g) = f(g)$ for all $\gamma \in \mathrm{Ker}(\mathrm{nr}: D^\times \to F^\times)$, all $g_x \in \mathrm{Ker}(\mathrm{nr}: D_x^\times \to F_x^\times)$ and all $g \in D_{\mathbb{A}}^\times$. (Note that $\mathrm{Ker}(\mathrm{nr}: D_x^\times \to F_x^\times) \subset D_x^\times \subset D_{\mathbb{A}}^\times$ is a normal subgroup). But

$$\mathrm{Ker}(\mathrm{nr}: D^\times \to F^\times)\,\mathrm{Ker}(\mathrm{nr}: D_x^\times \to F_x^\times) \subset \mathrm{Ker}(\mathrm{nr}: D_{\mathbb{A}}^\times \to \mathbb{A}^\times)$$

is dense (strong approximation theorem). We therefore have, by the admissibility of Π,

$$(\Pi(g')(f))(g) = f(gg') = f(gg'g^{-1}g) = f(g)$$

for any $g \in D_{\mathbb{A}}^\times$ and any $g' \in \mathrm{Ker}(\mathrm{nr}: D_{\mathbb{A}}^\times \to \mathbb{A}^\times)$, ($\mathrm{Ker}(\mathrm{nr}: D_{\mathbb{A}}^\times \to \mathbb{A}^\times)$ is a normal subgroup of $D_{\mathbb{A}}^\times$), and Π factors through $D_{\mathbb{A}}^\times \xrightarrow{\mathrm{nr}} \mathbb{A}^\times$.

Finally, if $\Pi \simeq \chi \circ \det$ for some character $\chi: F^\times \backslash \mathbb{A}^\times / \varpi_\infty^{\mathbb{Z}} \to \mathbb{C}^\times$, any $f \in \Pi \subset \mathscr{A}(D^\times \backslash D_A^\times / \varpi_\infty^{\mathbb{Z}})$ is of the form $f = \varphi \circ \mathrm{nr}$ for some $\varphi \in \mathscr{A}(F^\times \backslash \mathbb{A}^\times / \varpi_\infty^{d\mathbb{Z}})$ and $m(\Pi) = 1$ as $\mathscr{A}(F^\times \backslash \mathbb{A}^\times / \varpi_\infty^{d\mathbb{Z}})$ is multiplicity free. □

(14.9) Theorem. *Recall that we are assuming that ρ_∞ is the trivial representation of $D_\infty^\times$. Let Π be an automorphic irreducible representation of $D_{\mathbb{A}}^\times / \varpi_\infty^{\mathbb{Z}}$ and let π^∞ be an irreducible admissible representation of $(D^\infty)^\times$.*

(i) *If $\Pi_\infty \simeq 1_\infty$ (the trivial representation of $D_\infty^\times$), then there exists a character χ: $F^\times \backslash \mathbb{A}^\times / F_\infty^\times \to \mathbb{C}^\times$ such that $\Pi \simeq \chi \circ \mathrm{rn}$ (in particular, Π^∞ is 1-dimensional) and there exists at least one integer n such that $V_{\Pi\infty}^n \neq (0)$. Moreover, for almost all places $o \neq \infty$, $o \notin \mathrm{Bad}$, such that Π_o is unramified (i.e. $\chi_o(\mathcal{O}_o^\times) = \{1\}$), we have*

$$\mathrm{tr}(\mathrm{Frob}_o^r; V_{\Pi\infty}^*) = \chi_o(\varpi_o)^r(1 + q^{\deg(o)r} + \cdots + q^{\deg(o)r(d-1)}) \quad (\forall r \in \mathbb{Z})$$

(*cf.* (14.4) *and* (14.5)).

Conversely, if π^∞ is finite dimensional and if there exists at least one integer n such that $V_{\pi^\infty}^n \neq (0)$, then $1_\infty \otimes \pi^\infty$ is an automorphic representation of $D_{\mathbb{A}}^\times / \varpi_\infty^{\mathbb{Z}}$ (in particular π^∞ is 1-dimensional and there exists a character $\chi: F^\times \backslash \mathbb{A}^\times / F_\infty^\times \to \mathbb{C}^\times$ such that $\pi^\infty \simeq \chi^\infty \circ \mathrm{rn}$).

(ii) *If $\Pi_\infty \simeq \mathrm{St}_\infty$ (the Steinberg representation of $D_\infty^\times$), then there exists at least one integer n such that $V_{\Pi^\infty}^n \neq (0)$. Moreover, for almost all places $o \neq \infty$, $o \notin \mathrm{Bad}$, such that Π_o is unramified, we have*

$$\mathrm{tr}(\mathrm{Frob}_o^r; V_{\Pi^\infty}^*) = (-1)^{d-1} m(\Pi) q^{\deg(o)r(d-1)/2}(z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r) \ (\forall r \in \mathbb{Z})$$

(*cf.* (14.4) *and* (14.5)).

Conversely, if π^∞ is infinite dimensional and if there exists at least one n such that $V_{\pi^\infty}^n \neq (0)$, then $\mathrm{St}_\infty \otimes \pi^\infty$ is an automorphic representation of $D_{\mathbb{A}}^\times / \varpi_\infty^{\mathbb{Z}}$.

Proof. We closely follow [Kot 3]. Let $I \subset X \backslash (\{\infty\} \cup \mathrm{Bad})$ be a non empty closed subscheme such that $(\Pi^\infty)^{K_I^\infty} \neq (0)$ and $(\pi^\infty)^{K_I^\infty} \neq (0)$. Let $(\Pi_1, \ldots, \Pi_A)$ be a system of representatives of the isomorphism classes of automorphic representations of $D_{\mathbb{A}}^\times / \varpi_\infty^{\mathbb{Z}}$ which have a non-zero fixed vector under K_I^∞ and which have a local component at ∞ isomorphic to 1_∞ or to St_∞. We have $A < +\infty$ as $D^\times \backslash D^\times_{\mathbb{A}} / \varpi^{\mathbb{Z}}_\infty \mathscr{B}^o_\infty K^\infty_I$ is finite, where $\mathscr{B}^o_\infty$ is the Iwahori subgroup of $D^\times_\infty$. Similarly, let $(\pi^\infty_1, \dots, \pi^\infty_a)$ be a system of representatives of the isomorphism classes of admissible irreducible representations of $(D^\infty)^\times$ which have a non-zero fixed vector under K^∞_I and which occur in H^n_η for at least one integer n. We have $a < +\infty$ *as* $H^n_{\eta,I}$ is finite dimensional for every n and is zero for $n < 0$ or $n > 2d-2$. We can assume that $\Pi = \Pi_1$ (we are only considering Π's such that $\Pi_\infty \simeq 1_\infty$ or $\Pi_\infty \simeq \mathrm{St}_\infty$) and that $\pi^\infty_1 = \pi^\infty$. Let $F^\infty \in \mathscr{H}((D^\infty)^\times)$ (resp. $f^\infty \in \mathscr{H}((D^\infty)^\times)$) be K^∞_I-biinvariant and such that

$$\mathrm{tr}(\Pi^\infty_1(F^\infty)) = 1 \quad (\text{resp. } \mathrm{tr}(\pi^\infty_1(f^\infty)) = 1),$$

such that $\mathrm{tr}(\Pi^\infty_J(F^\infty)) = 0\ (\forall J = 2, \dots, A)$ (resp. $\mathrm{tr}(\pi^\infty_j(f^\infty)) = 0\ (\forall j = 2, \dots, a)$) and such that $\mathrm{tr}(\pi^\infty_j(F^\infty)) = 0$ unless π^∞_j is isomorphic to Π^∞ for all $j = 1, \dots, a$ (resp. $\mathrm{tr}(\Pi^\infty_J(f^\infty)) = 0$ unless Π^∞_J is isomorphic to π^∞ for all $J = 1, \dots, A$).

Now let $o \neq \infty$, $o \notin \mathrm{Bad}$, $o \notin I$ be a place of F such that we can split F^∞ (resp. f^∞) into

$$F^\infty = 1_{K_o} \cdot F^{\infty,o} \quad (\text{resp. } f^\infty = 1_{K_o} \cdot f^{\infty,o}).$$

Obviously, almost all places $o \neq \infty$, $o \notin \mathrm{Bad}$, of F such that Π_o (resp. π^∞_o) is unramified have these properties. Then we can apply (14.7) to $F^{\infty,o} \in \mathscr{H}((D^{\infty,o})^\times)$ (resp. $f^{\infty,o} \in \mathscr{H}((D^{\infty,o})^\times)$) and we get that there exists one (and only one) $j \in \{1, \dots, a\}$ (resp. $J \in \{1, \dots, A\}$) such that $\pi^\infty_j \simeq \Pi^\infty$ (resp. $\Pi^\infty_J \simeq \pi^\infty$) and such that

$$\chi(\Pi_\infty) m(\Pi)\, q^{\deg(o) r(d-1)/2} (z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r) = \mathrm{tr}(\mathrm{Frob}^r_o; V^*_{\pi^\infty_j})$$

(resp.

$$\mathrm{tr}(\mathrm{Frob}^r_o; V^*_{\pi^\infty}) = \chi(\Pi_{J,\infty}) m(\Pi_J) q^{\deg(o) r(d-1)/2} (z_1(\Pi_{J,o})^r + \cdots + z_d(\Pi_{J,o})^r))$$

for all $r \in \mathbb{Z}$.

Here we are using the fact that

$$\chi(\Pi_\infty) m(\Pi)\, q^{\deg(o) r(d-1)/2} (z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r)$$

(resp.

$$\mathrm{tr}(\mathrm{Frob}^r_o; V^*_{\pi^\infty}))$$

cannot be zero for all $r \in \mathbb{Z}$. Indeed, take $r = 0$ (resp. apply Deligne's purity theorem).

Therefore, the theorem follows from lemma 14.8 and the fact that St_∞ is infinite dimensional if $d > 1$. □

(14.10) Corollary. *Let $\chi : F^\times \backslash \mathbb{A}^\times / F^\times_\infty \to \mathbb{C}^\times$ be any character. Then $V^0_{\chi^\infty \circ \mathrm{rn}}$ is the one-dimensional representation of* $\mathrm{Gal}(\bar{F}/F)$ *which corresponds to χ by abelian class field theory (in particular, for each place x of F such that $\chi_x(\mathcal{O}^\times_x) = \{1\}$, $V^0_{\chi^\infty \circ \mathrm{rn}}$ is unramified at x and*

$$\mathrm{tr}(\mathrm{Frob}^r_x; V^0_{\chi^\infty \circ \mathrm{rn}}) = \chi_x(\varpi_x)^r$$

for all $r \in \mathbb{Z}$). For $m = 0, 1, \dots, d-1$, the representation

$$V^{2m}_{\chi^\infty \circ \mathrm{rn}}$$

of $\mathrm{Gal}(\bar{F}/F)$ *is isomorphic to*

$$V^0_{\chi^\infty \circ \mathrm{rn}}(-m).$$

If $n \notin \{0, 2, \ldots, 2d-2\}$, *we have*

$$V^n_{\chi^\infty \circ \mathrm{rn}} = (0).$$

Proof. Any character $\chi : F^\times \backslash \mathbb{A}^\times / F^\times_\infty \to \mathbb{C}^\times$ is of finite order as $F^\times \backslash \mathbb{A}^\times / F^\times_\infty$ is compact. Since $V^n_{\chi^\infty \circ \mathrm{rn}}$ is pure of weight n the assertion follows from (i) of the previous theorem. □

(14.11) Corollary. *If* Π *is any automorphic irreducible representation of* $D^\times_{\mathbb{A}}/\varpi^{\mathbb{Z}}_\infty$ *such that* $\Pi_\infty \simeq \mathrm{St}_\infty$, *then we have the following properties.*
(i) $V^n_{\Pi\infty} = 0$ *unless* $n \equiv d-1 \pmod 2$, *so that* $(-1)^{d-1} V^*_{\Pi\infty}$ *is the virtual representation associated to a true (graded) representation of* $\mathrm{Gal}(\bar{F}/F)$. *Put*

$$V^\bullet_{\Pi\infty} := \oplus V^n.$$

(ii) $\dim_{\bar{\mathbb{Q}}_\ell}(V^\bullet_{\Pi\infty}) = m(\Pi)d$.
(iii) *For each place* x *of* F, *there exists* $A_x \in \mathbb{C}^\times$ *and* $B_x \in \mathbb{Z}$ *such that*

$$\frac{L_x(V^\bullet_{\Pi\infty}, T)}{L_x(V^\bullet_{\Pi\infty}, q_x^{-d} T^{-1})} = A_x T^{B_x} \left[\frac{L_x(\Pi, q_x^{(d-1)/2} T)}{L_x(\Pi^\vee, q_x^{-(d+1)/2} T^{-1})} \right]^{m(\Pi)},$$

where $L_x(V^\bullet_{\Pi\infty}, T)$ *(resp.* $L_x(\Pi, T)$, $L_x(\Pi^\vee, T)$*) is the local Galois (resp. automorphic) L-factor at* x *of* $V^\bullet_{\Pi\infty}$ *(resp.* Π, *the contragredient* $\Pi^\vee$ *of* Π*) and where* $q_x = q^{\deg(x)}$.

We stress that $L(V^\bullet_{\Pi\infty}, T) = \prod_n L(V^n_{\Pi\infty}, T)$ is not an alternating product.

Proof. Thanks to the theorem, there exists a finite set S of places of F, containing ∞, Bad and all the ramified places of Π, such that

$$\prod_n L_x(V^{n+d-1}_{\Pi\infty}, T)^{(-1)^n} = L_x(\Pi, q_x^{(d-1)/2} T)^{m(\Pi)}$$

for all places $x \notin S$ of F.

The right hand side of this equality is the inverse of a polynomial of degree $m(\Pi)d$ in $1 + T\bar{\mathbb{Q}}_\ell[T]$. So the same must be true for the left hand side. But this left hand side is a quotient of two polynomials in $1 + T\bar{\mathbb{Q}}_\ell[T]$ and, thanks to Deligne's purity theorem, there cannot be any non trivial common factors in the numerator and the denominator of this rational function (recall that $V^n_{\Pi\infty}$ is unramified at any place $x \notin S$ of F for any integer n). Therefore, the numerator must be equal to 1 and the assertions (i) and (ii) follow immediately.

The assertion (iii) is obvious if $x \notin S$. Moreover, we have functional equations for the global L-functions

$$L(V^\bullet_{\Pi\infty}, T) = \prod_x L_x(V^\bullet_{\Pi\infty}, T^{\deg(x)})$$

and

$$L(\Pi, T) = \prod_x L_x(\Pi, T^{\deg(x)})$$

of the following form

$$\frac{L(V^{\bullet}_{\Pi\infty},T)}{L(V^{\bullet}_{\Pi\infty},q^{-d}T^{-1})}\in\mathbb{C}^{\times}T^{\mathbb{Z}}$$

and

$$\frac{L(\Pi,T)}{L(\Pi^{\vee},q^{-1}T^{-1})}\in\mathbb{C}^{\times}T^{\mathbb{Z}}.$$

(cf. [Gro] and [Go-Ja, (5.1)]). Here we are using the fact that the dual representation $\check{V}^{n}_{\Pi\infty}$ of $\mathrm{Gal}(\bar{F}/F)$ is canonically isomorphic to $V^{2d-2-n}_{\Pi\infty}(d-1)$ (Poincaré duality).

Now the assertion (iii) can be proved in the same way as [He 1 (4.1)] (cf. Remarks 1 and 2 following loc. cit.). Note that Henniart proves a stronger statement for which he needs the theory of local ε-factors for $V_{\Pi\infty}$; at this point in our argument this is not needed and Grothendieck's functional equation is sufficient, cf. however (15.13). □

(14.12) Theorem. *If Π is any automorphic irreducible representation of $D^{\times}_{\mathbb{A}}/\varpi^{\mathbb{Z}}_{\infty}$ such that $\Pi_{\infty}\simeq\mathrm{St}_{\infty}$, then we have the following properties.*
(i) *$V^{n}_{\Pi\infty}=0$ unless $n=d-1$ and $\dim_{\bar{\mathbb{Q}}_{\ell}}(V^{d-1}_{\Pi\infty})=m(\Pi)d$.*
(ii) *Let $o\neq\infty$, $o\notin\mathrm{Bad}$, be a place of F such that Π_o is unramified (cf. (14.5)), then $V^{d-1}_{\Pi\infty}$ is an unramified representation of $\mathrm{Gal}(\bar{F}_o/F_o)$ and*

$$\mathrm{tr}(\mathrm{Frob}^{r}_{o};\,V^{d-1}_{\Pi\infty})=m(\Pi)\,q^{\deg(o)r(d-1)/2}(z_1(\Pi_o)^{r}+\cdots+z_d(\Pi_o)^{r})$$

for each $r\in\mathbb{Z}$. Moreover, we have

$$|z_j(\Pi_o)|=1\quad(\forall j=1,\dots,d).$$

(iii) *The Frob_{∞}-semisimplification of the restriction of $V^{d-1}_{\Pi\infty}$ to $\mathrm{Gal}(\bar{F}_{\infty}/F_{\infty})$ is isomorphic to*

$$\sigma^{0}(\mathrm{St}_d)^{m(\Pi)}.$$

Here $\sigma^{0}(\mathrm{St}_d)$ (up to a Tate twist of $\frac{d-1}{2}$, the so-called d-dimensional special ℓ-adic representation of $\mathrm{Gal}(\bar{F}_{\infty}/F_{\infty})$), is the unique Frob_{∞}-semisimple indecomposable ℓ-adic representation of $\mathrm{Gal}(\bar{F}_{\infty}/F_{\infty})$ such that

$$gr^{M}_{j}\,\sigma^{0}(\mathrm{St}_d)=\begin{cases}\bar{\mathbb{Q}}_{\ell}(-\frac{j+d-1}{2}) & \text{if } j\equiv d-1 \ (\mathrm{mod}\ 2) \text{ and } |j|\leqq d-1\\ 0 & \text{otherwise}\end{cases}$$

(*cf.* [De 1]).

Remark. The proof of Theorem 14.12 is very simple if we know that the local component Π_o at the place o appearing in (ii) is generic (conjecturally this is automatically true). Indeed, in this case, Π_o being unitary and generic, we have [Ja–Sh 2, Ta]

$$|z_j(\Pi_o)|<q^{\deg(o)/2}(\forall j=1,\dots,d).$$

Combining this estimate with Deligne's purity theorem then allows one to deduce from Theorem 14.9 that $|z_j(\Pi_o)|=1, j=1,\dots,d$ and the assertions (i) and (ii) of (14.12). Then, (14.12) (iii) follows (cf. (14.11) (iii)).

Without the assumption of genericity, the proof is more delicate. Firstly, we shall classify all the "pure" and "integral" representations $V^\bullet$ of $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ having the properties (14.11) (i), (ii) and (iii). Then, we shall conclude using either Deligne's hard Lefschetz theorem or Tadić's classification of unitary representations of $\mathrm{GL}_d(F_o)$ for some unramified place $o \neq \infty$, $o \notin \mathrm{Bad}$.

(*14.13*) Let

$$V^\bullet = \bigoplus_{i=0}^{2d-2} V^i$$

be a graded Frob-semisimple ℓ-adic representation of $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$. We shall say that $V^\bullet$ is *pure* if for $i = 0, \dots, 2d-2$ and each j, $gr_j^M V^i$ is pure of weight $i + j$ (M is the monodromy filtration). We shall say that $V^\bullet$ *is integral* if

$$L_\infty(V^\bullet, T) = \prod_{i=0}^{2d-2} L_\infty(V^i, T)$$

has no pole at $T = q_\infty^n$ for each $n > 0$. We shall say that $V^\bullet$ is *selfdual* if

$$(V^i)^\vee = V^{2d-2-i}(d-1)$$

for each $i = 0, \dots, 2d-2$.

