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# A finiteness theorem in the Bruhat-Tits building: an application of Landvogt's embedding theorem

by M. Rapoport and Th. Zink

*Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany*
*Fakultät für Mathematik, Universität Bielefeld, POB 100131, 33501 Bielefeld, Germany*

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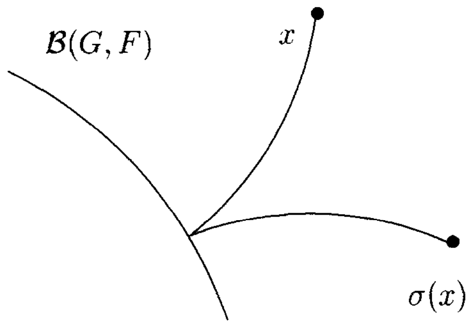
ABSTRACT

Let $G$ be a connected reductive group over $\mathbf{Q}_p$ and let $\mathcal{B}(G, \mathbf{Q}_p)$ be the extended Bruhat-Tits building of $G$ over $\mathbf{Q}_p$. Let $L$ be the completion of the maximal unramified extension of $\mathbf{Q}_p$ and let $\mathcal{B}(G, L)$ be the building of $G$ over $L$. By the theorem of Bruhat and Tits, $\mathcal{B}(G, \mathbf{Q}_p)$ may be identified with the fixed point set of the Frobenius automorphism $\sigma$ acting on $\mathcal{B}(G, L)$. A special case of our main result states that for any $c > 0$ there exists $C > 0$ with the property that any point $x \in \mathcal{B}(G, L)$ with distance $d(x, \sigma(x)) < c$ is at distance $< C$ from $\mathcal{B}(G, \mathbf{Q}_p)$. The results in this paper constitute a qualitative generalization of a result of Drinfeld.

Bruhat and Tits have associated to a semi-simple, or more generally a reductive algebraic group over a non-archimedean local field its *building*. They proved that in many respects the building has properties analogous to those of the symmetric space associated to a semi-simple Lie group. In particular there are many similarities between buildings and simply connected Riemannian manifolds of negative curvature.

The purpose of this note is to point out another such similarity, in a very special situation. Let $F$ be a finite extension of $\mathbf{Q}_p$, and let $L$ be a complete unramified extension of $F$ with algebraically closed residue field. Let $\sigma \in \mathrm{Aut}(L/F)$ be the relative Frobenius automorphism, so that $F = L^{<\sigma>}$. Let $G$ be a reductive group over $F$ and let $\mathcal{B}(G, F)$ resp. $\mathcal{B}(G, L)$ be the buildings of $G$ over $F$ resp. over $L$. By the Bruhat-Tits fixed point theorem, $\mathcal{B}(G, F)$ is the set of fixed points under $\sigma$ in $\mathcal{B}(G, L)$. A special case of the main result of this paper (corresponding to the case $b = 1$ in its statement (1.4)) asserts that the distance between the points $x$ and $\sigma(x)$ of $\mathcal{B}(G, L)$ increases with the distance of $x$ from $\mathcal{B}(G, F)$, and this in fact uniformly. The following picture is supposed to give a graphic description of the behaviour of the building in this respect.




A form of our main result, for $G = GL_n$, is used as an important step in the representability theorem of [RZ], and in fact our proof here, in Section 1, is by reduction to this case. In order to perform this reduction we need to embed the building of an arbitrary reductive group in an equivariant way in the building of a general linear group. Fortunately for us, Landvogt rose to the challenge and proved the embedding theorem on buildings which we needed for this purpose [L1].

Our method of proof entails that we do not know in how far the above picture remains valid for more general automorphisms of buildings than Frobenius automorphisms. In Section 2 we mention a natural conjecture in this direction. This conjecture is due to Rousseau and arose in discussions we had with him on our main theorem. We refer to [R1] for more details on this conjecture and for Rousseau's results in this direction. In the final section we explain the original question which gave rise to this paper, and which is related to a paper of Drinfeld [D].

