Appendix A. Accessible and weakly accessible period domains[1]

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0.1. **Introduction.** The goal of this appendix is to investigate in which situations the period maps from RZ spaces towards partial flag varieties are surjective. This question can be posed in two variants: One can either ask that the map is surjective on classical points, or surjective on *all* (adic, or equivalently, Berkovich) points. These questions can be translated into the question whether the weakly admissible, resp. admissible, locus inside the partial flag variety is the whole partial flag variety. We answer both of these questions below. It turns out that asking surjectivity for all points is significantly more restrictive, and occurs essentially only in the Lubin-Tate case.

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0.2. **Recollections on period domains.** Let $(G, b, \{\mu\})$ be a *PD-triple*[2] over the p-adic field F. This means that G is a reductive algebraic group over F, that $b \in G(\breve{F})$, and that $\{\mu\}$ is a conjugacy class of cocharacters of G. We will assume throughout that $\{\mu\}$ is minuscule. Two PD-triples $(G, b, \{\mu\})$ and $(G', b', \{\mu'\})$ are called equivalent if there is an isomorphism $G \simeq G'$ which takes $\{\mu\}$ into $\{\mu'\}$ and b into a σ-conjugate of b'. All concepts below depend only on the equivalence class of PD-triples. Let $E = E(G, \{\mu\})$ be the corresponding reflex field. We denote by $\mathcal{F}(G, \{\mu\})$ the corresponding partial flag variety defined over E, and by $\breve{\mathcal{F}}(G, \{\mu\})$ its base change to $\breve{E}$. We denote by $\mathcal{F}(G, \{\mu\})^{\mathrm{wa}}$ the *period domain* associated to the PD-triple $(G, b, \{\mu\})$, i.e., the *weakly admissible* subset of $\breve{\mathcal{F}}(G, \{\mu\})$, which we consider as an open adic subset. It is defined by the weak admissibility condition of Fontaine on the Lie algebra of G (*semi-stability*, cf. [4, Def. 9.2.14]) and the triviality of the *degree* in $\pi_1(G)_{\Gamma,\mathbb{Q}}$.

Definition 0.1. *A PD-triple $(G, b, \{\mu\})$ is weakly accessible if $\mathcal{F}(G, b, \{\mu\})^{\mathrm{wa}} = \breve{\mathcal{F}}(G, \{\mu\})$, i.e., the period domain associated to $(G, b, \{\mu\})$ is the whole partial flag variety.*

0.3. **The admissible set.** Let X_F be the Fargues-Fontaine curve relative to F (and some fixed algebraically closed perfectoid field of characteristic p). By Fargues, [5], there is a bijection

$$B(G) \to \{G\text{-bundles on } X_F\}/\simeq, \quad b \mapsto \mathcal{E}_b. \tag{1}$$

Restricted to basic elements, this yields even an equivalence of groupoids,

$$G(\breve{F})_{\mathrm{basic}} \to \{\text{semi-stable } G\text{-bundles on } X_F\}.$$

Here the LHS becomes a groupoid via the action by σ-conjugacy of $G(\breve{F})$. Also, a G-bundle $\mathcal{E}$ is called *semi-stable* if for all $\rho \in \mathrm{Rep}_G$ mapping the center of G into the center of GL_n, the vector bundle $\rho_*(\mathcal{E})$ on X_F is semi-stable in the sense of Mumford (recall that deg and rank are well-defined for vector bundles on X_F). It is enough to check this for ρ the adjoint representation of G.

Definition 0.2. *Fix a PD-triple $(G, b, \{\mu\})$ over F. Let C be an algebraically closed non-archimedean field extension of $\breve{F}$, and use the tilt $C^\flat$ of C to build X_F; denote by $\infty \in X_F(C)$ the corresponding distinguished point of X_F.*

[1]Appendix to: P. Scholze, *On the p-adic cohomology of the Lubin-Tate tower.*

[2]In [4, Ex. 9.1.22], to (G, b) is associated an *augmented affine group scheme* $\mathbb{G}$ over the category of F-isocrystals, and in [4, Def. 9.5.1] one considers the *PD-pair* associated to $(\mathbb{G}, \{\mu\})$, rather than the triple $(G, b, \{\mu\})$.

To any point $x \in \mathcal{F}(G,\{\mu\})(C)$, there is associated a G-bundle $\mathcal{E}_{b,x}$ on X_F which is called the modification of $\mathcal{E}_b$ at ∞ along x.