If $V^\bullet$ is pure, integral and selfdual, any direct summand of $V^\bullet$ isomorphic to

$$\sigma^0(\mathrm{St}_i)(-j)$$

occurs in degree $i + 2j - 1$ and such a direct summand can exist only if

$$1 \leqq i \leqq d$$

and

$$0 \leqq j \leqq d - 1.$$

(Recall that $\sigma^0(\mathrm{St}_i)(-j)$ is pure of weight $i + 2j - 1$ in the sense of the monodromy filtration, admits $1/(1 - q_\infty^j T)$ as local L-factor and admits $\sigma^0(\mathrm{St}_i)$ $(i + j - 1)$ as dual representation). The following three examples of graded Frob-semisimple ℓ-adic representations of $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ will be of particular importance for us. For each sequence $(i_0, \dots, i_s)$ of positive integers satisfying

$$2(i_0 + \cdots + i_{s-1}) + i_s = d$$

(resp.

$$2(i_0 + \cdots + i_s) = d,$$

resp.

$$i_0 + \cdots + i_s = d),$$

let us set

$$U'(i_0, \dots, i_s) = \bigoplus_{j=0}^{s-1} (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(-d + i_0 + \cdots + i_j))$$
$$\oplus \sigma^0(\mathrm{St}_{i_s})(-i_0 - \cdots - i_{s-1})$$

(resp.

$$U''(i_0, \dots, i_s) = \bigoplus_{j=0}^{s} (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(-d + i_0 + \cdots + i_j)),$$

resp.

$$\tilde{U}(i_0, \dots, i_s) = \bigoplus_{j=0}^{s} (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(-d + i_0 + \cdots + i_j)))$$

$(\sigma^0(\mathrm{St}_i)(-j)$ in degree $i + 2j - 1$ for each $i,j)$. Then $U'(i_0, \dots, i_s)$, $U''(i_0, \dots, i_s)$ and $\tilde{U}(i_0, \dots, i_s)$ are pure, integral and selfdual. The dimension of $U'(i_0, \dots, i_s)$ or $U''(i_0, \dots, i_s)$ (resp. $\tilde{U}(i_0, \dots, i_s)$) is d (resp. $2d$). Moreover, for

$$V^\bullet = U'(i_0, \dots, i_s) \text{ or } U''(i_0, \dots, i_s)$$

(resp.

$$V^\bullet = \tilde{U}(i_0, \dots, i_s))$$

we have

$$\frac{L_\infty(V^\bullet, T)}{L_\infty(V^\bullet, q_\infty^{-d} T^{-1})} \in \frac{1 - q_\infty^{-d} T}{1 - T} \bar{\mathbb{Q}}_\ell^\times T^{\mathbb{Z}}$$

(resp.

$$\frac{L_\infty(V^\bullet, T)}{L_\infty(V^\bullet, q_\infty^{-d} T^{-1})} \in \left(\frac{1 - q_\infty^{-d} T}{1 - T}\right)^2 \bar{\mathbb{Q}}_\ell^\times T^{\mathbb{Z}}.$$

If $f(T), g(T) \in \bar{\mathbb{Q}}_\ell(T)^\times$, we shall write

$$f(T) \sim g(T)$$

if the orders of the zero (or pole) at $T = q_\infty^n$ of $f(T)$ and $g(T)$ are equal for each $n \in \mathbb{Z}$.

(14.14) Lemma. *Let us fix non-negative integers m and m′ such that m′ divides m. Let* $V^\bullet$ *be a pure, integral, selfdual graded* Frob-*semisimple* ℓ-*adic representation of* $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ (*cf.* (14.13)). *We assume moreover that*

$$\frac{L_\infty(V^\bullet, T)}{L_\infty(V^\bullet, q_\infty^{-d} T^{-1})} \sim \left(\frac{1 - q_\infty^{-d} T}{1 - T}\right)^m$$

and that

$$L_\infty(V^i, T)^{-1} \in (1 + T\bar{\mathbb{Q}}_\ell[T])^{m'}$$

for each $i = 0, \dots, 2d - 2$. *Then, there exists a direct summand* $W^\bullet$ *of* $V^\bullet$ *of the form*

$$W^\bullet = \left(\bigoplus_{\alpha \in A} W_\alpha^\bullet\right) \oplus \left(\bigoplus_{\beta \in B} W_\beta^\bullet\right),$$

with $W_\alpha^\bullet$ (*resp.* $W_\beta^\bullet$) *isomorphic to* $U'(i_0, \dots, i_s)^{m'}$ *or* $U''(i_0, \dots, i_s)^{m'}$ (*resp.* $\tilde{U}(i_0, \dots, i_s)^{m'}$) *for some sequence* $(i_0, \dots, i_s)$ *for each* $\alpha \in A$ (*resp.* $\beta \in B$), *such that*

$$m = (|A| + 2|B|)m'.$$

Proof. We make an induction on m. For $m = 0$ there is nothing to prove. Let us assume $m > 0$ and therefore $m \geqq m'$. Then, the only thing that we need to prove is the existence of a direct summand $W^\bullet$ of $V^\bullet$ such that $W^\bullet$ is isomorphic to $U'(i_0, \dots, i_s)^{m'}$, $U''(i_0, \dots, i_s)^{m'}$ or $\tilde{U}(i_0, \dots, i_s)^{m'}$ for some sequence $(i_0, \dots, i_s)$ (then the lemma will follow by applying the induction hypothesis to $V^\bullet/W^\bullet$).
We consider $L_\infty(V^\bullet, T)^{-1}$. We have

$$L_\infty(V^\bullet, T)^{-1} \sim \prod_{n=0}^{d} (1 - q_\infty^n T)^{e_{0,n}}$$

with

$$e_{0,n} \in \mathbb{Z},\ e_{0,n} \geqq 0\ (\forall n),\ e_{0,0} = m$$

and

$$m' | e_{0,n}\ (\forall n).$$

From $e_{0,0} = m > 0$, we deduce that there exists a special representation $\sigma^0(\mathrm{St}_{i_0})$ occurring in $V^\bullet$. By purity, it necessarily occurs in V^{i_0-1} and, by integrality, we necessarily have $1 \leqq i_0 \leqq d$ (cf. (14.13)). As we are assuming

$$L_\infty(V^{i_0-1}, T)^{-1} \in (1 + T\bar{\mathbb{Q}}_\ell[T])^{m'}.$$

the multiplicity of $\sigma^0(\mathrm{St}_{i_0})$ in V^{i_0-1} is at least m'. Finally, by duality, V^{2d-1-i_0} contains a direct summand isomorphic to

$$(\sigma^0(\mathrm{St}_{i_0})\ (-d + i_0))^{m'}.$$

Now, we have three cases:
(1) $i_0 = d$;
(2) $i_0 = \frac{d}{2}$;
(3) $1 \leqq i_0 \leqq d - 1$, $i_0 \neq \frac{d}{2}$.

In the first case, we set

$$W^\bullet = (\sigma^0(\mathrm{St}_d))^{m'} = U'(d)^{m'}$$

and we are done. In the second case, we set

$$W^\bullet = (\sigma^0(\mathrm{St}_{\frac{d}{2}}) \oplus \sigma^0(\mathrm{St}_{\frac{d}{2}})(-\tfrac{d}{2}))^{m'} = U''(\tfrac{d}{2})^{m'}$$

(obviously, this case can occur only if d is even) and we are also done. In the third case, we set

$$W^\bullet = (\sigma^0(\mathrm{St}_{i_0}) \oplus \sigma^0(\mathrm{St}_{i_0})\ (-d + i_0))^{m'}$$

and we continue the proof.

Let us assume that we have found a sequence $(i_0, \dots, i_{j-1})$ of positive integers $(j \geqq 1)$ with

$$i_0 + \cdots + i_{j-1} < d$$

$$2(i_0 + \cdots + i_{j-2}) + i_{j-1} \neq d$$

and

$$i_0 + \cdots + i_{j-1} \neq \frac{d}{2}$$

and a direct summand $W^\bullet_{j-1}$ of $V^\bullet$ which is isomorphic to the m'-th power of the pure, integral, selfdual, graded, Frob-semisimple ℓ-adic representation

$$\begin{aligned}\sigma^0(\mathrm{St}_{i_0})\oplus\sigma^0(\mathrm{St}_{i_1})(-i_0)\oplus\cdots\oplus\sigma^0(\mathrm{St}_{i_{j-1}})(-i_0-\cdots-i_{j-2})\\ \oplus\sigma^0(\mathrm{St}_{i_0})(-d+i_0)\oplus\sigma^0(\mathrm{St}_{i_1})(-d+i_0+i_1)\\ \oplus\cdots\oplus\sigma^0(\mathrm{St}_{i_{j-1}})(-d+i_0+\cdots+i_{j-1})\end{aligned}$$

of $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$. Then, we have

$$\frac{L_\infty(V^\bullet/W^\bullet_{j-1},T)}{L_\infty(V^\bullet/W^\bullet_{j-1},q_\infty^{-d}T^{-1})}\sim\left(\frac{1-q_\infty^{-d}T}{1-T}\right)^{m-m'}\cdot\left(\frac{1-q_\infty^{d-i_0-\cdots-i_{j-1}}T}{1-q_\infty^{i_0+\cdots+i_{j-1}}T}\right)^{m'}$$

with no further simplifications as $i_0+\cdots+i_{j-1}<d$ and $i_0+\cdots+i_{j-1}\neq\frac{d}{2}$. It follows

$$L_\infty(V^\bullet/W^\bullet_{j-1},T)^{-1}\sim\prod_{n=0}^{d}(1-q_\infty^n T)^{e_{j,n}}$$

with

$$e_{j,n}\in\mathbb{Z},e_{j,n}\geqq 0\ (\forall n)$$

$$e_{j,0}=m-m'$$

$$e_{j,i_0+\cdots+i_{j-1}}\geqq m'$$

and

$$m'|e_{j,n}\ (\forall n).$$

From $e_{j,i_0+\cdots+i_{j-1}}\geqq m'>0$, we deduce that there exists a twisted special representation $\sigma^0(\mathrm{St}_{i_j})(-i_0-\cdots-i_{j-1})$ occurring in $V^\bullet/W^\bullet_{j-1}$.By purity, it necessarily occurs in degree $2(i_0+\cdots+i_{j-1})+i_j-1$ and, by integrality we necessarily have

$$1\leqq i_0+i_1+\cdots+i_j\leqq d.$$

$(i_j\geqq 1)$ (cf. (14.13)). As we are assuming

$$L_\infty(V^{2(i_0+\cdots+i_{j-1})+i_j-1},T)^{-1}\in(1+T\bar{\mathbb{Q}}_\ell[T])^{m'}$$

and as, obviously,

$$L_\infty(W^{2(i_0+\cdots+i_{j-1})+i_j-1}_{j-1},T)^{-1}\in(1+T\bar{\mathbb{Q}}_\ell[T])^{m'},$$

the multiplicity of $\sigma^0(\mathrm{St}_{i_j})(-i_0-\cdots-i_{j-1})$ in $V^\bullet/W^\bullet_{j-1}$ is at least m'. Finally, by duality $V^\bullet/W^\bullet_{j-1}$ contains a direct summand (in degree $2d-1-2(i_0+\cdots+i_{j-1})-i_j$) isomorphic to

$$(\sigma^0(\mathrm{St}_{i_j})(-d+i_0+\cdots+i_j))^{m'}.$$

Now, we have four cases:

(1) $2(i_0+\cdots+i_{j-1})+i_j=d$;
(2) $i_0+\cdots+i_j=\frac{d}{2}$;
(3) $i_0+\cdots+i_j=d$;
(4) none of the above.

In the first case, we set

$$\begin{aligned} W^\bullet &= W^\bullet_{j-1} \oplus (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}))^{m'} \\ &= U'(i_0, \cdots, i_j)^{m'} \end{aligned}$$

and we are done. In the second one, we set

$$\begin{aligned} W^\bullet &= W^\bullet_{j-1} \oplus (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(-\tfrac{d}{2}))^{m'} \\ &= U''(i_0, \cdots, i_j)^{m'} \end{aligned}$$

(obviously, this case can only occur if d is even) and we are also done. In the third case, we set

$$\begin{aligned} W^\bullet &= W^\bullet_{j-1} \oplus ((\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(0))^{m'} \\ &= \tilde{U}(i_0, \cdots, i_j)^{m'} \end{aligned}$$

and we are done too. In the last case, we set

$$W^\bullet_j = W^\bullet_{j-1} \oplus (\sigma^0(\mathrm{St}_{i_j})(-i_0 - \cdots - i_{j-1}) \oplus \sigma^0(\mathrm{St}_{i_j})(-d + i_0 + \cdots + i_j))^{m'}$$

and we conclude by induction on j. ☐

(14.15) Lemma. *Let W be a pure ℓ-adic representation of* $\mathrm{Gal}(\bar{F}/F)$ *and let m be a positive integer. We assume that*

$$L_o(W, T)^{-1} \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m,$$

for almost all places o of F where W is unramified. Let x be an arbitrary place of F and consider the monodromy filtration M for the restriction of W to $\mathrm{Gal}(\bar{F}_x/F_x)$ *and its decomposition into primitive parts, (cf.* [De 2, (1.6.4)])

$$gr_i^M(W) = \bigoplus_{\substack{j \geqq |i| \\ j \equiv i \,(\mathrm{mod}\ 2)}} P_j(-\tfrac{i+j}{2}).$$

Then we have

$$\det(1 - T \cdot \mathrm{Frob}_x;\, P_{-j}) \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m,$$

for all Frobenius elements $\mathrm{Frob}_x \in \mathrm{Gal}(\bar{F}_x/F_x)$, *and all $j \geqq 0$.*

Proof. It is clear that $\dim_{\bar{\mathbb{Q}}_\ell}(W) = me$ for some non-negative integer e. Let us consider the algebraic map

$$P : \mathbb{A}^e_{\bar{\mathbb{Q}}_\ell} \to \mathbb{A}^{me}_{\bar{\mathbb{Q}}_\ell}$$

given by

$$(1 + a_1 T + \cdots + a_e T^e)^m = 1 + \sum_{n=1}^{me} P_n(a_1, \ldots, a_e) T^n.$$

Then, P is a closed embedding (if $n = 1, \ldots, e$, we have

$$P_n(a_1, \ldots, a_e) = m a_n + Q_n(a_1, \ldots, a_{n-1})$$

for some polynomial $Q_n(a_1, \ldots, a_{n-1})$). Therefore, the set of $g \in \mathrm{Gal}(\bar{F}/F)$ such that

$$\det(1 - Tg; W) \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m$$

is closed for the Krull topology. But, by hypothesis, this set contains the Frobenius elements for almost all places of F. By Chebotarev's theorem, it follows that

$$\det(1 - Tg; W) \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m$$

for all $g \in \mathrm{Gal}(\bar{F}/F)$.

By Deligne's theorem on the purity of the monodromy filtration (cf. [De 2, (1.8.4)]), $gr_i^M(W)$ is pure of weight i. Therefore it follows that

$$\det(1 - T\cdot \mathrm{Frob}_x; gr_i^M(W)) \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m$$

for all Frobenius elements $\mathrm{Frob}_x \in \mathrm{Gal}(\bar{F}_x/F_x)$ and all i. The assertion now follows from the primitive decomposition by descending induction on j.

(*14.16*) Let Π be as in the statement of Theorem 14.12. We denote by $V^\bullet$ the Frob-semisimplification of the restriction of $V^\bullet_{\Pi^\infty}$ to $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$. Then $V^\bullet$ is a grξaded ℓ-adic representation which (by Deligne's theorem on the purity of the monodromy filtration) is pure in the sense of (14.13), is integral by Deligne's theorem on the integrality of L-functions (cf. [SGA 7, II, XXI, app., (5.2.2) and (5.3)(i)]) and is selfdual by Poincaré duality. Furthermore, specializing (14.10) (ii) to $x = \infty$ we obtain

$$\frac{L_\infty(V^\bullet, T)}{L_\infty(V^\bullet, q_\infty^{-d}T^{-1})} = A_\infty T^{B_\infty}\left[\frac{1 - q_\infty^{-d}T^{-1}}{1 - T}\right]^{m(\Pi)},$$

as the local L-factor of $\mathrm{St}_\infty \simeq \mathrm{St}_\infty^\vee$ is equal to

$$\frac{1}{1 - q_\infty^{(1-d)/2}T}.$$

We may therefore apply Lemma 14.14 to $V^\bullet$ with $m' = 1$, $m = m(\Pi)$. It follows that $V^\bullet$ contains a direct summand $W^\bullet$ of the form

$$W^\bullet = \left(\bigoplus_{\alpha\in A} W_\alpha^\bullet\right)\oplus\left(\bigoplus_{\beta\in B} W_\beta^\bullet\right)$$

(cf. (14.14)) with

$$m(\Pi) = |A| + 2\cdot|B|.$$

Since we have $\dim W_\alpha = d$ $(\alpha \in A)$ and $W_\beta = 2d$ $(\beta \in B)$ (cf. (14.13)) and since $\dim V^\bullet = m(\Pi)\cdot d$ (cf. (14.10) (ii)) we conclude that we have equality $V^\bullet = W^\bullet$. In particular each V^i is of the form

$$V^i = (\sigma^0(\mathrm{St}_{i+1}))^{f^i_{i+1}} \oplus (\sigma^0(\mathrm{St}_{i-1})(-1))^{f^i_{i-1}} \oplus \cdots$$

for suitable exponents f^i_j. Therefore the j-th primitive part of V^i is equal to

$$P^i_{-j} = \begin{cases} \bar{\mathbb{Q}}_\ell(-\frac{i-1}{2})^{f^i_{j+1}}, & 0 \leqq j \leqq i, i \equiv j (\mathrm{mod}\ 2) \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 14.9 (ii) and Deligne's purity theorem we may apply Lemma 14.15 to each $V^i_{\Pi^\infty}$. It follows that all exponents f^i_j are divisible by $m = m(\Pi)$ and hence $L(V^i, T)^{-1} \in (1 + T\bar{\mathbb{Q}}_\ell[T])^m$, all i. We may therefore reapply Lemma 14.14 to $V^\bullet$ with $m' = m = m(\Pi)$. It follows that $V^\bullet$ contains a direct summand $W^\bullet$ of the form

$$W^\bullet = \left(\bigoplus_{\alpha\in A} W_\alpha\right)\oplus\left(\bigoplus_{\beta\in B} W_\beta\right),$$

where dim $W_\alpha = md(\alpha \in A)$, dim $W_\beta = 2md(\beta \in B)$. Comparing dimensions we therefore obtain the following statement.

(14.17) Proposition. *Let Π be as in the statement of Theorem* 14.12. *Then the* Frob-*semisimplification of the restriction of $V^\bullet_{\Pi^\infty}$ to* $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ *is isomorphic to either*

$$V^\bullet = (V^\bullet_{\Pi^\infty})^{\mathrm{Frob-ss}} \simeq U'(i_0, \cdots, i_s)^{m(\Pi)}, 2(i_0 + \cdots + i_{s-1}) + i_s = d$$

or

$$V^\bullet = (V^\bullet_{\Pi^\infty})^{\mathrm{Frob-ss}} \simeq U''(i_0, \cdots, i_s)^{m(\Pi)}, 2(i_0 + \cdots + i_s) = d,$$

(cf. (14.13)).

(*14.18*) *Remark*. Assume that $s = 0$ in the above statement. Then the two alternatives given by the proposition are

$$V^\bullet \simeq \sigma^0(\mathrm{St}_d)^{m(\Pi)} \text{ or } V^\bullet \simeq [\sigma^0(\mathrm{St}_{\frac{d}{2}}) \oplus \sigma^0(\mathrm{St}_{\frac{d}{2}})(-\tfrac{d}{2})]^{m(\Pi)} .$$

We note that in the second case $V^\bullet$ has a non-trivial component in degrees $d/2 - 1$ and $3d/2 - 1$. The difference between these two numbers is equal to 2 if and only if $d = 2$ in which case

$$V^\bullet = [\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell\,(-1)]^{m(\Pi)}$$

in degrees 0 and 2.