In conclusion we wish to thank M. Aschbacher, E. Landvogt, J. de Jong and above all G. Rousseau for very instructive discussions.

## 1. THE FINITENESS THEOREM

**1.1.** In this section we let $k$ be an algebraically closed field of characteristic $p > 0$. Let $K$ be the fraction field of its ring of Witt vectors $W(k)$ and $\bar{K}$ be an algebraic closure of $K$. Let $F$ be a finite extension of $\mathbf{Q}_p$ contained in $\bar{K}$ and let $L$ be the compositum of $K$ and $F$ in $\bar{K}$. Let $\sigma \in \mathrm{Aut}(L/F)$ be the relative Frobenius automorphism ([K], 1.1). Then the fixed field of $\sigma$ in $L$ is $F$.

**1.2.** Let $G$ be a connected reductive algebraic group over $F$. An element $b \in G(L)$ defines a connected algebraic group $J$ over $F$, with values in a $F$-algebra $R$ ([RZ], (1.12))

$$J(R) = \{g \in G(R \otimes_F L);\;\; \sigma(g) = b^{-1}gb\}.$$

The element $b$ defines an associated slope homomorphism ([K], 4.2) defined over $L$,

$$\nu_b : \mathbf{D}_L \longrightarrow G_L.$$

Here $\mathbf{D}$ is the diagonizable pro-algebraic group over $\mathbf{Q}_p$ whose character group is $\mathbf{Q}$. The slope homomorphism is characterized by the fact that for every $F$-rational representation $(V, \varrho)$ of $G$ the $\mathbf{Q}$-filtration on $V \otimes_F L$ induced by $\nu_b$ is the slope filtration of the $\sigma$-$L$-space $(V \otimes_F L, \varrho(b) \cdot (\mathrm{id} \otimes \sigma))$ ([K], §3). For $s \in \mathbf{Q}^\times$ we use the notation $s\nu$ for the composite $\mathbf{D} \xrightarrow{s} \mathbf{D} \rightarrow G$. For suitable $s > 0$ the homomorphism $s\nu$ factors through the projection $\mathbf{D} \rightarrow \mathbf{G}_m$ induced by the inclusion $\mathbf{Z} \subset \mathbf{Q}$ of character modules. Let us assume that we have the following identity in the semi-direct product $G(L) \rtimes \langle\sigma\rangle$ (a *decency equation for b* ([RZ], (1.8)))

$$(b\sigma)^s = s\nu_b(\pi) \cdot \sigma^s, \quad s > 0.$$

Here $\pi$ denotes a uniformizer in $F$ and $s$ is sufficiently large so that $s\nu_b$ factors through $\mathbf{G}_m$. Let $F_s$ denote the unramified extension of degree $s$ in $\bar{K}$, i.e. the fixed field of $\sigma^s$ in $L$. Then ([RZ], (1.9)) $\nu_b$ is defined over $F_s$ and $J_{F_s}$ is the centralizer of the 1-parameter subgroup $s\nu_b$ of $G$, hence a Levi subgroup of $G_{F_s}$. We remark that by Kottwitz ([K], §4) any $\sigma$-conjugacy class in $G(L)$ contains elements satisfying a decency equation.

**1.3.** Let $\mathcal{B}(G, L)$ be the *extended* Bruhat-Tits building of $G$ over $L$, cf. [L1] (i.e. the center of $G$ contributes a euclidean space to $\mathcal{B}(G, L)$). We recall that this is a metric space which also has the structure of a polysimplicial complex. The group $G(L) \rtimes \langle\sigma\rangle$ operates on $\mathcal{B}(G, L)$, and this operation preserves these structures.