Remark 0.3. If $\mathcal{E}$ is a vector bundle of rank n on X_F, and $\{\mu\}$ is a minuscule cocharacter class of GL_n, then it is clear how to define the modification $\mathcal{E}_x$ for $x \in \mathcal{F}(\mathrm{GL}_n,\{\mu\})(C)$. On the other hand, for non-minuscule $\{\mu\}$, or general G (and then even for minuscule cocharacters), it is nontrivial to define the modification $\mathcal{E}_{b,x}$. Indeed, the definition involves the B_{dR}-Grassmannian $\mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$. One uses the *Bialynicki-Birula morphism*, valid for any $\{\mu\}$,

$$\mathrm{Gr}_{G,\{\mu\}}^{B_{\mathrm{dR}}^+} \to \breve{\mathcal{F}}(G,\{\mu\}),$$

which is an isomorphism if $\{\mu\}$ is minuscule. We refer to [2] for a precise discussion of this point. We note however that on points defined over a finite extension of $\breve{F}$, the Bialynicki-Birula morphism is a bijection (for all $\{\mu\}$).

Definition 0.4. *A point $x \in \mathcal{F}(G,\{\mu\})(C)$ is called admissible with respect to b if the associated G-bundle $\mathcal{E}_{b,x}$ is semi-stable. Equivalently, the image of $\mathcal{E}_{b,x}$ under the map in Corollary 0.10 is the unique basic class $[b^*]$ with $\kappa([b^*]) = \kappa([b]) - \mu^\natural$.*

Remarks 0.5. (i) An admissible point $x \in \mathcal{F}(G,\{\mu\})(C)$ is automatically weakly admissible. If x is defined over a finite extension of $\breve{F}$, the converse is true. For points defined over finite extensions of $\breve{F}$, these assertions can be reduced to the case of GL_n by using the adjoint representation, for which see [3]. Now the admissible locus is an open subset of $\mathcal{F}(G,\{\mu\})$ (cf. below) which on classical points agrees with the weakly admissible locus. As the weakly admissible locus is maximal among open subsets with given classical points, it follows that the admissible locus is contained in the weakly admissible locus.

(ii) Assume that $(G,\{\mu\}) \subset (\mathrm{GL}_n, \{\mu_{(1^{(r)},0^{(n-r)})}\})$, i.e., the PD-triple $(G,b,\{\mu\})$ is of Hodge type. Then Faltings and Hartl have defined the notion of admissibility of a point in $\mathcal{F}(G,\{\mu\})(C)$, cf. [4, ch. XI, §4] (Faltings' definition uses base change to $B_{\mathrm{cris}}(C)$; Hartl's definition uses the Robba ring $\tilde{B}_{\mathrm{rig}}^\dagger(C)$; Hartl has shown that these definitions coincide, comp. [4, Thm. 11.4.11]). The definition of admissibility above specializes in this case to their definition.

Definition 0.6. *Fix a PD-triple $(G,b,\{\mu\})$ over F. The admissible locus $\mathcal{F}(G,b,\{\mu\})^{\mathrm{a}}$ is the unique open adic subset of $\breve{\mathcal{F}}(G,\{\mu\})$ whose C-valued points are the admissible points of $\mathcal{F}(G,\{\mu\})(C)$, for any algebraically closed non-archimedean field extension of $\breve{F}$.*

It follows from [9] that the admissible set is indeed an open adic subset of $\breve{\mathcal{F}}(G,\{\mu\})$, again using the adjoint representation of G to reduce to the case $G = \mathrm{GL}_n$.

Remarks 0.7. Whereas we have a fairly accurate picture of what the *weakly admissible locus* looks like (and one of the main attractions of the corresponding theory is to determine explicitly this locus in specific cases, cf. [12, Ch. I]), the *admissible locus* seems quite amorphous, and is explicitly known in only very few cases. Here are two examples.

(i) Let $(G,b,\{\mu\}) = (\mathrm{GL}_n, b, \{\mu_{(1^{(1)},0^{(n-1)})}\})$, where $[b]$ is the unique basic element of $B(G,\{\mu\})$. This case is called the *Lubin-Tate case*. In this case, all points of $\breve{\mathcal{F}}(G,\{\mu\})$ are admissible. This follows by Gross/Hopkins [8] from Theorem 0.17 below. Another, more direct, proof is due to Hartl, comp. [4, Prop. 11.4.14]. The same holds for $(\mathrm{GL}_n, b, \{\mu_{(1^{(n-1)},0^{(1)})}\})$, where again $[b]$ is the unique basic element of $B(G,\{\mu\})$.