Assume now $s > 0$ and let $i_0, \cdots, i_s$ be a sequence of positive integers such that

$$2(i_0 + \cdots + i_{s-1}) + i_s = d \text{ (resp. } 2(i_0 + \cdots + i_s) = d).$$

Then we have a chain of inequalities

$$i_0 - 1 < 2i_0 + i_1 - 1 < \cdots < 2(i_0 + \cdots + i_{s-1}) + i_s - 1 = d - 1$$
$$< 2d - 1 - 2(i_0 + \cdots + i_{s-2}) - i_{s-1} < \cdots < 2d - 1 - 2i_0 - i_1 < 2d - 1 - i_0$$

(resp.

$$i_0 - 1 < 2i_0 + i_1 - 1 < \cdots < 2(i_0 + \cdots + i_{s-1}) + i_s - 1$$
$$< 2d - 1 - 2(i_0 + \cdots + i_{s-1}) - i_s < \cdots < 2d - 1 - 2i_0 - i_1 < 2d - 1 - i_0.)$$

Furthermore the difference between consecutive members of this chain is always of the form $i_j + i_{j-1}$ (resp. $i_j + i_{j-1}$ or $2i_s$). Therefore in both cases these differences are all equal to 2 if and only if $i_0 = i_1 = \cdots = i_s = 1$, in which case d is odd and $s = (d-1)/2$ (resp. d is even and $s = d/2$). Furthermore, in the first case we have

$$V^\bullet = [\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell\,(-1) \oplus \cdots \mathbb{Q}_\ell(-d+1)]^{m(\Pi)}$$

in degrees $0, 2, \ldots, 2d - 2$, whereas the second case does not in fact arise since when d is even the graded representation is concentrated in odd degree, cf. (14.10)(i).

(*14.19*) *Proof of Theorem 14.12* It suffices to prove the statement (iii). Indeed, it then follows from Proposition 14.17 that V_{Π^∞} only contributes to the middle degree cohomology from which (i) follows. The statement (ii) follows from (i) and (14.9) (ii) by Deligne's purity theorem. We shall present now two arguments for proving (iii), one based on the strong Lefschetz theorem and one based on the classification of unitarizable irreducible admissible representations of $\mathrm{GL}_d(F_o)$ due to Tadič [Ta].

First argument. In this argument we use the fact that there exists a class $h \in H_\eta^2(1)$ invariant under the action of $(D^\infty)^\times$ and of $\mathrm{Gal}(\bar{F}/F)$ such that the iterated cup product maps

$$h^i : V_{\Pi^\infty}^{d-1-i} \to V_{\Pi^\infty}^{d-1+i}(i), \quad i = 0, 1, \ldots$$

are isomorphisms. In fact, there is such a class induced by the canonical bundle on $\mathscr{E}\ell\ell_{X,\mathscr{D},I,\eta}/\mathbb{Z}$ which is ample for $\deg I \gg 0$. This last assertion can be roughly seen as follows (details omitted). A modification of the method of Drinfeld [Dr 4] allows one to formulate a moduli problem defining $\mathscr{E}\ell\ell_{X,\mathscr{D},I,\eta}/\mathbb{Z}$ over $X\backslash I$ and to show that its restriction to Spec $\mathcal{O}_\infty$ is represented by a finite disjoint sum of schemes of the form

$$\Gamma\backslash\hat{\Omega},$$

where $\hat{\Omega}$ is Drinfeld's upper half space of dimension $d - 1$ relative to the local field F_∞ and where $\Gamma \subset \mathrm{PGL}_d(F_\infty)$ is a (sufficiently small) cocompact discrete subgroup. For such varieties the ampleness of the canonical bundle of the generic fibre has been established by Mustafin [Mu]. Using the existence of h the proof of (14.12) (ii) is now very simple. By the "strong Lefschetz" property of h we have the implication

$$V^n \neq (0) \Rightarrow V^{n+2} \neq (0), \text{ if } n + 1 \leqq d - 1.$$

Therefore, Proposition 14.17 and the Remark 14.18 imply that if (14.12) (iii) does not hold, then the restriction of $V_{\Pi^\infty}^\bullet$ to $\mathrm{Gal}(\bar{F}_o/F_o)$ for a place $o \neq \infty$, $o \notin$ Bad such that Π_o is unramified is of the form

$$V_{\Pi^\infty}^\bullet \simeq [W^0 \oplus W^2 \oplus \cdots \oplus W^{2d-2}]^{m(\Pi)},$$

with W^{2i} of dimension 1 and pure of weight i, $0 \leqq i \leqq d - 1$. By the strong Lefschetz property we even have

$$W^i \simeq W^0(-i), \quad i = 0, \ldots, d - 1.$$

But then by (14.9) we can order the Hecke eigenvalues of Π_o in such a way that

$$\begin{aligned} z_1(\Pi_o) &= \alpha_o \cdot q_o^{(1-d)/2} \\ z_2(\Pi_o) &= \alpha_o \cdot q_o^{(3-d)/2} \\ &\vdots \\ z_d(\Pi_o) &= \alpha_o \cdot q_o^{(d-1)/2} \end{aligned}$$

where α_o is the eigenvalue of Frob_o on W^0. Thanks to the classification of unramified irreducible representations of $\mathrm{GL}_d(F_o)$ (cf. (14.5)) it follows that Π_o is the 1-dimensional representation

$$\chi_o \circ \det : \mathrm{GL}_d(F_o) \to \mathbb{C}^\times,$$

where χ_o is the unramified character of $F_o^\times$ such that $\chi_o(\varpi_o) = \alpha_0$. But $\Pi_\infty \simeq \mathrm{St}_\infty$ is infinite-dimensional ($d > 1$). So, by (14.8), Π_o cannot be finite-dimensional and we have derived a contradiction.

Second argument. In this argument we are going to use the classification theorem of Tadič [Ta]. In the case of positive characteristic the proof of Tadič uses the fact that the Zelevinsky involution carries irreducible representations again into irreducible representations. This has been announced by I. Bernstein but his proof was never published. Since, however, a proof of this assertion will be contained in a forthcoming paper by P. Schneider and one of us (U. Stuhler) there seems no harm in using it. According to this classification a unitarizable irreducible admissible representation Π_o of $\mathrm{GL}_d(F_o)$ for a local field F_o is of the form

$$\Pi_o = \mathrm{Speh}_{d_1}\,(\mathrm{St}_{b_1}(\rho_1))(\lambda_1) \times \cdots \times \mathrm{Speh}_{d_s}(\mathrm{St}_{b_s}(\rho_s))(\lambda_s),$$

(comp. [Mo-Wa, I. 10]). Here $a_1, \ldots, a_s, b_1, \ldots, b_s, d_1, \ldots, d_s$ are positive integers with $a_1 b_1 d_1 + \cdots + a_s b_s d_s = d$, ρ_i is a unitarizable irreducible supercuspidal representation of $\mathrm{GL}_{d_i}(F_o)$ and $\lambda_i \in]-1/2, 1/2[$ is a real number ($i = 1, \ldots, s$) and we used the notation introduced in (13.8). The product sign refers to the (normalized) induced representation from the standard parabolic $P_{a_1 b_1 d_1, \ldots, a_s b_s d_s}(F_o)$ to $\mathrm{GL}_d(F_o)$ which turns out to be irreducible. We apply this theorem to the local component of our automorphic representation Π at the place o of F with $o \neq \infty$, $o \notin \mathrm{Bad}$ where Π_o is unramified. Then by Deligne's purity theorem we have $\lambda_1 = \cdots = \lambda_s = 0$ and the fact that Π_o is unramified forces Π_o to be of the form

$$\begin{aligned}\Pi_o &= \mathrm{Speh}_{d_1}(\chi_1) \times \cdots \times \mathrm{Speh}_{d_s}(\chi_s)\\ &= \chi_1 \circ \det_{d_1} \times \cdots \times \chi_s \circ \det_{d_s}\end{aligned}$$

for suitable unitary unramified characters $\chi_1, \ldots, \chi_s$ of $F_o^\times$ and a partition $(d_1, \ldots, d_s)$ of d. Here $\det_{d_i}: \mathrm{GL}_{d_i}(F_o) \to F_o^\times$ denotes the determinant map. The Hecke eigenvalues of Π_o are $\chi_1(\varpi_o) \cdot q_o^{(d_1 - 1)/2}, \ldots, \chi_1(\varpi_o) \cdot q_o^{(1 - d_1)/2}, \ldots, \chi_s(\varpi_o) \cdot q_o^{(d_s - 1)/2}, \ldots, \chi_s(\varpi_o) \cdot q_o^{(1 - d_s)/2}$. By Deligne's purity theorem it is possible to determine the trace of Frob_o^r on $V_{\Pi^\infty}^n$ from $(\chi_1, \ldots, \chi_s)$ and $(d_1, \ldots, d_s)$. In particular we again have the implication

$$V^n \neq (0) \Rightarrow V^{n+2} \neq (0), \quad \text{if } n + 1 \leqq d - 1$$

and one concludes as before.

(*14.20*) We conclude this section with some remarks on the case of a general representation ρ_∞. They will not be used in the next sections.

We take up the notations and the assumptions introduced in (13.8). In this case (14.7) has to be modified in the obvious way: the sum of the right hand side is over the set of all automorphic irreducible representations of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ such that Π_∞ is either isomorphic to $\mathrm{Speh}_{2t+1}(\pi'_\infty)$, in which case $\chi(\Pi_\infty) = (-1)^{d-1-2t}$, or isomorphic to $\pi_\infty = \mathrm{St}_{2t+1}(\pi'_\infty)$, in which case $\chi(\Pi_\infty) = (-1)^{d-1}$.

(14.21) Conjecture *Let Π be an automorphic irreducible representation of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ and let π^∞ be an irreducible admissible representation of $(D^\infty)^\times$.*
(i) *If $\Pi_\infty \simeq \mathrm{Speh}_{2t+1}(\pi'_\infty)$, then $V_{\Pi^\infty}^n = 0$ unless $n = d - 1 + 2t - 2i$ for some $0 \leqq i \leqq 2t$,*

$$V_{\Pi^\infty}^{d-1+2t-2i} = V_{\Pi^\infty}^{d-1+2t}(i)$$

for any $i \in \mathbb{Z}$, $0 \leqq i \leqq 2t$. For all places $o \neq \infty$, $o \notin \mathrm{Bad}$, such that Π_o is unramified and such that Π_o is the unique irreducible quotient of the induced representation $\pi'_o(t) \times \pi'_o(t-1) \times \cdots \times \pi'_o(-t)$ where π'_o is an unramified unitary representation of $\mathrm{GL}_{d'}(F_o)$, we have

$$\mathrm{tr}(\mathrm{Frob}_o^r;\, V_{\Pi^\infty}^{d-1+2t}) = m(\Pi)\, q^{\deg(o) r (d-1+2t)/2}(z_1(\pi'_o)^r + \cdots + z_{d'}(\pi'_o)^r)$$

for all $r \in \mathbb{Z}$. Furthermore, $|z_{j'}(\pi'_o)| = 1$, $j' = 1, \ldots, d'$.

Conversely, if there exists at least one integer $n \neq d - 1$ *such that* $V^n_{\Pi^\infty} \neq (0)$, *then* $\mathrm{Speh}_{2t+1}(\pi'_\infty) \otimes \pi^\infty$ *is an automorphic irreducible representation of* $D^\times_{\mathbb{A}}/\varpi^{\mathbb{Z}}_\infty$.
(ii) *If* $\Pi_\infty \simeq \pi_\infty = \mathrm{St}_{2t+1}(\pi'_\infty)$, *then* $V^n_{\Pi^\infty} = 0$ *unless* $n = d - 1$. *For all places* $o \neq \infty, o \notin \mathrm{Bad}$, *such that* Π_o *is unramified,*

$$\mathrm{tr}(\mathrm{Frob}^r_o;\, V^{d-1}_{\Pi^\infty}) = m(\Pi)\, q^{\deg(o) r (d-1)/2} (z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r)$$

for all $r \in \mathbb{Z}$. *Furthermore,* $|z_j(\Pi_o)| = 1, j = 1, \ldots, d$.
Conversely, if $V^n_{\pi^\infty} = (0)$, *for all* $n \neq d - 1$ *and* $V^{d-1}_{\pi^\infty} \neq (0)$, *then* $\pi_\infty \otimes \pi^\infty$ *is an automorphic irreducible representation of* $D^\times_{\mathbb{A}}/\varpi^{\mathbb{Z}}_\infty$.

(*14.22*) The method that we have used to prove (14.9), (14.10) and (14.12), i.e. the case ρ_∞ trivial of the above conjecture, does not extend to the non trivial ρ_∞'s. But, under the local assumptions of (13.8), it is not difficult to deduce the conjecture from the following hypothesis (we leave the details to the reader).

(14.23) Hypothesis. *Let* Π *be an automorphic irreducible representation of* $D^\times_{\mathbb{A}}/\varpi^{\mathbb{Z}}_\infty$ *and let* $o \neq \infty, o \notin \mathrm{Bad}$ *be a place of* F *such that* Π_o *is unramified.*
(i) *If* $\Pi_\infty \simeq \mathrm{Speh}_{2t+1}(\pi'_\infty)$, *then* Π_o *is the unique irreducible quotient of the induced representation* $\pi'_o(t) \times \pi'_o(t-1) \times \cdots \times \pi'_o(-t)$ *where* π'_o *is an unramified generic unitary representation of* $\mathrm{GL}_{d'}(F_o)$.
(ii) *If* $\Pi_\infty \simeq \pi_\infty = \mathrm{St}_{2t+1}(\pi'_\infty)$, Π_o *is generic.*

(*14.24*) *Remark.* In the extreme case where π_∞ is supercuspidal, (14.23) (i) and (14.23) (ii) coincide. In the other extreme case where $\pi_\infty = \mathrm{St}_\infty(t = \frac{d-1}{2}, d' = 1, \pi'_\infty$ is the trivial character of $F^\times_\infty$ and $\mathrm{Speh}_{2t+1}(\pi'_\infty) = 1_\infty)$, (14.23) (i) is trivially satisfied (cf. (14.8)) and (14.23) (ii) is a consequence of (14.12) (ii) (we have even proved that Π_o is tempered).

We also note that, if a fully worked out global Jacquet-Langlands correspondance existed between $D^\times$ and $\mathrm{GL}_d(F)$, (14.23) (ii) would hold and (14.23) (i) would follow from the results of [Mo-Wa IV]. Indeed, any automorphic irreducible representation of $\mathrm{GL}_d(\mathbb{A})/\varpi^{\mathbb{Z}}_\infty$ (or $\mathrm{GL}_{d'}(\mathbb{A})/\varpi^{(2t+1)\mathbb{Z}}_\infty$) with a discrete series local component at ∞ is cuspidal and all unramified local components of any cuspidal automorphic irreducible representation of $\mathrm{GL}_d(\mathbb{A})/\varpi^{\mathbb{Z}}_\infty$ (or $\mathrm{GL}_{d'}(\mathbb{A})/\varpi^{(2t+1)\mathbb{Z}}_\infty$) are generic (cf. [Sh]). In particular, thanks to the results of [Ja-Pi-Sh 1], Conjecture 14.21 holds true in the cases $d = 2$ and $d = 3$.

15 A global proof of the local Langlands conjecture in characteristic *p*

In this section we shall make use of the results of the previous chapter to give a proof of the local Langlands conjecture in a strong form. In the beginning we shall depart from the notations used elsewhere in the paper. We now let F denote a local field of characteristic p. We denote by $\mathcal{O}, \varpi, \kappa \simeq \mathbb{F}_q$ the ring of integers, a uniformizer and the residue field of F. Recall from the list of notations that we have fixed an isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$.

(*15.1*) We start by recalling some facts about ℓ-adic representations. Let $\mathcal{G}_F(d)$ be the set of isomorphism classes of ℓ-adic representations of dimension d of $\mathrm{Gal}(\bar{F}/F)$ with determinant character of finite order and let $\mathcal{G}^0_F(d) \subset \mathcal{G}_F(d)$ be the subset of isomorphism classes of irreducible representations.

(*15.2*) *Remark.* Each $\sigma \in \mathscr{G}_F^0(d)$ is automatically Frob-semisimple (cf. [De 1, Sect. 8]). Indeed, σ factors through a finite quotient of $\mathrm{Gal}(\bar{F}/F)$. Conversely, any ℓ-adic representation σ whose associated Frob-semisimple representation σ' is irreducible is itself irreducible. Indeed, since σ' is irreducible, the restriction of σ' and therefore of σ to the inertia group of $\mathrm{Gal}(\bar{F}/F)$ factors through a finite quotient. But in this case, σ' is just the associated semi-simple representation to σ, hence if σ' is irreducible then so is σ and $\sigma' \simeq \sigma$.

(*15.3*) We denote by $\mathrm{Sp}_n(1) = \sigma^0(\mathrm{St}_n)\,(\frac{n-1}{2})$ (cf. (14.12)) the special representation of dimension n of $\mathrm{Gal}(\bar{F}/F)$. It is an indecomposable representation which is a successive extension of one-dimensional representations $\bar{\mathbb{Q}}_\ell(\frac{1-n}{2})$, $\bar{\mathbb{Q}}_\ell(\frac{1-n}{2}+1), \ldots, \bar{\mathbb{Q}}_\ell(\frac{n-1}{2})$ with $\bar{\mathbb{Q}}_\ell(\frac{n-1}{2})$ as the unique irreducible submodule and with $\bar{\mathbb{Q}}_\ell(\frac{1-n}{2})$ as the unique irreducible quotient module. For any $\sigma \in \mathscr{G}_F(d)$ its Frob-semisimplification $\sigma^{\mathrm{Frob-ss}}$ can be written in a unique way as a direct sum

$$\sigma^{\mathrm{Frob-ss}} = \bigoplus_{n \geqq 1} \bigoplus_{1 \leqq d' \leqq d} \bigoplus_{\rho' \in \mathscr{G}_F^0(d')} (\mathrm{Sp}_n(1) \otimes \rho')^{m_{n,\rho'}}$$

(cf. [De 3, (3.1.3) (ii)]).