Similarly, we denote by $\mathcal{B}(J, L)$ resp. $\mathcal{B}(J, F_s)$ the Bruhat-Tits buildings of $J$ over $L$ resp. over $F_s$. Since $F_s$ is the fixed field of $\sigma^s$ in $L$ and $L$ is a unramified extension of $F_s$, the building $\mathcal{B}(J, F_s)$ may be identified with the fixed point set of $\sigma^s$ in $\mathcal{B}(J, L)$ (theorem of Bruhat and Tits),

$$\mathcal{B}(J, F_s) = \mathcal{B}(J, L)^{\langle\sigma^s\rangle}.$$

Since $J_{F_s}$ is a Levi subgroup of $G_{F_s}$, there is a canonical $J(L) \rtimes \langle\sigma^s\rangle$-equivariant injective map ([L1])

$$\mathcal{B}(J, L) \longrightarrow \mathcal{B}(G, L).$$

We identify $\mathcal{B}(J, L)$ with the image under this injection. We may now formulate our main result.

**Theorem 1.4.** *Fix $b \in G(L)$ and $s > 0$ such that a decency equation holds for $b$ relative to $s$, cf.* (1.2). *Let $F_s$ be the corresponding unramified extension of degree $s$ of $F$. Let $c > 0$. Then there exists $C > 0$ with the following property. If $x \in \mathcal{B}(G, L)$ is such that $d(x, b\sigma(x)) < c$ then there exists $x_0 \in \mathcal{B}(J, F_s)$ with $d(x, x_0) < C$.*

Here $d$ denotes the metric on $\mathcal{B}(G, L)$.

**1.5.** We are going to deduce this proposition from a similar statement about lattices in $\sigma$-$L$-spaces. Let $(V, \varphi)$ be a $\sigma$-$L$-space, i.e. $V$ is a finite-dimensional $L$-vector space and $\varphi$ is a bijective $\sigma$-linear endomorphism of $V$. Let $V$ be isotypic of slope $\mu = r/s$, $s > 0$. Then the fixed vectors of $\pi^{-r} \cdot \varphi^s$ in $V$ define a $F_s$-form $V_0$ of $V$ ([K], §3),

$$V_0 \otimes_{F_s} L \xrightarrow{\sim} V.$$

More generally, let $V$ be any $\sigma$-$L$-space and let $s > 0$ be such that $s \cdot \mu \in \mathbf{Z}$ for all slopes of all isotypical components $V^\mu$ of $V$. Let $\nu : \mathbf{D}_L \to GL(V)$ be the corresponding slope homomorphism. Then $s\nu$ factors through $\mathbf{G}_{mL}$ and the fixed vectors of $s\nu(\pi)^{-1} \cdot \varphi^s$ in $V$ define a $F_s$-form $V_0$ of $V$ compatible with the slope decomposition,

$$V_0 \otimes_{F_s} L = V, \quad V_0^\mu \otimes_{F_s} L = V^\mu.$$

**Proposition 1.6.** *Let $c > 0$. Then there exists $C > 0$ with the following property. If $M$ is a lattice in $V$ with*

$$\pi^c \varphi(M) \subset M \subset \pi^{-c} \varphi(M)$$

*then there exists a lattice $M_0 \subset V_0$ with*

$$M_0 = \bigoplus_\mu (M_0 \cap V_0^\mu)$$

*and with*

$$\pi^C (M_0 \otimes_{O_{F_s}} O_L) \subset M \subset \pi^{-C} (M_0 \otimes_{O_{F_s}} O_L).$$

We note that for $M$ as above we have

$$\pi^{c'} \cdot \varphi^s(M) \subset M \subset \pi^{-c'} \varphi^s(M),$$

for some constant $c'$ depending on $c$ and the slopes of $V$. Conversely any lattice $M_0$ as above satisfies

$$\pi^{c_0} \varphi^s (M_0 \otimes_{O_{F_s}} O_L) \subset M_0 \otimes_{O_{F_s}} O_L \subset \pi^{-c_0} \varphi^s (M_0 \otimes_{O_{F_s}} O_L)$$

for some constant $c_0$ depending on the slopes of $V$.