(ii) Let $(G,b,\{\mu\}) = (D_{\frac{1}{n}}, b, \{\mu_{(1^{(1)},0^{(n-1)})}\})$, where $[b]$ is the unique basic element of $B(G,\{\mu\})$. This case is called the *Drinfeld case*. In this case, all weakly admissible points of $\breve{\mathcal{F}}(G,\{\mu\})$ are admissible. They form the Drinfeld halfspace inside $\mathbb{P}^{n-1}$. This follows by Faltings' theorem [12, ch. 5] from Theorem 0.17 below, but has also been shown by Hartl, comp. [4, Prop. 11.4.14]. The same holds for $(D_{-\frac{1}{n}}, b, \{\mu_{(1^{(n-1)},0^{(1)})}\})$, where again $[b]$ is the unique basic element of $B(G,\{\mu\})$.

Definition 0.8. *A PD-triple $(G,b,\{\mu\})$ is accessible if $\mathcal{F}(G,b,\{\mu\})^{\mathrm{a}} = \breve{\mathcal{F}}(G,b,\{\mu\})$, i.e., the admissible set associated to $(G,b,\{\mu\})$ is the whole partial flag variety.*

From Remarks 0.5, (i) it follows that an accessible PD-triple is weakly accessible.

Proposition 0.9. *Associating to a G-bundle its isomorphism class, we obtain from* (1) *a bijection*

$$\big\{ \text{ iso-classes of } G\text{-bundles of the form } \mathcal{E}_{1,x} \mid x \in \mathcal{F}(G,\{\mu^{-1}\})\big\} \to B(G,\{\mu\}).$$

Proof. Let $b \in G(\breve{F})$. If $[b]$ lies in the image of the map, it follows from the construction of $\mathcal{E}_{1,x}$ that $\kappa([b]) = \mu^\natural$ in $\pi_1(G)_\Gamma$. Now b represents an element of the image of the map if and only if $\mathcal{E}_b$ is of the form $\mathcal{E}_{1,x}$; equivalently, if and only if $\mathcal{E}_{b,x^*}$ is the trivial G-bundle for some $x^* \in \mathcal{F}(G,\{\mu\})$. In other words, this holds if and only if there exists x^* such that $\mathcal{E}_{b,x^*}$ is a semi-stable G-bundle. Hence this is equivalent to $\mathcal{F}(G,b,\{\mu\})^{\mathrm{a}} \neq \emptyset$. This in turn is equivalent to the condition that $\mathcal{F}(G,b,\{\mu\})^{\mathrm{wa}} \neq \emptyset$, as these are two open sets with the same classical points. By [4, Thm. 9.5.10] this is equivalent to $[b] \in A(G,\{\mu\})$. Since we saw already the equality $\kappa(b) = \mu^\natural$, this is equivalent to $[b] \in B(G,\{\mu\})$. □

Corollary 0.10. *Let $b \in G(\breve{F})$ be basic. Then there is a bijection*

$$\big\{ \text{ iso-classes of } G\text{-bundles of the form } \mathcal{E}_{b,x} \mid x \in \mathcal{F}(G,\{\mu^{-1}\})\big\} \to B(J_b,\{\mu\}+\nu_b).$$

Here ν_b is the central cocharacter associated to the basic element b.

Proof. This follows by translation with b from the previous proposition, cf. [10, 4.18]. Alternatively, one can apply the functor $\mathcal{H}om(\mathcal{E}_b, \ )$ to the assertion of the corollary, to reduce to the previous proposition. □

0.4. **Weakly accessible PD-triples.** Our first aim is to determine all weakly accessible PD-Pairs. The following lemma reduces this problem to the core cases. We always make the assumption that the period domain associated to any PD-triple considered below is non-empty.

Lemma 0.11. (i) *$(G,b,\{\mu\})$ is weakly accessible if and only if $(G_{\mathrm{ad}}, b_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is weakly accessible.*

(ii) *$\big(G_1 \times G_2, (b_1,b_2), \{(\mu_1,\mu_2\}\big)$ is weakly accessible if and only if $(G_1,b_1,\{\mu_1\})$ and $(G_2,b_2,\{\mu_2\})$ are both weakly accessible.*

(iii) *If $\{\mu\}$ is central, then $(G,b,\{\mu\})$ is weakly accessible.*

Proof. (i) Let $\pi : \breve{\mathcal{F}}(G,\{\mu\}) \to \breve{\mathcal{F}}(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\})$ denote the natural morphism. Then the assertion follows from

$$\mathcal{F}(G,\{\mu\})^{\mathrm{wa}} = \pi^{-1}\big(\mathcal{F}(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\})^{\mathrm{wa}}\big)$$

(recall that we are assuming both period domains to be non-empty).