Here $m_{n,\rho'} \in \mathbb{N}$, and $m_{n,\rho'} = 0$ for all but finitely many pairs (n, ρ'). Then for the L-function (which we regard here as a function of a complex parameter) there is the expression (cf. [De 1, 8.12])

$$\begin{aligned} L(\sigma, s) = L(\sigma^{\mathrm{Frob-ss}}, s) &= \prod_{n \geqq 1} \prod_{1 \leqq d' \leqq d} \prod_{\rho' \in \mathscr{G}_F^0(d')} L(\mathrm{Sp}_n(1) \otimes \rho', s)^{m_{n,\rho'}} \\ &= \prod_{n \geqq 1} \prod_{1 \leqq d' \leqq d} \prod_{\rho' \in \mathscr{G}_F^0(d')} L\left(\rho', s + \frac{n-1}{2}\right)^{m_{n,\rho'}} \end{aligned}$$

Note that for $\rho' \in \mathscr{G}_F^0(d')$ we have $L(\rho', s) \equiv 1$ unless $d' = 1$ and $\rho' = \chi$ is an unramified character of finite order. In the latter case, the L-function $L(\chi, s)$ has no zero's and all poles of $L(\chi, s)$ are on the line $\mathrm{Re}(s) = 0$ and there is in fact a pole at $s = 0$ if and only if χ is trivial. Applying the above formula to $\sigma = \rho_1 \otimes \rho_2$ it therefore follows that for ρ_1, $\rho_2 \in \mathscr{G}_F^0(d_1)$, $\mathscr{G}_F^0(d_2)$ we have $L(\rho_1 \otimes \rho_2, s) \not\equiv 1$ if and only if $d_1 = d_2$ and if there exists an unramified character χ of finite order with $\rho_2 \simeq \check{\rho}_1 \cdot \chi$. Here and in the sequel we denote by $\check{\rho}$ the contragredient of a representation ρ and $\rho \cdot \chi$ the tensor product $\rho \otimes \chi$. We are going to use the formula [De 2, (1.6.11)]

$$\mathrm{Sp}_{n_1}(1) \otimes \mathrm{Sp}_{n_2}(1) = \bigoplus_{j=0}^{\inf(n_1, n_2) - 1} \mathrm{Sp}_{n_1 + n_2 - 1 - 2j}(1).$$

It follows that for $\sigma_1, \sigma_2 \in \mathscr{G}_F(d)$ with their Frob-semisimplifications decomposed as above,

$$\begin{aligned} L(\sigma_1 \otimes \sigma_2, s) = &\prod_{n_1, n_2 \geqq 1} \prod_{\substack{1 \leqq d'_1, d'_2 \leqq d \\ \rho'_1 \in \mathscr{G}_F^0(d'_1), \rho'_2 \in \mathscr{G}_F^0(d'_2)}} \prod_{j=0}^{\inf(n_1, n_2) - 1} \\ &L\left(\rho'_1 \otimes \rho'_2, s + \frac{n_1 + n_2}{2} - 1 - j\right)^{m_{1,n_1,\rho'_1}\ m_{2,n_2,\rho'_2}} \\ = &\prod_{n_1, n_2 \geqq 1} \prod_{\substack{1 \leqq d' \leqq d \\ \rho' \in \mathscr{G}_F^0(d')}} \prod_{\chi} \prod_{j=0}^{\inf(n_1, n_2) - 1} L\left(\chi, s + \frac{n_1 + n_2}{2} - 1 - j\right)^{m_{1,n_1,\rho'} \cdot m_{2,n_2,\check{\rho}'\chi}}. \end{aligned}$$

Here χ ranges over the unramified characters of finite order. In particular, since $\mathrm{Sp}_n(1)^\vee \simeq \mathrm{Sp}_n(1)$ [De 2, (1.6.11)], we obtain for $\sigma \in \mathscr{G}_F(d)$,

$$L(\sigma \otimes \check{\sigma}; s) = \prod_{n_1, n_2 \geqq 1} \prod_{\substack{1 \leqq d' \leqq d \\ \rho' \in \mathscr{G}_F^0(d')}} \prod_{\chi} \prod_{j=0}^{\inf(n_1, n_2) - 1} L\left(\chi, s + \frac{n_1 + n_2}{2} - 1 - j\right)^{m_{n_1,\rho'} \, m_{n_2,\rho'\chi^{-1}}}.$$

(15.4) Corollary. *Let $\sigma \in \mathscr{G}_F(d)$. Then $\sigma \in \mathscr{G}_F^0(d)$ if and only if the L-function $L(\sigma \otimes \check{\sigma}, s)$ has all its poles on the line* $\mathrm{Re}(s) = 0$ *and a simple pole at $s = 0$.*

Proof. The part "only if" has already been proved. Let us check the part "if". Let σ' be the associated Frob-semisimple representation. Then $\sigma' \otimes \check{\sigma}'$ is the associated Frob-semisimple representation to $\sigma \otimes \check{\sigma}$ and $L(\sigma \otimes \check{\sigma}, s) = L(\sigma' \otimes \check{\sigma}', s)$. From the previous formula we deduce that σ' is irreducible, and the part "if" of the corollary follows from Remark 15.2. □

(*15.5*) We next recall some facts from the theory of local L-functions of pairs [Ja-Pi-Sh, Sects. 8,9]. For every pair (π, π') consisting of an irreducible admissible representation π of $\mathrm{GL}_d(F)$ and an irreducible admissible representation π' of $\mathrm{GL}_{d'}(F)$ there is an L-function $L(\pi \times \pi', s)$, and an ε-factor $\varepsilon(\pi \times \pi', \psi, s)$ depending on the choice of a non-trivial additive character ψ of F. We note that implicit here is the choice of a Haar measure on F which will always be taken to be selfdual with respect to ψ. We shall need the following analytic properties of the L-functions. Assume that π and π' are both supercuspidal representations with central character of finite order. Then

$$L(\pi \times \pi', s) = \prod_{\chi} L(\chi, s),$$

where the product ranges over all unramified characters of finite order such that $\pi' \simeq \check{\pi}\chi$. Here $\pi\chi = \pi \otimes (\chi \circ \det)$. In particular, $L(\pi \times \pi', s) \equiv 1$, if $d \neq d'$ and $L(\pi \times \check{\pi}, s)$ has all its poles on the line $\mathrm{Re}(s) = 0$, and there is in fact a simple pole at $s = 0$. In what follows we let $\mathscr{A}_F^0(d)$ be the set of isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_d(F)$ with central character of finite order.

(15.6) Proposition. *We assume given for any $\pi \in \mathscr{A}_F^0(d)$ a non-empty subset $\Sigma_\pi \subset \mathscr{G}_F(d)$ with the following properties:*

(i) *For all π, $\pi' \in \mathscr{A}_F^0(d)$ and $\sigma \in \Sigma_\pi$, $\sigma' \in \Sigma_{\pi'}$* we have

$$L(\sigma \otimes \sigma', s) = L(\pi \times \pi', s).$$

(ii) *For all $\pi \in \mathscr{A}_F^0(d)$, if $\sigma \in \Sigma_\pi$ then $\check{\sigma} \in \Sigma_{\check{\pi}}$.*

Then for all $\pi \in \mathscr{A}_F^0(d)$ the set Σ_π consists of one element σ_π and $\sigma_\pi \in \mathscr{G}_F^0(d)$. Furthermore, the map

$$\mathscr{A}_F^0(d) \to \mathscr{G}_F^0(d) : \pi \mapsto \sigma_\pi$$

is injective.

Proof. Let $\pi \in \mathscr{A}_F^0(d)$ and $\sigma \in \Sigma_\pi$. Then

$$L(\sigma \otimes \check{\sigma}, s) = L(\pi \times \check{\pi}, s)$$

has all its poles on the line $\mathrm{Re}(s) = 0$ and a simple pole at $s = 0$ (cf. (15.5)). Therefore, by (15.4), $\sigma \in \mathscr{G}_F^0(d)$.

Let $\sigma_1, \sigma_2 \in \Sigma_\pi$ for $\pi \in \mathscr{A}_F^0(d)$. Then σ_1, $\sigma_2 \in \mathscr{G}_F^0(d)$ and

$$L(\sigma_1 \otimes \check{\sigma}_2, s) = L(\pi \times \check{\pi}, s) = L(\sigma_1 \otimes \check{\sigma}_1, s).$$

Both L-functions, on the left and on the right, are of the form $\Pi L(\chi, s)$. For the left L-function the product is over all unramified characters χ of finite order such that $\sigma_2 \simeq \sigma_1 \chi$, and for the right L-function the product is over all χ such that $\sigma_1 \simeq \sigma_1 \chi$. The identity of L-functions implies that these two index sets coincide. Since the trivial character appears in the product on the right, it follows that $\sigma_1 = \sigma_2$.

To prove the injectivity of the constructed map from $\mathscr{A}_F^0(d)$ to $\mathscr{G}_F^0(d)$, let π_1, $\pi_2 \in \mathscr{A}_F^0(d)$ and let $\sigma \in \Sigma_{\pi_1} \cap \Sigma_{\pi_2}$. Then

$$L(\pi_1 \times \check{\pi}_2, s) = L(\sigma \otimes \check{\sigma}, s) = L(\pi_1 \times \check{\pi}_1, s).$$

An identical argument to the one just employed shows that $\pi_1 \simeq \pi_2$. □

The following theorem is the main result of this section.

(15.7) Theorem. *For each $d \geq 1$ there exists a bijective map*

$$\mathscr{A}_F^0(d) \to \mathscr{G}_F^0(d) : \pi \mapsto \sigma_\pi$$

with the following properties.

(i) *For any π, $\pi' \in \mathscr{A}_F^0(d)$,*

$$L(\sigma_\pi \otimes \sigma_{\pi'}, s) = L(\pi \times \pi', s)$$

(ii) *For any $\pi \in \mathscr{A}_F^0(d)$,*

$$\sigma_{\check{\pi}} = \check{\sigma}_\pi.$$

Furthermore, this collection of maps (for variable d) has the following properties.

(iii) *For any $\pi \in \mathscr{A}_F^0(d)$, $\pi' \in \mathscr{A}_F^0(d')$,*

$$L(\sigma_\pi \otimes \sigma_{\pi'}, s) = L(\pi \times \pi', s)$$

$$\varepsilon(\sigma_\pi \otimes \sigma_{\pi'}, \psi, s) = \varepsilon(\pi \times \pi', \psi, s)$$

(iv) *For any $\pi \in \mathscr{A}_F^0(d)$, the determinant of σ_π corresponds to the central character of π under local class field theory.*

(v) *For any $\pi \in \mathscr{A}_F^0(d)$ and any character χ of finite order of $F^\times$,*

$$\sigma_{\pi\chi} = \sigma_\pi \chi$$

(correspondence under local class field theory).

(15.8) Remark. The restriction we imposed on the central character (resp. the determinant character) is merely made to simplify our exposition and allows us to avoid the use of the Weil group. Also, it is well-known [He 4] how to extend this map to include all irreducible admissible representations of $\mathrm{GL}_d(F)$ on the one hand and all d-dimensional Frob-semisimple representations of the Deligne-Weil group on the other hand. As a matter of fact, we shall need some version of this extension below, cf. (15.18). We refer to the appendix by Henniart [He 5] for a proof of the fact that there is at most one collection of such maps. This appendix also contains a discussion of the influence of the choice of the isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$ on this correspondence.

(*15.9*) We note that by Proposition 15.6 the map appearing in (15.7), if it exists, is automatically injective. Our next objective will be to use global methods to construct subsets $\Sigma_\pi \subset \mathcal{G}^0_F(d)$ as in (15.6). We therefore return, for the global arguments, to the notation used elsewhere in this paper. We let F denote a global field and x_0 a place of F such that F_{x_0} is isomorphic to the local field under consideration. Let $\infty \neq x_0, x_1 \neq x_0, \infty \neq x_1$ be two other places. Let D be a central division algebra of dimension d^2 over F with invariants

$$\mathrm{inv}_x(D) = \begin{cases} 1/d & \text{if } x = x_0 \\ -1/d & \text{if } x = x_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be a sheaf of maximal orders of D over the smooth projective model X of F. The corresponding moduli scheme $\mathcal{E}\ell\ell_{X,\mathcal{D},I}/\mathbb{Z}$ over $\mathrm{Spec}(F)$ has good reduction outside $I \cup \{\infty, x_0, x_1\}$ for every non-empty $I \subset X \backslash \{\infty\}$.

Let π be a fixed representative of the isomorphism class of a given element of $\mathcal{A}^0_{F_{x_0}}(d)$.

(15.10) Lemma. *Let x_2 be a place of F with $x_2 \notin \{\infty, x_0, x_1\}$. There exists a cuspidal automorphic representation $\Pi \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_d(F)F^\times_\infty \backslash \mathrm{GL}_d(\mathbb{A}))$ such that*

$$\Pi_\infty \simeq \mathrm{St}_\infty$$

$$\Pi_{x_0} \simeq \pi$$

Π_{x_1} and Π_{x_2} are supercuspidal admissible irreducible representations.

Proof (cf. [Ar-Cl, (I.6.5)] for a closely related result in the number field case). Let $x_3 \notin \{\infty, x_0, x_1, x_2\}$ be another place of F. We wish to apply the Deligne-Kazhdan simple trace formula to a function $f = f_\infty f_{x_0} f_{x_1} f_{x_2} f_{x_3} f^{\infty, x_0, x_1, x_2, x_3} \in \mathcal{C}^\infty_c(\mathrm{GL}_d(\mathbb{A}))$. We take f_∞ = the weakly cuspidal Euler-Poincaré function considered in Sect. 13. We fix supercuspidal irreducible representations $\pi_{x_0} = \pi$, π_{x_1}, π_{x_2} at the places x_i, $i = 0, 1, 2$. We choose open compact subgroups $K_{x_i} \subset \mathrm{GL}_d(F_{x_i})$ such that $\pi^{K_{x_i}}_{x_i} \neq (0)$. Then by [Be-Ze 1 2.4.2], there exists a function $f_{x_i} \in \mathcal{C}^\infty_c(\mathrm{GL}_d(F_{x_i}))$ such that for the induced operator on any irreducible admissible representation π' of $\mathrm{GL}_d(F_{x_i})$

$$\pi'(f_{x_i}) = \begin{cases} \pi_{x_i}(1_{K_{x_i}}) & \text{if } \pi' \simeq \pi_{x_i} \\ 0 & \text{if } \pi' \not\simeq \pi_{x_i}. \end{cases}$$

In particular

$$\mathrm{tr}\, \pi_{x_i}(f_{x_i}) = \dim \pi^{K_{x_i}}_{x_i} \neq 0.$$

We recall briefly the construction of f_{x_i}. It is based on the two maps of $\mathrm{GL}_d(F_{x_i}) \times \mathrm{GL}_d(F_{x_i})$-modules

$$\mathrm{End}(V)^\infty \xrightarrow{\varphi} \mathcal{C}^\infty_c(\mathrm{GL}_d(F_{x_i})) \xrightarrow{\pi_{x_i}} \mathrm{End}(V)^\infty.$$

Here V denotes the representation space of π_{x_i} and the upper index ∞ are the smooth vectors and

$$\varphi(A)(g) = \mathrm{tr}(\pi_{x_i}(g^{-1}) \circ A), \quad A \in \mathrm{End}(V)^\infty.$$

The composition $\pi_{x_i} \circ \varphi$ is a scalar $c \neq 0$. Then

$$f_{x_i} = c^{-1} \cdot \varphi(\pi_{x_i}(1_{K_{x_i}}))$$

has the asserted properties. From this expression we conclude that

$$f_{x_i}(1) = c^{-1} \cdot \operatorname{tr} \pi_{x_i}(1_{K_{x_i}}) = c^{-1} \cdot \dim \pi_{x_i}^{K_{x_i}} \neq 0.$$

This fixes the choice of f_{x_i} for $i = 0, 1, 2$. For $f_{x_3} \in \mathscr{C}_c^\infty(\mathrm{GL}_d(F_{x_3}))$ we take a function with support contained in the regular elliptic set. Then for any $f^{\infty, x_0, x_1, x_2, x_3} \in \mathscr{C}_c^\infty(\mathrm{GL}_d(\mathbb{A}^{\infty, x_0, x_1, x_2, x_3}))$ we have the simple trace formula, as follows

$$\sum_{\Pi \subset \mathscr{A}_{\mathrm{cusp}}(\mathrm{GL}_d(F) F_\infty^\times \backslash \mathrm{GL}_d(\mathbb{A}))} \operatorname{tr}\left(\Pi(f), \frac{dg}{dz_\infty}\right)$$

$$= \sum_{\substack{\gamma \in \mathrm{GL}_d(F)_\natural \\ \gamma \text{ elliptic regular}}} \operatorname{vol}\left(\mathrm{GL}_d(F)_\gamma F_\infty^\times \backslash \mathrm{GL}_d(\mathbb{A})_\gamma, \frac{dg_\gamma}{dz_\infty}\right) \cdot O_\gamma\left(f, \frac{dg}{dg_\gamma}\right)$$

(cf. [De-Ka-Vi] and [He 3]).

It suffices to prove that both sides of this identity are non-zero. Indeed, any Π on the left hand side with $\operatorname{tr}(\Pi(f)) \neq 0$ will have, by the choice of the function f_{x_i}, $i = 0, 1, 2$, component $\Pi_{x_i} \simeq \pi_{x_i}$. Furthermore $\Pi_\infty \simeq \mathrm{St}_\infty$ by Theorem 13.2 since the other possibility for a unitary representation, Π_∞ = trivial representation, is excluded for a cuspidal representation Π.

First step. For $i = 0$, 1, 2 there exists a non-empty open subset V_i of $\mathrm{GL}_d(F_{x_i})$ consisting of regular elliptic elements and invariant under conjugation by $\mathrm{GL}_d(F_{x_i})$, such that $O_{\gamma_i}\left(f_{x_i}, \frac{dg}{dg_{\gamma_i}}\right) \neq 0$ for all $\gamma_i \in V_i$.

Indeed, fix an elliptic maximal torus $T \subset \mathrm{GL}_d(F_{x_i})$ and denote by T' the subset of regular elements of $T(F)$. Then there is the germ expansion (comp. [He 3, A.3])

$$O_t\left(f, \frac{dg}{dt}\right) = \sum_o \Gamma_o(t) \cdot O_o(f), \quad t \in T'$$

valid for all $t \in T'$ close to 1. Here o ranges over the unipotent orbits. Furthermore, there is the following homogeneity property for the germ Γ_o. If $t = 1 + y \in T'$ is regular elliptic sufficiently close to 1, then [He 3, A.3.4]

$$\Gamma_o(1 + ay) = |a|^{-d(o)/2} \cdot \Gamma_o(1 + y), \quad a \in \mathcal{O}_{F_{x_i}}.$$

Here $d(o)$ denotes the dimension of the unipotent orbit o. Furthermore, $o = \{1\}$ is the unique orbit with $d(o) = 0$. Therefore, if $O_t(f_{x_i}, \frac{dg}{dt}) = 0$ then $\Gamma_1(t) f_{x_i}(1) = 0$. However, $\Gamma_1(t)$ is constant $\neq 0$ for all t sufficiently close to 1 ([He 3, A.3]). On the other hand we know that $f_{x_i}(1) \neq 0$, whence a contradiction. It follows that there exists a neighbourhood U_i of 1 in T such that for all $t \in T' \cap U_i$ we have $O_t(f_{x_i}, \frac{dg}{dg_t}) \neq 0$. However, the map

$$\mathrm{GL}_d(F_{x_i}) \times T' \to \mathrm{GL}_d(F_{x_i})$$

$$(x, t) \mapsto x^{-1} \cdot tx$$

is submersive and hence open [H-C], therefore $T' \cap U_i$ generates by conjugation under $\mathrm{GL}_d(F_{x_i})$ an open subset V_i with the required properties.

Second step. There exists an element $\gamma \in \mathrm{GL}_d(F)$ *such that* γ *is regular elliptic in* $\mathrm{GL}_d(F_\infty)$ *and in* $\mathrm{GL}_d(F_{x_i})$ *for* $i = 0, 1, 2, 3$ *and such that*

$$O_\gamma\left(f_\infty, \frac{dg_\infty}{dg_{\gamma_\infty}}\right) \neq 0, \quad O_\gamma\left(f_{x_i}, \frac{dg_{x_i}}{dg_{\gamma x_i}}\right) \neq 0 \quad i = 0, 1, 2.$$

Indeed, let $P_i(T) \in F_{x_i}[T]$ be the characteristic polynomial of an element $\gamma_i \in V_i$, $i = 0, 1, 2$. The fact that γ_i is regular elliptic in $\mathrm{GL}_d(F_{x_i})$ is equivalent to the fact that $P_i(T)$ is separable and irreducible. Let $P(T) \in F[T]$ be a polynomial which is close to $P_i(T)$ in $F_{x_i}[T]$, $i = 0, 1, 2$, and which is an Eisenstein polynomial in $F_\infty[T]$ and in $F_{x_3}[T]$. Then any $\gamma \in \mathrm{GL}_d(F)$ with characteristic polynomial $P(T)$ will lie in V_i, $i = 0, 1, 2$ and also will be regular elliptic in $\mathrm{GL}_d(F_\infty)$ and in $\mathrm{GL}_d(F_{x_3})$. The assertion about the orbital integral of f_∞ follows therefore from Theorem 13.2 and the assertion about the orbital integrals of f_{x_i}, $i = 0, 1, 2$ from the definition of V_i.