**Proof.** We only sketch the proof which is essentially contained in [RZ], §2 (2.17 - 2.19). Suppose first that $V$ is isotypic of slope $\mu = r/s$. Then $\tau = \pi^{-r} \varphi^s$ is the relative Frobenius of $V$ with respect to $V_0$. In loc.cit. it is shown that the lattice in $V$

$$M + \tau(M) + \ldots + \tau^{d-1}(M), \quad d = \dim V$$

is $\tau$-invariant, i.e. of the form $M_0 \otimes_{O_{F_s}} O_L$ for a unique lattice $M_0 \subset V_0$ ($k$ is algebraically closed). The index of $M$ in $M_0 \otimes_{O_{F_s}} O_L$ is bounded in terms of $c, r, s$ and $d$.

In the general case one proceeds by induction over the number of isotypical components of $V$. We write $V = V' \oplus V''$ where $V'$ is the isotypical component of maximal slope. We obtain an exact sequence of $\sigma$-$L$-spaces and an induced exact sequence of lattices,

$$
\begin{array}{ccccccccc}
 & 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\
(1) & & & \cup & & \cup & & \cup & & \\
 & 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0.
\end{array}
$$

By induction hypothesis we may assume that $M''$ resp. $M'$ is of the form $M_0'' \otimes_{O_{F_s}} O_L$ resp. $M_0' \otimes_{O_{F_s}} O_L$ where

$$M_0'' = \bigoplus (M_0'' \cap V_0''^{\mu}).$$

We write the slope of $V'$ in the form $\mu = r/s$, $s > 0$. Replace the exact sequence (1) by its push-out by $p^r : M' \to M'$. Then, by [RZ] Lemma 2.19 the sequence splits, which gives a decomposition $M = M' \oplus M''$.
It follows that putting $M_0 = M_0' \oplus M_0''$ we have

$$M_0 = \bigoplus (M_0 \cap V_0^{\mu})$$

and

$$M = M_0 \otimes_{F_s} O_L,$$

whence the assertion. $\square$

**1.7.** For the proof of the theorem we need a general fact on Bruhat-Tits buildings. Let $(\mathcal{B}, d)$ be the complete metric space given by the Bruhat-Tits building of a connected reductive group over a local field (= discretely valued field with perfect residue field). Then for $x, y \in \mathcal{B}$ there is a unique point $m \in \mathcal{B}$ satisfying $d(x,m) = d(y,m) = \frac{1}{2} d(x,y)$, the midpoint of the unique geodesic $[x,y]$ between $x$ and $y$. If $z \in \mathcal{B}$ is yet another point there is the inequality

$$(2) \qquad d(x,z)^2 + d(y,z)^2 \geq 2 \cdot d(m,z)^2 + \frac{1}{2} d(x,y)^2.$$

The following lemma seems to be well-known to the specialists. We refer to [R1] for a proof.

**Lemma 1.8.** *Let $\emptyset \neq C \subset \mathcal{B}$ be a convex closed subset.*

(i) *For every $z \in \mathcal{B}$ there exists a unique point $z_0 \in C$ with minimal distance. Let*

$$\pi = \pi_C : \mathcal{B} \longrightarrow C$$

*be the corresponding projection map.*

(ii) *We have*

$$d(\pi(x),\ \pi(y)) \leq d(x,y), \quad x, y \in \mathcal{B}.$$

*In particular, $\pi$ is continuous.*

**1.9.** We wish to reduce Theorem 1.4 to Proposition 1.6. Obviously the assertion of Theorem 1.4 only depends on the metric $d$ on $\mathcal{B}(G, L)$ 'in the large', i.e. its large scale structure. The correct language to express this is due to Bernstein [B]. We recall some concepts from [B]. A *semimetric space* is a set $M$ with a distance function $d(x,y)$ such that

$$d(x,y) = d(y,x), \quad d(x,x) = 0 \quad \text{for } x, y \in M$$

and satisfying the triangular inequality. Two distance functions $d_1$ and $d_2$ on the same set $M$ are called *equivalent* if there exists $C > 0$ such that

$$C^{-1}(d_1 + 1) \leq (d_2 + 1) \leq C(d_1 + 1).$$

A *large scale space* is a set with an equivalence class of distance functions. A *large scale map* between two large scale spaces $M$ and $N$ is a map $f : M \to N$ such that for some $C > 0$

$$d(f(x), f(y)) \leq C(d(x,y) + 1), \quad x, y \in M.$$

Two such maps $f_1, f_2$ are called *equivalent* ($f_1 \sim f_2$) if the distance $d(f_1(x), f_2(x))$ is bounded for all $x \in M$. A large scale map $f : M \to N$ is called a *large scale equivalence* if there exists a large scale map $h : N \to M$ such that $f \circ h \sim \mathrm{id}_N$, $h \circ f \sim \mathrm{id}_M$.