Finally, (ii) and (iii) are obvious. □

After the previous reduction steps, the following proposition gives the complete classification of all weakly accessible PD-triples.

Proposition 0.12. *Let $(G,b,\{\mu\})$ be a PD-triple defining a non-empty period domain, where G is F-simple adjoint and $\{\mu\}$ is non-trivial. Then the PD-triple $(G,b,\{\mu\})$ is weakly accessible if and only if the F-group J_b is anisotropic, in which case $[b]$ is basic.*

Proof. We note that, G being of adjoint type, weak admissibility is equivalent to semi-stability in the sense of [4], i.e, $\mathcal{F}(G,b,\{\mu\})^{\mathrm{wa}} = \mathcal{F}(G,b,\{\mu\})^{\mathrm{ss}}$, cf. [4, top of p.272]. We also note that the last sentence follows because if J is anisotropic, then b is basic. Indeed, if b is not basic, then the slope vector ν_b is a non-trivial cocharacter of J defined over F, cf. [10, after (3.4.1)].

Assume that there exists a point $x \in \mathcal{F}(G,\{\mu\}) \setminus \mathcal{F}(G,b,\{\mu\})^{\mathrm{ss}}$. Then, applying [4, Thm. 9.7.3], we obtain a 1-PS λ of J_b defined over F which violates the Hilbert-Mumford inequality. In particular, λ is non-trivial, and J_b is not anisotropic.

Conversely, assume that $\mathcal{F}(G,b,\{\mu\})^{\mathrm{ss}} = \mathcal{F}(G,\{\mu\})$. We claim that then J_b is anisotropic. To prove this, we may change b within its σ-conjugacy class $[b]$, since this leaves the isomorphism class of J_b unchanged. We argue by contradiction. So, let us assume that T is a maximal torus of J_b such that $X_*(T)^\Gamma \neq (0)$. Here $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Then $T \otimes_F \breve{F}$ is also a maximal torus of $G \otimes_F \breve{F}$. By assumption, for any $\mu \in X_*(T)$ defining an element $x \in \mathcal{F}(G,\{\mu\})$, the pair $(b, \mathcal{F}_x)$ is semi-stable. To apply the Hilbert-Mumford inequality, we fix an invariant inner product $(\ ,\ )$ on G, cf. [4, Def. 6.2.1]. Hence by the Hilbert-Mumford inequality [4, Thm. 9.7.3], we obtain

$$(\lambda, \mu - \nu_b) \geq 0, \quad \forall \lambda \in X_*(T)^\Gamma,$$

where $\nu_b \in X_*(T)_{\mathbb{Q}}$ denotes the slope vector of b. Indeed, the LHS is equal to $\mu^{\mathcal{L}}(x,\lambda)$, by [4, Lemma 11.1.3] (in loc. cit., the situation over a finite field is considered; but the lemma holds in the present situation *mutatis mutandum*). Replacing λ by its negative, we see that $(\lambda, \mu - \nu_b) = 0$. Hence (λ, μ) is independent of $\mu \in X_*(T)$ in its geometric conjugacy class. It follows that for any w, w' in the geometric Weyl group W of T in G,

$$(\lambda, w\mu - w'\mu) = 0. \tag{2}$$

We wish to show that this implies that $\lambda = 0$, which would yield the desired contradiction. We write $G = \mathrm{Res}_{F'/F}(G')$, where G' is an absolutely simple adjoint group over the extension field F' of F. Let F'_0 be the maximal unramified subextension of F'/F. Then

$$G(\breve{F}) = \prod_{i \in \mathbb{Z}/f\mathbb{Z}} G'(\breve{F}'), \tag{3}$$

where $\mathbb{Z}/f\mathbb{Z}$ denotes the Galois group of F'_0/F, and where $\breve{F}$, resp. $\breve{F}'$, denotes the completion of the maximal unramified extension of F, resp. F'. Furthermore, it is easy to see that any $b \in G(\breve{F})$ is σ-conjugate to an element in the product on the RHS of (3) of the form $(b'_0, 1, \ldots, 1)$, and that then