Third step. We may choose the functions f_{x_3} *and* $f^{\infty, x_0, x_1, x_2, x_3}$ *such that in the sum on the right hand side of the simple trace formula there is precisely one non-vanishing term.*

We choose a $\gamma \in \mathrm{GL}_d(F)$ as in the previous step. Let f_{x_3} be the characteristic function of an open compact neighbourhood of γ in the set of regular elliptic elements of $\mathrm{GL}_d(F_{x_3})$ and let $f^{\infty, x_0, x_1, x_2, x_3}$ be the characteristic function of an open compact neighbourhood of γ in $\mathrm{GL}_d(\mathbb{A}^{\infty, x_0, x_1, x_2, x_3})$ which in all but finitely many places is the canonical maximal compact subgroup. Now the set of non-vanishing terms in the sum is finite and contains at least γ. By shrinking the support of f_{x_3} or $f^{\infty, x_0, x_1, x_2, x_3}$ we can arrange that the term corresponding to γ is the only one non-vanishing. □

To pass from GL_d to $D^\times$ we quote from [He 1, A.4], the following result.

(15.11) Lemma. *Let* Π *be as in* (15.10). *Then there is one and up to isomorphism only one* $\tilde{\Pi} \subset \mathscr{A}(D^\times F_\infty^\times \backslash D_{\mathbb{A}}^\times)$ *such that*

$$\tilde{\Pi}_y \simeq \Pi_y \quad y \neq x_0, x_1.$$

Furthermore, the multiplicity $m(\tilde{\Pi})$ *in* $\mathscr{A}(D^\times F_\infty^\times \backslash D_{\mathbb{A}}^\times)$ *is* 1.

(15.12) Theorem. *Let* Π *be a cuspidal representation in* $\mathscr{A}_{\mathrm{cusp}}(\mathrm{GL}_d(F)\, F_\infty^\times \backslash \mathrm{GL}_d(\mathbb{A}))$ *with* $\Pi_\infty \simeq \mathrm{St}_\infty$ *and which satisfies the conclusions of Lemma* 15.11. *There exists a semi-simple* ℓ*-adic representation, unique up to isomorphism,*

$$\Sigma : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_d(\bar{\mathbb{Q}}_\ell)$$

such that

(i) Σ *is unramified outside* $\{\infty, x_0, x_1\} \cup T$ *where* $T = \{x \in |X|;\ \Pi_x^{\mathrm{GL}_d(\mathcal{O}_x)} = (0)\}$ (*note that* $x_2 \in T$ *if* Π *is as in* (15.10)).

(ii) *For every* $r \geqq 0$,

$$\mathrm{tr}(\mathrm{Frob}_x^r; \Sigma) = z_1(\Pi_x)^r + \cdots + z_d(\Pi_x)^r$$

with

$$|z_j(\Pi_x)| = 1, \quad j = 1, \ldots, d$$

for any $x \notin \{\infty, x_0, x_1\} \cup T$.

Proof. Let $\tilde{\Pi}$ be the automorphic representation of $D_{\mathbb{A}}^{\times}$ defined by Lemma 15.11. Since $\tilde{\Pi}_\infty \simeq \Pi_\infty \simeq \mathrm{St}_\infty$ we can take $\Sigma = V_{\tilde{\Pi}^\infty}^{d-1}(\frac{d-1}{2})$, cf. (14.12). The uniqueness follows from the Chebotarev density theorem. □

Remark. We note that by [Sh] the local components $\tilde{\Pi}_y$, $y \neq x_0, x_1$ are generic. We are therefore in the range of applicability of the remark after the statement of (14.12), which allows us to avoid in our proof of the local Langlands conjecture the use of the more subtle arguments needed for establishing (14.12).

(15.13) Proposition. *Let* $\pi \in \mathcal{A}_{F_{x_0}}^0(d)$ *and denote by* $\Pi(\pi)$ *the set of cuspidal automorphic representations* $\Pi \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_d(F)F_\infty^\times \backslash \mathrm{GL}_d(\mathbb{A}))$ *such that there exists* $x_2 \notin \{\infty, x_0, x_1\}$ *for which* Π *satisfies the conditions of Lemma* 15.10. *For* $\Pi \in \Pi(\pi)$ *denote by* $\Sigma(\Pi)$ *the d-dimensional* ℓ*-adic representation associated to* Π *by Theorem* 15.12. *For a place x, denote by* $\Sigma(\Pi)_x$ *the restriction of* $\Sigma(\Pi)$ *to the decomposition group at x.*
(i) *The determinant character of* $\Sigma(\Pi)$ *is of finite order, and corresponds via global class field theory to the central character of* Π.
(ii) *If* $\Pi \in \Pi(\pi)$, *then* $\check{\Pi} \in \Pi(\check{\pi})$ *and* $\Sigma(\check{\Pi}) = \check{\Sigma}(\Pi)$.
(iii) *If* $\Pi \in \Pi(\pi)$ *and* χ *is a character of* $F^\times F_\infty^\times \backslash \mathbb{A}^\times$, *then* $\Pi \cdot \chi \in \Pi(\pi\chi_{x_0})$ *and* $\Sigma(\Pi \cdot \chi) = \Sigma(\Pi)\chi$ *(correspondence via global class field theory).*
(iv) *Let* $\pi \in \mathcal{A}_{F_{x_0}}^0(d)$, $\pi' \in \mathcal{A}_{F_{x_0}}^0(d')$ *and* $\Pi \in \Pi(\pi)$, $\Pi' \in \Pi(\pi')$. *Then we have*

$$L(\pi \times \pi', s) = L(\Sigma(\Pi)_{x_0} \otimes \Sigma(\Pi')_{x_0}, s)$$

$$\varepsilon(\pi \times \pi', \psi_{x_0}, s) = \varepsilon(\Sigma(\Pi)_{x_0} \otimes \Sigma(\Pi')_{x_0}, \psi_{x_0}, s),$$

for any non-trivial additive character ψ_{x_0} *of* F_{x_0}.

Proof. Part (*i*) The second statement implies the first. To prove the second statement it suffices by the Chebotarev density theorem to prove the desired equality locally at almost all places of F. However at an unramified place x (i.e. in the notation of theorem 15.12 $x \notin \{\infty, x_0, x_1\} \cup T$) the central character of Π is given by

$$\omega_{\Pi_x}(\varpi_x) = \prod_{j=1}^{d} z_j(\Pi_x)$$

(here ϖ_x denotes a uniformizer at x) and the determinant character of $\Sigma(\Pi)$ at x is given by (15.12) (ii), as

$$\det \Sigma(\Pi)(\mathrm{Frob}_x) = \prod_{j=1}^{d} z_j(\Pi_x) \, .$$

Parts (*ii*) *and* (*iii*) Just as for assertion (i) it suffices to prove these statements locally at the unramified places where they again follow from (15.12) (ii).

Part (*iv*) By (15.12) (ii) for every place $x \notin \{\infty, x_0, x_1\}$ where Π and Π' are unramified we have

$$L(\Pi_x, s) = L(\Sigma(\Pi)_x, s)$$

$$L(\Pi'_x, s) = L(\Sigma(\Pi')_x, s) \, .$$

Therefore, by [He 1, 4.1] we obtain for any non-trivial additive character ψ_{x_0} of F_{x_0},

$$\frac{\varepsilon(\Pi_{x_0}\times\Pi'_{x_0},\psi_{x_0},s)\cdot L(\check{\Pi}_{x_0}\times\check{\Pi}'_{x_0},1-s)}{L(\Pi_{x_0}\times\Pi'_{x_0},s)}$$
$$=\frac{\varepsilon(\Sigma(\Pi)_{x_0}\otimes\Sigma(\Pi')_{x_0},\psi_{x_0},s)\cdot L(\Sigma(\check{\Pi})_{x_0}\otimes\Sigma(\check{\Pi}')_{x_0},1-s)}{L(\Sigma(\Pi)_{x_0}\otimes\Sigma(\Pi')_{x_0},s)}\,.$$

Here we make use of the fact that for the ℓ-adic representations $\Sigma(\Pi)$ and $\Sigma(\Pi')$ a local theory of ε-factors exists [De 1], so that the proof of loc. cit. goes through (compare Remark 1 immediately following 4.1 in loc. cit.). We wish to apply Lemma 4.4 of loc. cit. through Proposition 4.5 of loc. cit. to deduce the assertion (iv). We have to show (considering the L-functions as inverse polynomials in an indeterminate T) that $L(\Pi_{x_0}\times\Pi'_{x_0},T)$ and $L(\check{\Pi}_{x_0}\times\check{\Pi}'_{x_0},q^{-1}T^{-1})$ have no common pole and similarly for $L(\Sigma(\Pi)_{x_0}\otimes\Sigma(\Pi')_{x_0},T)$ and $L(\Sigma(\check{\Pi})_{x_0}\otimes\Sigma(\check{\Pi}')_{x_0}, q^{-1}T^{-1})$. For the first pair this follows from loc. cit. 4.5(ii) since the representations $\pi\simeq\Pi_{x_0}$, $\pi'\simeq\Pi'_{x_0}$ are supercuspidal. Here we used the notation $L(\Pi_x,T)=L_x(\Pi,T^{\deg(x)})$ (cf. Sect. 14) and similarly for the L-functions of pairs. Put $\Sigma=\Sigma(\Pi)\otimes\Sigma(\Pi')$. Then Σ is a dd'-dimensional ℓ-adic representation of $\mathrm{Gal}(\bar{F}/F)$ unramified outside a finite set S of places of F and for $y\notin S$ the restriction Σ_y to the decomposition group at y is an unramified representation pure of weight 0 (all eigenvalues of Frob_y have complex absolute value 1). But then by [De 2, (1.8.1)] (a form of the "purity of the monodromy filtration") we conclude that all eigenvalues of Frob_{x_0} operating on the invariants under the inertia subgroup at x_0 have absolute values $\leqq 1$. It follows that all poles of the L-function $L(\Sigma_{x_0},T)$ have absolute value $\geqq 1$. For the same reason all poles of the L-function $L(\check{\Sigma}_{x_0},q^{-1}T^{-1})$ have absolute value $\leqq q_{x_0}^{-1}$. Therefore common poles cannot occur. □

(15.14) Corollary. *Let* $\pi\in\mathcal{A}^0_{F_{x_0}}(d)$ *and define* $\Sigma_\pi\subset\mathcal{G}_{F_{x_0}}(d)$ *by*

$$\Sigma_\pi=\{\Sigma(\Pi)_{x_0};\quad \Pi\in\Pi(\pi)\}\,.$$

Then Σ_π *satisfies the conditions* (i) *and* (ii) *of Proposition* 15.6, *and hence consists of one element* $\sigma_\pi\in\mathcal{G}^0_F(d)$. *The map* $\pi\mapsto\sigma_\pi$ *is injective and has all properties* (i)–(v) *of Theorem* 15.7.

(15.15) To finish the proof of Theorem 15.7 it remains to prove the surjectivity of the constructed map. Since our proof of this will be purely local we switch back to the notation used in the beginning of this section (hence F is again a local field etc.). We are going to reproduce the relevant parts of Henniart's proof of the numerical Langlands conjecture [He 2]. (This method, introduced by J. Tunnel and by H. Koch, partitions the source and the target of the map in question into finite subsets which are mapped into each other and consists in showing that corresponding finite subsets have the same number of elements.)

(15.16) We recall that to a representation $\sigma\in\mathcal{G}_F(d)$ there is associated its Artin exponent $a(\sigma)\geqq 0$ and its Swan conductor.

$$\mathrm{sw}(\sigma)=a(\sigma)-(\dim\sigma-\dim\sigma^{I_F})\,.$$

(Here I_F denotes the inertia subgroup.) There is a close relation to the ε-factor. Fix an additive character of conductor 0. Then (comp. [Ta (3.4.6)]),

$$\varepsilon(\sigma, \psi, s) = c \cdot q^{-a(\sigma)s}, \quad c \in \mathbb{C}^\times .$$

If χ is an unramified character, $\mathrm{sw}(\rho\chi) = \mathrm{sw}(\rho)$. We are also going to use the formula

$$\mathrm{sw}(\mathrm{Sp}_n(1) \otimes \rho') = n \cdot \mathrm{sw}(\rho'), \quad \rho' \in \mathscr{G}_F^0(d') .$$

There is an analogous theory for irreducible admissible representations π of $\mathrm{GL}_d(F)$. Again for ψ of conductor 0, write

$$\varepsilon(\pi, \psi, s) = c \cdot q^{-a(\pi)s}, \quad c \in \mathbb{C}^\times .$$

Then $a(\pi) \in \mathbb{Z}, \geqq 0$ and depends on π alone [Ja-Sh 1], and is called the Artin exponent of π. Put again

$$\mathrm{sw}(\pi) = a(\pi) - (d - \deg_T(L(\pi, T)^{-1})) .$$

There is a compatibility with the formation of the generalized Steinberg module

$$\mathrm{sw}(\mathrm{St}_n(\rho')) = n \cdot \mathrm{sw}(\rho'), \quad \rho' \in \mathscr{A}_F^0(d') .$$

Again for an unramified character χ we have $\mathrm{sw}(\pi\chi) = \mathrm{sw}(\pi)$. Furthermore, if π is square-integrable (more generally, if π is generic), then $a(\pi) \leqq j$ for some integer $j \geqq 0$ if and only if π has a non-zero invariant vector under a certain open compact subgroup $K_j \subset \mathrm{GL}_d(F)$, [Ja-Sh 1, (5.1)] ($K_j$ = congruence subgroup modulo ϖ^j of the standard "mirahoric" subgroup).

Let D be a central division algebra of dimension d^2 over F with invariant $1/d$. Let ρ be an irreducible admissible representation of $D^\times$. There is again an ε-factor $\varepsilon(\rho, \psi, s)$ associated to ρ[Go-Ja] and an Artin exponent $a(\rho)$ and a Swan conductor $\mathrm{sw}(\rho)$ as before. Let $j \geqq d - 1$. Then $a(\rho) \leqq j$ if and only if $\rho \mid 1 + \mathfrak{p}_D^{j-(d-1)}$ is trivial (as usual $1 + \mathfrak{p}_D^0 := \mathscr{D}^\times$.). Here $\mathfrak{p}_D$ denotes the maximal ideal in the maximal order $\mathscr{D}$ of D.

Since we have established compatibility of the constructed map $\mathscr{A}_F^0(d) \to \mathscr{G}_F^0(d)$ with twisting by characters and with ε-factors (even ε-factors of pairs) the following theorem may be applied to finish the proof of Theorem 15.7.

(15.17) Theorem (Henniart [He 2, 1.2]). *Every injective map*

$$\mathscr{A}_F^0(d) \to \mathscr{G}_F^0(d)$$

which preserves conductors and is compatible with twisting by unramified characters of finite order is bijective.

For the convenience of the reader, we are going to reproduce the combinatorial part of Henniart's proof reducing this theorem to its key points. The proof will be based on Propositions 15.18 and 15.20 below. We introduce the following notations. Let $I_F \subset \mathrm{Gal}(\bar{F}/F)$ be the inertia subgroup and let $\mathrm{Frob} \in \mathrm{Gal}(\bar{F}/F)$ be a fixed representative of the geometric Frobenius so that

$$\mathrm{Gal}(\bar{F}/F) = I_F \rtimes \mathrm{Frob}^{\hat{\mathbb{Z}}} .$$

For an integer $j \geqq 1$ we let $\mathscr{G}_F^2(d)^j \subset \mathscr{G}_F(d)$ be the subset of isomorphism classes of indecomposable Frob-semisimple d-dimensional ℓ-adic representations σ with

$\det \sigma(\mathrm{Frob}) = 1$ and $\mathrm{sw}(\sigma) \leqq j$. (Note that the condition $\det \sigma(\mathrm{Frob}) = 1$ automatically implies that $\det \sigma$ is of finite order.) We also put $\bar{\mathscr{G}}_F^0(d)^j = \mathscr{G}_F^0(d) \cap \bar{\mathscr{G}}_F^2(d)^j =$ the set of isomorphism classes of irreducibles in $\bar{\mathscr{G}}_F^2(d)^j$.

Similarly, we introduce the set $\bar{\mathscr{A}}_F^2(d)^j$ of isomorphism classes of square-integrable irreducible admissible representations π of $\mathrm{GL}_d(F)$ with $\omega_\pi(\varpi) = 1$ and with $\mathrm{sw}(\pi) \leqq j$. (Here ω_π denotes the central character of π.) We put also $\bar{\mathscr{A}}_F^0(d)^j = \mathscr{A}_F^0(d) \cap \bar{\mathscr{A}}_F^2(d)^j$.

Finally, let $\bar{\mathscr{D}}_F(d)^j$ be the set of isomorphism classes of irreducible admissible representations ρ of $D^\times$ with $\omega_\rho(\varpi) = 1$ and with $\mathrm{sw}(\rho) \leqq j$. Here D denotes the central division algebra of dimension d^2 over F with invariant $1/d$.

(15.18) Proposition. *The sets $\bar{\mathscr{A}}_F^0(d)^j$, $\bar{\mathscr{A}}_F^2(d)^j$ and $\bar{\mathscr{D}}_F(d)^j$ are finite. For their cardinality there is the formula*

$$\sum_{d'|d} \frac{d}{d'} \cdot |\bar{\mathscr{A}}_F^0(d')^{[\frac{jd'}{d}]}| = |\bar{\mathscr{A}}_F^2(d)^j| = |\bar{\mathscr{D}}_F(d)^j| = \sum_{t|d} \frac{d}{t^2} \sum_{s|t} \mu\left(\frac{t}{s}\right) \cdot (q^s - 1) \cdot q^{s \cdot [j/t]} .$$

Here μ denotes the Möbius function.

Proof. The fact that $\bar{\mathscr{D}}_F(d)^j$ is finite is easy to see (for any open compact normal subgroup K of $D^\times$, the factor group $D^\times / K \cdot \varpi^{\mathbb{Z}}$ is finite) and the formula for its cardinality is due to Koch [Ko]. In fact, Koch showed the above formula for the cardinality of $\bar{\mathscr{D}}_F(d)^j$ for any local field F of arbitrary characteristic equipped with a uniformizer ϖ.

We use the Bernstein-Zelevinsky classification which sets up a bijection between the set of isomorphism classes of square-integrable irreducible representations of $\mathrm{GL}_d(F)$ with central character of finite order and the set $\coprod_{d'|d} \mathscr{A}_F^0(d')$, given by $\rho' \mapsto \mathrm{St}_{d/d'}(\rho')$. Since the central character of the Steinberg module $\mathrm{St}_n(\rho')$ is $\omega_{\rho'}^n$ and by the compatibility of the formation of the Steinberg module with the Swan conductor, cf. (15.16), we obtain a surjection

$$\left(\coprod_{d'|d} \bar{\mathscr{A}}_F^0(d')^{[\frac{jd'}{d}]} \right) \times \{\chi \colon \mathrm{Gal}(\bar{F}/F)/I_F \to \bar{\mathbb{Q}}_\ell^\times ;\ \chi^d = 1\} \to \bar{\mathscr{A}}_F^2(d)^j$$

$$(\rho', \chi) \mapsto \mathrm{St}_{d/d'}(\rho') \cdot \chi .$$

Furthermore, two elements (ρ_1', χ_1) and (ρ_2', χ_2) of the left hand side have the same image if and only if $d_1' = d_2'$ and $\rho_2' = \rho_1' \chi_1^{-1} \chi_2$. It follows that

$$|\bar{\mathscr{A}}_F^2(d)^j| = \sum_{d'|d} \frac{d}{d'} |\bar{\mathscr{A}}_F^0(d')^{[\frac{jd'}{d}]}| .$$

Let now F' be a local field of characteristic zero, with the same residue field as F, and equipped with a uniformizer ϖ'. Then, if the valuation of the prime number p in F' is sufficiently large, there is a bijection

$$\bar{\mathscr{A}}_{F'}^2(d)^j \simeq \bar{\mathscr{A}}_F^2(d)^j$$

(cf. [He 2, (2.7)]; this is based on the idea of Kazhdan of comparing representations of groups over close local fields and on the determination of certain Hecke algebras by Howe). By the local Jacquet-Langlands correspondence [De-Ka-Vi, Ro], proved for any local field of characteristic zero, there is a bijection

$$\bar{\mathscr{A}}_{F'}^2(d)^j \simeq \bar{\mathscr{D}}_{F'}(d)^j .$$

Taking into account the remark made previously that $|\bar{\mathscr{D}}_F(d)^j| = |\bar{\mathscr{D}}_{F'}(d)^j|$ the result follows. □

(15.19) Remark. If the Jacquet–Langlands correspondence were proved in characteristic p the recourse to characteristic zero could be avoided and our proof of the local Langlands correspondence would proceed purely in characteristic p. This is in contrast to Henniart's proof of the numerical Langlands conjecture where the cases of characteristic p and characteristic zero are intricately intertwined.