**1.10.** Let $\mathcal{B}$ be the Bruhat-Tits building of a connected reductive group $G$ over a local field $K$. As an example of the preceding concepts the reader checks that when $G$ is semi-simple the semimetric defined by the combinatorial distance $d^c$ between points is large-scale equivalent to the metric $d$,

$$d^c(x,y) = \text{length of a minimal gallery joining } x \text{ and } y.$$

This remark will not be used in the sequel.

Let $G = GL(V)$ where $V$ is a finite-dimensional vector space over the local field $K$. Then the Bruhat-Tits building of $G$ may be identified with the space of norms on $V$, i.e. maps $\alpha : V \to \mathbf{R} \cup \{\infty\}$ such that

$$\begin{aligned} \alpha(\lambda v) &= \alpha(v) + \operatorname{ord}(\lambda), \quad \lambda \in K \\ \alpha(v + v') &\geq \inf(\alpha(v), \alpha(v')), \\ \alpha(v) &= \infty \Leftrightarrow v = 0. \end{aligned}$$

Let $\mathrm{Latt}(V)$ be the set of lattices in $V$. Then any $M \in \mathrm{Latt}\, V$ defines a norm $\alpha_M$ by the rule

$$\alpha_M(v) = \inf\{\nu \in \mathbf{Z};\ \pi^\nu \cdot v \in M\}.$$

Here $\pi$ denotes a uniformizer in $K$. On $\mathrm{Latt}(V)$ there is the following semimetric which we used in Proposition 1.6,

$$d^{\mathcal{L}}(M,N) = \min\{\nu \in \mathbf{Z};\ \pi^\nu N \subset M \subset \pi^{-\nu} N\}.$$

It is easy to see (by putting two lattices in one apartment) that this semimetric is equivalent to the following one

$$d'^{\mathcal{L}}(M,N) = \lg(M/M \cap N) + \lg(N/M \cap N).$$

**Lemma 1.11.** *Let $G = GL(V)$. The inclusion*

$$f : \mathrm{Latt}(V) \longrightarrow \mathcal{B}(G, K)$$

*is a large scale equivalence.*

**Proof.** We define a map $h$ in the opposite direction by associating to a point of $\mathcal{B}(G, K)$ one of the lattices of closest distance. By putting two points in one apartment one checks easily that $f \circ h \sim \mathrm{id}$ and $h \circ f \sim \mathrm{id}$. □

**Proof of Theorem 1.4 for** $G = GL(V)$, where $V$ is a finite-dimensional vector space. In this case $J_{F_s}$ is the Levi subgroup which is the product of the general linear groups of the various isotypic components of $V \otimes_F F_s$. We note that the $F_s$-form $V \otimes_F F_s$ of $V \otimes_F L$ is the one associated in (1.5) to the $\sigma$-$L$-space $(V \otimes_F L,\ b\sigma)$. Consider the natural embeddings of Bruhat-Tits buildings

$$\begin{array}{ccc} \mathcal{B}(J, L) & \subset & \mathcal{B}(G, L) \\ \cup & & \cup \\ \mathcal{B}(J, F_s) & \subset & \mathcal{B}(G, F_s). \end{array}$$

An element $M \in \mathrm{Latt}(V \otimes_F L)$ lies in $\mathcal{B}(J, L)$ iff $M = \bigoplus\limits_{\mu}(M \cap (V \otimes_F L)^{\mu})$, and similarly for $M_0 \in \mathrm{Latt}(V \otimes_F F_s)$. By (Lemma 1.11) we therefore deduce from Proposition 1.6 that the assertion of Theorem 1.4 holds in this case.