$$J_b = \mathrm{Res}_{F'/F} J'_{b'_0}.$$

Correspondingly, $T = \mathrm{Res}_{F'/F}(T')$, where T' is a maximal torus of $J'_{b'_0}$ defined over F'. Hence

$$X_*(T)_{\mathbb{Q}} = \prod_{\tau \in \mathrm{Hom}_F(F', \bar{F})} X_*(T')_{\mathbb{Q}}, \tag{4}$$

with its action by Γ induced by the action of $\Gamma' = \mathrm{Gal}(\bar{F}/F')$ on $X_*(T')_{\mathbb{Q}}$. Since $0 \neq \lambda \in X_*(T)^\Gamma$, all components λ_τ of λ in the product decomposition (4) are non-zero, and are determined by any one of them. Now $T' \otimes_{F'} \breve{F}'$ is a maximal torus of $G' \otimes_{F'} \breve{F}'$ and, since G' is absolutely simple, its geometric Weyl group W' acts irreducibly on $X_*(T')_{\mathbb{Q}}$, cf. [1, Cor. of Prop. 5 in VI, §1.2]. Furthermore, the geometric Weyl group of T is the product of copies of W' over the same index set as in (4). Hence the identity (2) implies that any time the component μ_τ of μ is non-trivial, the component λ_τ is zero. Hence the assumption $\lambda \in X_*(T)^\Gamma$ implies $\lambda = 0$, since the assumption $\mu \neq 0$ implies that $\mu_\tau \neq 0$ for some τ. This yields the desired contradiction. $\square$

Corollary 0.13. *In Proposition 0.12, assume that G is absolutely simple adjoint and that $\{\mu\}$ is non-trivial. Then $(G, b, \{\mu\})$ satisfies the condition of Proposition 0.12 if and only if G is the algebraic group associated to a simple central algebra D of some rank n^2 over F, $[b]$ is basic, and the difference between the Hasse invariant of D in $\mathbb{Z}/n\mathbb{Z} \simeq \pi_1(G)_\Gamma$ and the class $\kappa([b])$ lies in $(\mathbb{Z}/n)^\times$.* □

Remark 0.14. Note that the class $\{\mu\}$ does not intervene in Proposition 0.12. It does, however, enter in the condition that the period domain $\mathcal{F}(G, b, \{\mu\})^{\mathrm{wa}}$ be non-empty. Indeed, this condition is equivalent to the condition that $[b] \in A(G, \{\mu\})$, cf. [4, Thm. 9.5.10], i.e., that $[b]$ be acceptable with respect to $\{\mu\}$ in the sense of [11].

0.5. **Accessible PD-triples.** Here the classification is much more narrow.

Proposition 0.15. *A PD-triple $(G, b, \{\mu\})$ is accessible if and only if b is basic, and the pair $(J_b, \{\mu\})$ is uniform in the sense of* [10, §6]*, i.e. $B(J_b, \{\mu\})$ contains precisely one element.*

Proof. The accessibility of $(G, b, \{\mu\})$ implies its weak accessibility, cf. Remark 0.5, (i); hence b is basic by Proposition 0.12. The assumption that $(G, b, \{\mu\})$ is accessible is equivalent to saying that any modification $\mathcal{E}_{b,x}$ for $x \in \mathcal{F}(G, \{\mu\})$ is semi-stable. Hence, by Corollary 0.10, the set $B(J_b, \{\mu^{-1}\} + \nu_b)$ contains only one element, i.e., $(J_b, \{\mu^{-1}\} + \nu_b)$ is uniform. The assertion follows since $(J_b, \{\mu^{-1}\} + \nu_b)$ is uniform if and only if $(J_b, \{\mu^{-1}\})$ is uniform, if and only if $(J_b, \{\mu\})$ is uniform. □

Kottwitz [10, §6] has given a complete classification of uniform pairs $(G, \{\mu\})$. Applying his result, we obtain the following corollary.