(15.20) Proposition. *The sets $\bar{\mathscr{G}}_F^2(d)^j$, $\bar{\mathscr{G}}_F^0(d)^j$ are finite. For their cardinalities there is the formula*

$$\sum_{d'|d} \frac{d}{d'} \cdot |\bar{\mathscr{G}}_F^0(d')^{[\frac{jd'}{d}]}| = |\bar{\mathscr{G}}_F^2(d)^j| = \sum_{t|d} \frac{d}{t^2} \cdot \sum_{s|t} \mu\left(\frac{t}{s}\right) \cdot (q^s - 1) \cdot q^{s[j/t]} .$$

Proof. The restrictions of all elements σ of $\bar{\mathscr{G}}_F^2(d)^j$ to a certain ramification group are trivial. Since $\det \sigma(\mathrm{Frob}) = 1$ the finiteness assertions follow easily. The first identity is proved just as the corresponding identity in Proposition 15.18, replacing in its proof the Steinberg module $\mathrm{St}_n(\rho')$ by the special representation $\mathrm{Sp}_n(1) \otimes \rho'$.

We introduce the notation

$$\bar{\mathscr{G}}_F^0(d, s)^j = \{\sigma \in \bar{\mathscr{G}}_F^0(d)^j;\ \sigma | I_F \text{ has precisely } s \text{ irreducible components}\} .$$

Note that by Frobenius reciprocity the s irreducible components of $\sigma | I_F$ have the same dimension d/s. Then

$$\bar{\mathscr{G}}_F^0(d)^j = \coprod_{s|d} \bar{\mathscr{G}}_F^0(d, s)^j .$$

Let F_s be the unique unramified extension of degree s of F in the fixed algebraic closure of F. Let

$$\iota_s\colon \mathrm{Gal}(\bar{F}/F_s) \to \mathrm{Gal}(\bar{F}/F_s)\colon g \mapsto \mathrm{Frob}^{-1}\, g\, \mathrm{Frob} .$$

Note that $(\iota_s)^s$ is an inner automorphism.

We introduce, for each integer $t \geqq 1$ dividing s,

$$\bar{\mathscr{G}}_{F_s}^0(d, 1)_t^j = \{\sigma \in \bar{\mathscr{G}}_{F_s}^0(d, 1)^j;\ \sigma \circ \iota_s^t \simeq \sigma \text{ and } t \text{ is minimal among integers} \geqq 1 \text{ with this property}\} .$$

Therefore

$$\bar{\mathscr{G}}_{F_s}^0(d, 1)^j = \coprod_{t|s} \bar{\mathscr{G}}_{F_s}^0(d, 1)_t^j .$$

(15.21) Lemma. *For every $s | d$ we have*

$$|\bar{\mathscr{G}}_F^0(d, s)^j| = \frac{1}{s} \left| \bar{\mathscr{G}}_{F_s}^0\left(\frac{d}{s}, 1\right)_s^{[j/s]} \right|$$

and hence

$$|\bar{\mathscr{G}}_F^0(d)^j| = \sum_{s|d} \frac{1}{s} \left| \bar{\mathscr{G}}_{F_s}^0\left(\frac{d}{s}, 1\right)_s^{[j/s]} \right| .$$

Proof. We have a surjective map

$$\bar{\mathscr{G}}_{F_s}^0\left(\frac{d}{s}, 1\right)_s^{[j/s]} \to \bar{\mathscr{G}}_F^0(d, s)^j$$

$$\sigma' \mapsto \mathrm{Ind}_{F_s}^F(\sigma') .$$

The fibre of this map through σ' is equal to

$$\{\sigma', \sigma' \circ \iota_s, \ldots, \sigma' \circ (\iota_s)^{s-1}\}$$

and hence has always s elements. For all these assertions one uses Frobenius reciprocity for $\mathrm{Gal}(\bar{F}/F) = \mathrm{Gal}(\bar{F}/F_d) \rtimes \mathbb{Z}/d\mathbb{Z}$. Details are left to the reader. □

(15.22) Lemma. *For $t|s|d$ we have*

$$|\bar{\mathscr{G}}^0_{F_s}(d,1)^j_t| = |\bar{\mathscr{G}}^0_{F_t}(d,1)^j_t| .$$

Proof. Define $\tilde{\mathscr{G}}^0_{F_t}(d,1)^j_t$ just as $\bar{\mathscr{G}}^0_{F_t}(d,1)^j_t$, except that the condition on $\det(\mathrm{Frob}^t) = 1$, is changed into

$$\det(\mathrm{Frob}^t)^{s/t} = 1 .$$

Then there is a surjection

$$\tilde{\mathscr{G}}^0_{F_t}(d,1)^j_t \to \bar{\mathscr{G}}^0_{F_s}(d,1)^j_t$$
$$\sigma \mapsto \sigma | \mathrm{Gal}(\bar{F}/F_s)$$

(the irreducibility of $\sigma | \mathrm{Gal}(\bar{F}/F_s)$ follows from the irreducibility of $\sigma | I_{F_t} = \sigma | I_{F_s}$). Furthermore, the elements of the fibre of this map through σ correspond in a one-to-one way to the intertwining maps

$$B\colon \sigma' \circ (\iota_s)^t \simeq \sigma'$$

with given (s/t)-th power

$$B^{s/t} = \sigma'(\mathrm{Frob}^s)\colon \sigma' \circ (\iota_s)^s \simeq \sigma' .$$

Here $\sigma' = \sigma | \mathrm{Gal}(\bar{F}/F_s)$. ($B$ = image of Frob^t). It follows that

$$|\bar{\mathscr{G}}^0_{F_s}(d,1)^j_t| = \frac{1}{s/t} \cdot |\tilde{\mathscr{G}}^0_{F_t}(d,1)^j_t| .$$

On the other hand, it is easy to see that

$$|\tilde{\mathscr{G}}^0_{F_t}(d,1)^j_t| = \frac{s}{t}\, |\bar{\mathscr{G}}^0_{F_t}(d,1)^j_t| .$$

The lemma is proved. □

End of the proof of Proposition 15.20 We obtain from Lemma 15.22 and the decomposition of $\bar{\mathscr{G}}^0_{F_s}(d,1)^j$ preceding Lemma 15.21 for any d'', any j'' and any s

$$|\bar{\mathscr{G}}^0_{F_s}(d'',1)^{j''}| = \sum_{t|s} |\bar{\mathscr{G}}^0_{F_t}(d'',1)^{j''}_t| .$$

Applying the Möbius inversion formula we obtain

$$|\bar{\mathscr{G}}^0_{F_t}(d'',1)^{j''}_t| = \sum_{s|t} \mu\left(\frac{t}{s}\right) \cdot |\bar{\mathscr{G}}^0_{F_s}(d'',1)^{j''}| ,$$

for any d'', any j'' and any t.

Plugging this into Lemma 15.21 we get for any d' and any j'

$$|\bar{\mathscr{G}}^0_F(d')^{j'}| = \sum_{t|d'} \frac{1}{t} \cdot \sum_{s|t} \mu\left(\frac{t}{s}\right) \cdot \left|\bar{\mathscr{G}}^0_{F_s}\left(\frac{d'}{t}, 1\right)^{[j'/t]}\right| .$$

By the first identity of Proposition 15.20 (which is already proved) we obtain

$$\begin{aligned} |\bar{\mathscr{G}}_F^2(d)^j| &= \sum_{d' \mid d} \frac{d}{d'} \cdot \sum_{t \mid d'} \frac{1}{t} \cdot \sum_{s \mid t} \mu\left(\frac{t}{s}\right) \left| \bar{\mathscr{G}}_{F_s}^0\left(\frac{d'}{t}, 1\right)^{[[\frac{jd'}{d}]/t]} \right| \\ &= \sum_{t \mid d} \frac{d}{t^2} \sum_{s \mid t} \mu\left(\frac{t}{s}\right) \cdot \sum_{\bar{d}' \mid \frac{d}{t}} \frac{1}{\bar{d}'} \cdot |\bar{\mathscr{G}}_{F_s}^0(\bar{d}', 1)^{[[j/t] \frac{\bar{d}'}{d/t}]}| \, . \end{aligned}$$

($\bar{d}' = d'/t$).

Here we used the fact that for $x \in \mathbb{R}$, $x > 0$ and $t \in \mathbb{Z}$, $t \geqq 1$, $\left[\frac{x}{t}\right] = \left[\frac{[x]}{t}\right]$.

Now we use the main theorem of [He 2, 1.3] which yields

$$\sum_{d' \mid d} \frac{1}{d'} |\bar{\mathscr{G}}_F^0(d', 1)^{[\frac{jd'}{d}]}| = (q-1) \cdot q^j \, .$$

(The proof of this formula uses the geometric Fourier transform [Lau 1]; to deduce the above formula from the one of loc. cit. note that the expression appearing in loc. cit.

$$\sum_{\substack{k \geqq 1 \\ kd \leqq d'j}} |\mathscr{X}_F \backslash \mathscr{G}_F^{00}(d', k)|$$

is, in our notation, equal to $\frac{1}{d'} |\bar{\mathscr{G}}_F^0(d', 1)^{[\frac{jd'}{d}]}|$, cf. [He 2, 2.2(c)].)

Applying this identity to F_s instead of F and d/t instead of d and $[j/t]$ instead of j and inserting the resulting expression in the sum above we obtain the desired formula. □

Proof of theorem 15.17 From Propositions 15.18 and 15.20 we deduce that

$$\sum_{d' \mid d} \frac{d}{d'} \cdot |\bar{\mathscr{G}}_F^0(d')^{[\frac{jd'}{d}]}| = \sum_{d' \mid d} \frac{d}{d'} |\bar{\mathscr{A}}_F^0(d')^{[\frac{jd'}{d}]}| \, .$$

Let X_F be the set of unramified characters of finite order, either of the multiplicative group $F^\times$ or of the Galois group $\mathrm{Gal}(\bar{F}/F)$. Let $\mathscr{G}_F^0(d)^j$ resp. $\mathscr{A}_F^0(d)^j$ be the set of elements in $\mathscr{G}_F^0(d)$ and $\mathscr{A}_F^0(d)$ with Swan conductor $\leqq j$. Then X_F acts on $\mathscr{G}_F^0(d)^j$ and $\mathscr{A}_F^0(d)^j$. There is a surjective map

$$\bar{\mathscr{G}}_F^0(d)^j \times X_F \to \mathscr{G}_F^0(d)^j \colon (\sigma, \chi) \mapsto \sigma\chi$$

and the elements in the fibre through an element $\sigma \in \bar{\mathscr{G}}_F^0(d)^j$ are parametrized by the factor group

$$\{\chi \in X_F;\ \chi^d = 1\} / \{\chi \in X_F;\ \sigma\chi = \sigma\} \, .$$

It follows that the quotient $X_F \backslash \mathscr{G}_F^0(d)^j$ is finite. The same holds for $\mathscr{A}_F^0(d)^j$ instead of $\mathscr{G}_F^0(d)^j$. However, we have, by the hypotheses of (15.17), inclusions

$$\mathscr{A}_F^0(d)^j \subset \mathscr{G}_F^0(d)^j, \ X_F \backslash \mathscr{A}_F^0(d)^j \subset X_F \backslash \mathscr{G}_F^0(d)^j \, .$$

and the number of elements in the inverse image under the above map of an element of $X_F \backslash \mathscr{A}_F^0(d)^j$ and of the corresponding element of $X_F \backslash \mathscr{G}_F^0(d)^j$ is the same. Therefore we deduce from the equality of the two sums above that there has to be termwise equality, yielding

$$X_F \backslash \mathscr{A}_F^0(d)^j = X_F \backslash \mathscr{G}_F^0(d)^j\ .$$

Since any element of $\mathscr{G}_F^0(d)$ lies in $\mathscr{G}_F^0(d)^j$ for some j the surjectivity of the map $\mathscr{A}_F^0(d) \to \mathscr{G}_F^0(d)$ follows. □

16 Further remarks on the global Galois representations

In this section we give some complements on the global Galois representations $V_{\Pi\infty}^{d-1}$ associated to an automorphic representation Π of $D_{\mathbb{A}}^\times/\varpi_\infty^{\mathbb{Z}}$ with $\Pi_\infty \simeq \mathrm{St}_\infty$, cf. (14.12). Note that, by definition, $V_{\Pi\infty}^{d-1}$ is a semi-simple ℓ-adic representation of $\mathrm{Gal}(\bar{F}/F)$. The arguments will all be based on the fact that the Frob_∞-semisimplification of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ is isomorphic to $\sigma^0(\mathrm{St}_d)^{m(\Pi)}$, where $\sigma^0(\mathrm{St}_d)$ is, up to a Tate twist, the special representation of dimension d, cf. (14.12)(iii). Throughout this section a representation Π as above is fixed.

(16.1) Proposition. *Assume that $m(\Pi) = 1$. Then $V_{\Pi\infty}^{d-1}$ is an irreducible representation of* $\mathrm{Gal}(\bar{F}/F)$.

Proof. Since the special representation is indecomposable the hypothesis implies that the Frob_∞-semisimplification of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ is indecomposable. But then the same is true of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ and therefore the semi-simplicity of $V_{\Pi\infty}^{d-1}$ as $\mathrm{Gal}(\bar{F}/F)$-module implies the result. □

We note that we have encountered in (15.11) representations Π satisfying the hypothesis of the previous proposition. Also, for $d = 2$ and $d = 3$ the hypothesis is automatically satisfied [Ja-Pi-Sh 1].

(16.2) Proposition. *Let E/F be a finite extension contained in the fixed algebraic closure $\bar{F}$ of F. Assume $d \geqq 2$. Then for all i*

$$(V_{\Pi\infty}^{d-1}(i))^{\mathrm{Gal}(\bar{F}/E)} = (0)$$

(Galois invariants).

Proof. We argue by contradiction. If $(V_{\Pi\infty}^{d-1}(i))^{\mathrm{Gal}(\bar{F}/E)} \neq 0$, then there exists an injective $\mathrm{Gal}(\bar{F}/E)$-module homomorphism

$$\mathbb{Q}_\ell(-i) \to V_{\Pi\infty}^{d-1}$$

whose image is a direct summand (semi-simplicity). Let E_∞ be the completion of E in a place dividing $\infty \in |F|$. But the Frob_∞-semisimplification of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}/E_\infty)$ is, up to a Tate twist, a power of the special representation $\mathrm{Sp}_d(1)$. (The restriction of the special representation of $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ to $\mathrm{Gal}(\bar{F}_\infty/E_\infty)$ is the special representation of $\mathrm{Gal}(\bar{F}_\infty/E_\infty)$.) It follows easily that the Frob_∞-semisimplication of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}_\infty/E_\infty)$ cannot contain a direct summand of dimension one and therefore the same is true of the restriction of $V_{\Pi\infty}^{d-1}$ to $\mathrm{Gal}(\bar{F}_\infty/E_\infty)$ itself. This contradiction proves the claim. □

We note that it is not difficult to deduce from the previous proposition and standard facts on poles of L-functions the Tate conjectures on algebraic cycles of arbitrary codimension over F on $\mathscr{E}\ell\ell_{X,\mathscr{D},I}/\mathbb{Z}$.

(16.3) Let S be a finite set of places of F containing Bad and all the places $x \notin \text{Bad}$ such that Π_x is ramified (in particular, $\infty \in S$). Then, for each $x \notin S$, the local L-factor

$$L(\Pi_x \times \check{\Pi}_x, T) = \prod_{i,j=1}^{d} \frac{1}{1 - z_i(\Pi_x) z_j(\Pi_x)^{-1} T^{\deg(x)}}$$

is well-defined and we can form the partial L-function

$$L^S(\Pi \times \check{\Pi}, T) = \prod_{x \notin S} L(\Pi_x \times \check{\Pi}_x, T) .$$

Since

$$|z_i(\Pi_x)| = 1, \quad i = 1, \ldots, d ,$$

for all $x \notin S$ (cf. (14.12) (ii)), this product is absolutely convergent for $|T| < q^{-1}$ (we have

$$|\{x \notin S; \deg(x) = n\}| = O(q^n)) .$$

The following lemma must be well known to the specialists.

(16.4) Lemma. *The holomorphic function $L^S(\Pi \times \check{\Pi}, T)$ on the disk $\{T \in \mathbb{C};$ $|T| < q^{-1}\}$ has a meromorphic extension to the whole complex plane.*

Proof. Let $G = \mathrm{GL}_2(D)$ and let $P = MN$ its obvious standard parabolic subgroup ($M \simeq D^\times \times D^\times$ and $N \simeq D$). Then $\Pi \otimes \Pi$ is a "cuspidal" representation of $M(\mathbb{A}) \simeq D_{\mathbb{A}}^\times \times D_{\mathbb{A}}^\times$. Let ω_s: $M(\mathbb{A}) \to \mathbb{C}^\times$ be the quasicharacter defined by

$$\omega_s(g_1, g_2) = |\mathrm{rn}(g_1)/\mathrm{rn}(g_2)|^s ,$$

for each $s \in \mathbb{C}$. We consider the Eisenstein series which are parabolically induced from $(\Pi \otimes \Pi)\,\omega_s$ and let $M(s, \Pi \otimes \Pi)$ be the corresponding global intertwining operator which is a priori only defined for $\mathrm{Re}(s) \gg 0$. For each place x of F, we have a corresponding local intertwining operator

$$M(s, \Pi_x \times \Pi_x)\colon \mathrm{Ind}_{P(F_x)}^{G(F_x)}((\Pi_x \otimes \Pi_x)\omega_s)$$
$$\to \mathrm{Ind}_{P(F_x)}^{G(F_x)}((\Pi_x \otimes \Pi_x)\omega_{-s}) ,$$

a priori defined for $\mathrm{Re}(s) \gg 0$. If $x \notin S$, the above induced representation is unramified. Let f_x^0 be its unique $\mathrm{GL}_{2d}(\mathcal{O}_x)$-fixed vector, up to a scalar ($\mathrm{GL}_2(D_x) \simeq \mathrm{GL}_{2d}(F_x)$). A standard computation [La] shows that

$$M(s, \Pi_x \times \Pi_x)(f_x^0) = \frac{L(\Pi_x \times \check{\Pi}_x, q_x^{-s})}{L(\Pi_x \times \check{\Pi}_x, q_x^{-(1+s)})} f_x^0$$

(comp. [Mo-Wa, I.1]). It follows that, for any

$$f = \left(\prod_{x \in S} f_x\right) \times \left(\prod_{x \notin S} f_x^0\right)$$

in the global induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}((\Pi \otimes \Pi)\omega_s) ,$$

we have ($\mathrm{Re}(s) \gg 0$)

$$M(s, \Pi \times \Pi)(f) = \frac{L^S(\Pi \times \check{\Pi}, q^{-s})}{L^S(\Pi \times \check{\Pi}, q^{-(1+s)})} \left(\prod_{x \in S} M(s, \Pi_x \times \Pi_x)(f_x) \right) \times \left(\prod_{x \notin S} f_x^0 \right).$$

Then, the lemma follows from two fundamental results:

(1) the local intertwining operators $M(s, \Pi_x \times \Pi_x)$ have a meromorphic continuation to the whole complex plane (comp. [Sha, Corollary to Theorem 2.2.2]).
(2) the global intertwining operator $M(s, \Pi \times \Pi)$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$M(-s, \Pi \times \Pi) \circ M(s, \Pi \times \Pi) = \mathrm{id} ,$$

(cf. [Mo]). □

(16.5) Proposition. *There exists an irreducible* ℓ*-adic representation* $\Xi(\Pi)$, *unique up to isomorphism, with*

$$V_{\Pi\infty}^{d-1}\left(\tfrac{d-1}{2}\right) = \Sigma(\Pi) \simeq \Xi(\Pi)^{m(\Pi)} .$$

Moreover, $\Xi(\Pi)$ *has the following properties:*
(i) *for any place* $o \notin S$, $\Xi(\Pi)$ *is an unramified representation of* $\mathrm{Gal}(\bar{F}_o/F_o)$ *with*

$$\mathrm{tr}(\mathrm{Frob}_o^r; \Xi(\Pi)) = z_1(\Pi_o)^r + \cdots + z_d(\Pi_o)^r, \quad \forall r ;$$

(ii) *the restriction of* $\Xi(\Pi)$ *to* $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ *is isomorphic to the special representation* $\mathrm{Sp}_d(1)$.