**Proof of Theorem 1.4 in the general case.** We fix a faithful representation over $F$,

$$\varrho : G \longrightarrow G' = GL(V).$$

We denote by $b' = \varrho(b)$ the image of $b$ in $G'(L)$. Then $\nu_{b'} = \varrho \circ \nu_b$ is the slope homomorphism of $b'$ which again satisfies a decency equation $(b'\sigma)^s = s\nu_{b'}(\pi) \cdot \sigma^s$. Let $J'$ be the corresponding form over $F$ of the Levi subgroup $J'_{F_s}$ of $G'_{F_s}$.

According to the theorem of Landvogt [L1] there exists an injective isometric map of Bruhat-Tits buildings

$$\mathcal{B}(G, L) \longrightarrow \mathcal{B}(G', L)$$

which is equivariant with respect to the action of $G(L) \rtimes \langle \sigma \rangle$. Identifying $\mathcal{B}(G, L)$ with its image under this map we obtain a diagram of closed convex subsets of $\mathcal{B}(G', L)$,

$$\begin{array}{ccc} \mathcal{B}(G, L) & \subset & \mathcal{B}(G', L) \\ \cup & & \cup \\ \mathcal{B}(J, L) & \subset & \mathcal{B}(J', L) \\ \cup & & \cup \\ \mathcal{B}(J, F_s) & \subset & \mathcal{B}(J', F_s). \end{array}$$

Fix $x \in \mathcal{B}(G, L)$ as in the statement of Theorem 1.4. By Theorem 1.4 applied to $G', b'$ we find a constant $C$ and a point $x'_0 \in \mathcal{B}(J', F_s)$ such that

$$d(x, x_0') < C.$$

Let $x_0 = \pi_{\mathcal{B}(G,L)}(x_0')$ be the closest point to $x_0'$ in $\mathcal{B}(G, L)$. Since $x \in \mathcal{B}(G, L)$ we have by Lemma 1.8, (ii) that

$$d(x, x_0) \leq d(x, x_0') < C.$$

By the uniqueness of the closest point mapping, $\pi_{\mathcal{B}(G,L)}$ is equivariant. However, $x_0'$ is fixed by $\sigma^s$ and by $s\nu(w)$ where $w \in O_L^\times$ is any unit. Hence

$$x_0 \in \mathcal{B}(G, L)^{\langle \sigma^s, s\nu(O_L^\times)\rangle} = \mathcal{B}(J, F_s).$$

Here the last equality sign combines the theorem of Rousseau ([R], (5.3.2)) asserting that $\mathcal{B}(J, L) = \mathcal{B}(G, L)^{s\nu(O_L^\times)}$ and the theorem of Bruhat and Tits, cf. (1.3). $\square$

## 2. A VARIANT AND A CONJECTURE

**2.1.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $K = k((T))$ be the field of Laurent series in one variable over $k$ and let $\bar{K}$ be an algebraic closure of $K$. Let $F$ be a finite extension of $\mathbf{F}_p((T))$ contained in $\bar{K}$ and let $L$ be the compositum of $K$ and $F$ in $\bar{K}$. Let $\sigma \in \mathrm{Aut}(L/F)$ be the relative Frobenius automorphism. With these notational changes the main theorem holds as stated in Theorem 1.4. The proof is essentially the same.

**2.2.** It is natural to ask for generalizations of our main result. The following conjecture is due to Rousseau (comp. [R1], 4.5). Let $L$ be a local field. Let $\sigma \in \mathrm{Aut}(L)$ be an automorphism of $L$ and put $F = L^{<\sigma>}$. Let $G$ be a reductive group over $F$. Let $J$ be the centralizer of a one-parameter subgroup $\lambda : \mathbf{G}_m \to G$. We obtain a canonical inclusion of buildings,

$$\mathcal{B}(J, F) \longrightarrow \mathcal{B}(G, L).$$

For a point $x \in \mathcal{B}(G, L)$ we denote by $x_0$ its projection onto the convex subset $\mathcal{B}(J, F)$. The question is whether there exists a constant $c > 0$ such that

$$d(x, \lambda(z)\sigma(x))^2 \geq d(x_0, \lambda(z)x_0)^2 + c \cdot d(x, x_0)^2,$$

for varying $z \in L^\times$.