Corollary 0.16. *Let $(G, b, \{\mu\})$ be a PD-triple. Assume that G is absolutely simple adjoint, that $\{\mu\}$ is non-trivial, and that $[b] \in B(G, \{\mu\})$. Then $(G, b, \{\mu\})$ is accessible if and only if $G \simeq \mathrm{PGL}_n$, and $\{\mu\}$ corresponds to $(1, 0, \ldots, 0)$ or $(1, 1, \ldots, 1, 0)$.* □

0.6. **An application to the crystalline period map.** Let $(G, b, \{\mu\})$ be a *local Shimura datum* over F, cf. [11], i.e., a PD-triple such that $\{\mu\}$ is *minuscule* and such that $[b] \in B(G, \{\mu\})$. Conjecturally, there is an associated local Shimura variety, i.e., a tower of rigid-analytic spaces over $\breve{E}$, with members enumerated by the open compact subgroups of $G(\mathbb{Q}_p)$,

$$\{\mathbb{M}_K\}_K = \{\mathbb{M}(G, b, \{\mu\})_K\}_K, \tag{5}$$

on which $G(\mathbb{Q}_p)$ acts as Hecke correspondences. The tower comes with a compatible system of morphisms

$$\phi_K \colon \mathbb{M}_K \to \breve{\mathcal{F}}(G, \{\mu\}). \tag{6}$$

The morphism ϕ_K is called the *crystalline period morphism* at level K of the local Shimura variety attached to $(G, b, \{\mu\})$.

Theorem 0.17. *Assume that the local Shimura variety associated to $(G, b, \{\mu\})$ comes from an RZ-space of type EL or PEL, in which case the local Shimura variety exists. Then the image of the crystalline period morphisms coincides with the admissible locus $\mathcal{F}(G, b, \{\mu\})^{\mathrm{a}}$.*

Proof. See [7] (which uses [6]) and [13]. □

Example 0.18. (i) In the Lubin-Tate case (see Remarks 0.7, (i)), Gross and Hopkins [8] have shown that the image of the crystalline period morphism is the whole projective space $\mathcal{F}(G, \{\mu\})$.

(ii) In the Drinfeld case (see Remarks 0.7, (ii)), the image of the crystalline period map is the Drinfeld half-space, cf. [12, ch. 5].

Corollary 0.19. *Assume that the local Shimura variety associated to $(G, b, \{\mu\})$ comes from an RZ-space of type EL or PEL, in which case the local Shimura variety exists. Also, assume that G is absolutely simple. Then the crystalline period morphisms are surjective if and only if the local Shimura variety is of Lubin-Tate type.* □

0.7. **Open questions.** Here we list some open questions.

Question 0.20. When is $\mathcal{F}(G, b, \{\mu\})^{\mathrm{a}} = \mathcal{F}(G, b, \{\mu\})^{\mathrm{wa}}$?

This question was answered by Hartl in the case when $G = \mathrm{GL}_n$. Besides the Lubin-Tate case and the Drinfeld case, there is one essentially new case related to GL_4. B. Gross asks whether the PD-triples formed by an adjoint orthogonal group G, its natural minuscule coweight $\{\mu\}$ (the one attached to a Shimura variety for $\mathrm{SO}(n-2, 2)$) and the unique basic element in $B(G, \{\mu\})$ give further examples.

For the next question, recall that for any standard parabolic P^* in the quasi-split form G^* of G, there is a subset $B(G)_{P^*}$ defined in terms of the Newton map on $B(G)$. If $P^* = G^*$, then $B(G)_{G^*} = B(G)_{\mathrm{basic}}$. We call the inverse image of $B(G)_{P^*}$ under the map in Corollary 0.10 the HN-stratum $\mathcal{F}(G, b, \{\mu\})_{P^*}$ attached to P^*. Hence for $P^* = G^*$ the corresponding HN-stratum is the admissible set.

Question 0.21. For which P^* is the HN-stratum non-empty? Does the decomposition into disjoint sets $\mathcal{F}(G, b, \{\mu\})_{P^*}$ of $\breve{\mathcal{F}}(G, \{\mu\})$ have the stratification property? Which strata $\mathcal{F}(G, b, \{\mu\})_{P^*}$ have classical points?

The first question is non-empty, as is shown by the Lubin-Tate case, in which only $\mathcal{F}(G, b, \{\mu\})_{G^*}$ is non-empty. There are examples of strata $\mathcal{F}(G, b, \{\mu\})_{P^*}$ without classical points: One gets these by looking at cases of weakly accessible, but non-accessible, PD-triples, in which case all strata with $P^* \neq G^*$ have no classical points, but some of them are nonempty.

There is also a HN-decomposition of $\breve{\mathcal{F}}(G, \{\mu\})$ in the sense of [4]. It does not have the stratification property. Here we have an understanding of the structure of the individual strata, in terms of period domains of PD-triples of smaller dimension. However, even for these simpler strata, the question of the non-emptiness of strata is only partially solved (by Orlik).

Question 0.22. What is the relation between the two stratifications?

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