Proof. Let

$$\Sigma(\Pi) = \bigoplus_{i=1}^{s} V_i^{\oplus m_i}, \quad m_i > 0 ,$$

be the decomposition of $\Sigma(\Pi)$ into irreducibles, where V_i are pairwise non-isomorphic irreducible ℓ-adic representations of $\mathrm{Gal}(\bar{F}/F)$. Since the Frob_∞-semisimplication of the restriction of $\Sigma(\Pi)$ to $\mathrm{Gal}(\bar{F}_\infty/F_\infty)$ is a power of the special representation $\mathrm{Sp}_d(1)$, the same is true for each V_i, $i = 1, \ldots, s$ ($\mathrm{Sp}_d(1)$ is indecomposable), i.e.

$$V_i | \mathrm{Gal}(\bar{F}_\infty/F_\infty) \simeq \mathrm{Sp}_d(1)^{n_i}$$

for $n_i > 0$, $i = 1, \ldots, s$. Counting dimensions, we get the relation

$$\sum_{i=1}^{s} m_i n_i = m(\Pi) .$$

On the other hand, we have

$$L(\Sigma(\Pi)_x \otimes \check{\Sigma}(\Pi)_x, T) = L(\Pi_x \times \check{\Pi}_x, T)^{m(\Pi)^2}$$

for all $x \notin S$ (cf. (14.12) (ii)), so that we get the equality of partial L-functions

$$L^S(\Sigma(\Pi) \otimes \check{\Sigma}(\Pi), T) = L^S(\Pi \times \check{\Pi}, T)^{m(\Pi)^2}.$$

Therefore it follows from Lemma 16.4 that

$$\prod_{i,j=1}^{s} L^S(V_i \otimes \check{V}_j, T)^{m_i m_j} = L^S(\Sigma(\Pi) \otimes \check{\Sigma}(\Pi), T)$$

has a pole of order divisible by $m(\Pi)^2$ at $T=q^{-1}$. But, each V_i defines a smooth ℓ-adic sheaf $\mathscr{V}_i$ on $X\backslash S$, pure of weight 0 (cf. (14.12) (ii)) and, for each $i, j = 1, \ldots, s$, we have (cf. [De 2, (1.4) and (3.3.4)])

$$L^S(V_i \otimes \check{V}_j, T) = \frac{\det(1 - T\,\mathrm{Frob};\, H_c^1)}{\det(1 - T\,\mathrm{Frob};\, H_c^2)}$$

with

$$H_c^1 = H_c^1((X\backslash S)\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q, \mathscr{V}_i\otimes\check{\mathscr{V}}_j)$$

mixed of weights $\leqq 1$ and

$$\begin{aligned} H_c^2 &= H_c^2((X\backslash S)\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q, \mathscr{V}_i\otimes\check{\mathscr{V}}_j) \\ &= \mathrm{Hom}_{\mathrm{Gal}(\bar{F}/F\cdot\mathbb{F}_q)}(V_i, V_j)^{\vee}(-1)\,. \end{aligned}$$

Therefore, the order of the pole at $T=q^{-1}$ of $L^S(V_i\otimes\check{V}_j, T)$ is equal to the multiplicity of the eigenvalue 1 for Frob acting on

$$\mathrm{Hom}_{\mathrm{Gal}(\bar{F}/F\cdot\mathbb{F}_q)}(V_i, V_j)\,,$$

i.e. to 1 if $i=j$ and 0 otherwise (recall that the irreducibles V_i are pairwise non-isomorphic). Then, we have proved that $m(\Pi)^2$ divides

$$\sum_{j=1}^{s} m_i^2 \leqq \sum_{i,j=1}^{s} m_i n_i m_j n_j = m(\Pi)^2$$

and this implies that $s=1$, $n_1=1$ and $m_1=m(\Pi)$. Putting

$$\Xi(\Pi) = V_1$$

it follows that $\Xi(\Pi)$ satisfies all the requirements of the proposition. The uniqueness assertion follows from the Chebotarev density theorem. □

Remark. We let the reader check that a stronger form of the Lemma 16.4 (including the definition of $L(\Pi\times\Pi', s)$ and its functional equation for Π, Π' automorphic representations of $D_{\mathbb{A}}^{\times}/F_{\infty}^{\times}$) would imply that, for each $x_0 \notin \{\infty\} \cup \mathrm{Bad}$ such that Π_{x_0} is supercuspidal, the restriction of $\Xi(\Pi)$ to $\mathrm{Gal}(\bar{F}_{x_0}/F_{x_0})$ is isomorphic to $\sigma_{\Pi_{x_0}}$ (use the techniques of Sect. 15).

A φ-spaces and Dieudonné F_x-modules (following Drinfeld)

The definitions and the results of this appendix are taken from [Dr 7]. Let F be a function field with field of constants $\mathbb{F}_q$. Let k be an algebraic closure of the field $\mathbb{F}_q$ and let us denote by frob_q the arithmetic Frobenius element in $\mathrm{Gal}(k/\mathbb{F}_q)$. For each positive integer n, k contains a unique subfield $\mathbb{F}_{q^n}$ with q^n elements (the fixed field of frob_q^n in k).

(A.1) Definition. *A φ-space* (over k) is a finite-dimensional $F\otimes_{\mathbb{F}_q}k$-vector space which is endowed with a bijective $F\otimes_{\mathbb{F}_q}\mathrm{frob}_q$-semilinear map $\varphi\colon V\to V$. *A morphism α between two φ-spaces* (V_1, φ_1) *and* (V_2, φ_2) is a $F\otimes_{\mathbb{F}_q} k$-linear map $V_1 \xrightarrow{\alpha} V_2$ such that $\varphi_2\circ\alpha = \alpha\circ\varphi_1$.

Obviously, the φ-spaces and their morphisms form a category which is F-linear, abelian, noetherian and artinian.

(A.2) Definition. A *φ-pair* is a pair $(\tilde{F}, \tilde{\Pi})$ where $\tilde{F}$ is a commutative finite-dimensional F-algebra and where $\tilde{\Pi} \in \tilde{F}^{\times} \otimes \mathbb{Q}$ satisfies the following property: for any proper F-subalgebra F' of $\tilde{F}$, $\tilde{\Pi}$ does not belong to $F'^{\times} \otimes \mathbb{Q} \subset \tilde{F}^{\times} \otimes \mathbb{Q}$.

(A.3) Lemma. *If $(\tilde{F}, \tilde{\Pi})$ is a φ-pair and if N is any non zero integer such that $\tilde{\Pi}^N \in \tilde{F}^{\times}$ (more correctly, $\tilde{\Pi}^N$ belongs to the image of $\tilde{F}^{\times}$ in $\tilde{F}^{\times} \otimes \mathbb{Q}$), then $\tilde{F} = F[\tilde{\Pi}^N]$.*

Proof. Let $F' = F[\tilde{\Pi}^N] \subset \tilde{F}$, then $\tilde{\Pi} \in F'^{\times} \otimes \mathbb{Q}$. Therefore $F' = \tilde{F}$. □

(A.4) To each non zero φ-space (V, φ) Drinfeld associates a φ-pair $(F_{(V,\varphi)}, \Pi_{(V,\varphi)})$ in the following way.

Any φ-space (V, φ) (over k) is defined over $\mathbb{F}_{q^n}$ for n divisible enough. So, we can choose a positive integer n', a finite-dimensional $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n'}}$-vector space V', a bijective $F \otimes_{\mathbb{F}_q}$ frob_q-semilinear map φ': $V' \to V'$ and an isomorphism of φ-spaces

$$(V, \varphi) \simeq (V', \varphi') \otimes_{\mathbb{F}_{q^{n'}}} k\ .$$

Then $\varphi'^{n'}$: $V' \to V'$ is a bijective $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n'}}$-linear map and the F-subalgebra

$$F[\varphi'^{n'}] \subset \mathrm{End}_{F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}}(V')$$

is a commutative finite-dimensional F-algebra. If Π': $V \to V$ is the $F \otimes_{\mathbb{F}_q} k$-linear extension of $\varphi'^{n'}(\Pi' = \varphi'^{n'} \otimes_{\mathbb{F}_{q^n}} k)$, the F-subalgebra

$$F' = F[\Pi'] \subset \mathrm{End}_{F \otimes_{\mathbb{F}_q} k}(V)$$

is isomorphic to $F[\varphi'^{n'}]$ and therefore is also a commutative finite-dimensional F-algebra. Moreover, Π' commutes with φ, so

$$F' \subset \mathrm{End}(V, \varphi) \subset \mathrm{End}_{F \otimes_{\mathbb{F}_q} k}(V)$$

(ring of endomorphisms in the category of φ-spaces.)

Now, if $(n'', V'', \varphi'', (V, \varphi) \simeq (V'', \varphi'') \otimes_{\mathbb{F}_{q^{n'}}} k)$ is another choice, there exists some positive integer m such that $\Pi'^{n''m} = \Pi''^{n'm}$. Therefore, the F-algebra

$$\tilde{F} = \bigcap_{N \geqq 1} F[\Pi'^N]$$

is independent of the choices of n', V', φ' and $(V, \varphi) \simeq (V', \varphi') \otimes_{\mathbb{F}_{q^n}} k$. As $\tilde{F} \subset F'$, $\tilde{F}$ is also a commutative finite-dimensional F-algebra and there exists a positive integer N such that $\tilde{F} = F[\Pi'^N]$. Let us choose one such N. As Π'^N: $V \to V$ is bijective, $\Pi'^N \in \tilde{F}^{\times}$ and we can set

$$\tilde{\Pi} = (\Pi'^N)^{1/n'N} \in \tilde{F}^{\times} \otimes \mathbb{Q}\ .$$

Again it is easy to see that $\tilde{\Pi}$ is independent of the choices.

The pair $(\tilde{F}, \tilde{\Pi})$ is clearly a φ-pair. We set $(F_{(V,\varphi)}, \Pi_{(V,\varphi)}) = (\tilde{F}, \tilde{\Pi})$.

By definition, $F_{(V,\varphi)}$ is a F-subalgebra of the center of $\mathrm{End}(V, \varphi)$ ($F[\varphi'^{n'}]$ is contained in the center of $\mathrm{End}(V', \varphi') \subset \mathrm{End}_{F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}}(V')$).

Drinfeld uses the construction (A.4) to prove that the category of φ-spaces is semi-simple and to classify its simple objects.

(A.5) Lemma. *If $(\tilde{F}, \tilde{\Pi})$ is a φ-pair, $\tilde{F}$ is an étale F-algebra, i.e. is a product of finitely many separable finite field extensions of F.*

Proof. It suffices to check that $\tilde{F} = F[\tilde{F}^p]$. But, if we set $F' = F[\tilde{F}^p]$, F' is a F-subalgebra of $\tilde{F}$, $\tilde{\Pi}^p \in F'^{\times} \otimes \mathbb{Q}$ and $\tilde{\Pi} \in F'^{\times} \otimes \mathbb{Q}$. Therefore, $F' = \tilde{F}$. □

(A.6) Theorem. (i) *The abelian category of φ-spaces (over k) is semi-simple.*
(ii) *The map $(V, \varphi) \mapsto (F_{(V,\varphi)}, \Pi_{(V,\varphi)})$ induces a bijection between the set of isomorphism classes of irreducible φ-spaces and the set of isomorphism classes of φ-pairs $(\tilde{F}, \tilde{\Pi})$ where $\tilde{F}$ is a field.*
(iii) *If $\tilde{F}$ is a finite field extension of F and if $\tilde{\Pi} \in \tilde{F}^{\times} \otimes \mathbb{Q}$, let us denote by $d(\tilde{\Pi})$ the common denominator of the rational numbers* $\deg(\tilde{x})\tilde{x}(\tilde{\Pi})$ *where $\tilde{x}$ runs through the set of places of $\tilde{F}$ and where* $\deg(\tilde{x})$ *is the degree of the residue field of $\tilde{x}$ over $\mathbb{F}_q$. Then, for any irreducible φ-space (V, φ), we have*

$$\dim_{F \otimes_{\mathbb{F}_q} k}(V) = [F_{(V,\varphi)} : F]\, d(\Pi_{(V,\varphi)})$$

and $\operatorname{End}(V, \varphi)$ *is a central division algebra over $F_{(V,\varphi)}$ of dimension $d(\Pi_{(V,\varphi)})^2$ with invariant*

$$\operatorname{inv}_{\tilde{x}}(\operatorname{End}(V, \varphi)) \equiv -\deg(\tilde{x})\tilde{x}(\Pi_{(V,\varphi)}) \quad (\text{modulo } \mathbb{Z})$$

at each place $\tilde{x}$ of $F_{(V,\varphi)}$.

Proof. Let us begin with two remarks.

If (V, φ) is a non zero φ-space, any non trivial idempotent of $F_{(V,\varphi)}$ is also a non trivial central idempotent of $\operatorname{End}(V, \varphi)$. Therefore, if (V, φ) is indecomposable, $F_{(V,\varphi)}$ is a field. Let

$$0 \to (V_1, \varphi_1) \to (V, \varphi) \to (V_2, \varphi_2) \to 0$$

be a short exact sequence of non zero φ-spaces. We can find a positive integer n' such that this exact sequence is defined over $\mathbb{F}_{q^{n'}}$. Moreover, if

$$0 \to (V'_1, \varphi'_1) \to (V', \varphi') \to (V'_2, \varphi'_2) \to 0$$

is an $\mathbb{F}_{q^{n'}}$-structure for the above exact sequence we can assume that

$$(F_{(V,\varphi)}, \Pi_{(V,\varphi)}) = (F[\Pi'], \Pi'^{1/n'})$$

and

$$(F_{(V_i,\varphi_i)}, \Pi_{(V_i,\varphi_i)}) = (F[\Pi'_i], \Pi_i'^{1/n'})$$

where $\Pi' = \varphi'^{n'} \otimes_{\mathbb{F}_{q^{n'}}} k$ and $\Pi'_i = \varphi_i'^{n} \otimes_{\mathbb{F}_{q^n}} k$ $(i = 1, 2)$. It follows that we have a surjective homomorphism of F-algebras $F_{(V,\varphi)} \twoheadrightarrow F_{(V_1,\varphi_i)}$ which maps $\Pi_{(V,\varphi)}$ into $\Pi_{(V_i,\varphi_i)}$ (Π'_i is induced by Π') $(i = 1, 2)$. In particular, if $F_{(V,\varphi)}$ is a field, the φ-pair $(F_{(V_i,\varphi_i)}, \Pi_{(V_i,\varphi_i)})$ is canonically isomorphic to the φ-pair $(F_{(V,\varphi)}, \Pi_{(V,\varphi)})$ $(i = 1, 2)$.

Now, let $(\tilde{F}, \tilde{\Pi})$ be a φ-pair such that $\tilde{F}$ is a field and let $\mathscr{E}$ be the following full subcategory of the category of φ-spaces: the objects of $\mathscr{E}$ are the φ-spaces (V, φ) such that either $V = 0$ or $(F_{(V,\varphi)}, \Pi_{(V,\varphi)})$ is isomorphic to $(\tilde{F}, \tilde{\Pi})$. Thanks to the above two remarks, to prove the theorem it suffices to prove that, for any φ-pair as before, $\mathscr{E}$ has the following properties:

(1) any indecomposable object of $\mathscr{E}$ is an irreducible object of $\mathscr{E}$,
(2) there is one and only one isomorphism class of irreducible objects in $\mathscr{E}$,
(3) if (V, φ) is an irreducible object of $\mathscr{E}$, the dimension of V and $\operatorname{End}(V, \varphi)$ satisfy the requirements of the theorem.

To prove these properties of $\mathscr{E}$, Drinfeld gives a new description of $\mathscr{E}$. Let I be the set of pairs (n', Π'), where n' is a positive integer and $\Pi' \in \tilde{F}^{\times}$, such that $\Pi' \otimes 1 = \tilde{\Pi}^{n'}$ in $\tilde{F}^{\times} \otimes \mathbb{Q}$. On I we have the partial order which is defined by

$$(n_1', \Pi_1') \leqq (n_2', \Pi_2')$$

if and only if n_1' divides n_2' and $\Pi_2' = \Pi_1'^{n_2'/n_1'}$.

If $(n_1', \Pi_1'), (n_2', \Pi_2') \in I$, then there exists $(n', \Pi') \in I$ such that

$$(n', \Pi') \geqq (n_i', \Pi_i')$$

for $i = 1, 2$. For each $(n', \Pi') \in I$, let $\mathscr{E}_{(n', \Pi')}$ be the category of finite-dimensional $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n'}}$-vector spaces V', which are endowed with a bijective $F \otimes_{\mathbb{F}_q}$ frob_q-semilinear map φ': $V' \to V'$, such that either $V' = 0$ or the pair $(F[\varphi'^{n'}], \varphi'^{n'})$ is isomorphic to the pair $(\tilde{F}, \Pi')$. If $(n_1', \Pi_1') \leqq (n_2', \Pi_2')$, we have a functor

$$\mathscr{E}_{(n_1', \Pi_1')} \to \mathscr{E}_{(n_2', \Pi_2')}, \quad (V_1', \varphi_1') \mapsto (V_1', \varphi_1') \otimes_{\mathbb{F}_{q^{n_1'}}} \mathbb{F}_{q^{n_2'}}$$

and we get in this way a direct system of categories. The functors

$$\mathscr{E}_{(n', \Pi')} \to \mathscr{E}, \quad (V', \varphi') \mapsto (V', \varphi') \otimes_{\mathbb{F}_{q^{n'}}} k$$

$((n', \Pi') \in I)$ induce an equivalence of categories

$$\varinjlim_{I} \mathscr{E}_{(n', \Pi')} \xrightarrow{\sim} \mathscr{E} .$$

But, for each $(n', \Pi') \in I$, $\mathscr{E}_{(n', \Pi')}$ is equivalent to the category of left modules of finite type over the ring

$$R_{(n', \Pi')} = (\tilde{F} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n'}}[\tau])/((1 \otimes \tau)^{n'} - (\Pi' \otimes 1))$$

($\mathbb{F}_{q^{n'}}[\tau]$ is the non-commutative polynomial ring over $\mathbb{F}_{q^{n'}}$ in the variable τ with commutation rule $\tau \cdot \lambda = \lambda^q \cdot \tau$ if $\lambda \in \mathbb{F}_{q^{n'}}$). The ring $R_{(n', \Pi')}$ is clearly a central simple algebra of dimension n'^2 over $\tilde{F}$ with invariant

$$\mathrm{inv}_{\tilde{x}}(\mathrm{R}_{(n', \Pi')}) \equiv -\frac{\deg(\tilde{x})\,\tilde{x}(\Pi')}{n'} = -\deg(\tilde{x})\,\tilde{x}(\tilde{\Pi}) \quad (\text{modulo } \mathbb{Z})$$

at each place $\tilde{x}$ of $\tilde{F}$. Therefore, the category $\mathscr{E}_{(n', \Pi')}$ is semi-simple, has only one irreducible object (V', φ') up to isomorphism and $\mathrm{End}(V', \varphi')$ is a central division algebra over $\tilde{F}$ with invariant

$$\mathrm{inv}_{\tilde{x}}(\mathrm{End}(V', \varphi')) \equiv -\deg(\tilde{x})\tilde{x}(\tilde{\Pi}) \quad (\text{modulo } \mathbb{Z})$$

at each place $\tilde{x}$ of $\tilde{F}$. Moreover

$$\dim_{\tilde{F}}(V') = n'(\dim_{\tilde{F}}(\mathrm{End}(V', \varphi')))^{1/2}$$

i.e.

$$\dim_{\tilde{F}}(V') = n' d(\tilde{\Pi})$$

and

$$\dim_{F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n'}}}(V') = [\tilde{F} : F] d(\tilde{\Pi}) .$$

The theorem follows. □

B Dieudonné F_x-modules (following Drinfeld)

The definition and the results of this appendix are taken from [Dr 7]. Let F and k be as in Appendix A. Let x be a place of F. We denote by F_x the completion of F at x, by $\mathcal{O}_x$ the ring of integers of F_x, by $\kappa(x)$ the residue field of $\mathcal{O}_x$ and by $\deg(x)$ the degree of $\kappa(x)$ over $\mathbb{F}_q$.