Our hope in raising these questions is that one can give a purely geometric proof of our main result. We refer to [R1] for encouraging results in this direction.

## 3. THE ORIGIN OF THIS PAPER

Let us return to the notation of Section 1. Let $(V, \varphi)$ be a $\sigma$-$L$-space. Let $d = \dim V$. We consider maximal periodic lattice chains $\mathcal{M}$ in $V$,

$$\ldots \subset M_i \subset M_{i+1} \subset \ldots \quad ; \quad M_{i-d} = \pi \cdot M_i .$$

Let us fix an integer $r$ with $0 \leq r \leq d$. We consider the following condition on the lattice chain $\mathcal{M}$.

$(*_r)$ For all $i$ we have

$$\pi M_{i+r} \subset \varphi(M_i) \subset M_{i+r} \quad \text{with } \dim_k M_{i+r}/\varphi(M_i) = r.$$

If $r = 0$ or $r = d$, then if $\mathcal{M}$ satisfies $(*_r)$, then $\varphi(M_i) = M_i$ for all $i$. Hence all slopes of $\varphi$ are equal to zero, the fixed vectors of $\varphi$ in $V$ define an $F$-form $V_0$ of $V$ and all lattices $M_i$ of $\mathcal{M}$ are rational, i.e. of the form

$$M_i = M_{i,0} \otimes_{O_F} O_L$$

with $M_{i,0} = M_i \cap V_0$.

Assume now that $r = 1$ or $r = d - 1$. Then it is a striking observation of Drinfeld [D] that there exists at least one $i$ with $\varphi(M_i) = M_i$. Indeed, the inclusions $M_j \subset M_{j+1}$ induce a chain of morphisms of 1-dimensional vector spaces over $k$:

$$M_j/\varphi(M_{j-1}) \to M_{j+1}/\varphi(M_j) \to \ldots \to \pi^{-1}M_j/\varphi(M_{j+d-1}).$$

The composition of the arrows in this chain is zero because of the condition $\pi M_{j+d} \subset \varphi(M_{j+d-1})$. Hence one of the morphisms in the chain must be zero, which implies the existence of an index $i$ such that $\varphi(M_i) = M_i$. Such indices are called *critical*. Hence in this case again all slopes of $\varphi$ are equal to zero and the lattices $M_i$ for critical indices $i$ are rational.

The question which arises in this context is whether similar statements hold for arbitrary $r$. It turns out that this is not the case: there are examples of periodic lattice chains $\mathcal{M}$ which satisfy $(*_r)$ but where not all slopes of $\varphi$ are zero. Here is an example where $d = 4$, $r = 2$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \sigma.$$

The slope vector of $\varphi$ is $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. The following maximal periodic lattice chain $\mathcal{M}$ satisfies$(*_2)$ (we use the customary notation for lattices in the standard apartment).

$$\begin{array}{llllllll} \mathcal{M}_\bullet : & \ldots & (1111) & (1110) & (1100) & (1000) & (0000) & \ldots \\ \varphi\mathcal{M}_{\bullet-2} : & \ldots & (2121) & (2120) & (2110) & (2010) & (1010) & \ldots \end{array}$$

There still remains the question whether, if all slopes of $\varphi$ are equal to zero, and $\mathcal{M}$ satisfies $(*_r)$, then there exists $i$ with $\varphi(M_i) = M_i$. We would expect that such an index $i$ does not exist in general. But the construction of an example seems to be more difficult.

The qualitative sense of Drinfeld's observation is that since $\varphi$ moves very little the point in the Bruhat-Tits building corresponding to a lattice $M_i$ in $\mathcal{M}$, then this point is very close to a rational point. Our main result is the corresponding statement on general buildings.

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