(B.1) Definition. A *Dieudonné F_x-module* (*over k*) is a $F_x \hat{\otimes}_{\mathbb{F}_q} k$-module N of finite type which is equipped with a bijective $F_x \hat{\otimes}_{\mathbb{F}_q} \mathrm{frob}_q$-semilinear map $\psi: N \to N$. A *morphism α between two Dieudonné F_x-modules* (N_1, ψ_1) *and* (N_2, ψ_2) is a $F_x \hat{\otimes}_{\mathbb{F}_q} k$-linear map $N_1 \xrightarrow{\alpha} N_2$ such that $\psi_2 \circ \alpha = \alpha \circ \psi_1$.

Obviously, the Dieudonné F_x-modules and their morphisms form a category which is F_x-linear, abelian, noetherian and artinian.

We have an exact and F-linear functor

$$(V, \varphi) \mapsto (V_x, \varphi_x) = (F_x \hat{\otimes}_F V, F_x \hat{\otimes}_F \varphi)$$

from the category of φ-spaces to the category of Dieudonné F_x-modules.

(B.2) Remark. Let $\iota_0: \kappa(x) \hookrightarrow k$ be a fixed embedding of $\mathbb{F}_q$-algebras. Put $\iota_j = \mathrm{frob}_q^j \circ \iota_0 (j \in \mathbb{Z}/\deg(x)\mathbb{Z})$. Then we have a canonical splitting

$$F_x \hat{\otimes}_{\mathbb{F}_q} k = \prod_{j \in \mathbb{Z}/\deg(x)\mathbb{Z}} F_x \hat{\otimes}_{\kappa(x), \iota_j} k$$

($\kappa(x)$ is naturally embedded in F_x) and each factor is a field. Therefore, to give a Dieudonné F_x-module (N, ψ) is equivalent to giving finite-dimensional vector spaces N_j over $F_x \hat{\otimes}_{\kappa(x), \iota_j} k$ and bijective semilinear maps $\psi_j: N_j \to N_{j+1}$ over

$$F_x \hat{\otimes}_{\kappa(x)} \mathrm{frob}_q: F_x \hat{\otimes}_{\kappa(x), \iota_j} k \xrightarrow{\sim} F_x \hat{\otimes}_{\kappa(x), \iota_{j+1}} k$$

for all $j \in \mathbb{Z}/\deg(x)\mathbb{Z}$ which, in turn, is equivalent to giving a finite-dimensional vector space N_0 over $F_x \hat{\otimes}_{\kappa(x), \iota_0} k$ and a bijective $F_x \hat{\otimes}_{\kappa(x), \iota_0} \mathrm{frob}_q^{\deg(x)}$-semilinear map

$$\Psi_0 = \psi_{\deg(x)-1} \circ \psi_{\deg(x)-2} \circ \cdots \circ \psi_0 : N_0 \to N_0 . \qquad \square$$

Let us fix an embedding ι_0 as in (B.2) and a uniformizer ϖ_x of $\mathcal{O}_x$. Let d, r be two integers such that $d \geqq 1$ and $(d, r) = 1$ (if $r = 0$, this means that $d = 1$). Let us consider the finite dimensional vector space

$$N_0 = (F_x \hat{\otimes}_{\kappa(x), \iota_0} k)^d$$

over $F_x \hat{\otimes}_{\kappa(x), \iota_0} k$ with its standard basis $e_1, \ldots, e_d$ and let $\Psi_0 : N_0 \to N_0$ be the bijective $F_x \hat{\otimes}_{\kappa(x), \iota_0} \mathrm{frob}_q^{\deg(x)}$-semilinear map which is defined by

$$\Psi_0(e_i) = \begin{cases} \varpi_x^r e_d & \text{if } i = 1 \\ e_{i-1} & \text{if } i = 2, \ldots, d \end{cases} .$$

The pair (N_0, Ψ_0) defines a Dieudonné F_x-module (see (B.2)). We will denote it by $(N_{d,r}, \psi_{d,r})$. Its isomorphism class does not depend on the choices of ι_0 and ϖ_x.

(B.3) Theorem. (i) *The abelian category of Dieudonné F_x-modules* (*over k*) *is semi-simple.*

(ii) *The Dieudonné F_x-modules* $(N_{d,r}, \psi_{d,r})$ $(d, r \in \mathbb{Z}, d \geqq 1, (d, r) = 1)$ *are irreducible and any irreducible Dieudonné F_x-module is isomorphic to one and only one of them.*
(iii) *For each* $d, r \in \mathbb{Z}$, $d \geqq 1$, $(d, r) = 1$, $\mathrm{End}(N_{d,r}, \psi_{d,r})$ *is a central division algebra over F_x with invariant* $-r/d$ (modulo $\mathbb{Z}$).

(B.4) Proposition. *Let (V, φ) be an irreducible φ-space and let $(\tilde{F}, \tilde{\Pi}) = (F_{(V,\varphi)}, \Pi_{(V,\varphi)})$ be the corresponding φ-pair. For each place $\tilde{x}$ of $\tilde{F}$ which divides x, let*

$$(V_{\tilde{x}}, \varphi_{\tilde{x}}) = \tilde{F}_{\tilde{x}} \otimes_{\tilde{F}} (V, \varphi) .$$

The canonical splitting $F_x \otimes_F \tilde{F} = \prod_{\tilde{x}|x} \tilde{F}_{\tilde{x}}$ induces a splitting

$$(V_x, \varphi_x) = \bigoplus_{\tilde{x}|x} (V_{\tilde{x}}, \varphi_{\tilde{x}})$$

of (V_x, φ_x) as a Dieudonné F_x-module. Then, for each place $\tilde{x}$ of $\tilde{F}$ which divides x, $(V_{\tilde{x}}, \varphi_{\tilde{x}})$ is (non canonically) isomorphic to

$$(N_{d_{\tilde{x}}, r_{\tilde{x}}}, \psi_{d_{\tilde{x}}, r_{\tilde{x}}})^{s_{\tilde{x}}}$$

where the integers $d_{\tilde{x}}, r_{\tilde{x}}$ and $s_{\tilde{x}}$ are uniquely determined by the following relations

$$\begin{cases} d_{\tilde{x}}, s_{\tilde{x}} \geqq 1 \\ (d_{\tilde{x}}, r_{\tilde{x}}) = 1 \\ r_{\tilde{x}}/d_{\tilde{x}} = \deg(\tilde{x})\tilde{x}(\tilde{\Pi})/[\tilde{F}_{\tilde{x}}: F_x] \\ d_{\tilde{x}} s_{\tilde{x}} = d(\tilde{\Pi})[\tilde{F}_{\tilde{x}}: F_x]. \end{cases}$$

Proof. Let $s_{\tilde{x}}(d, r)$ be the multiplicity of the irreducible Dieudonné F_x-module $(N_{d,r}, \psi_{d,r})$ in the Dieudonné F_x-module $(V_{\tilde{x}}, \varphi_{\tilde{x}})$ for each place $\tilde{x}$ of $\tilde{F}$ which divides x and each pair (d, r) of integers with $d \geqq 1$ and $(d, r) = 1$. It follows from (A.6) (iii) that

$$(*) \qquad \sum_{\tilde{x}|x} \sum_{(d,r)} d s_{\tilde{x}}(d, r) = \dim_{F_x \hat{\otimes}_{\mathbb{F}_q} k}(V_x) = d(\tilde{\Pi})[\tilde{F}: F] .$$

It also follows from (A.6) (iii) that $\tilde{F}_{\tilde{x}} \otimes_{\tilde{F}} \mathrm{End}(V, \varphi)$ is a central simple algebra over $\tilde{F}_{\tilde{x}}$ of dimension $d(\tilde{\Pi})^2$ and with invariant $-\deg(\tilde{x})\tilde{x}(\tilde{\Pi})$ (modulo $\mathbb{Z}$), for each place $\tilde{x}$ of $\tilde{F}$ dividing x.

Now, let us consider the natural homomorphism of $\tilde{F}_{\tilde{x}}$-algebras

$$\tilde{F}_{\tilde{x}} \otimes_{\tilde{F}} \mathrm{End}(V, \varphi) \to \mathrm{End}_{\tilde{F}_{\tilde{x}}}(V_{\tilde{x}}, \varphi_{\tilde{x}}) ,$$

where $\mathrm{End}_{\tilde{F}_{\tilde{x}}}(V_{\tilde{x}}, \varphi_{\tilde{x}})$ is the commutant of $\tilde{F}_{\tilde{x}}$ in $\mathrm{End}(V_{\tilde{x}}, \varphi_{\tilde{x}})$. It is automatically injective. If we compare the dimensions over $\tilde{F}_{\tilde{x}}$ of the source and the target, we get the inequality

$$d(\tilde{\Pi})^2 \leqq \dim_{\tilde{F}_{\tilde{x}}}(\mathrm{End}_{\tilde{F}_{\tilde{x}}}(V_{\tilde{x}}, \varphi_{\tilde{x}})) .$$

But, we have

$$\begin{aligned} \dim_{\tilde{F}_{\tilde{x}}}(\mathrm{End}_{\tilde{F}_{\tilde{x}}}(V_{\tilde{x}}, \varphi_{\tilde{x}})) &= \frac{\dim_{F_x}(\mathrm{End}(V_{\tilde{x}}, \varphi_{\tilde{x}}))}{[\tilde{F}_{\tilde{x}}: F_x]^2} \\ &= \sum_{(d,r)} (d s_{\tilde{x}}(d, r)/[\tilde{F}_{\tilde{x}}: F_x])^2 \end{aligned}$$

(see [Re]). Therefore, we have

$$(**) \qquad d(\tilde{\Pi})^2 \leqq \sum_{(d,r)} (ds_{\tilde{x}}(d,r)/[\tilde{F}_{\tilde{x}}: F_x])^2$$

for each place $\tilde{x}$ which divides x.

Finally, it is an elementary exercise to prove that the equality $(*)$ and the inequalities $(**)$ are compatible if and only if, for each place $\tilde{x}$ of $\tilde{F}$ which divides x, there is at most one (d, r) with $s_{\tilde{x}}(d, r) \neq 0$. The proposition follows. □

(B.5) Let x be a place of F and let (N, ψ) be a Dieudonné F_x-module (over k). A *lattice* M in N is a free $\mathcal{O}_x \hat{\otimes}_{\mathbb{F}_q} k$-submodule of N of finite type which generates N as a $F_x \hat{\otimes}_{\mathbb{F}_q} k$-module. It follows from (B.2) that N is a free $F_x \hat{\otimes}_{\mathbb{F}_q} k$-module (of finite type). Therefore, it always contains lattices. Let

$$N^{\psi} = \{n \in N \mid \psi(n) = n\} .$$

It is easy to see that N^{ψ} is a finite dimensional F_x-subvector-space of N and that the canonical map $N^{\psi} \hat{\otimes}_{\mathbb{F}_q} k \to N$ is injective. In fact, for each pair of integers (d, r) with $d \geqq 1$ and $(d, r) = 1$, we have

$$(N_{d,r})^{\psi_{d,r}} = \begin{cases} F_x & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the canonical map

$$N_{d,r}^{\psi_{d,r}} \hat{\otimes}_{\mathbb{F}_q} k \to N_{d,r}$$

is either an isomorphism (if $r = 0$) or zero (otherwise).

(B.6) Lemma. *The following properties of (N, ψ) are equivalent:*

(i) *there exists a lattice M in N with $\psi(M) = M$;*
(ii) *the canonical map $N^{\psi} \hat{\otimes}_{\mathbb{F}_q} k \to N$ is bijective;*
(iii) *any irreducible Dieudonné F_x-submodule of (N, ψ) is isomorphic to $(N_{1,0}, \psi_{1,0})$ (i.e. (N, ψ) is isomorphic to $(N_{1,0}, \psi_{1,0})^d$ for some integer $d \geqq 0$).*

Moreover, if these conditions are fulfilled then for any lattice M in N with $\psi(M) = M$, $M^{\psi} = M \cap N^{\psi}$ is a lattice in the finite dimensional F_x-vector space N^{ψ} and the canonical map

$$M^{\psi} \hat{\otimes}_{\mathbb{F}_q} k \to M$$

is an isomorphism.

(B.7) Lemma. *The following properties of (N, ψ) are equivalent:*

(i) *there exists a lattice M in N such that $\psi(M) \subset M$ (resp. $M \subset \psi(M)$), $\psi^n(M) \subset \varpi_x M$ (resp. $M \subset \varpi_x \psi^n(M)$) for any uniformizer ϖ_x of $\mathcal{O}_x$ and some positive integer n and*

$$\dim_k(M/\psi(M)) = 1 \quad (\textit{resp. } \dim_k(\psi(M)/M) = 1)$$

(ii) *(N, ψ) is isomorphic to $(N_{d,1}, \psi_{d,1})$ (resp. $(N_{d,-1}, \psi_{d,-1})$) for some positive integer d.*

(B.8) Lemma. *The following properties of (N, ψ) are equivalent:*

(i) *there exists a lattice M in N such that $\psi(M) \subset M$ (resp. $M \subset \psi(M)$) and*

$$\dim_k(M/\psi(M)) = 1 \quad (\textit{resp.}\ \dim_k(\psi(\mathrm{M})/\mathrm{M}) = 1)$$

(ii) *(N, ψ) is isomorphic to*

$$(N_{1,0}, \psi_{1,0})^{d-h} \oplus (N_{h,1}, \psi_{h,1}) \quad (\textit{resp.}\ (N_{1,0}, \psi_{1,0})^{d-h} \oplus (N_{h,-1}, \psi_{h,-1}))$$

for some integers d, h with $d \geqq h > 0$.

Moreover, if these conditions are fulfilled, then any lattice M in N with $\psi(M) \subset M$ (resp. $M \subset \psi(M)$) has a unique decomposition into a direct sum of two free $\mathcal{O}_x \hat{\otimes}_{\mathbb{F}_q} k$-submodules

$$M = M^{\text{ét}} \oplus M^{\text{c}}, \quad \textit{such that}\ \psi(M^{\text{ét}}) = M^{\text{ét}}$$

and

$$\psi(M^{\text{c}}) \subset M^{\text{c}} \quad (\textit{resp.}\ M^{\text{c}} \subset \psi(M^{\text{c}}))\,,$$

$$\psi^n(M^{\text{c}}) \subset \varpi_x M^{\text{c}} \quad (\textit{resp.}\ M^{\text{c}} \subset \varpi_x \psi^n(M^{\text{c}}))$$

for any uniformizer ϖ_x of $\mathcal{O}_x$ and some positive integer n and

$$\dim_k(M^{\text{c}}/\psi(M^{\text{c}})) = 1 \quad (\textit{resp.}\ \dim_k(\psi(M^{\text{c}})/M^{\text{c}}) = 1)\,.$$

(B.9) Remark. If the equivalent conditions (i) and (ii) of (B.8) are fulfilled and if M is any lattice in N such that $\psi(M) \subset M$ (resp. $M \subset \psi(M)$), we automatically have

$$\dim_k(M/\psi(M)) = 1 \quad (\text{resp. } \dim_k(\psi(M)/M) = 1)\,,$$

$$(N^{\text{ét}}, \psi^{\text{ét}}) := (F_x \otimes_{\mathcal{O}_x} M^{\text{ét}}, \psi \,|\, F_x \otimes_{\mathcal{O}_x} M^{\text{ét}})$$

is the sum of all Dieudonné F_x-submodules of (N, ψ) which are isomorphic to $(N_{1,0}, \psi_{1,0})$ and

$$(N^{\text{c}}, \psi^{\text{c}}) := (F_x \otimes_{\mathcal{O}_x} M^{\text{c}}, \psi \,|\, F_x \otimes_{\mathcal{O}_x} M^{\text{c}})$$

is the sum of all Dieudonné F_x-submodules of (N, ψ) which are isomorphic to $(N_{d,r}, \psi_{d,r})$ for some pair of integers (d, r) with $d \geqq 1, (d, r) = 1$ and $r > 0$ (resp. $r < 0$). We will say that $(N^{\text{ét}}, \psi^{\text{ét}})$ is the *etale part* of (N, ψ) and that $(N^{\text{c}}, \psi^{\text{c}})$ is its *connected part.* □

(B.10) Proposition. *Let us assume that there exists a lattice M in N such that*

$$\begin{cases} \psi(M) \subset M \\ \psi^n(M) \subset \varpi_x M \\ \dim_k(M/\psi(M)) = 1 \end{cases} \quad (\textit{resp.}\ \begin{cases} M \subset \psi(M) \\ M \subset \varpi_x \psi^n(M) \\ \dim_k(\psi(M)/M) = 1) \end{cases}$$

for any uniformizer ϖ_x of $\mathcal{O}_x$ and some positive integer n. Then any other lattice M' in N such that

$$\psi(M') \subset M' \quad (\textit{resp.}\ M' \subset \psi(M'))$$

is equal to $\psi^m(M)$ for one and only one $m \in \mathbb{Z}$.

In other words, if (N, ψ) is isomorphic to $(N_{d,1}, \psi_{d,1})$ (resp. $(N_{d,-1}, \psi_{d,-1})$) for some positive integer d, the set of lattices M in N such that

$$\psi(M) \subset M \quad (\textit{resp.}\ M \subset \psi(M))$$

is a principal homogeneous space over $\mathbb{Z}$ *(*$m \in \mathbb{Z}$ *acts on this set of lattices by* $M \mapsto \psi^m(M)$*). Moreover any lattice in this set satisfies*

$$\begin{cases}\psi^n(M) \subset \varpi_x M \\ \dim_k(M/\psi(M)) = 1\end{cases} \quad \textit{(resp.} \begin{cases}M \subset \varpi_x \psi^n(M) \\ \dim_k(\psi(M)/M) = 1)\end{cases}$$

for any uniformizer ϖ_x *of* $\mathcal{O}_x$ *and some positive integer n.*

(B.11) Remark. Under the hypothesis of (B.10), $\mathrm{End}(N, \psi)$ is a central division algebra over F_x with invariant $-1/d$(resp. $1/d$) (see (B.7) and (B.3)). Now, the natural action of the multiplicative group of $\mathrm{End}(N, \psi)$ on the set of lattices M in N such that

$$\psi(M) \subset M \quad (\text{resp. } M \subset \psi(M))$$

can be described in the following way. We have a group homomorphism

$$\mathrm{End}(N, \psi)^\times \xrightarrow{\mathrm{rn}} F_x^\times \xrightarrow{\deg(x)\, x(-)} \mathbb{Z}\,,$$

$$(\text{resp.} \qquad \mathrm{End}(N, \psi)^\times \xrightarrow{\mathrm{rn}} F_x^\times \xrightarrow{-\deg(x)\, x(-)} \mathbb{Z})$$

where rn is the reduced norm and $\delta \in \mathrm{End}(N, \psi)^\times$ maps the lattice M into the lattice $\psi^m(M)$ where

$$m = \deg(x)x(\mathrm{rn}(\delta)) \quad (\text{resp. } m = -\deg(x)x(\mathrm{rn}(\delta)))\,.$$

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